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# ISOTROPIC PROBABILITY MEASURES IN INFINITE DIMENSIONAL SPACES

(Inverse Problems/Prior Information/Stochastic Inversion)

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**Abstract.** Every isotropic probability measure on the space $R^\infty$ of real sequences $\mathbf{x}=(x_1,x_2,\ldots)$ is a convex combination of the measure concentrated at $\mathbf{0}$ and a member of $I_0(R^\infty)$, the set of all isotropic probability measures $p_\infty$ on $R^\infty$ with $p_\infty(\{0\})=0$. Each $p_\infty \in I_0(R^\infty)$ is completely determined by any one of its finite-dimensional marginal distributions $p_n$. Each $p_n$ has a density function $f_n$ with $dp_n(x_1,\ldots,x_n)=dx_1 \cdots dx_n f_n(x_1^2+\cdots+x_n^2)$. Each $f_n$ is completely monotone in $0<\xi<\infty$ (hence analytic in the right complex $\xi$ half-plane), and

$$\pi^{n/2}\Gamma(n/2)^{-1}\int_0^\infty d\xi\,\xi^{n/2-1}f_n(\xi)=1\,.$$

Every $f$ which satisfies these two conditions is $f_n$ for a unique $p_\infty \in I_0(R^\infty)$. Hence the equation

$$\pi\int_\xi^\infty d\zeta f_2(\zeta)=\int_0^\infty d\mu(t)e^{-t\xi}$$

defines a bijection between $I_0(R^\infty)$ and the set of all probability measures $\mu$ on $0\le t<\infty$. If $p_\infty \in I_0(R^\infty)$ then $p_\infty(\{\mathbf{x}:\sum_{i=1}^\infty x_i^2<\infty\})=0$, so $p_\infty$ is not a "softened" or "fuzzy" version of the inequality $\sum_{i=1}^\infty x_i^2 \le 1$. If the prior information in a linear inverse problem consists of this inequality and nothing else, stochastic inversion and Bayesian inference are both unsuitable inversion techniques.

**Introduction.** Let $R$ be the real numbers, $R^n$ the linear space of all real $n$-tuples, and $R^\infty$ the linear space of all infinite real sequences $\mathbf{x} = (x_1, x_2, \ldots)$. Let $P_n : R^\infty \to R^n$ be the projection operator with $P_n(\mathbf{x}) = (x_1, \ldots, x_n)$. Let $p_\infty$ be a probability measure on the smallest $\sigma$-ring of subsets of $R^\infty$ which includes all of the cylinder sets $P_n^{-1}(B_n)$, where $B_n$ is an arbitrary Borel subset of $R^n$. Let $p_n$ be the marginal distribution of $p_\infty$ on $R^n$, so $p_n(B_n) = p_\infty(P_n^{-1}(B_n))$ for each $B_n$. A measure on $R^n$ is "isotropic" if it is invariant under all orthogonal transformations of $R^n$. The measure $p_\infty$ will be called isotropic if all its marginal distributions $p_n$ are isotropic. The set of all isotropic probability distributions on $R^\infty$ will be written $I(R^\infty)$. The present note describes all members of $I(R^\infty)$. The result calls into question both stochastic inversion and Bayesian inference, as currently used in many geophysical inverse problems.

**Necessary Conditions for Isotropy.** Let $\mathbf{0} = (0, 0, \cdots)$ and let $p_\infty^0$ be the member of $I(R^\infty)$ such that $p_\infty^0(\{\mathbf{0}\}) = 1$. If $p_\infty \in I(R^\infty)$ and $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$, then $\alpha p_\infty + \beta p_\infty^0 \in I(R^\infty)$. Conversely, if $p_\infty \in I(R^\infty)$ and $p_\infty(\{\mathbf{0}\}) = \beta$, then $p_\infty = (1-\beta)\tilde{p}_\infty + \beta p_\infty^0$ where $\tilde{p}_\infty \in I(R^\infty)$ and $\tilde{p}_\infty(\{\mathbf{0}\}) = 0$. Therefore it is necessary to study only those $p_\infty \in I(R^\infty)$ for which $p_\infty(\{\mathbf{0}\}) = 0$. They constitute the subset $I_0(R^\infty)$ of $I(R^\infty)$.

