Applications of Nonlinear Systems Theory to Control Design

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ABSTRACT. For most applications in the control area, the standard practice is to approximate a nonlinear mathematical model by a linear system. Since the feedback linearizable systems contain linear systems as a subclass, we examine the procedure of approximating a nonlinear system by a feedback linearizable one. Because many physical plants (e.g. aircraft at the NASA Ames Research Center) have mathematical models which are "close" to feedback linearizable systems, such approximations are certainly justified. We introduce results and techniques for measuring the "gap" between the model and its "truncated linearizable part." The topic of pure feedback systems is important in our study.

I. INTRODUCTION. In control design for nonlinear systems, the most common method is to approximate the nonlinear system by a linear system using the Taylor series truncation. Thus, we approximate a nonlinear system by a linear one and design with respect to the linear system. Recent advances have shown that control design can be achieved using a much larger class (containing the linear ones) of systems, which are called "feedback linearizable." These are nonlinear systems which are feedback equivalent to controllable linear systems [1], [2], [3].

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Concerning applications, the feedback linearization approximation for totally automatic flight control is used in [4] and [5]. In [5] the particular case of the UH-1H helicopter is studied, and successful flight test results are discussed in [6].

The first author and R. Su [7] have examined the procedure of approximating a general nonlinear system with a feedback linearizable one by introducing the concept of a pure feedback system. Every pure feedback system is feedback linearizable, and for each feedback linearizable system there exists a state space coordinate system in which it is pure feedback (assuming generic controllability assumptions). These coordinates are called the s-coordinates and they are generated geometrically (see also [1] and [8]). In the s-coordinates we can quickly ascertain if the system is feedback equivalent to a linear system or not. If approximation is necessary, the "pure feedback part" is easily recognizable.

Suppose we take a nonlinear system in its s-coordinates

$$\dot{s} = f(s) + \sum_{i=1}^{m} u_i g_i(s) \tag{1}$$

with analytic vector fields $f, g_1, \ldots, g_m$ on $\mathbb{R}^n$ (or say an open set in $\mathbb{R}^n$ containing the origin), $\dot{s} = \frac{ds}{dt}$, and $u_1, u_2, \ldots, u_m$ as controls. In general, the Taylor series approximation of a single input system, m=1 (we shall analyze multi-input also), has error $O((s_1, s_2, \ldots, s_n)^2)$ in terms of vector field differences. By $O((s_1, s_2, \ldots, s_n)^2)$ we mean no linear terms in these variables. However, the pure feedback approximation has error $O((s_3, s_4, \ldots, s_n)^2)$. Moreover, if $g, [f,g], \ldots, (ad^k f, g)$ are involutive, $0 \le k \le n-3$, the pure feedback approximation error is $O((s_3, s_4, \ldots, s_{n-k})^2)$, and the vector fields in the original system and the pure feedback part agree when $s_3 = 0,\ s_4 = 0, \ldots, s_{n-k} = 0$. If $k = n-2$, the system is pure feedback, and no approximation is necessary.

In approximating a nonlinear system by its pure feedback part, it is of interest to compare the state time responses of the system and its approximation. For this purpose we propose the Volterra series expansion of Fliess, Lamnabhi, and Lamnabhi-Lagarrigue [9].

An interesting class of partial differential equations consists of those failing to be elliptic, but possessing many desirable properties of elliptic equations. We consider the possibility of introducing the s-coordinates to find interesting coordinates for those hypoelliptic operators studied in [10] and [11].

In keeping with the purpose of this Engineering Foundation Conference, we are presenting an overview of recent work and our thoughts and ideas about interesting problems and future directions. This concept is certainly reflected in the style and intent of this paper.

Sections II and III contain definitions, results, and examples concerning pure feedback approximation for the single input and multi-input systems, respectively. A discussion of geometrically generated coordinates for the study of partial differential equations is the topic of Section IV. Future research directions are mentioned in the final section.

