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Dynamic Response of Laminated Composite Plates Using a Three-Dimensional Hybrid-Stress Finite-Element Formulation

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In this paper a method of analysis of dynamic response of laminated composite plates is presented. The analysis is carried by using a hybrid-stress finite-element numerical technique established by the authors in their earlier publication. By using this approach the response of simply supported laminated plates subject to sinusoidal loading are investigated. For the solution of the finite-element equations of motion of free vibrations and dynamic response problems, two effective methods of solution, the space iteration method and the Newmark direct integration method are used. These two methods are discussed in this paper.

INTRODUCTION

Since Pian [1] first established the assumed stress hybrid finite element model and derived the corresponding element stiffness matrix in 1964, the hybrid stress model has been shown highly accurate, and easy to fulfill the compatibility condition of the finite element method. Laminated thick plate element has been developed by Mau et al. [2] by using hybrid stress method. In the comparison of results with elasticity solution [3,4], they observed excellent accuracy in predicting both displacements and stresses. In their assumption for the stress field, transverse normal stress was not included. Constant transverse displacement through the laminate thickness was also assumed. These assumptions did not agree well with the actual mechanism of deformation of laminated plates in bending. Spilker [5] developed an eight-node isoparametric multilayer plate element for the analysis of thin to thick fiber-reinforced composite plates. This model has the generality in describing laminate response and can be easily used to implement to attack complex laminated plate problems, but the assumption of constant transverse displacement through laminate thickness still remains.

The hybrid stress model is based on the modified complementary energy principle. An optimum choice of the number of the assumed stress modes for given boundary displacement approximation can be made, which give greater flexibility in the descriptions of the stress field. The detail of the development of this method is documented in [6].

In the present investigation, a three-dimensional eight-node hybrid stress element has been developed to analyze free vibrations of laminated plates. All six stress components are included and assumed independently within each layer through stress polynomials with 55 unknown stress parameters. The stress field within each layer satisfies the dynamic equilibrium equations of free vibration. The interface traction continuity and laminate upper/lower surface traction-free conditions are also enforced. The displacement field is interpolated in terms of

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nodal displacements through shape functions. The displacements (u,v,w) are assumed to vary linearly through the thickness of each lamina.

To solve the governing finite element equations of motion for a linear dynamic analysis without damping the well-known Newmark direct integration method [7] will be used to integrate the following equation

$$[M]\{\ddot{q}\} + [k]\{q\} = [Q]$$

step by step.

The dynamic response of simply supported laminated composite plates under a dynamic sinusoidally distributed load

$$Q = Q_o \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} H(t)$$

is analyzed. Numerical results of [90/0] antisymmetric cross-ply laminate and [0/90/0] symmetric cross-ply laminate are presented. Center deflection, bending stresses σ_x, σ_y, transverse shear stresses τ_{xz} and τ_{yz} and normal stress σ_z for both laminates are plotted as a function of time. Fast convergence is observed .

SUBSPACE ITERATION METHOD

the free vibration finite element equations of motion with damping neglected are

$$M\ddot{q} + K q = 0 \tag{1}$$

where K is the structure stiffness matrix and M is the structure mass matrix. Equation (1) can be solved by expressing the field variable as

$$q = \phi e^{i\omega t} \tag{2}$$

where ϕ is a nodal vector of order n, t the time variable, and ω the natural frequency of vibration of the plate in the mode described by the vector ϕ. Substituting equation (2) into Equation (1) yields the generalized eigenvalue problem

$$K\phi - \omega^2 M \phi = 0 \tag{3}$$

from which ϕ and ω can be determined. For matrices of dimension n x n, there will be n eigensolutions (ω_1^2, ϕ_1), (ω_2^2, ϕ_2),......., (ω_n^2, ϕ_n), An important property of the eigenvectors is that they satisfy the orthogonality conditions, i.e.