If $p_\infty \in I_0(R^\infty)$, for every $\xi$ in $0 \leq \xi < \infty$ define

$$F_n(\xi) = p_\infty(\{\mathbf{x} : x_1^2 + \cdots + x_n^2 > \xi\}). \qquad [1]$$

Then $F_n$ is right semi-continuous, and

$$F_n(0) = 1 \qquad [2a]$$

$$F_n(\infty) = \lim_{\xi \to \infty} F_n(\xi) = 0. \qquad [2b]$$

Also, if $n \leq N$ and $\alpha \leq A$, then

$$0 \leq F_n(A) \leq F_n(\alpha) \leq F_N(\alpha) \leq 1. \qquad [2c]$$

Properties sufficient to characterize the members of $I_0(R^\infty)$ are given in

*Theorem 1:* Suppose $p_\infty \in I_0(R^\infty)$ and $F_n$ given by [1]. Then for each integer $n \ge 1$, $F_n(\xi)$ is analytic in the open right half plane of complex $\xi$. There is a function $f_n(\xi)$, also analytic there, such that for every Borel subset $B_n$ of $R^n$

$$p_n(B_n) = \int_{B_n} dx_1 \cdots dx_n \, f_n(x_1^2 + \cdots + x_n^2). \qquad [3a]$$

In particular, if $0 \le \alpha < \infty$ then

$$F_n(\alpha) = \pi^{n/2}\Gamma(n/2)^{-1} \int_\alpha^\infty d\xi \, \xi^{n/2-1} f_n(\xi). \qquad [3b]$$

The $f_n$ are related by

$$f_n(\xi) = \int_\xi^\infty d\eta \, (\eta-\xi)^{-1/2} f_{n+1}(\eta) \qquad [3c]$$

$$f_{n+1}(\xi) = -\pi^{-1}\partial_\xi \int_\xi^\infty d\eta \, (\eta-\xi)^{-1/2} f_n(\eta) \qquad [3d]$$

$$f_n(\xi) = \pi \int_\xi^\infty d\eta \, f_{n+2}(\eta) \qquad [3e]$$

$$f_{n+2}(\xi) = -\pi^{-1}\partial_\xi f_n(\xi) \qquad [3f]$$

For every $\beta$ in $0 \le \beta < \infty$

$$\lim_{n\to\infty} F_n(\beta) = 1. \qquad [3g]$$

*PROOF:* Let $S(n-1)$ denote the unit sphere in $R^n$, and let $|S(n-1)|$ be its $(n-1)$-dimensional Euclidean content, $2\pi^{n/2}\Gamma(n/2)^{-1}$. Let $|S(n-1)|\phi_n(w)$ be the content of the part of $S(n-1)$ where $x_n^2 \le 1-w$. Then

$$\phi_{n+1}(w) = 1 - |S(n-1)|\,|S(n)|^{-1} \int_0^w d\zeta \, \zeta^{n/2}(1-\zeta)^{-1/2}.$$

Since $p_n$ is the marginal distribution on $R^n$ of $p_{n+1}$ on $R^{n+1}$,

$$F_n(\xi) = -\int_\xi^\infty dF_{n+1}(\eta)\,\phi_{n+1}(\xi/\eta), \qquad [4a]$$

the right side being a Stieltjes integral. For any $\beta$ and $B$ satisfying $\xi < \beta < B$, $\partial_\eta \phi_{n+1}(\xi/\eta)$ is continuous in $\beta \le \eta \le B$, so integration by parts (1) permits the conclusion

$$\int_\beta^B dF_{n+1}(\eta)\phi_{n+1}(\xi/\eta)+\int_\beta^B d\eta\, F_{n+1}(\eta)\partial_\eta\phi_{n+1}(\xi/\eta)$$
$$=F_{n+1}(B)\phi_{n+1}(\xi/B)-F_{n+1}(\beta)\phi_{n+1}(\xi/\beta).$$

Here let $\beta\to\xi+$ and $B\to\infty$. The integrated parts tend to zero, so the Lebesque bounded convergence theorem permits [4a] to be rewritten

$$\xi^{-n/2}F_n(\xi)=|S(n-1)|\,|S(n)|^{-1}\int_\xi^\infty d\eta\,\eta^{-(n+1)/2}F_{n+1}(\eta)(\eta-\xi)^{-1/2}.$$