<u>II. SINGLE INPUT SYSTEMS</u>. We begin with a single input system

$$\dot{x} = f(x) + ug(x), \tag{2}$$

where f and g are real analytic vector fields on some open set in $\mathbb{R}^n$ containing the origin.

<u>Definition 2.1</u>. A system of the form

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2, x_3) \\
&\vdots \\
\dot{x}_{n-1} &= f_{n-1}(x_1, x_2, \ldots, x_n) \\
\dot{x}_n &= f_n(x_1, x_2, \ldots, x_n) + u g_n(x_1, x_2, \ldots, x_n)
\end{aligned} \tag{3}$$

is called a <u>pure feedback system</u>.

Let $[f,g]$, $(ad^2 f, g) = [f,[f,g]], \ldots, (ad^k f, g) = [f, (ad^{k-1} f, g)]$ denote Lie brackets involving the vector fields f and g. By $L_f y$ we mean the Lie derivative of a function y with respect to f; i.e. $L_f y = \langle dy, f \rangle$, $\langle \cdot, \cdot \rangle$ denoting the duality between one forms and gradients. We assume for the remainder of this section that $g, [f,g], \ldots, (ad^{n-1} f, g)$ are linearly independent.

For a pure feedback system (3) we define new state space coordinates $y_1, y_2, \ldots, y_n$ and a new control v by

$$\begin{aligned}
y_1 &= x_1 \\
y_2 &= L_f y_1 \\
y_3 &= L_f y_2 \\
&\vdots \\
y_n &= L_f y_{n-1} \\
v &= L_f y_n + u L_g y_n .
\end{aligned} \tag{4}$$

We obtain the controllable linear system

$$\begin{aligned}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= y_3
\end{aligned}$$

(5)

$$\vdots$$

$$\dot{y}_{n-1} = y_n$$

$$\dot{y}_n = v.$$

Hence system (3) is feedback linearizable or is feedback equivalent to the controllable linear system (5). In general, for a feedback transformation we include nonsingular state space coordinate changes (e.g. $y_1, y_2, \ldots, y_n$), additive state feedback (e.g. $L_f y_n$), and nonsingular state dependent input space coordinate changes (e.g. $uL_g y_n$).

In moving toward general results, the following two examples should be helpful.

Example 2.2. On $\mathbb{R}^3$ the system

$$\dot{x}_1 = x_2 + x_3^2$$

$$\dot{x}_2 = x_3 + x_2^2 \qquad (6)$$

$$\dot{x}_3 = x_1^2 + u$$

is not a pure feedback system. Moreover, there is no coordinate system on $\mathbb{R}^3$ in which this system appears as a pure feedback system.

The usual Taylor series approximation about $0 \epsilon R^3$ is

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3 \qquad (7)$$

$$\dot{x}_3 = u,$$

and the error between the vector fields in (6) and (7) is $O(\,(x_1)^2, (x_2)^2, (x_3)^2\,)$. Approximation by a pure feedback system

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3 + x_2^2 \qquad (8)$$

$$\dot{x}_3 = x_1^2 + u$$

yields an error difference in systems (6) and (8) of $O(\,(x_3)^2\,)$. Moreover, the approximation of (6) by (8) is exact when $x_3 = 0$.

Example 2.3. The system

$$\dot{x}_1 = x_2 + x_2^2 + x_3^2 + u$$

$$\dot{x}_2 = x_3 + \sin\,(x_1 - x_3) \qquad (9)$$

$$\dot{x}_3 = x_3^2 + u$$

on $\mathbb{R}^3$ is not a pure feedback system. However, it is shown in [6] that near the origin there exists state space coordinates in which we do have a pure feedback system.

Hence a pure feedback system is not invariant under coordinate changes on state space. Those nonlinear systems that can be reduced to pure feedback form can be classified, and, in fact, are the feedback linearizable systems.

