NASA Contractor Report 181620

ICASE REPORT NO. 88-11

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(NASA-CR-181620) DOMAIN DECOMPOSITION PRECONDITIONERS FOR THE SPECTRAL COLLOCATION METHOD Final Report (NASA) 38 p CSCL 12A

N88-18345

Unclas

G3/64 0126002

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Contract No. NAS1-18107

January 1988

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Operated by the Universities Space Research Association

National Aeronautics and Space Administration

Langley Research Center
Hampton, Virginia 23665

DOMAIN DECOMPOSITION PRECONDITIONERS FOR THE SPECTRAL COLLOCATION METHOD

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ABSTRACT

We propose and analyze several block iteration preconditioners for the solution of elliptic problems by spectral collocation methods in a region partitioned into several rectangles. It is shown that convergence is achieved with a rate which does not depend on the polynomial degree of the spectral solution. The iterative methods here presented can be effectively implemented on multiprocessor systems due to their high degree of parallelism.

Research was supported under the National Aeronautics and Space Administration under NASA Contract No. NAS1-18107 while the first author was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23665.

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INTRODUCTION

We consider the numerical solution of elliptic problems by spectral collocation methods in regions partitioned into several rectangles (also called subdomains). The interaction between the subdomains is handled by proper iterative methods.
Such iterative domain decomposition methods are particulary attractive in the context of spectral approximations. Actually, they allow the reduction of a problem set in a complicated geometry to a sequence of problems of similar type but with smaller size in every subdomain. The latter can therefore be faced by standard monodomain spectral solvers. Besides, due to their (hopefully high) degree of parallelism, these methods can be advantageously implemented on multiprocessor systems.
To be effective, the rate of convergence of the iterative method should not deteriorate as the polynomial degree N of the numerical solution in each subdomain increases. This is what we prove for our method which alternates the solution of Dirichlet problems on the odd subdomains to that of mixed Neumann-Dirichlet problems on the even ones. Our proof applies to any domain Ω decomposed into rectangles without internal crosspoints, i.e., whose vertices lie on the boundary of Ω (see fig.0.1). In the last section we consider a decomposition with internal cross points; in this case it is possible to prove the convergence of the iterative method for fixed N, although the rate of convergence decreases if N increases. This convergence behavior with respect to the decomposition is shared by other multidomain approaches based on finite elements or finite differences. We refer, e.g., to [4], [5].

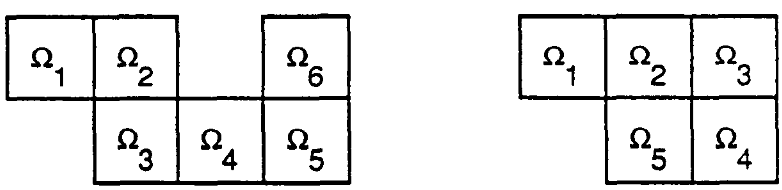


FIG.0.1 a) Decomposition without internal cross points b) Decomposition with an internal cross point

The results we obtain generalize those of [15] which were established with a different proof technique and were limited to the case of a rectangle divided into two subdomains only.
The outline of this paper is the following. In section 1 we write the differential problem in a variational form which is set up on the multidomain decomposition of the physical domain.

In section 2 we introduce the multidomain collocation problem. This amounts to collocating the differential problem at the Chebyshev (or Legendre) collocation points internal to each subdomain, and to enforcing, at the interface-boundary points, the continuity of the solution as well as of a suitable combination of the residual of the equation and the normal flux. This method has been proposed by Funaro [13] and retains the same convergence properties of the more classical patching method introduced by Orszag [23]. In particular, as shown in [13] and [19], the numerical solution converges with spectral accuracy to the physical solution as the polynomial degree N grows on each subdomain. In section 2 we also define an "iteration-by-subdomains" method for solving the discrete multidomain decomposition problem. Each step of this iterative method is based on the solution of Dirichlet problems in the odd subdomains, and mixed problems in the even ones. In section 3 we give a variational formulation of both the multidomain decomposition problem and the iteration-by-subdomain method. These variational formulations allow us to prove, in section 4, the convergence of the iterative method, the rate of convergence being independent of the degree of the polynomial solution in each subdomain. In section 5 we show that the iteration-by-subdomain method here analyzed amounts to solving the capacitance system governing the interface unknowns by a linear iteration procedure using a proper block diagonal preconditioner. Going further along this equivalence, we propose in section 6 other preconditioners for the capacitance system, which in turn gives rise to some new iteration-by-subdomains algorithms. Finally in section 7 we consider a decomposition of the domain with internal cross points and we state how our method can be formulated and analyzed in this case.

Numerical experiments on this iteration-by-subdomain method are presented in [13], [28].

Earlier works on algorithms of this nature for finite element and finite difference domain decomposition methods are reviewed and analyzed in [26].