$$\begin{aligned} \phi_i^T M \phi_j &= \delta_{ij} \\ \phi_i^T K \phi_j &= \omega_i^2 \delta_{ij} \end{aligned} \tag{4}$$

and $0 < \omega_1^2 < \omega_2^2 \, ... < \omega_n^2$

There are many different techniques existing for the solution of eigenvalue problems. Since the procedures for the eigenvalues problems are time consuming, the choice of an appropriate and effective method is an important factor for the general application, especially in the large eigenvalue problem. The subspace iteration method suggested by Bathe [8] will be adopted to conduct the investigation. This method has been used extensively in a number of generalpurpose finite element analysis programs and has proven cost-effective and reliable. In structure analysis, the lowest few eigenvalues (natural frequencies) are the main concern of investigators. The basic objective in the subspace iteration method is to solve for the p smallest eigenvalues and corresponding eigenvectors, which satisfy

$$K \phi = M \phi \Lambda \tag{5}$$

where $\phi = [\phi_1, \phi_2, \ldots\ldots\ldots, \phi_p]$

and Λ is a diagonal matrix of ω_i^2 and the eigenvector ϕ_i also satisfies the orthogonality conditions (Equation (4)).

The subspace iteration method consists of three steps [8]:

1. Establish q starting iteration vectors; $q > p$, $q = \min(2p, p+8)$ is a proper selection, where p is the number of eigenvalues and eigenvectors to be calculated.
2. Use simultaneous inverse iteration on the q vectors and Ritz analysis to extract the "best" eigenvalue and eigenvector approximations from the q iteration vectors.

For $k = 1, 2, \ldots.,$

$$K \bar{X}_{k+1} = M X_k \tag{6}$$

where X_1 is the starting iteration vector.

Find the projections of the operators K and M,

$$K_{k+1} = \bar{X}_{k+1}^T K \bar{X}_{k+1}$$

$$M_{k+1} = \bar{X}_{k+1}^T M \bar{X}_{k+1} \tag{7}$$

Solve for the eigensystem of the projected operators,

$$K_{k+1} Q_{k+1} = M_{k+1} Q_{k+1} \Lambda_{k+1} \tag{8}$$

Find an improved approximation to the eigenvectors,

$$K_{k+1} = \bar{X}_{k+1} Q_{k+1} \tag{9}$$

As $k \to \phi$

$$\Lambda_{k+1} \to \Lambda \text{ and } X_{k+1} \to \phi \tag{10}$$

3. After iteration convergence, use the Sturm sequence check to verify that the required eigenvalues and corresponding eigenvectors have been calculated.

Reference [8] has presented very detailed descriptions about the subspace iteration method and is a good reference to use to become familiar with this method.

NEWMARK DIRECT INTEGRATION METHOD

The governing finite element equations of motion for a linear dynamic analysis are

$$M \ddot{q} + K q = 0 \tag{11}$$

where M and K are the mass and stiffness matrices, q and q are the nodal displacement and acceleration vector, and Q is the load (nodal force) vector of the finite element system. In finite element analysis, there exist many effective numerical procedures to solve the linear differential equations, Equation (11). Basically, they can be divided into two methods of solution: the direct integration method and the mode superposition method. In the present study, the Newmark direct integration method [7] will be followed to integrate Equation (11) step by step. The following assumptions are made in the numerical analysis:

$$\dot{q}_{n+1} = \dot{q}_n + [(1-\delta)\ddot{q}_n+\delta\ddot{q}_{n+1}]\Delta t \tag{12}$$

$$q_{n+1} = q_n + \dot{q}_n\Delta t + [(\frac{1}{2} -\alpha)\ddot{q}_n+\alpha\ddot{q}_{n+1}]\Delta t^2 \tag{13}$$

where Δt is the time step size, n is the step number, and the parameters δ and α control integration accuracy and stability. At time $t_{n+1} = (n+1)\Delta t$, the finite element equations of motion (Equation (11)) are described as:

$$M \ddot{q}_{n+1} + K q_{n+1} = Q_{n+1} \tag{14}$$

Solving from Equation (13) for $\ddot{q}_{n+1}$ in terms of q_{n+1}, and then substituting into Equation (14) and rearranging the terms transforms the equations to the form

$$\hat{K} q_{n+1} = \hat{Q}_{n+1} \tag{15}$$

where

$$\begin{aligned} \hat{K} &= a_o M + K \\ \hat{Q}_{n+1} &= Q_{n+1} + M(a_o q_n + a_1 \dot{q}_n + a_2 \ddot{q}_n) \end{aligned} \tag{16}$$

and

$$\begin{aligned} a_o &= 1/\alpha(\Delta t)^2 \\ a_1 &= 1/\alpha(\Delta t) \\ a_2 &= (1 - 2\alpha)/2\alpha \end{aligned} \tag{17}$$