Given a general nonlinear system

$$\dot{x} = f(x) + ug(x) \qquad (2)$$

with $g, [f,g], \ldots, (ad^{n-1}f, g)$ linearly independent, we shall find a coordinate system (called the s coordinates) so that

i) if the system can be put in pure feedback form, it appears in this coordinate system,

ii) if the system cannot be put in pure feedback form, we approximate it by that part of the system in the s coordinates appearing in the form (3).

For every system (2) we have s coordinates and we can expand in a power series in these coordinates.

Definition 2.4. The pure feedback part of a nonlinear system is that part in the s coordinates which appears in the form (3).

We introduce the following coordinate system. Solve in order the following systems of ordinary differential equations with the indicated initial conditions.

$$\begin{aligned}
\frac{dx}{ds_1} &= (ad^{n-1}f, g)\ , \quad x(0) = 0 \\
\frac{dx}{ds_2} &= (ad^{n-2}f, g)\ , \quad x(s_1, 0) = x(s_1) \\
&\ \vdots \\
\frac{dx}{ds_{n-1}} &= [f,g]\ , \quad x(s_1, s_2, \ldots, s_{n-2}, 0) = (x_1, s_2, \ldots, s_{n-2}) \\
\frac{dx}{ds_n} &= g\ , \quad x(s_1, s_2, \ldots, s_{n-1}, 0) = x(s_1, s_2, \ldots, s_{n-1}).
\end{aligned} \qquad (10)$$

By the inverse function theorem we invert (locally) to find

$$\begin{aligned}
&s_1(x_1, x_2, \ldots, x_n) \\
&s_2(x_1, x_2, \ldots, x_n) \\
&\ \vdots \\
&s_n(x_1, x_2, \ldots, x_n)\ .
\end{aligned} \qquad (11)$$

We set $\quad X_1 = (ad^{n-1}f, g)$

$$X_2 = (ad^{n-2}f,g)$$
$$(12) \qquad \vdots$$
$$X_n = g\ ,$$

and define

$$(13) \quad \begin{aligned} S_0 &= \{0\varepsilon\mathbb{R}^n\} \\ S_k &= \{s = (s_1,s_2,\ldots,s_n)\ \mathbb{R}^n:\ s_m = 0,\ k+1 \le m \le n\} \end{aligned}$$

for $k = 1,2,\ldots,n$.

In the s coordinates, usual derivatives can be replaced by derivatives with respect to the $X_j, j = 1,2,\ldots,n$.

Theorem 2.5[7]. In terms of the s coordinates the system (2) assumes the form

$$(14) \quad \dot{s} = f(0) + \sum_{i=1}^{\infty}\sum_{j=1}^{n} \frac{(s_j)^i}{i!}(ad^iX_j,f)\Big|_{S_{j-1}} + ue_n\ ,$$

where $\Big|_{S_{j-1}}$ denotes restriction to S_{j-1} and e_n is an n vector whose only nonzero entry is a 1 in the n^{th} component.

It is very easy to recognize the pure feedback part of system (14).

For a definition of an involutive set of vector fields and a statement of the Frobenius Theorem we refer the reader to [2].

Theorem 2.6. If $g,\ [f,g],\ldots,(ad^kf,g)$ are involutive, k an integer, $0\le k\le n-3$, then the vector field difference between system (14) and its pure feedback part is $O(\ (s_3,s_4,\ldots,s_{n-k})^2)$. If $k = n-2$, then (14) is a pure feedback system.

Proof. In the s coordinates, the manifolds S_k are linear subspaces of $\mathbb{R}^n$. Moreover, on each S_k the vector field X_k takes for form $X_k\Big|_{S_k} = \frac{\partial}{\partial s_k}$, or in column vector notation

$$X_k\Big|_{S_k} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow k^{th} \text{ place}.$$

Letting

$$f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

we find that $(ad^1X_n,f)\Big|_{S_{n-1}} = -[f,g]\Big|_{S_{n-1}} = -X_{n-1}\Big|_{S_{n-1}}$.