1. MULTIDOMAIN FORMULATION OF THE DIFFERENTIAL PROBLEM

Let Ω be an open two dimensional domain with boundary $\partial\Omega$. We consider the boundary-value-problem

(1.1) $$Lu = f \quad \text{in } \Omega \;, \quad u = 0 \quad \text{on } \partial\Omega$$

where f is a given function of $L^2(\Omega)$ and

(1.2) $$Lu := -\Delta u + \alpha_0 u \quad , \quad \alpha_0(x) \geq 0 \quad \forall x \in \Omega \;, \quad \alpha_0 \in C^0(\Omega)$$

In variational form (1.1) reads as

(1.3) $$u \in H_0^1(\Omega): \quad a(u,v) := \int_\Omega (\nabla u\, \nabla v + \alpha_0 uv)dx = \int_\Omega fv dx \quad \forall\, v \in H_0^1(\Omega) \;.$$

The Lax-Milgram theorem ensures that there exists a unique solution to (1.3); moreover, $u \in H^2(\Omega)$, hence (1.1) holds almost everywhere in Ω (see, e.g., [18]).

Let us consider a partition of Ω by non intersecting open subdomains Ω_i, i = 1,...,M, as in fig.1.1, and denote by Γ_i the common boundary between Ω_i and Ω_{i+1}, $i \leq M-1$. Then set $\Gamma = \cup\Gamma_i$, and, for each i, we denote with $\Phi := \prod_{i=1}^{M-1} H_{00}^{1/2}(\Gamma_i)$ the space of functions defined on Γ, which are the traces on Γ of functions of $H^1{}_0(\Omega)$ (see [18]). Note that these functions vanish at the endpoints of Γ_i for i=1,...,M-1.

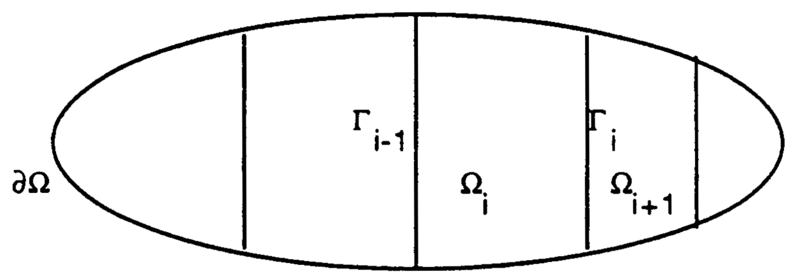


Fig.1.1: The partition of Ω

For any $\varphi \in \Phi$, let $\varphi \in H^1{}_0(\Omega)$ denote any extension of φ to Ω. An example is furnished by the "harmonic" extension $R\varphi$, whose restriction $R_i\varphi$ to Ω_i satisfies the Dirichlet problem:

$$(1.4) \qquad \begin{cases} a_i(R_i\varphi, v) = 0 & \forall\, v \in H^1_0(\Omega_i) \quad , \\ R_i\varphi = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \quad , \\ R_i\varphi = \varphi & \text{on } \Gamma_{i-1} \cup \Gamma_i \quad , \end{cases}$$

where we have set

$$a_i(u,v) = \int_{\Omega_i} (\nabla u \, \nabla v + \alpha_0 \, uv)\, dx \quad .$$

The following equivalence statement is readily proven.

<u>Proposition 1.1</u> <u>Problem</u> (1.3) <u>is equivalent to find</u> $u \in H^1{}_0(\Omega)$ <u>so that</u> $u_i := u_{|\Omega_i}$, $1 \le i \le M$, <u>are solutions to</u>:

$$(1.5) \qquad a_i(u_i, v) = \int_{\Omega_i} f v dx \qquad \forall\, v \in H^1_0(\Omega_i) \;,\; 1 \le i \le M \quad ,$$

$$(1.6) \qquad u_i = u_{i+1} \qquad \text{on } \Gamma_i \qquad , \; 1 \le i \le M\text{-}1 \, ,$$

$$(1.7) \qquad u_i = 0 \qquad \text{on } \partial\Omega_i \cap \partial\Omega \quad , \; 1 \le i \le M \quad ,$$

$$(1.8) \qquad \sum_{\substack{i=1 \\ i \text{ odd}}}^{M} \left\{ a_i(u_i, \varphi) - \int_{\Omega_i} f\varphi dx \right\} = - \sum_{\substack{j=2 \\ j \text{ even}}}^{M} \left\{ a_j(u_j, \varphi) - \int_{\Omega_j} f\varphi dx \right\} \quad \forall \varphi \in \Phi \, .$$

Clearly, (1.5) amounts to require that

$$(1.9) \qquad Lu_i = f \quad \text{in } \Omega_i \quad , \quad 1 \le i \le M$$

while (1.8) is equivalent to the *transmission condition*

$$(1.10) \qquad \frac{\partial u_i}{\partial \upsilon} = \frac{\partial u_{i+1}}{\partial \upsilon} \quad \text{on } \Gamma_i \quad , \quad 1 \le i \le M\text{-}1$$

(υ is the outward unit vector to Γ_i).