Once the displacements q_{n+1} at time step n+1 are known from Equation (15), the velocities and accelerations can be computed by using Equations, (12) and (13) and expresses as

$$\ddot{q}_{n+1} = a_o(q_{n+1} - q_n) - a_1\dot{q}_n - a_2\ddot{q}_n \tag{18}$$

$$\dot{q}_{n+1} = \dot{q}_n + a_3\ddot{q}_n + a_4\ddot{q}_{n+1} \tag{19}$$

where

$$a_3 = (1 - \delta)(\Delta t) \tag{20}$$

$$a_4 = \delta(\Delta t)$$

A special scheme originated by Newmark with $\delta = 0.5$ and $\alpha = 0.25$ is used here to integrate the equations step by step. These values correspond to the constant-average-acceleration method, which gives an unconditionally stable numerical scheme [7].

DYNAMIC RESPONSE OF A SIMPLY SUPPORTED LAMINATED SQUARE PLATE

The dynamic response of simply supported laminated plates is presented in this section. The laminates are subjected to suddenly applied sinusoidally distributed pulse loading,

$$q(x, y, t) = (q_o \sin\pi x/a \sin\pi y/b)H(t) \tag{21}$$

where H(t) is the Heavyside step function. The following two laminated plates are considered:

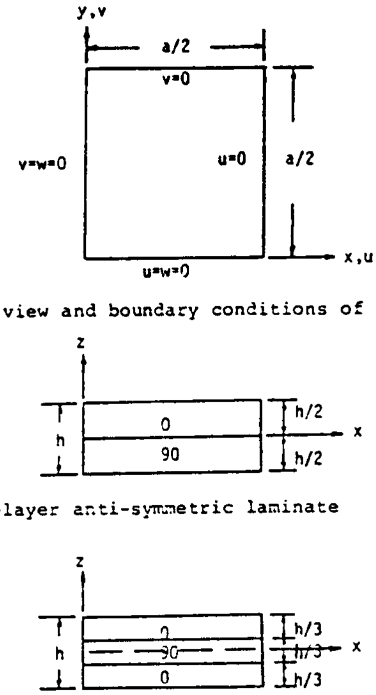
1. A two-layer anti-symmetric cross-ply (90/0) square laminate with layers of equal thickness.

2. A three-layer symmetric cross-ply (0/90/0) square laminate with layers of equal thickness.

In both problems, the same material properties as in Putcha and Reddy [9] are employed for each individual layer.

$$\begin{aligned} E_L &= 525 \text{ GPa} \\ E_T &= 21 \text{ GPa} \\ G_{LT} &= G_{TT} = 10.5 \text{ GPa} \\ \nu_{LT} &= \nu_{TT} = 0.25 \\ \rho &= 0.8 \text{ g/cm}^3 \end{aligned} \tag{22}$$

Owing to the biaxial symmetry of the laminate geometry, only one quadrant of the laminate is analyzed. The geometry configurations and boundary conditions of the finite element model are shown in Figure 1.



(a) Top view and boundary conditions of laminate

(b) Two-layer anti-symmetric laminate

(c) Three-layer symmetric laminate

Figure 1 Laminate geometry configurations and boundary conditions

The normalized deflection and stresses are described as

$$\bar{w} = 1000E_T h^3 w / q_o a^4$$

$$(\bar{\sigma}_x, \bar{\sigma}_y) = 10(\sigma_x, \sigma_y)/q_o S^2 \tag{23}$$

$$(\bar{\sigma}_z, \bar{\tau}_{xz}, \bar{\tau}_{yz}) = 10(\sigma_z, \tau_{xz}, \tau_{yz})/q_o S$$

