

NASA Contractor Report 181619

ICASE REPORT NO. 88-10

ICASE

ON THE INTERACTION OF TOLLMIEIN-SCHLICHTING
WAVES IN AXISYMMETRIC SUPERSONIC FLOWS

(NASA-CR-181619) ON THE INTERACTION OF
TOLLMIEIN-SCHLICHTING WAVES IN AXISYMMETRIC
SUPERSONIC FLOWS Final Report (NASA) 21 p

CSCD 20D

N88-17955

G3/34 Unclass
0126001

P. W. Duck

P. Hall

Contract No. NAS1-18107
January 1988

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association



National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

**On the interaction of Tollmien-Schlichting
waves in axisymmetric supersonic flows.**

by P.W. Duck,
Department of Mathematics
University of Manchester,
Oxford Road, Manchester, M13 9PL
and P. Hall,
Department of Mathematics,
University of Exeter,
North Park Road, Exeter. EX4 4QE

ME 205550
E 860-1956

Abstract.

It is known that two-dimensional lower branch Tollmien-Schlichting waves described by triple-deck theory are always stable for planar supersonic flows. Here the possible occurrence of axisymmetric unstable modes in the supersonic flow around an axisymmetric body is investigated. In particular flows around bodies with typical radii comparable with the thickness of the upper deck are considered. It is shown that such unstable modes exist below a critical nondimensional radius of the body a_0 . At values of the radius above a_0 all the modes are stable whilst if unstable modes exist they are found to occur in pairs. The interaction of these modes in the nonlinear regime is investigated using a weakly nonlinear approach and it is found that, dependent on the frequencies of the imposed Tollmien-Schlichting waves, either of the modes can be set up.

This work was supported under the National Aeronautics and Space Administration under NASA Contract No. NAS1-18107 while the authors were in residence at ICASE, NASA Langley Research Center, Hampton VA 23665.

1. Introduction.

It is now well-known that lower branch Tollmien-Schlichting waves in incompressible boundary layers are governed by triple-deck theory. The linear description of Tollmien-Schlichting waves near the lower branch of the neutral curve in the wavenumber-Reynolds number plane (in the limit of large Reynolds number) was given by Smith (1) whilst the extension into the corresponding weakly nonlinear regime was given by Smith (2). It was shown in the latter paper that nonlinear effects stabilize the linear growth of the waves so that as they propagate downstream a local equilibrium state is reached. Both two- and three- dimensional modes can be described by this approach and it was shown by Hall and Smith (3) that when nonlinear effects first become important nonparallel effects cannot be ignored. Further downstream the waves adjust in a quasi-parallel manner to the local flow conditions.

As the disturbance moves downstream its amplitude grows and ultimately the large amplitude high frequency state of Smith and Burggraf (4) is set up. Here the disturbance has a Stokes layer at the wall and resonant triad interactions can occur (Smith and Stewart (5)).

For supersonic planar flows the complex eigenrelation for two-dimensional Tollmien-Schlichting waves has only stable solutions. Here the possible existence of unstable axisymmetric modes in supersonic flows is discussed. We show that such modes exist when the nondimensional radius of the body about which the fluid flows falls below a critical size. This is consistent with the planar result which is achieved in the limit of large body radius.

In fact we find that unstable modes occur in pairs so that in the finite amplitude regime a nonlinear interaction problem must be set up in order to see which is the preferred mode. This is done using the approach of Hall and Smith (3) and we find that the interaction is characterized by a pair of coupled cubic nonlinear ordinary differential amplitude equations. The solutions of these equations depend on the values of the coefficients appearing in them. The latter coefficients depend on the radius of the body so that a different bifurcation structure is found as the radius varies. We show that, depending on the radius and frequencies of the Tollmien-Schlichting waves, two types of stable solution are possible.

In the next section we formulate the triple-deck problem which governs lower branch Tollmien-Schlichting waves in supersonic flows. In §3 these equations are solved for disturbances so small that linear theory is a valid approximation. In §4 the weakly nonlinear nonparallel interaction of the two possible instability modes of §3 is discussed. In §5 we discuss the results of the interaction problem of §4 and draw some conclusions.

2. Formulation

In this study we are concerned with the stability of an axisymmetric boundary layer which forms on a cylindrical body of radius a^* , in a uniform supersonic stream of velocity U_∞ directed along the axis of the cylinder.

Suppose that L^* denotes a typical streamwise lengthscale, ν_∞^* the kinematic viscosity of the fluid, then the Reynolds number Re is defined to be

$$Re = U_\infty L^* / \nu_\infty^*; \quad (2.1)$$

this will be taken to be large throughout this paper, and so the parameter

$$\varepsilon = Re^{-1/8} \quad (2.2)$$

may be taken to be small.

It will be assumed that

$$\bar{a} = \frac{a^*}{\varepsilon^3 L^*} = O(1) \quad (2.3)$$

which defines the scale of the radius of the body (this follows the order of a^* considered by Kluwick et al (6)).

The scalings of the various components of the solution then fall broadly into the triple-deck model. The following non-dimensional variables are defined

$$\begin{aligned} \bar{X} &= \frac{x^* - L^*}{\varepsilon^3 L^*} = \frac{x - 1}{\varepsilon^3}, & \bar{r} &= \frac{r^*}{\varepsilon^3 L^*}, \\ \bar{p} &= \frac{p^* - p_\infty^*}{\rho_\infty^* U_\infty^{*2}}, & \bar{u} &= \frac{u^*}{U_\infty^*}, \\ v &= \frac{v^*}{U_\infty^*}, & c &= \frac{c^*}{U_\infty^*}, \\ \bar{t} &= \frac{U_\infty^* t^*}{\varepsilon^2 L^*}. \end{aligned} \quad (2.4)$$

Here x^*, r^* denote streamwise and radial co-ordinates respectively, u^*, v^* the corresponding velocity components, c^* the speed of sound, ρ_∞^* the fluid density in the external flow, $p^*(p_\infty^*)$ the pressure (in the external flow), and t^* denotes time.

The stability of the flow is investigated at a downstream location. Here the boundary layer thickness is $O(\varepsilon^4 L^*)$, which is somewhat thinner than the radius of the body; we are thus concerned with streamwise locations relatively close to the leading edge of the body. At the axial position in question, we suppose that the skin friction is $\lambda \varepsilon^{-4} / L^*$, and in Section 4 we shall incorporate non-parallel effects into our study by allowing streamwise variations of λ .

The first layer of the triple-deck to be considered is the upper deck, wherein $\bar{r} = O(1)$. The solution develops (Kluwick et al (6)) as follows

$$\begin{aligned} u &= 1 + \varepsilon^2 u_1(\bar{X}, \bar{r}, \bar{t}) +, \\ v &= \varepsilon^2 v_1(\bar{X}, \bar{r}, \bar{t}) +, \\ p &= \varepsilon^2 p_1(\bar{X}, \bar{r}, \bar{t}) +, \\ c &= M_\infty^{-1} + \varepsilon^2 c_1(\bar{X}, \bar{r}, \bar{t}) +, \end{aligned} \quad (2.5)$$

where M_∞ is the Mach number of the external flow. The problem may be reduced to the solution for the pressure, which is governed by the axisymmetric form of the Prandtl Glauert equation, viz

$$(1 - M_\infty^2)p_{1\bar{X}\bar{X}} + \frac{1}{\bar{r}}p_{1\bar{r}} + p_{1\bar{r}\bar{r}} = 0. \quad (2.6)$$

The solution to this will be discussed later.