Hence $\frac{\partial f_1}{\partial s_n}, \frac{\partial f_2}{\partial s_n}, \ldots, \frac{\partial f_{n-2}}{\partial s_n}$ must vanish on S_{n-1}, i.e. when $s_n = 0$. The lowest power of s_n that can appear in $f_1,f_2,\ldots,f_{n-2}$ in the expansion (14) is two.

Computing we find that $(ad^1X_{n-1},f)\Big|_{S_{n-2}} = -(ad^2f,g)\Big|_{S_{n-2}} = -X_{n-2}\Big|_{S_{n-2}}$. Thus $\frac{\partial f_1}{\partial s_{n-1}}, \frac{\partial f_2}{\partial s_{n-1}}, \ldots, \frac{\partial f_{n-3}}{\partial s_{n-1}}$ must vanish on S_{n-2}, i.e. when s_{n-1} and $s_n = 0$.

We continue this process with $(ad^1X_{n-2}f)\Big|_{S_{n-3}}, \ldots, (ad^1X_3,f)\Big|_{S_2}$. For example, in the last step, $(ad^1X_3,f)\Big|_{S_2} = -X_2\Big|_{S_2}$, implying that $\frac{\partial f_1}{\partial s_3}$ must vanish on S_2, i.e. when $s_3 = s_4 = \ldots = s_n = 0$.

Thus it is clear that the vector field error difference between system (14) and its pure feedback part is at worst $O(\ (s_3,s_4,\ldots,s_n)^2)$.

We now turn our attention to the assumption that the set $\{g,[f,g],\ldots,\ (ad^kf,g)\}$ is involutive.

From Lemma 4 of [7] we have that

$$X_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix},\ X_{n-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ * \end{bmatrix},\ \ldots,\ X_{n-k} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ * \\ \vdots \\ * \end{bmatrix} \text{ on } \mathbb{R}^n\ ,$$

where * denotes possible nonzero entries. Now

$$X_{n-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ * \end{bmatrix} = [f,g] \text{ implies } f_1,f_2,\ldots,f_{n-2} \text{ are independent of } s_n,$$

$$X_{n-2} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ * \\ * \end{bmatrix} = (ad^2f,g) \text{ implies } f_1,f_2,\ldots,f_{n-3} \text{ are independent of } s_{n-1} \text{ and } s_n,$$

$$\vdots$$

$$X_{n-k} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ * \\ \vdots \\ * \end{bmatrix} = (ad^k f, g) \text{ implies } f_1, f_2, \ldots, f_{n-(k+1)} \text{ are independent of } s_{n-k+1}, \ldots, s_n.$$

Hence the error between system (14) and its pure feedback part is clearly $O((s_3, s_4, \ldots, s_{n-k})^2)$. If $k = n-2$, we have a pure feedback system for (14). $\square$

It is very important that a method be devised to compare the time responses of the state evolutions of system (14) and its pure feedback part. Given an initial condition s_0 in the s coordinates and an input u we examine the difference between these time responses by applying a formal Volterra series expansion.

We begin with system (2) and assume that we are in the s coordinates (i.e. $x = s$). To

$$\dot{x} = f(x) + ug(x) \tag{2}$$

we add a real analytic output function $y = h(x)$ and obtain the single input, single output system

$$\begin{aligned} \dot{x} &= f(x) + ug(x) \\ y &= h(x). \end{aligned} \tag{15}$$

By the formula of Fliess et.al. [9] the formal Volterra series expansion of the system (15) is