$$S = a/h \qquad \bar{z} = z/h$$

In the present study q_o is taken to be 100 and the time step size is equal to 5 microseconds. The normalized central transverse deflection, $\bar{w}(a/2, a/2, 0)$, as a function of time for a two-layer anti-symmetric simply supported cross-ply square laminate under sinusoidal loading is shown in Figure 2. Throughout Figures 3, 4, 5 and 6, the stresses with respect to time for a two-layer laminate are plotted. In Figures 3 and 4, it is observed that the normalized normal stress $\bar{\sigma}_x$ at center top surface of the laminate is close to the normalized normal stress $\bar{\sigma}_y$ at center bottom surface of the laminate, except $\bar{\sigma}_x$ is in tension and $\bar{\sigma}_y$ is in compression. As shown in Figure 5, the variation of normalized shear stress $\bar{\tau}_{xz}$ is similar to $\bar{\tau}_{yz}$. From the plots, it is seen that the periods of the transient response for $\bar{w}$, $\bar{\sigma}_x$, $\bar{\sigma}_y$, $\bar{\tau}_{xz}$, and $\bar{\tau}_{yz}$ are very closely related. This fact agrees with the results of Putcha and Reddy [9]. The period for the normalized transverse normal stress $\bar{\sigma}_z(a/2, a/2, h/2)$, is much shorter when compared with others. A shorter time step size ($\Delta t = 1$ microsecond) is employed to observe the periodic response of the normalized transverse normal stress $\bar{\sigma}_z$ as shown in Figure 6.

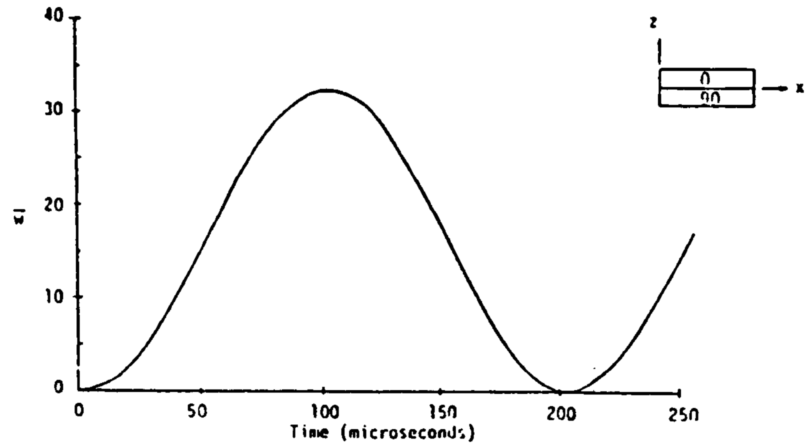


Figure 2 Center deflection versus time for a 2-layer cross-ply (90/0) simply supported square laminate under suddenly applied sinusoidal loading

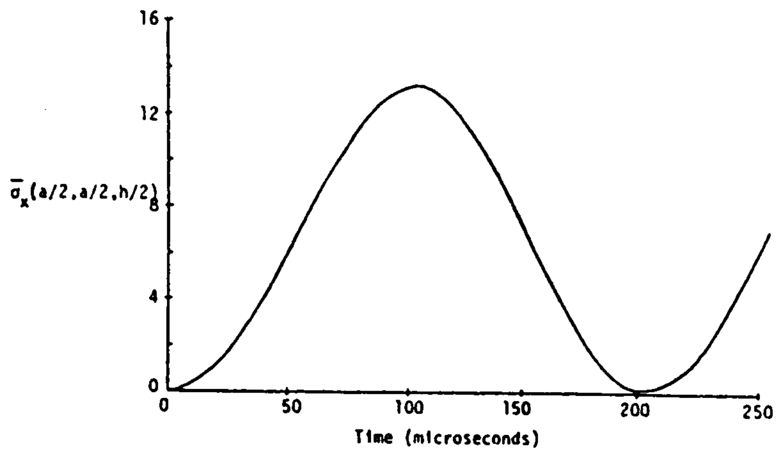


Figure 3 Normal stress versus time for a 2-layer (90/0) laminate under suddenly applied sinusoidal loading