The main deck corresponds to the transverse scale

$$y = \frac{r^* - a^*}{\epsilon^4 L^*} = 0(1), \quad (2.7)$$

and is the same as planar flow to leading order, with the solution taking the form

$$\begin{aligned} u &= U_0(y) + \epsilon \bar{A}(\bar{X}, \bar{t}) U'_0(y) +, \\ v &= -\epsilon^2 \bar{A}_{\bar{X}}(\bar{X}, \bar{t}) U_0(y) +, \\ \rho &= R_0(y) + \epsilon \bar{A}(\bar{X}, \bar{t}) R_{0y}(y), \\ p &= \epsilon^2 \bar{P}(\bar{X}, \bar{t}) +. \end{aligned} \quad (2.8)$$

$U_0(y)$ and $R_0(y)$ are the axial velocity and density distributions respectively, in the undisturbed boundary layer.

Taking the limit of the v expansion as $y \rightarrow \infty$ in (2.8) yields an inner boundary condition for (2.6) namely

$$p_{1\bar{r}}|_{\bar{r}=\bar{a}} = \bar{A}_{\bar{X}}\bar{X}, \quad (2.9)$$

whilst if $y \rightarrow 0$, the no-slip condition on the wall is violated, and this demands the presence of a lower deck wherein

$$\bar{Y} = y/\epsilon = 0(1),$$

and

$$\begin{aligned} u &= \lambda \bar{Y} + \epsilon \bar{U}(\bar{X}, \bar{Y}, \bar{t}) +, \\ v &= \epsilon^2 \bar{V}(\bar{X}, \bar{Y}, \bar{t}) +, \\ p &= \epsilon^2 \bar{P}(\bar{X}, \bar{Y}, \bar{t}) +, \\ \rho &= \bar{R}_0(0) + \epsilon \rho_1, (\bar{X}, \bar{Y}, \bar{t}) +. \end{aligned} \quad (2.10)$$

At this stage it is convenient to scale out a number of the physical parameters. Following Kluwick et al (6), this is achieved with the following transformations

$$\begin{aligned} \bar{X} &= C^{\frac{3}{8}} |M_\infty^2 - 1|^{-\frac{3}{8}} (T_w/T_\infty)^{\frac{3}{8}} X, \\ \bar{Y} &= C^{\frac{5}{8}} |M_\infty^2 - 1|^{-\frac{5}{8}} (T_w/T_\infty)^{\frac{3}{8}} Y, \\ \bar{P} &= C^{\frac{1}{4}} |M_\infty^2 - 1|^{-\frac{1}{4}} P, \\ \bar{U} &= C^{\frac{1}{8}} |M_\infty^2 - 1|^{-\frac{1}{8}} (T_w/T_\infty)^{\frac{1}{8}} U, \\ \bar{V} &= C^{\frac{3}{8}} |M_\infty^2 - 1|^{\frac{1}{8}} (T_w/T_\infty)^{\frac{1}{8}} V, \\ \bar{A} &= C^{\frac{5}{8}} |M_\infty^2 - 1|^{-\frac{5}{8}} (T_w/T_\infty)^{\frac{3}{8}} A, \\ \bar{a} &= C^{\frac{3}{8}} |M_\infty^2 - 1|^{-\frac{7}{8}} (T_w/T_\infty)^{\frac{3}{8}} a, \\ \bar{t} &= C^{\frac{1}{4}} |M_\infty^2 - 1|^{-\frac{1}{4}} (T_w/T_\infty) t. \end{aligned} \quad (2.11)$$

Here C is the Chapman constant which arises from the linear viscosity law

$$\mu^*/\mu_\infty^* = C(T/T_\infty), \quad (2.12)$$

and the nondimensional wall temperature is denoted by T_w . The governing equations in the lower deck reduce to the planar incompressible form at leading order, namely

$$\begin{aligned} \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} &= \frac{\partial^2 U}{\partial Y^2} - \frac{\partial P}{\partial X}, \\ \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} &= 0, \\ \frac{\partial P}{\partial Y} &= 0, \end{aligned} \quad (2.13)$$

with boundary conditions

$$\begin{aligned} U = V = 0 \quad \text{on} \quad Y = 0, \\ U \rightarrow \lambda[Y + A(X, t)] \text{ as } Y \rightarrow \infty. \end{aligned} \quad (2.14)$$

The problem is closed by means of solving (2.6) subject to (2.9), giving (symbolically)

$$P = \mathcal{L}(A). \quad (2.15)$$

In general, the system is nonlinear, and requires a numerical treatment; however, progress may be made by seeking solutions which are slight perturbations about the undisturbed state $U = \lambda Y, V = P = A = 0$, and this will form the basis for the following section.

3. Linear Stability

Here the basic flow is slightly perturbed by a small amount $h \ll 1$, and the solution is thus written

$$\begin{aligned} U &= \lambda Y + hU_1 + \dots, \\ V &= hV_1 + \dots, \\ P &= hP_1 + \dots, \\ A &= hA_1 + \dots. \end{aligned} \quad (3.1)$$

A solution is sought of the form

$$\begin{aligned} U_1 &= \hat{U}_1(Y)E_1 + \text{c.c.}, \\ V_1 &= \hat{V}_1(Y)E_1 + \text{c.c.}, \\ P_1 &= \hat{P}_1 E_1 + \text{c.c.}, \\ A_1 &= \hat{A}_1 E_1 + \text{c.c.} \end{aligned} \quad (3.2)$$

Here

$$E_1 = \exp[i(\alpha X - \Omega t)], \quad (3.3)$$

and 'c.c.' denotes a complex conjugate, $\hat{U}_1(Y)$ and $\hat{V}_1(Y)$ are complex functions of Y alone, and $\hat{P}_1, \hat{A}_1, \alpha$ and Ω are, in general, complex constants.

The problem for $\hat{U}_1(Y)$ then reduces to

$$i(\alpha Y - \Omega)\hat{U}_{1Y} = \hat{U}_{1YYY}, \quad (3.4)$$

with

$$\begin{aligned} \hat{U}_1 &= 0 & \text{on} & Y = 0 \\ \hat{U}_1 &\rightarrow \hat{A}_1 & \text{as} & Y \rightarrow \infty. \end{aligned} \quad (3.5)$$

At this stage it is necessary to determine the solution of (2.6), subject to (2.9) which yields:

$$\hat{P}_1 = -\frac{i\alpha K_0(i\alpha a)\hat{A}_1}{K_1(i\alpha a)}. \quad (3.6)$$

This solution (see Ward (7) and Kluwick et al (6)) is chosen such that information propagates downstream along the characteristics of (2.6). $K_n(z)$ denotes the modified Bessel function of order n argument z .