$$\begin{aligned} y(t) = {} & \sum_{\nu\geq 0} L_f^{\nu} h(x_0) \frac{t^{\nu}}{\nu!} + \\ & \int_0^t \sum_{\nu_0,\nu_1\geq 0} L_f^{\nu_0} L_g L_f^{\nu_1} h(x_0) \frac{(t-\tau_1)^{\nu_1}}{\nu_1!\,\nu_0!} \tau_1^{\nu_0} u(\tau_1)\, d\tau_1 + \\ & \int_0^t \int_0^{\tau_2} \sum_{\nu_0,\nu_1,\nu_2\geq 0} L_f^{\nu_0} L_g L_f^{\nu_1} L_g L_f^{\nu_2} h(x_0) \cdot \\ & \frac{(t-\tau_2)^{\nu_2} (\tau_2-\tau_1)^{\nu_1} \tau_1^{\nu_0}}{\nu_2!\,\nu_1!\,\nu_0!} u(\tau_2) u(\tau_1)\, d\tau_2 d\tau_1 \\ & + \ldots + \int_0^t \int_0^{\tau_k} \cdots \int_0^{\tau_2} \Big[\sum_{\nu_0,\ldots,\nu_k\geq 0} L_f^{\nu_0} L_g L_f^{\nu_1} \ldots L_g L_f^{\nu_k} h(x_0) \cdot \\ & \frac{(t-\tau_k)^{\nu_k} \ldots (\tau_1)^{\nu_0}}{\nu_k! \ldots \nu_0!} u(\tau_k) \ldots u(\tau_1) \Big] d\tau_k \ldots d\tau_1 \end{aligned} \tag{16}$$

$$+ \ldots,$$

where $x_0 \in R^n$ is a point at which the system is defined.

Our recommended method proceeds as follows: Take in order the outputs $h = x_1$, $h = x_2, \ldots, h = x_n$, compute the Volterra series expansions for the system (15) and the system

$$\begin{aligned} \dot{x} &= \tilde{f}(x) + ug(x) \\ y &= h(x) \end{aligned} \tag{17}$$

given by the pure feedback part, and compare the results for corresponding state time responses.

We examine the effect of the involutivity assumptions of Theorem 2.6 on the Volterra series (again taking $x = s$). From the proof of that result and the successive application of the formula

$$\begin{aligned} L_f\langle dh, g\rangle &= \langle dL_f h, g\rangle + \langle dh, [f,g]\rangle \\ &= L_g L_f h + \langle dh, [f,g]\rangle \end{aligned}$$

we find that

$$\begin{aligned} L_g L_f^{\nu} x_1 &= 0, \quad 0 \leq \nu \leq k \\ L_g L_f^{\nu} x_2 &= 0 \quad 0 \leq \nu \leq k-1 \\ &\vdots \\ L_g x_{k+1} &= 0 \end{aligned} \tag{18}$$

For any initial condition x_0 and with $h = x_1$, $h = x_2, \ldots, h = x_{k+1}$ in turn, the terms in the expansion (16) corresponding to the Lie derivatives in (18) must vanish.

Since the dynamical equation in (17) is pure feedback, it is feedback linearizable. This implies the set $\{g, [f,g], \ldots, (ad^{n-2} f, g)\}$ is involutive (see [12] and [2]) and

$$\begin{aligned} L_g L_{\tilde{f}}^{\nu} x_1 &= 0, \quad 0 \leq \nu \leq n-2 \\ L_g L_{\tilde{f}}^{\nu} x_2 &= 0, \quad 0 \leq \nu \leq n-2 \\ &\vdots \\ L_g x_{n-1} &= 0. \end{aligned} \tag{19}$$

As before, given a point x_0 and $h = x_1$, $h = x_2, \ldots, h = x_{n-1}$ in turn, the terms in the expansion (16) (with f replaced by $\tilde{f}$) corresponding to the Lie derivatives in (19) must vanish.

The conditions in equation (19) common to those in (18) illustrate the importance of the involutivity assumptions in improving the errors in the corresponding Volterra series expansion differences for (15) and (17). Again, the outputs are taken successively to be $h = x_1$, $h = x_2, \ldots, h = x_n$.