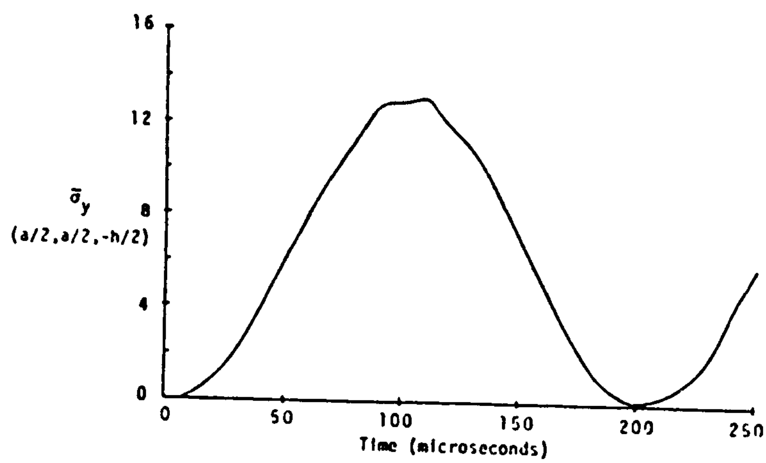


Figure 4 Normal stress versus time for a 2-layer (90/0) laminate under suddenly applied sinusoidal loading

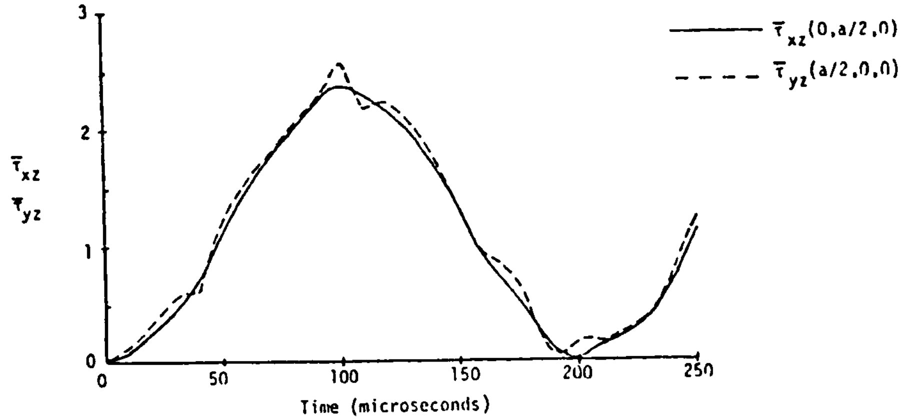


Figure 5 Transverse shear stresses versus time for a 2-layer (90/0) laminate under suddenly applied sinusoidal loading

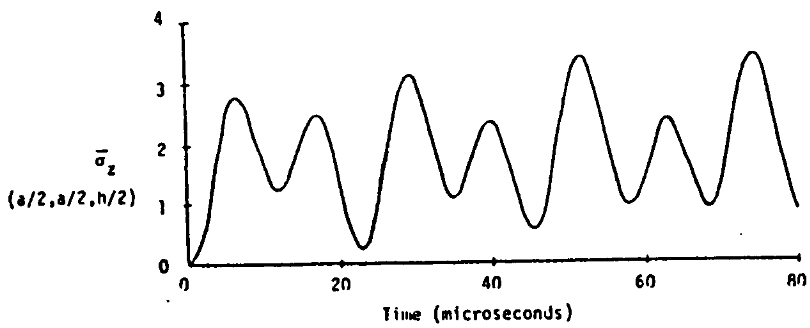


Figure 6 Transverse normal stress versus time for a 2-layer (90/0) laminate under suddenly applied sinusoidal loading

The normalized central transverse deflection with respect to time for a three-layer simply supported cross-ply (0/90/0) square laminate under sinusoidal loading is shown in Figure 7. Figure 8 contains the normalized maximum normal stresses, $\bar{\sigma}_x$ and $\bar{\sigma}_y$, as a function of time. The normalized shear stresses, $\bar{\tau}_{xz}$ and τ_{yz}, are shown in Figure 9. The periods for the normalized deflection and stresses are similar. In Figure 9, the amplitude of the response is larger for $\bar{\tau}_{xz}$ than for $\bar{\tau}_{yz}$; it is because the bending stiffness is higher in the x-direction than in the y-direction for a three-layer (0/90/0) laminate with layers of equal thickness. The normalized central normal stress distribution, $\bar{\sigma}_x$, through the thickness of the laminate for time from 20 to 80 microseconds is shown in Figure 10.