The system comprising (3.4), (3.5) is closed by utilizing (3.6) with

$$\hat{U}_{1YY} |_{Y=0} = i\alpha\hat{P}_1, \quad (3.7)$$

which arises from evaluating (2.13) on $Y = 0$.

The solution for \hat{U}_{1Y} may be written down in terms of Airy functions, and a non-trivial solution is only possible if the following dispersion relationship is satisfied

$$\frac{-\lambda^3 Ai'_0}{\int_{\xi_0}^{\infty} Ai(\xi) d\xi} = \frac{(i\alpha\lambda)^{\frac{4}{3}} K_0(i\alpha a)}{K_1(i\alpha a)}, \quad (3.8)$$

where

$$\begin{aligned} \xi_0 &= -(i\alpha)^{\frac{1}{3}} \Omega / \alpha \lambda^{\frac{2}{3}}, \\ Ai'_0 &= Ai'(\xi_0). \end{aligned} \quad (3.9)$$

It is well known that neutrally stable Tollmien-Schlichting waves arise from (3.8) when $Imag\{\alpha\} = Imag\{\Omega\} = 0$. It is also worth noting at this stage that in the limit of $a \rightarrow \infty$, i.e. the flat plate limit, (α fixed), (3.6) degenerates to the plane supersonic interaction condition

$$\hat{P}_1 \sim -i\alpha\hat{A}_1, \quad (3.10)$$

which is known to exhibit only decaying solutions (see for example Ryzhov and Zhuk (8)). On the other hand, in the limit of a very thin cylinder (or needle), namely $a \rightarrow 0$, then

$$\hat{P}_1 \sim -\alpha^2 a \hat{A}_1 \log \alpha, \quad (3.11)$$

which is akin to the interaction condition found in certain jet and channel flows, (see Smith (9)), which does admit growing Tollmien-Schlichting instabilities (Bogdanova and Ryzhov (10)). From these somewhat qualitative observations, we may suspect that prior to a thorough investigation of the roots of (3.8), growing Tollmien-Schlichting instabilities are possible (at least over a range of a).

A search was carried out (using Newton iteration) for neutral roots of (3.8), i.e. for which $\text{Imag}\{\alpha\} = \text{Imag}\{\Omega\} = 0$, with $\lambda = 1$ (indeed, λ may here be scaled from the problem, but is retained for the purposes of the following section). No roots of this type were found for $a > a_0 = 0.003$ (approximately, although it is likely that an infinite number of exponentially decaying roots do exist in this regime). For $a < a_0$ it was found that two neutrally stable roots exist for each value of a . Figure 1 shows the variation of neutral frequency with a , whilst Figure 2 shows the variation of wavenumber with a .

Notice that both Ω and α become unbounded at $a \rightarrow 0$, and this can be confirmed analytically. First consider the lower branch, marked L in Figures 1 and 2. We seek an asymptotic solution to (3.8) in the limit as $a \rightarrow 0$ with $\xi_0 = 0(1)$, and $\alpha = 0(a^{-1})$. Then it follows that

$$\frac{K_0(i\alpha a)}{K_1(i\alpha a)} \sim -i\alpha a \left\{ \ln(\alpha) + \frac{i\pi}{2} + \gamma + \dots \right\}, \quad (3.12)$$

where $\gamma = 0.577215\dots$, and so

$$\frac{\lambda^3 Ai(\xi_0)}{\int_{\xi_0}^{\infty} Ai'(\xi) d\xi} \sim \lambda^{\frac{4}{3}} (i\alpha)^{\frac{7}{3}} a \left[\ln(\alpha) + i\frac{\pi}{2} + \dots \right]. \quad (3.13)$$

Hence the leading order term on the right-hand-side of this expression has argument $\pi/6$, and it is well known from planar incompressible stability studies (see Smith (1), Hall and Smith (3)) that the right-hand-side of (3.13) has this same argument when

$$\xi_0 \simeq -2.298i^{\frac{1}{3}}, \quad (3.14)$$

which corresponds to

$$\frac{Ai'(\xi_0)}{\int_{\xi_0}^{\infty} Ai(\xi) d\xi} \simeq i^{\frac{1}{3}}. \quad (3.15)$$

At leading order (3.13) then becomes

$$\alpha^{\frac{7}{3}} a |\log(\alpha a)| \sim \lambda^{\frac{5}{3}}. \quad (3.16)$$

This is a transcendental equation linking α to a (see Duck (11) for a similar type of situation); however it may be solved by successive approximations for $\alpha \gg 1$.

The first approximation gives

$$\alpha = 0(a^{-\frac{3}{7}}). \quad (3.17)$$

The second approximation gives

$$\alpha = \left(\frac{7\lambda^{\frac{5}{3}}}{4a |\log a|} \right)^{\frac{3}{7}}, \quad (3.18)$$

(this process may be continued further, with the inclusion of terms like $\log |\log a|$ etc., but we shall terminate the approximation at this stage).

Taking (3.18), we obtain an asymptotic expression to Ω , namely

$$\Omega \sim 2.298 \left(\frac{7\lambda^{\frac{5}{3}}}{4a |\log a|} \right)^{\frac{2}{7}}. \quad (3.19)$$

The asymptotic results (3.18) and (3.19) are indicated on Figures 1 and 2 by broken lines, and the agreement with the computed results as $a \rightarrow 0$ is seen to be satisfactory. (We cannot expect better agreement in view of the likely largeness of the correction terms to (3.18), (3.19) which have been omitted.)

We now turn to consider the asymptotic form of the upper branch (denoted by U on Figures 1 and 2) as $a \rightarrow 0$. Since we are concerned with neutral modes, it is convenient to set

$$\xi_0 = -i^{\frac{1}{3}} R, \quad (3.20)$$

$$R = \Omega/\alpha^{\frac{2}{3}}, \quad (3.21)$$

and to seek solutions to (3.8) for which $R = |\xi_0| \rightarrow \infty$, $a \rightarrow 0$, $\alpha = 0(a^{-1})$. With these restrictions, (3.8) becomes

$$\begin{aligned} & -\lambda^{\frac{4}{3}} (i\alpha)^{\frac{7}{3}} [\ln(\alpha a) + \frac{i\pi}{2} + \gamma + \dots] \\ & \sim \lambda^3 [-i^{\frac{1}{3}} R + e^{-\frac{7i\pi}{12}} R^{-\frac{1}{2}}], \end{aligned} \quad (3.22)$$

i.e.

$$\alpha^{\frac{7}{3}} a [\ln(\alpha a) + \frac{i\pi}{2} + \gamma + \dots] \sim \lambda^{\frac{5}{3}} [-R + e^{-\frac{3i\pi}{4}} R^{-\frac{1}{2}} + \dots]. \quad (3.23)$$

We now take the leading order real and imaginary parts of this equation separately to yield two (real) equations linking α , $R(\Omega)$ and a , namely

$$\alpha^{\frac{7}{3}} a |\ln(\alpha a)| \sim \lambda^{\frac{5}{3}} R, \quad (3.24)$$

$$\alpha^{\frac{7}{3}} a \frac{\pi}{2} \sim \frac{\lambda^{\frac{5}{3}}}{\Omega} R^{-\frac{1}{2}}. \quad (3.25)$$