III. MULTI-INPUT SYSTEMS. We consider the nonlinear system

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x) \tag{20}$$

with $f, g_1, \ldots, g_m$ being real analytic vector fields on some open set in $\mathbb{R}^n$ containing the origin, and $g_1, g_2, \ldots, g_m$ being linearly independent. Assume there exists a set of positive integers $\kappa_1, \kappa_2, \ldots, \kappa_m$ such that:

i) the set $C = \{g_1, (ad^1 f, g_1), \ldots, (ad^{\kappa_1 - 1} f, g_1),$
$g_2, (ad^1 f, g_2), \ldots, (ad^{\kappa_2 - 1} f, g_2), \ldots, g_m, (ad^1 f, g_m), \ldots,$
$(ad^{\kappa_m - 1} f, g_m)\}$ spans $\mathbb{R}^n$ on our open set containing $0 \in \mathbb{R}^n$,

ii) the span of $C_j \cap C$ = the span of C_j, $j = 1, 2, \ldots, m$,
where $C_j = \{g_1, \ldots, (ad^{\kappa_j - 2} f, g_1), g_2, \ldots, (ad^{\kappa_j - 2} f, g_2)$
$, \ldots, g_m, \ldots, (ad^{\kappa_j - 2} f, g_m)\}$,

iii) $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_m$ (renumber $g_1, g_2, \ldots, g_m$ if necessary).

We fill a $\kappa_1 \times m$ array by putting from top to bottom

1) $g_1, [f, g_1], \ldots, (ad^{\kappa_1 - 1} f, g_1)$ in the first column

2) $g_2, [f, g_2], \ldots, (ad^{\kappa_2 - 1} f, g_2)$ and 0's (if needed) in the second column

$\vdots$

m) $g_m, [f, g_m], \ldots, (ad^{\kappa_m - 1} f, g_m)$ and 0's (if needed) in the m^{th} column.

Let $X_1 = (ad^{\kappa_1 - 1} f, g_1)$, the entry in the last row and first column,

X_2 = the vector field entry in the last row and 2nd column if it is nonzero, or the vector field entry in the $\kappa_1 - 1^{th}$ row and 1st column if $(ad^{\kappa_1 - 1} f, g_1)$ is the only nonzero entry in the last row.

$\vdots$

$X_n = g_m$.

Thus we start at the κ_1^{th} row and first column and move from left to right among the nonzero entries. Encountering a zero, we move up one row and return to the first column.

The s coordinates are defined by solving in order the system of o.d.e.'s with initial conditions

$$\begin{aligned} \frac{dx}{ds_1} &= X_1, \quad x(0) = 0 \\ \frac{dx}{ds_2} &= X_2, \quad x(s_1, 0) = x(s_1) \\ &\vdots \\ \frac{dx}{ds_n} &= X_n, \quad x(s_1, s_2, \ldots, s_{n-1}, 0) = x(s_1, s_2, \ldots, s_{n-1}), \end{aligned} \tag{21}$$

and inverting (locally) to obtain

$$\begin{aligned} &s_1(x_1, x_2, \ldots, x_n) \\ &s_2(x_1, x_2, \ldots, x_n) \\ &\vdots \\ &s_n(x_1, x_2, \ldots, x_n) . \end{aligned} \tag{22}$$

The manifolds

$$\begin{aligned} S_0 &= 0 \in \mathbb{R}^n \\ S_k &= \{s = (s_1, s_2, \ldots, s_n) \in \mathbb{R}^n : \; s_i = 0, \; k+1 \leq i \leq n\} \end{aligned} \tag{23}$$

are essential in the following result, which is a multi-input analogue to Theorem 2.5.