Iterating this formula once, reversing orders of integration, and invoking the identity

$$\int_\xi^\zeta d\eta\,(\zeta-\eta)^{-1/2}(\eta-\xi)^{-1/2}=\pi$$

leads to

$$\xi^{-n/2}F_n(\xi)=(n/2)\int_\xi^\infty d\zeta\,\zeta^{-(n+2)/2}F_{n+2}(\zeta). \qquad [4b]$$

By induction on $n$, it follows that $F_n(\xi)$ is infinitely differentiable in $0<\xi<\infty$. If we define

$$f_n(\xi)=-\pi^{-n/2}\Gamma(n/2)\xi^{1-n/2}\partial_\xi F_n(\xi), \qquad [5a]$$

then $f_n$ is also infinitely differentiable in $0<\xi<\infty$ and [2b] yields [3b]. Then [3a] follows by straightforward integration theory. Then the definition of marginal distributions implies

$$f_n(x_1^2+\cdots+x_n^2)=\int_{-\infty}^\infty dx_{n+1}\,f_{n+1}(x_1^2+\cdots+x_{n+1}^2), \qquad [5b]$$

which is [3c] with $\xi=x_1^2+\cdots+x_n^2,\eta=x_1^2+\cdots+x_{n+1}^2$. Also,

$$f_n(x_1^2+\cdots+x_n^2)=\int_{-\infty}^\infty dx_{n+1}\int_{-\infty}^\infty dx_{n+2}\,f_{n+2}(x_1^2+\cdots+x_{n+2}^2), \qquad [5c]$$

which is [3e]. Then [3f] follows from [3e], and [3d] follows from [3f] and [3c] with $n$ replaced by $n-1$. To prove analyticity, note that if $q$ is an integer $\geq 0$ and if $0<\alpha<\beta$, then by Taylor's theorem with remainder

$$F_2(\alpha)-F_2(\beta)=\sum_{i=1}^q\frac{(\beta-\alpha)^i}{i!}(-\partial_\xi)^iF_2(\beta)+\frac{1}{q!}\int_\alpha^\beta d\xi(\xi-\alpha)^q(-\partial_\xi)^{q+1}F_2(\xi). \qquad [6a]$$

But $(-\partial_\xi)^i F_2 = \pi^i f_{2i}$, so by [3b]

$$\frac{1}{q!}\int_\alpha^\beta d\xi\, \xi^q (-\partial_\xi)^{q+1} F_2(\xi) = F_{2q+2}(\alpha) - F_{2q+2}(\beta)\,. \tag{6b}$$

Hence, the Lebesque bounded convergence theorem implies that as $\alpha \to 0$ the integral in [6a] converges to $1 - F_{2q+2}(\beta)$. Therefore

$$F_{2q+2}(\beta) - F_2(\beta) = \sum_{i=1}^{q} \frac{\beta^i}{i!} (-\partial_\xi)^i F_2(\beta)\,. \tag{6c}$$

All terms in the sum [6c] are nonnegative, and $F_{2q+2}(\beta) \le 1$, so the series

$$\sum_{i=1} \frac{(-\beta)^i}{i!} F_2^{(i)}(\beta) \tag{6d}$$

converges absolutely (here $F_2^{(i)} = \partial_\xi^i F_2$). Therefore, the power series for $F_2(\xi)$ at $\xi = \beta$ converges absolutely for all complex $\xi$ in the closed disk $|\xi - \beta| \le \beta$. Since $\beta$ is arbitrary, $F_2(\xi)$ is analytic for all complex $\xi$ with positive real part. By [5a], so is $f_2(\xi)$ and then by [3c,d] so is $f_n(\xi)$ for every $n \ge 1$. Hence so is $F_n(\xi)$ for every $n \ge 1$. Furthermore, since [6d] converges, Abel's theorem (2) implies that

$$F_2(0) - F_2(\beta) = \sum_{i=1}^{\infty} \frac{\beta^i}{i!} (-\partial_\xi)^i F_2(\beta)\,. \tag{6e}$$

Together, [6e], [6c] and [2a] imply [3g].

*COROLLARY 1:* If one of the marginal distributions $p_n$ is known, $p_\infty$ is completely determined.