The leading order terms in the solution for this system are

$$\begin{aligned} \alpha & \sim 2^{-\frac{1}{7}} \lambda^{\frac{5}{7}} 7^{\frac{1}{7}} a^{-\frac{3}{7}} \pi^{-\frac{2}{7}} (|\ln a|)^{-\frac{1}{7}} \left[1 + \frac{1}{28} \frac{|\ln |\ln a||}{|\ln a|} \right] \\ \Omega & \sim 2^{\frac{11}{7}} \lambda^{\frac{10}{21}} 7^{-\frac{4}{7}} \pi^{-\frac{6}{7}} a^{-\frac{2}{7}} (|\ln a|)^{\frac{4}{7}} \left[1 - \frac{1}{7} \frac{|\ln |\ln a||}{|\ln a|} \right]. \end{aligned} \quad (3.26)$$

These asymptotic results are shown as 'dot-dashed' lines on Figures 1 and 2. The agreement is not as satisfactory as that for the lower branch asymptotic behaviour considered previously; here the presence of the many log, log log etc. terms considerably degrades the convergence of the asymptotic series. However results obtained by the authors (not shown here) for $a \ll 10^{-4}$, indicate the proposed upper branch structure to be correct. The existence of two modes of instability, found above, suggests that interaction between them must be taken into account when studying the nonlinear problem. We consider this aspect in the following section.

4. Wave interactions in the weakly nonlinear regime

Here we shall discuss the nonlinear nonparallel interaction of two axisymmetric Tollmien-Schlichting waves at a value of a less than the cut-off value a_0 . The interaction is essentially similar to that given by Hall and Smith (3) for three-dimensional interactions in incompressible flow so all of the details will not be repeated here. We first note that the Tollmien-Schlichting waves vary on the triple-deck length scale ϵ^{-3} in the linear regime. Now we are concerned with their possible equilibration on a longer length scale centred on the downstream position where they are locally neutrally stable. It was shown by Hall and Smith that the appropriate length scale where nonparallel effects are most important is $O(\epsilon^{\frac{3}{2}})$. At distances large compared to $\epsilon^{\frac{3}{2}}$ downstream of the neutral position the disturbances suffer no nonparallel effects to the order given in our calculation.

The linearized form of the evolution equation for a Tollmien-Schlichting wave of amplitude B in an $O(\epsilon^{\frac{3}{2}})$ neighbourhood of the neutral position x_n can be derived from (3.8) by writing

$$\tilde{X} = \frac{x - x_n}{\epsilon^{\frac{3}{2}}}, i\alpha = i\alpha + \epsilon^{\frac{3}{2}} \frac{\partial}{\partial \tilde{X}},$$

and expanding to give

$$\frac{dB}{d\tilde{X}} + e\tilde{X}B = 0, \quad (4.1)$$

where

$$e = \frac{\frac{2}{3}\lambda'(x_n)\lambda^{-1}(x_n)\frac{Ai_0\xi_0}{\chi_0}\left[1 - \frac{\xi_0\chi_0}{Ai_0'}\right]}{\frac{1}{3i\alpha} - \frac{aK_1}{K_0(i\alpha a)} - \frac{aK_0}{K_1(i\alpha a)} - \frac{2i}{3}\xi_0\frac{Ai_0}{\alpha\chi_0} + \frac{2i}{3}\xi_0^2\frac{Ai_0}{\alpha Ai_0'}}, \quad (4.2)$$

where

$$Ai_0 = Ai(\xi_0), \chi_0 = \int_{\xi_0}^{\infty} Ai(\xi)d\xi$$

Thus on the basis of linear theory it follows that in an $O(\epsilon^{\frac{3}{2}})$ neighbourhood of x_n , B is given by

$$B = \text{constant } e^{-\frac{e\tilde{X}^2}{2}} \quad (4.3)$$

This linear growth will take the size of the initial disturbance to a size where nonlinearity cannot be neglected. This occurs when the x velocity component in the lower deck is of order $\epsilon^{\frac{1}{4}}$.

Suppose then that at $x = x_n$ the two modes with $(\alpha, \Omega) = (\alpha_j, \Omega_j), j = 1, 2$ are neutrally stable. We define $E_j, j = 1, 2$ by

$$E_j = \exp i \left\{ \frac{\alpha_j(x - x_n)}{\varepsilon^3} - \Omega_j t \right\}, \quad (4.4)$$

where k_j and Ω_j are now real quantities. We must expand α_j and Ω_j in the form

$$\begin{aligned} \alpha_j &= \alpha_{j0} + \varepsilon \alpha_{j1} + \dots, \\ \Omega_j &= \Omega_{j0} + \varepsilon \Omega_{j1} + \dots, \end{aligned} \quad (4.5)$$

where $(\alpha_{j0}, \Omega_{j0})$ and $(\alpha_{j1}, \Omega_{j1})$ are chosen such that the waves are neutrally stable at x_n . Thus $(\alpha_{j0}, \Omega_{j0})$ are as given in §3 whilst $(\alpha_{j1}, \Omega_{j1})$ can be found by continuing the expansion procedure of that section to next order. For the purposes of this calculation we do not need to know $(\alpha_{j1}, \Omega_{j1})$ so we give no further details of their determination here. We suppose that the waves have grown sufficiently for nonlinear effects to be important so that in the lower deck the appropriate expansion is

$$\begin{aligned} (u, v, p) &= (\varepsilon \lambda Y, 0, 0) + \{ A [\varepsilon^{\frac{7}{4}} U_{11}, \varepsilon^{\frac{15}{4}} V_{11}, \varepsilon^{\frac{13}{4}} P_{11}] [1 + O(\varepsilon)] E_1 \\ &\quad + B [\varepsilon^{\frac{7}{4}} U_{12}, \varepsilon^{\frac{15}{4}} V_{12}, \varepsilon^{\frac{11}{4}} P_{12}] [1 + O(\varepsilon)] E_2 + A^2 [\varepsilon^{\frac{5}{2}} U_{21}, \varepsilon^{\frac{9}{2}} V_{21}, \varepsilon^{\frac{7}{2}} P_{21}] [1 + O(\varepsilon)] \\ &\quad + B^2 [\varepsilon^{\frac{5}{2}} U_{22}, \varepsilon^{\frac{9}{2}} V_{22}, \varepsilon^{\frac{7}{2}} P_{22}] [1 + O(\varepsilon)] E_2^2 + A \bar{B} [\varepsilon^{\frac{5}{2}} U_{24}, \varepsilon^{\frac{9}{2}} V_{24}, \varepsilon^{\frac{7}{2}} P_{24}] [1 + O(\varepsilon)] \\ &\quad + AB [\varepsilon^{\frac{5}{2}} U_{23}, \varepsilon^{\frac{9}{2}} V_{23}, \varepsilon^{\frac{7}{2}} P_{23}] [1 + O(\varepsilon)] E_1 E_2 \} + \text{c.c.} \\ &\quad + |A|^2 [\varepsilon^{\frac{5}{2}} U_{25}, 0, 0] [1 + O(\varepsilon)] + |B|^2 [\varepsilon^{\frac{5}{2}} U_{26}, 0, 0] + \dots \end{aligned} \quad (4.6)$$