In the present study, a four-node isoparametric plate element with 48 assumed stress parameters for each lamina is used. Fast convergence is observed; only a 5 x 5 mesh is modeled in a quadrant of the laminate.

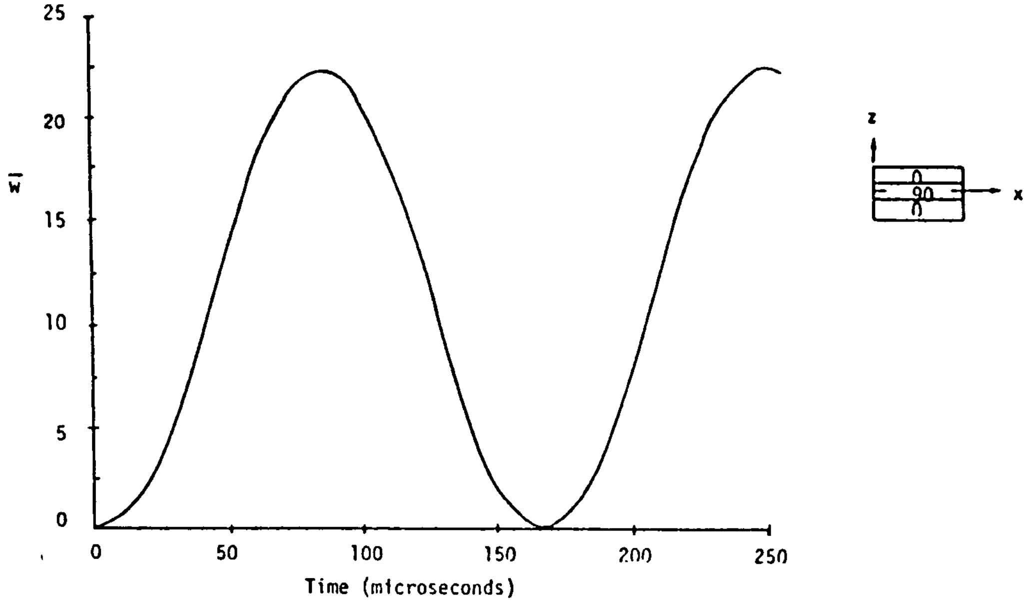


Figure 7 Center deflection versus time for a 3-layer cross-ply (0/90/0) laminate under suddenly applied sinusoidal loading

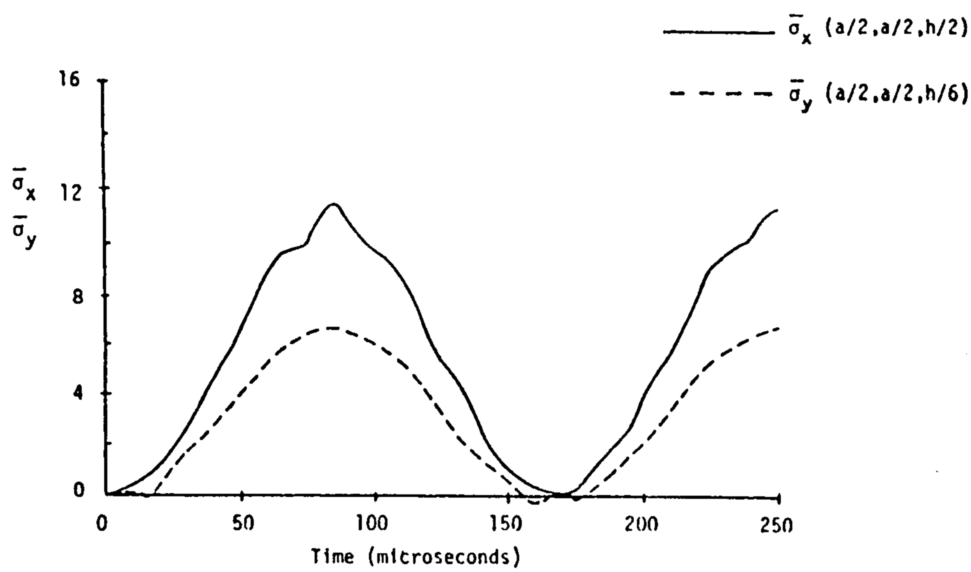


Figure 8 Normal stresses versus time for a 3-layer (0/90/0) laminate under suddenly applied sinusoidal loading