*COROLLARY 2:* Let $H(\alpha)$ be the set of $\mathbf{x}$ in $R^\infty$ with $\sum_{i=1}^{\infty} x_i^2 < \alpha$. Then $p_\infty(H(\infty)) = 0$. This follows immediately from [3g] and the fact that $H(\infty)$ is the monotone limit of the sets $H(\alpha)$ (3).

**Sufficient Conditions for Isotropy.** Let $M(n)$ be the set of infinitely differentiable real-valued functions $f$ on the open half-line $0 < \xi < \infty$ such that

$$\pi^{n/2}\Gamma(n/2)^{-1} \int_0^\infty d\xi\, \xi^{n/2-1} f(\xi) = 1 \tag{7a}$$

and also for every integer $q \geq 0$ and every $\xi$ in $0<\xi<\infty$

$$(-\partial_\xi)^q f(\xi) \geq 0. \qquad [7b]$$

Note that if $p_\infty \in I_0(R^\infty)$ and $f_n$ comes from $p_\infty$ via [3a] then $f_n \in M(n)$. The converse is also true, and to prove it we need

*LEMMA 1:* Suppose $n \geq 1$ and $f \in M(n)$. Then

$$\lim_{\xi\to\infty} \xi^{n/2} f(\xi) = 0 \qquad [8a]$$

$$\lim_{\xi\to 0} \xi^{n/2} f(\xi) = 0 \qquad [8b]$$

$$f(\xi) = \int_\xi^\infty d\eta\,[-\partial_\eta f(\eta)] \qquad [8c]$$

$$(n/2)\int_0^\infty d\xi\,\xi^{n/2-1} f(\xi) = \int_0^\infty d\xi\,\xi^{n/2}[-\partial_\xi f(\xi)] \qquad [8d]$$

$$-\pi^{-1}\partial_\xi f \in M(n+2). \qquad [8e]$$

*PROOF:* Let $m = n/2-1$ and let $0<\alpha<A<\infty$. Integration by parts gives

$$(m+1)\int_\alpha^A d\xi\,\xi^m f(\xi) = A^{m+1} f(A) - \alpha^{m+1} f(\alpha) + \int_\alpha^A d\xi\,\xi^{m+1}[-\partial_\xi f(\xi)]. \qquad [9a]$$

Fix $\alpha$. The integral on the right in [9a] increases as $A \to \infty$ and yet is bounded, so it has a limit. Therefore $\lim_{A\to\infty} A^{m+1} f(A)$ exists. By [7a] it cannot be positive, so we have [8a], and hence [8c], and also

$$(m+1)\int_\alpha^\infty d\xi\,\xi^m f(\xi) = -\alpha^{m+1} f(\alpha) + \int_\alpha^\infty d\xi\,\xi^{m+1}[-\partial_\xi f(\xi)]. \qquad [9b]$$

As $\alpha$ decreases to 0, the integral on the right in [9b] ]increases, and that on the left has a finite limit, so $\alpha^{m+1} f(\alpha)$ approaches either $+\infty$ or a nonnegative limit. Then [7a] requires [8b], and [9b] converges to [8d]. Then [8e] follows from [8d] and [7b].

Now we can prove

*THEOREM 2:* Suppose $n$ is a nonnegative integer and $f \in M(n)$. Then there is a $p_\infty \in I_0(R^\infty)$ whose marginal distribution $p_n$ on $R^n$ is given by [3a] with $f_n = f$.

*PROOF:* For every integer $q \geq 0$, define $f_{n+2q}(\xi)=\pi^{-q}(-\partial_\xi)^q f(\xi)$. If $N-n$ is a nonnegative even integer, induction on [8c] implies

$$f_N(x_1^2+\cdots+x_N^2)=\int_{-\infty}^{\infty} dx_{N+1}\int_{-\infty}^{\infty} dx_{N+2} f_{N+2}(x_1^2+\cdots+x_{N+2}^2)\,. \qquad [10a]$$

If $N-n$ is a nonnegative odd integer, define $f_N$ from $f_{N+1}$ via [3c]. Then

$$f_N(x_1^2+\cdots+x_N^2)=\int_{-\infty}^{\infty} dx_{N+1} f_{N+1}(x_1^2+\cdots+x_{N+1}^2)\,. \qquad [10b]$$

That [10b] also holds when $N-n$ is nonnegative and even follows from [10a]. Therefore [10b] holds for all $N \geq n$. Use it inductively to define $f_N$ for $1 \leq N < n$. For $N=n$, [7a] implies

$$\int_{R^N} dx_1 \cdots dx_N\, f_N(x_1^2+\cdots+x_N^2)=1\,, \qquad [10c]$$

and then [10b] implies [10c] for all $N \geq 1$. Thus the probability distributions $p_N$ on $R^N$ given by $f_N$ via [3a] satisfy the Kolmogorov consistency condition. Then the existence of $p_\infty$ follows from Kolmogorov's Fundamental Theorem (4).