Here the factor $[1 + O(\varepsilon)]$ is needed because α_j and Ω_j are 'neutral' to order ε and the terms represented by \dots include the effect of the interactions between first harmonic, mean and fundamental terms. These interactions produce terms proportional to E_1 and E_2 and it is the solvability condition on the equations for the coefficients of these quantities which give the required amplitude equations for $A(\tilde{X})$ and $B(\tilde{X})$. We further expand λ in the form

$$\lambda = \lambda(x_n) + \varepsilon^{\frac{3}{2}} \frac{\lambda'(x_n)}{\lambda(x_n)} \tilde{X} + \dots, \quad (4.7)$$

to account for the variation in the shear stress away from the neutral position. In the main and upper decks similar expansions can be written down but there is no interaction between the modes so that the solutions of the equations are essentially unchanged from the linear analysis of §3. For that reason we shall not write down the expansions in these layers but merely quote the required matching conditions for the lower deck where needed. It is a routine matter to substitute the expansions (4.6), (4.7) into the Navier-Stokes equations and equate like powers of ε for each Fourier component.

The zeroth order equations for the fundamentals show that $(U_{11}, V_{11}, P_{11}), (U_{12}, V_{12}, P_{12})$ are simply the eigenfunctions of the linear problem appropriate to $(\alpha, \Omega) =$

$(\alpha_{10}, \Omega_{10})$ and $(\alpha, \Omega) = (\alpha_{20}, \Omega_{20})$ respectively. At next order the equations for the functions proportional to $E_1^2, E_2^2, E_1 E_2, E_1 \bar{E}_2$ are obtained. We find that (U_{21}, V_{21}, P_{21}) satisfies

$$\begin{aligned} \{-2i\Omega_{10} + 2i\alpha_{10}\lambda(x_n)y\}U_{21} + V_{21}\lambda &= -2i\alpha_{10}P_{21} + U_{21}'' - \{i\alpha_{10}U_{11}^2 + V_{11}U_{11}'\}, \\ 2i\alpha_{10}U_{21} + V_{21}' &= 0, \\ U_{21} = V_{21} = 0, Y &= 0, \end{aligned} \quad (4.8a, b, c)$$

and the required matching conditions imposed by the main deck are

$$U_{21} \rightarrow A_{21}\lambda(x_n), Y \rightarrow \infty; 2\alpha_{10}A_{21} = \frac{iP_{21}K_1(2i\alpha_{10}a)}{K_0(2i\alpha_{10}a)}. \quad (4.9)$$

The function $(U_{22}, V_{22}, P_{22}), (U_{23}, V_{23}, P_{23})$ and (U_{24}, V_{24}, P_{24}) satisfy similar inhomogeneous differential systems with the wavenumber and frequency changed appropriately and the nonlinear term on the right hand side of (4.8a) replaced by $-\{i\alpha_{20}U_{12}^2 + V_{12}U_{12}'\}$, $-\{i(\alpha_{10} + \alpha_{20})U_{11}V_{12} + V_{12}U_{11}' + V_{11}U_{12}'\}$ and $-\{i(\alpha_{10} - \alpha_{20})U_{11}\bar{U}_{12} + \bar{V}_{12}U_{11}' + V_{11}\bar{U}_{12}'\}$ respectively. In the absence of the inhomogeneous term the above systems would of course not have a solution. Finally at this order the mean flow correction functions U_{25} and U_{26} are found to satisfy

$$\begin{aligned} U_{25}'' &= V_{11}\bar{U}_{11}' + \bar{V}_{11}U_{11}', & U_{26}'' &= V_{12}\bar{U}_{12}' + \bar{V}_{12}U_{12}', \\ U_{25} = U_{26} &= 0, & y &= 0; & U_{25}', U_{26}' &\rightarrow 0, & Y &\rightarrow \infty. \end{aligned}$$

The next order system in the expansion procedure leads to equations which determine α_{11}, α_{21} the correction terms in the expansions of the neutral wavenumbers. However the system obtained next involves terms proportional to E_1 and E_2 produced by nonlinear interactions and nonparallel effects. Thus, unless the required orthogonality condition is satisfied, there will not be a solution of the equations obtained at this order. After some manipulation the required solvability conditions yield

$$\begin{aligned} \frac{dA}{d\tilde{X}} &= -c_1\tilde{X}A + a_1A|A|^2 + b_1A|B|^2, \\ \frac{dB}{d\tilde{X}} &= -c_2\tilde{X}B + a_2B|B|^2 + b_2B|A|^2. \end{aligned} \quad (4.10a, b)$$

Here c_1 and c_2 are given by (4.2) with $(\Omega, \alpha) = (\Omega_{10}, \alpha_{10})$ and $(\Omega, \alpha) = (\Omega_{20}, \alpha_{20})$ respectively whilst a_1, b_1, a_2, b_2 are defined by

$$\begin{aligned} a_1 &= \int_0^\infty r_1(Y) F'_1(Y) dY/d_1, \quad b_1 = \int_0^\infty r_1(Y) F'_2(Y) dY/d_1, \\ d_1 &= \{-r_1(0) \frac{K_0(i\alpha_{10}a)}{K_1(i\alpha_{10}a)} [2i\alpha_{10} + \alpha_{10}^2 a (\frac{K_1(i\alpha_{10}a)}{K_0(i\alpha_{10}a)} + \frac{K'_1(i\alpha_{10}a)}{K_1(i\alpha_{10}a)})] U_{11}(\infty) \\ &\quad + \int_0^\infty (Y \lambda(x_n) U_{11})' r_1(Y) dY\}, \\ a_2 &= \int_0^\infty r_2(Y) G'_1(Y) dY/d_2, \quad b_2 = \int_0^\infty r_2(Y) G'_2(Y) dY/d_2, \\ d_2 &= \{-r_2(0) \frac{K_0(i\alpha_{20}a)}{K_1(i\alpha_{20}a)} [2i\alpha_{20} + \alpha_{20}^2 a (\frac{K_1(i\alpha_{20}a)}{K_0(i\alpha_{20}a)} + \frac{K'_1(i\alpha_{20}a)}{K_1(i\alpha_{20}a)})] U_{12}(\infty) \\ &\quad + \int_0^\infty (Y \lambda(x_n) U_{12})' r_2(Y) dY\}. \end{aligned} \tag{4.11a-f}$$