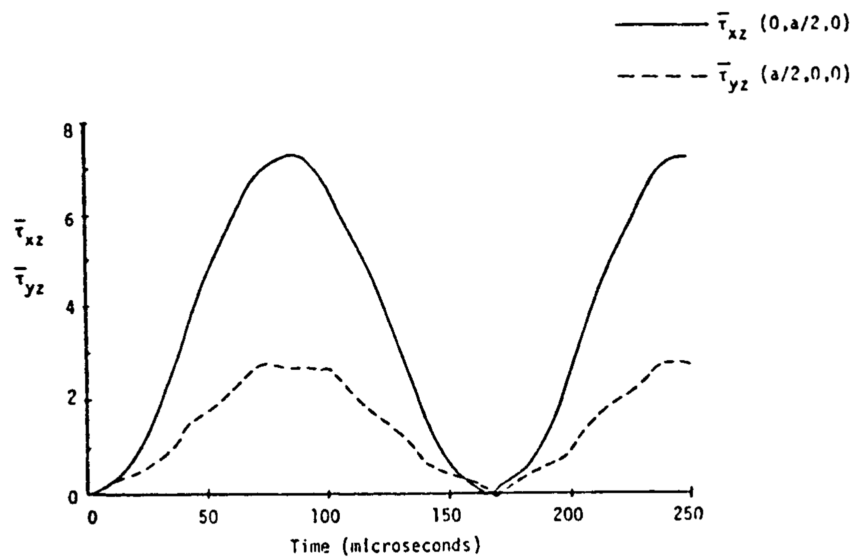


Figure 9 Transverse shear stresses versus time for a 3-layer (0/90/0) laminate under suddenly applied sinusoidal loading

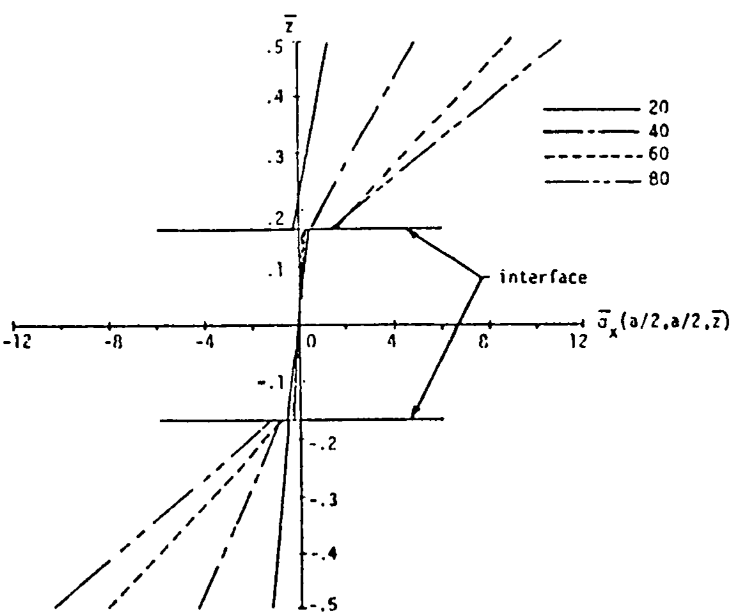


Figure 10 Through-thickness center normal stress versus $\bar{z}$ for a 3-layer (0/90/0) laminate under suddenly applied sinusoidal loading for time from 20 to 80 microseconds

CONCLUDING REMARKS

The proposed three-dimensional hybrid stress finite element method in conjunction with the Newmark's direct integration method seems to be a powerful technique for analyzing laminated composite plates under dynamic loading. By using this approach the transverse deflection, the in-plane bending stresses and the interlaminar shear stresses and normal stress can be evaluated very easily without consuming too much computation time. This method can also be used to analyze forced vibration problems of laminated composite under impact loading. The result will be published in the near future.

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