*COROLLARY 1:* If $f \in M(n)$, $f(\xi)$ is analytic in the open right half-plane of complex $\xi$.

*COROLLARY 2:* The equation $F_2(\xi)=\int_0^\infty d\mu(t)e^{-\xi t}$ furnishes a bijection between the members of $I_0(R^\infty)$ and the probability measures $\mu$ on $0 \leq t < \infty$.

*PROOF:* Demanding that $f_2 \in M(2)$ is equivalent to demanding that $F_2(\xi)$ be completely monotonic on $0 \leq \xi < \infty$ (5).

**Examples and Applications.** Setting $f_2(\xi)=\pi^{-1}e^{-\xi}$ gives $f_n(\xi)=\pi^{-n/2}e^{-\xi}$. This $p_\infty$ is the gaussian with independent $x_1, x_2, \ldots$, each having mean 0 and variance 1. Setting $f_2(\xi)=\pi^{-1}\nu[\xi^{\nu-1}-(1+\xi)^{\nu-1}]$ with $0<\nu<1$ gives a $p_\infty$ for which $\lim_{\xi\to 0} f_n(\xi)=\infty$ if $n \leq 2$ and also if $n=1$ and $1/2 \leq \nu < 1$. Thus the densities $f_n(\xi)$ need not remain finite as $\xi \to 0$.

The geophysical application is to inverse theory. An infinite dimensional linear space $X$ of earth models $\mathbf{x}$ is given, along with a finite number of linear functionals, $g_j : X \to R$,

$j=1,\ldots,D+1$. An observer measures $D$ data $y_i = g_i(\mathbf{x}_E)+\varepsilon_i$ for $i=1,\ldots,D$. Here $\mathbf{x}_E$ is the correct earth model and $\varepsilon_i$ is the error in observing $y_i$. The observer wants to predict the value of $z = g_{D+1}(\mathbf{x}_E)$. Since $\dim X = \infty$, the problem is hopeless unless $g_{D+1}$ is a linear combination of $g_1,\ldots,g_D$, or unless the observer has some prior information about $\mathbf{x}_E$ not included among the data (6,7). One common sort of prior information is a quadratic bound on $\mathbf{x}_E$, a quadratic form $Q$ on $X$ such that $\mathbf{x}_E$ is known to satisfy

$$Q(\mathbf{x}_E,\mathbf{x}_E) \le 1 . \qquad [11]$$

Often [11] is a bound on energy content or dissipation rate (8). In stochastic inversion and Bayesian inference, such a bound is often "softened" to a prior personal probability distribution $p_\infty$ on $X$ (8–10). In practice, $X$ is truncated to an $R^n$, and $p_n$ is used in the inversion.

To see why this process is questionable, complete $X$ to a Hilbert space with the inner product $\mathbf{x}\cdot\mathbf{x}' = Q(\mathbf{x},\mathbf{x}')$. Let $\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \cdots$ be an orthonormal basis for $X$, and write $\mathbf{x} = \sum_{i=1}^{\infty} x_i \hat{\mathbf{x}}_i$. Then $X$ becomes the subset $H(\infty)$ of $R^\infty$ defined in corollary 2 to theorem 1. The prior information [11] can now be written

$$\sum_{i=1}^{\infty} x_i^2 \le 1 . \qquad [12]$$

If the observer wants to soften [12] to a probability distribution $p_\infty$, without introducing new information not implied by [12], then clearly he should take $p_\infty \in I(R^\infty)$. He is unlikely to assign nonzero probability to 0, so $p_\infty \in I_0(R^\infty)$. But then $p_\infty(X)=0$ by corollary 2 to theorem 1. Any prior personal probability distribution obtained by softening [12] without adding new information must deny [12] with probability 1.

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