Theorem 3.1[7]. In terms of the s coordinates, the system (20) assumes the form

$$\begin{aligned} \dot{s} = f(0) &+ \sum_{i=1}^{\infty} \sum_{j=1}^{} \frac{(s_j)^i}{i!} (ad^i X_j, f) \Big|_{S_{j-1}} \\ &+ \sum_{k=n-m+1}^{n} \Big\{ e_k + \sum_{i=1}^{\infty} \sum_{j=k+1}^{n} \frac{(s_j)^i}{i!} (ad^i X_j, X_k) \Big|_{S_{j-1}} \Big\} u_{k-n+m}, \end{aligned} \tag{24}$$

where e_k is an n vector whose only nonzero entry is a 1 in the k^{th} component.

We need a definition of pure feedback system for the multi-input case. Take the $\kappa_1 \times m$ array that we constructed above. For $i, k = 1, 2, \ldots, \kappa_1$, let n_i = number of nonzero elements in the $(\kappa_1 - i + 1)^{th}$ row

$\beta_k = n_1 + n_2 + \ldots + n_k$.

For $j = 2, 3, \ldots, \kappa_1$ set

$y_j = (x_1, x_2, \ldots, x_{\beta_j})$

$y_{\kappa_1} = (x_1, x_2, \ldots, x_n)$.

Denote by g_{ij} the j^{th} component of the vector field g_i.

Definition 3.2. The nonlinear system (20) is a pure feedback system if it is of the form

$\dot{x}_j = f_j(y_2), \quad j = 1, 2, \ldots, \beta_1$

$\dot{x}_j = f_j(y_3), \quad j = \beta_1 + 1, \ldots, \beta_2$

$\vdots$

(25) $$\dot{x}_j = f_j(y_{\kappa_1}), \quad j = \beta_{\kappa_1-2}+1,\ldots,n-m$$
$$\dot{x}_j = f_j(y_{\kappa_1}) + \sum_{i=1}^{m} g_{ij}(y_{\kappa_1})u_i, \quad j=n-m+1,\ldots,n.$$

The block triangular systems found in [4] are a particular subset of the pure feedback systems.

For a general nonlinear system (20) (perhaps not pure feedback), we can move to the s coordinates as shown in equations (24).

Definition 3.3. The pure feedback part of a nonlinear system (20) is that part in the s coordinates which appears in the form (25).

Without proof we give a multi-input version of Theorem 2.6. The sets C_j are defined in the assumptions following equation (20).

Theorem 3.4. If each of the sets $\{g_1,[f,g_1],\ldots,(ad^k f,g_1), g_2,[f,g_2],\ldots,(ad^k f,g_2), g_m,[f,g_m],\ldots,(ad^k f,g_m)\}$ and C_j are involutive, where k is an integer, $0 \le k \le \kappa_1-3$, and j is any positive integer with $\kappa_j-2 \le k$, then the vector field difference between system (24) and its pure feedback part is $O((s_{\beta_2+1},\ldots,s_p)^2)$. Here p is the largest subscript on a nonzero vector field X_i in the $(k+1)^{th}$ row of our $\kappa_1 x m$ array. If $k = \kappa_1-2$, then (24) is a pure feedback system.

We can compare the time state responses of the system (24) and its pure feedback part by using multi-input versions of the Volterra series expansions of [9].

IV. PARTIAL DIFFERENTIAL EQUATIONS. Let L be a linear partial differential operator with real C^∞ coefficients and, for simplicity, assume L is second order. We also suppose that the principal part of L is the sum of squares of vector fields on an open set containing the origin in $\mathbb{R}^n$.