Here F_1, F_2, G_1, G_2 are given by

$$\begin{aligned} F_1 &= -\{i\alpha_{10}(U_{24}U_{11} + U_{21}\bar{U}_{11}) + V_{11}U'_{25} + V_{21}\bar{U}'_{11} + \bar{V}_{11}U'_{21}\}, \\ F_2 &= -\{i\alpha_{10}U_{12}U_{24} + i\alpha_{10}U_{26}U_{11} + i\alpha_{10}U_{23}\bar{U}_{12} + V_{12}U'_{24} \\ &\quad + V_{23}\bar{U}'_{12} + V_{24}U'_{12} + V_{11}U'_{26} + \bar{V}_{12}U'_{23}\}, \\ G_1 &= -\{i\alpha_{20}(U_{26}U_{12} + U_{22}\bar{U}_{12}) + V_{12}U'_{26} + V_{22}\bar{U}'_{12} + \bar{V}_{12}U'_{22}\}, \\ G_2 &= -\{i\alpha_{20}U_{11}\bar{U}_{24} + i\alpha_{20}U_{25}U_{12} + i\alpha_{20}U_{23}\bar{U}_{11} + V_{11}\bar{U}'_{24} \\ &\quad + V_{23}\bar{U}'_{11} + \bar{V}_{24}U'_{11} + \bar{V}_{11}U'_{23} + V_{12}U'_{25}\}, \end{aligned} \tag{4.12a, b, c, d}$$

whilst $r_1(Y)$ and $r_2(Y)$ are the adjoint eigenfunctions defined by

$$\frac{d}{dY} \begin{pmatrix} p_j \\ q_j \\ r_j \end{pmatrix} = - \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -i\Omega_{j0} + \lambda(x_n)i\alpha_{j0}Y \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} p_j \\ q_j \\ r_j \end{pmatrix}$$

$$q_j = 0, \text{ on } y = 0; \quad r_j = 0, \text{ as } Y \rightarrow \infty, \quad p_j(\infty) = \frac{\alpha_{j0}^2 K_0(i\alpha_{j0}a) r_j(0)}{\lambda K_1(i\alpha_{j0}a)}, \quad j = 1, 2. \tag{4.13}$$

5. Results and Discussion

Following Hall and Smith (3) it is an easy matter to generalize (4.10) to the case where the wave has moved a distance $O(h^2)$ where h is small compared to unity but large compared to $\varepsilon^{\frac{5}{4}}$. Here the variable \tilde{X} is now defined by

$$\tilde{X} = \frac{(x - x_n)}{\varepsilon^3} h^2,$$

and the size of the disturbances discussed in §4 are now changed from, for example, $u \sim \varepsilon^{\frac{7}{4}}$ in the lower deck to $u \sim \varepsilon h$. We then find that (4.10) becomes

$$\begin{aligned}\frac{dA}{d\tilde{X}} &= -c_1\Delta A + a_1A |A|^2 + b_1A |B|^2, \\ \frac{dB}{d\tilde{X}} &= -c_2\Delta B + a_2B |B|^2 + b_2B |A|^2,\end{aligned}\tag{5.1a, b}$$

where $\lambda = \lambda(x_n) + \Delta h^2 + \dots$. A further generalization of (5.1) is obtained by detuning the frequency of either of the Tollmien-Schlichting waves by an amount sufficient to introduce further linear terms in these equations. Without any loss of generality we suppose that the 'A' mode is determined so that (5.1) becomes

$$\begin{aligned}\frac{dA}{d\tilde{X}} &= -c_1\Delta A + g_1A + a_1A |A|^2 + b_1A |B|^2, \\ \frac{dB}{d\tilde{X}} &= -c_2\Delta B + a_2B |B|^2 + b_2B |A|^2.\end{aligned}\tag{5.2}$$

Here g_1 is a complex constant with $g_{1r} > 0$ if the A mode is the most unstable and $g_{1r} < 0$ otherwise. In order to discuss the solution of (5.1) we write

$$\delta = |A|^2, \quad \psi = |B|^2,$$

so that

$$\begin{aligned}\frac{1}{2} \frac{d\delta}{d\tilde{X}} &= (-c_{1r}\Delta + g_{1r})\delta + a_{1r}\delta^2 + g_{1r}\delta\psi, \\ \frac{1}{2} \frac{d\psi}{d\tilde{X}} &= -c_{2r}\Delta\psi + a_{2r}\psi^2 + b_{2r}\delta\psi.\end{aligned}\tag{5.3a, b}$$

Apart from the trivial solution $\delta = \psi = 0$ the following finite amplitude solutions are possible.

- a. $\delta = 0 \quad \psi = c_{2r}\Delta a_{2r}^{-1}$
- b. $\delta = (c_{1r}\Delta - d_{1r})a_{1r}^{-1}, \quad \psi = 0,$
- c. $\delta = [a_{2r}(c_{1r}\Delta - d_{1r}) - c_{2r}b_{1r}][a_{1r}a_{2r} - b_{1r}b_{2r}]^{-1},$
 $\psi = [c_{2r} - \delta b_{2r}]a_{2r}^{-1}.$

Here a subscript r denotes the real part of a quantity. We refer to a, b, c as the 'pure B', 'pure A' and the mixed mode respectively. The range of values of Δ for which the different solutions exist and their stability properties depend on the coefficients in (5.3). This problem has been discussed by Keener (12) for the case when $c_{1r}, c_{2r}, a_{1r}, a_{2r}, b_{1r}$ and b_{2r} are all negative.

The results in Table I show that $\frac{a_{2r}}{b_{1r}} < \frac{c_{2r}}{c_{1r}} < \frac{b_{2r}}{a_{1r}}$ for all the values of a used in the calculations. It is straightforward to show that in this case whichever mode bifurcates first remains stable at all values of Δ and suffers no secondary bifurcation to the mixed mode. The second 'pure' mode to bifurcate is initially unstable but then undergoes secondary

bifurcation to the mixed mode. The mixed mode is always unstable but the pure mode from which it bifurcates is stable after the bifurcation. Thus at sufficiently high values of Δ either pure mode is a possible stable state. However if the modes are set up by increasing Δ we would of course expect to observe just the most unstable linear mode. These results are illustrated in Figure 3 for the case where the pure A mode bifurcates first. When a tends to its critical value from above the two modes become identical so that for example $a_1 \rightarrow b_2$ and $2a_1 \rightarrow b_1$. This served as a useful check on the calculations.

References

1. Smith, F.T. 1979 Proc. Roy. Soc. (A) 368, 573.
2. Smith, F.T. 1979 Proc. Roy. Soc. (A) 366, 91.
3. Hall, P. and Smith, F.T. 1984 Studies in Applied Maths 70, 91.
4. Smith, F.T. and Burggraf, O. 1985 Proc. Roy. Soc. (A) 399, 25.
5. Smith, F.T. and Stewart, P. 1987 Proc. Roy. Soc. (A) 409, 229.
6. Kluwick, A., Gittler, P. and Bodonyi, R. 1984 J. Fluid Mech. 140, 281.
7. Ward, G.N. 1955 Linearized theory of high speed flows, CUP.
8. Ryzhov, O. and Zhuk, V.I. 1980 J. de Mecanique 19, 561.
9. Smith, F.T. 1982 JIMA 28, 207.
10. Bogdanova, E.V. and Ryzhov, O.S. QJMAM 36, 271.
11. Duck, P.W. 1984 QJMAM 37, 57.
12. Keener, J. 1976 Studies in Applied Maths 55, 187.