We let the principal part of L be $f^2 + g_1^2 +\ldots+g_m^2$, where

$$f = \alpha_1(x)\frac{\partial}{\partial x_1} + \alpha_2(x)\frac{\partial}{\partial x_2} +\ldots+ \alpha_n(x)\frac{\partial}{\partial x_n}$$
$$g_1 = \beta_{11}(x)\frac{\partial}{\partial x_1} + \beta_{12}(x)\frac{\partial}{\partial x_2} +\ldots+ \beta_{1n}(x)\frac{\partial}{\partial x_n}$$
$$\vdots$$
$$g_m = \beta_{m1}(x)\frac{\partial}{\partial x_1} + \beta_{m2}(x)\frac{\partial}{\partial x_2} +\ldots+ \beta_{mn}(x)\frac{\partial}{\partial x_n},$$

and the squares mean that an operator is applied twice.

In most studies the operator L is taken to be elliptic, but we are interested in the case that f can vanish on certain sets and $g_1,g_2,\ldots,g_m$ are linearly independent with $m<n$. We assume the existence of integers $\kappa_1,\kappa_2,\ldots\kappa_m$, sets C, and C_j as in section III.

Definition 4.1. An operator L is said to be hypoelliptic if Lu = f, where f is C^∞ on an open set U of $\mathbb{R}^n$, implies that u is C^∞ on U.

By the results of Hörmander [10] (with extensions due to Rothschild and Stein [11] we have that our operator L with principal part $f^2+g_1^2+\ldots+g_m^2$ is hypoelliptic since the vector fields in C are linearly independent.

We remark that the s coordinates in Sections II and III for our real analytic systems of ordinary differential equations are also applicable for C^∞ systems. Thus we can view our operator L in the s coordinates as generated in section III.

The relationship between controllability of systems of o.d.e.'s and hypoellipticity of p.d.e.'s has been well established in the literature. However, perhaps the special coordinates (e.g. the s coordinates) and the equivalence results from o.d.e.'s have not been applied to yield nice coordinates and equivalence criteria for operators like L. For example, the theory of Krener [8] and Respondek [13] for state space equivalence of systems when applied to p.d.e.'s produces the following theorem in the m=1 (with $g=g_1$) case.

Theorem 4.2. For the second order partial differential operator with principal part f^2+g^2 on $\mathbb{R}^n$, there exist a (local) coordinate system on $\mathbb{R}^n$ in which the principal part appears as $(Ax)^2+b^2$, where

A is a real constant n by n matrix
b is a constant vector

if and only if $[(ad^r f,g), (ad^s f,g)] = 0$ for $0 \le r,s \le n$. Here Ax and b denote the coefficients of $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2},\ldots,\frac{\partial}{\partial x_n}$.

In fact, under the assumptions of the theorem, A can be in rational canonical form and

$$b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Perhaps we can find "geometrically interesting canonical forms" for the class of partial differential equations under consideration.

V. FUTURE DIRECTIONS. The s coordinates of section II and III are generated by solving systems of ordinary differential equations. For feedback linearizable systems, symbolic and numerical methods exist in some cases for constructing the feedback linearizing transformations [4], [14], the major task being to find the s coordinates. G. Blankenship et. al. at the University of Maryland are developing an expert system to construct such transformations. The process of finding the s coordinates for general systems seems quite difficult. However, for many physical systems the mathematical model is often "near" pure feedback form and "close" to being in the s coordinates. In this case, a "good" initial guess for the s coordinates is easily obtained and an appropriate iteration scheme is recommended. Efforts to handle more difficult nonlinear systems are presently underway by G. Meyer of the NASA Ames Research Center.

When an approximation method is used for nonlinear systems of ordinary differential equations, the ultimate goal is to achieve close time responses between the original system and the approximating system. The Volterra series expansions mentioned in Section III seem ideal. We shall develop a symbolic manipulation program to generate and compare Volterra series expansions for nonlinear control systems and their pure feedback approximations.

An important numerical method in the engineering and mathematical approach to controlled systems of partial differential equations is the finite element method. How do the geometric hypoellipicity conditions of Section IV influence the use of finite elements? Moreover, one often does have a system of partial differential equations to model a physical system, but a computer generated finite element model. Is it possible to find the desired geometric conditions in these finite element models?

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