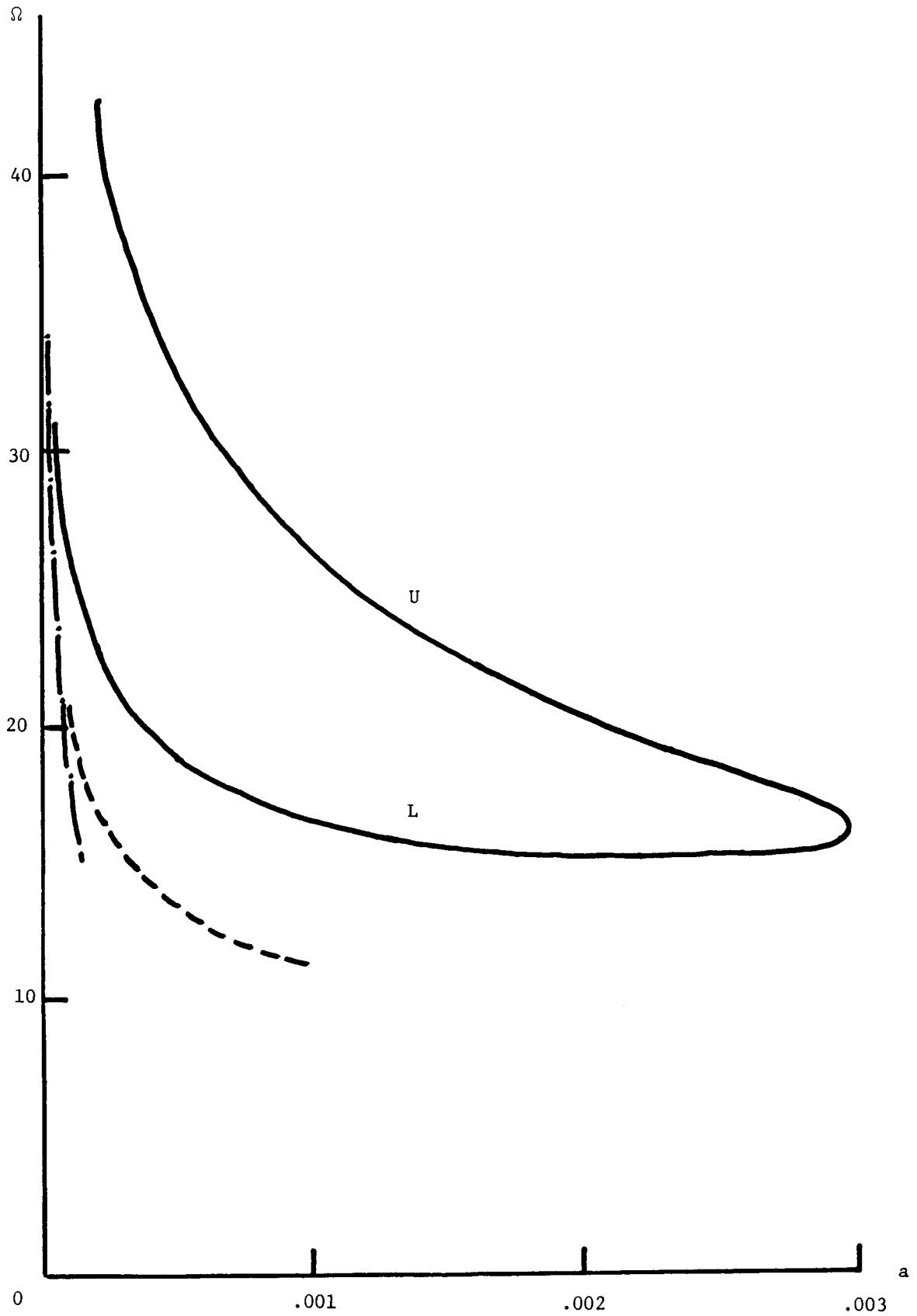


Figure 1. The variation of the neutral wavenumber with a .

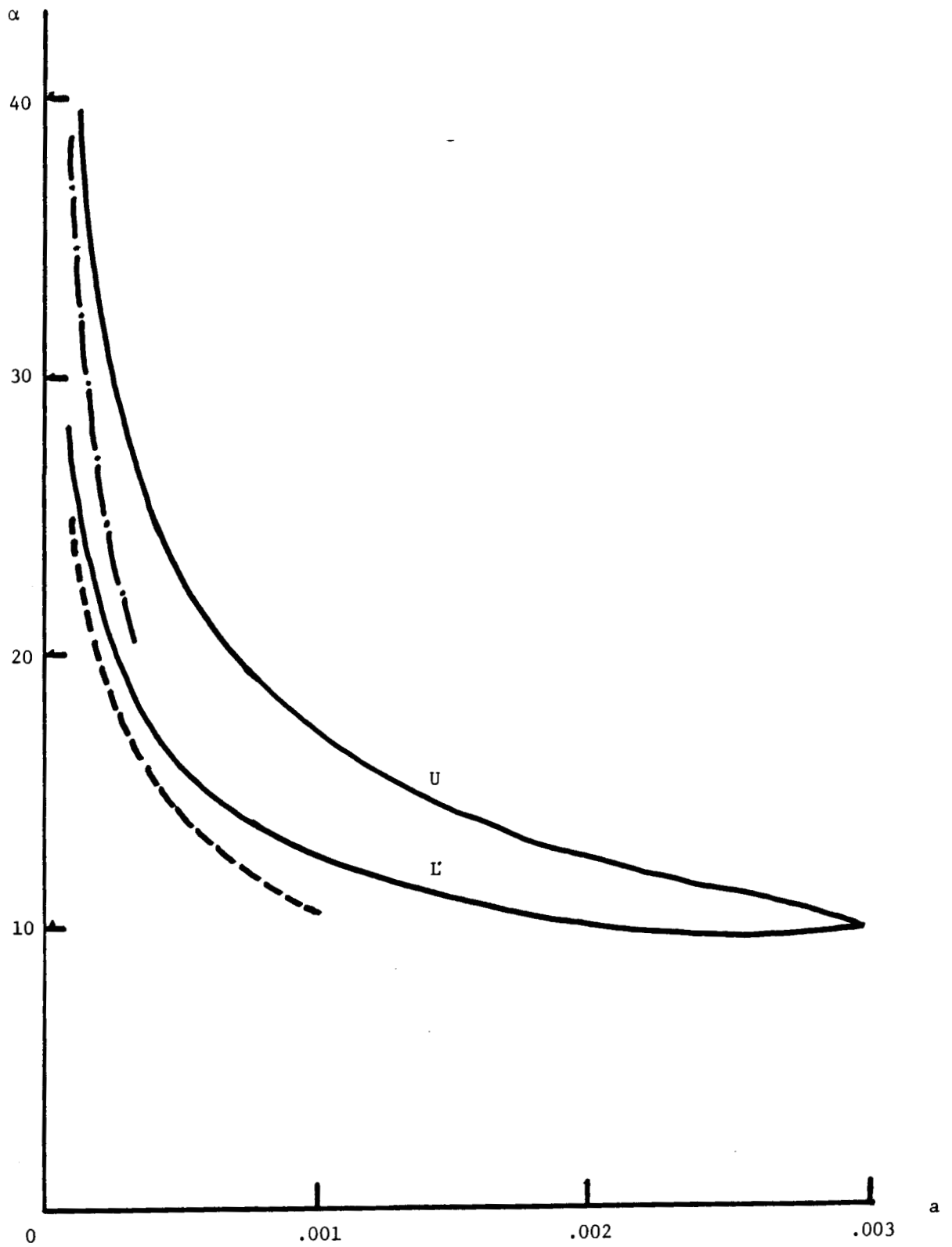


Figure 2. The variation of the neutral frequency with a .

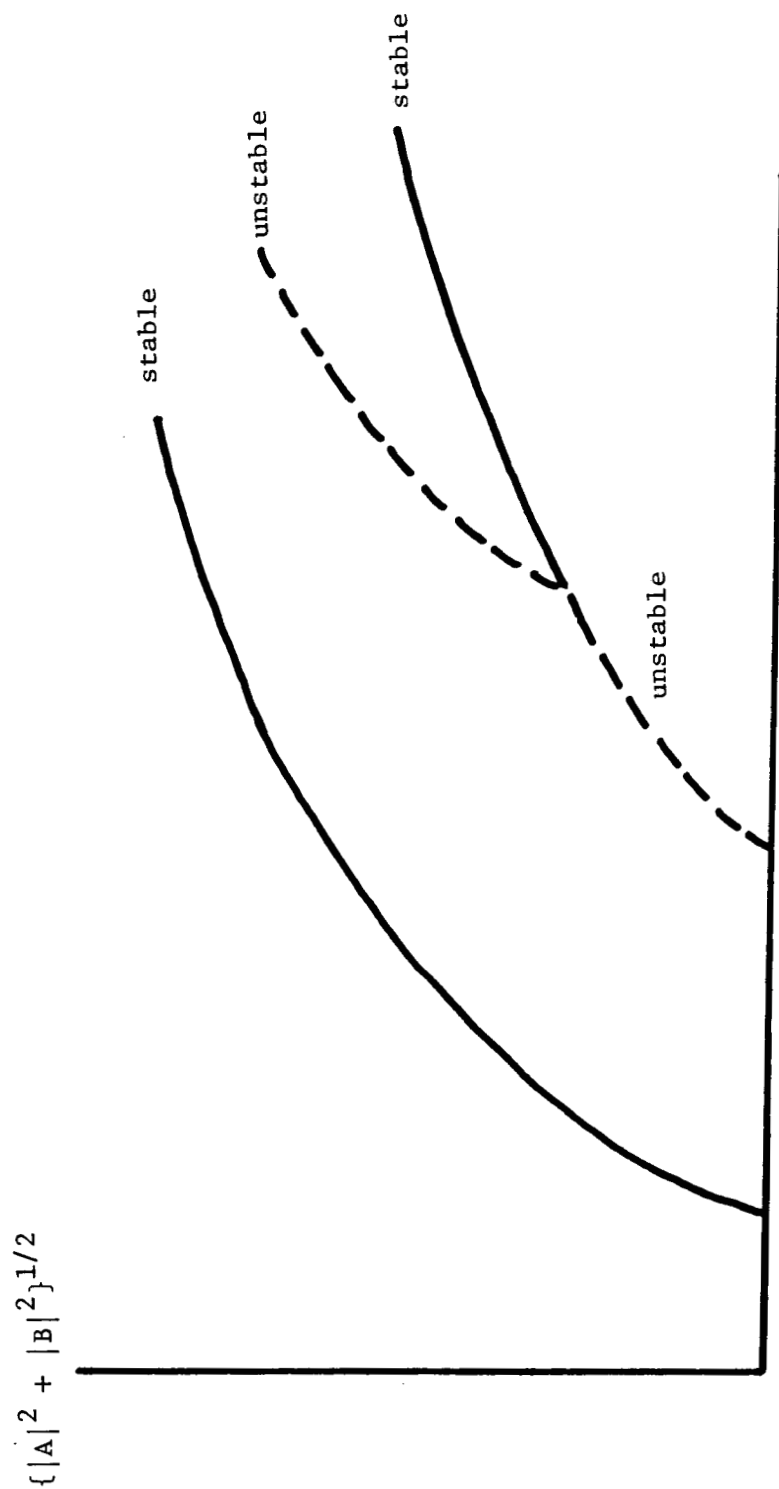


Figure 3. The equilibrium solutions of (5.3).

TABLE I

a	a_r/b_r	b_r/a_r	c_r/c_r
.0028	.590	1.684	.935
.0026	.592	1.672	.916
.0024	.592	1.666	.907
.0022	.589	1.660	.903
.0020	.585	1.655	.903
.0018	.580	1.651	.905
.0016	.574	1.647	.904
.0014	.567	1.645	.915
.0012	.559	1.643	.922
.0010	.549	1.642	.931
.0008	.537	1.643	.943
.0006	.526	1.642	.952



Report Documentation Page

1. Report No. NASA CR-181619 ICASE Report No. 88-10		2. Government Accession No.		3. Recipient's Catalog No.	
4. Title and Subtitle ON THE INTERACTION OF TOLLMIE-SCHLICHTING WAVES IN AXISYMMETRIC SUPERSONIC FLOWS				5. Report Date January 1988	
				6. Performing Organization Code	
7. Author(s) P. W. Duck and P. Hall				8. Performing Organization Report No. 88-10	
				10. Work Unit No. 505-90-21-01	
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665-5225				11. Contract or Grant No. NAS1-18107	
				13. Type of Report and Period Covered Contractor Report	
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Langley Research Center Hampton, VA 23665-5225				14. Sponsoring Agency Code	
15. Supplementary Notes Langley Technical Monitor: Richard W. Barnwell Final Report Submitted to Quarterly Journal of Mechanics and Applied Math.					
16. Abstract It is known that two-dimensional lower branch Tollmien-Schlichting waves described by triple-deck theory are always stable for planar supersonic flows. Here the possible occurrence of axisymmetric unstable modes in the supersonic flow around an axisymmetric body is investigated. In particular flows around bodies with typical radii comparable with the thickness of the upper deck are considered. It is shown that such unstable modes exist below a critical nondimensional radius of the body a_0 . At values of the radius above a_0 all the modes are stable whilst if unstable modes exist they are found to occur in pairs. The interaction of these modes in the nonlinear regime is investigated using a weakly nonlinear approach and it is found that, dependent on the frequencies of the imposed Tollmien-Schlichting waves, either of the modes can be set up.					
17. Key Words (Suggested by Author(s)) axisymmetric, supersonic, Tollmien-Schlichting			18. Distribution Statement 34 - Fluid Mechanics and Heat Transfer Unclassified - unlimited		
19. Security Classif. (of this report) Unclassified		20. Security Classif. (of this page) Unclassified		21. No. of pages 20	22. Price A02