2. SPECTRAL COLLOCATION APPROXIMATION TO THE MULTIDOMAIN PROBLEM

In this section we consider a spectral collocation approximation of problem (1.5)-(1.8) which consists of collocating the differential equations (1.9) at Gaussian collocation points internal to Ω_i and to enforcing the interface conditions at some selected points of Γ. Either Gaussian-Legendre and Gaussian-Chebyshev points will be considered. We also give a variational formulation of the multidomain problem, then we define an "iteration-by-subdomains" procedure to solve effectively this problem.

We confine ourselves to the case of a rectangular domain Ω partitioned into M non intersecting rectangles Ω_i without internal cross points (see fig.2.1), but what we are proving can be extended to more general situations as well (cf. remarks 3.1 and 4.2).

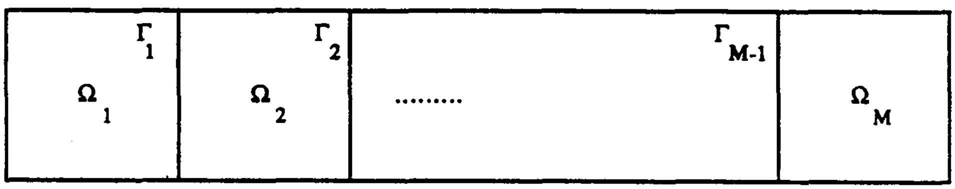


FIG.2.1 Decomposition by aligned subdomains

Let D be the reference domain $(-1,1)^2$ and let N_x, N_y be two given natural numbers. The Legendre Gauss Lobatto collocation points in D are the roots $\{\zeta_{km},\ 0 \le k \le N_x,\ 0 \le m \le N_y\}$ of the polynomial

$$\frac{\partial}{\partial x} L_{N_x}(x) \frac{\partial}{\partial y} L_{N_y}(y) \ ,$$

where $L_k(t)$ is the Legendre polynomial of degree k in $[-1,1]$. In the Chebyshev case, the Gauss Lobatto collocation points in D are given by

$$\zeta_{km} = \left(-\cos \frac{\pi k}{N_x}, -\cos \frac{\pi m}{N_y}\right) \ , \quad \text{for } 0 \le k \le N_x \ , \quad 0 \le m \le N_y \ .$$

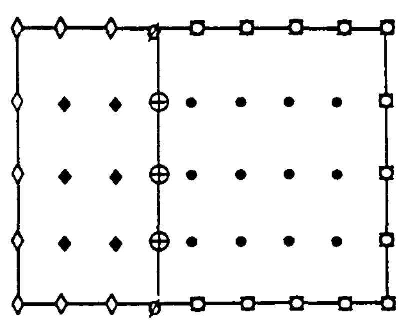
We remind that, in both Legendre and Chebyshev cases, ζ_{km} lies on the boundary of D if either k equals 0 or N_x, or m equals 0 or N_y. Moreover

$$C_i := \{(x_k^i , y_m) , 0 \le k \le N_x^i , 0 \le m \le N_y\} \text{ , for } i=1,...,M \text{ ,}$$

will denote the set of collocation points in the domain Ω_i. Given M natural numbers N^i_x, i = 1,...,M, we take the points of C_i as the images of the Legendre (or Chebyshev) points in D, with $N_x = N^i_x$, through the affine transformation which maps D into Ω_i. For the convenience of exposition we set

$$C_i^{int} := C_i \cap \Omega_i \text{ , } C_i^b := C_i \cap \partial\Omega \text{ , } C^{\Gamma_i} := C_i \setminus (C_i^{int} \cup C_i^b) \text{ for } i=1,...,M \text{ .}$$

See fig.2.2 for an example with M=2.



◊ points of C^b_1

□ points of C^b_2

♦ points of C_1^{int}

• points of C_2^{int}

⊕ points of C^{Γ_1}

ø points of $C^b_1 \cap C^b_2$

Fig.2.2 Interior and Boundary Collocation Points

For i=1,...,M we denote by $P_N(\Omega_i)$ the space of polynomials of degree N^i_x with respect to the x variable and degree N_y with respect to the y one. Moreover we set

$$P_N^0 (\Omega_i) := \{p \in P_N (\Omega_i) : p = 0 \text{ on } \partial\Omega_i\} \text{ .}$$

For the convenience of the reader we recall the Legendre and Chebyshev Gauss-Lobatto formulae that will be extensively used in the sequel (see also [10]).

Legendre Gauss-Lobatto formula

$$(2.1)\qquad \int_{\Omega_i} g(x,y)\,dxdy \cong \sum_{k=0}^{N_x^i} \sum_{m=0}^{N_y} g(x_k^i, y_m)\, \omega_k^i\, \omega_m \quad ,$$

where

$$(2.2)\qquad \omega_k^i = \frac{a_i}{N_x^i\,(N_x^i+1)} \cdot \frac{1}{[L_{N_x}(x_k^i)]^2} \quad \text{and} \quad \omega_m = \frac{2}{N_y(N_y+1)} \cdot \frac{1}{[L_{N_y}(y_m)]^2} \quad ,$$

We have set $a_i = x_i - x_{i-1}$, where x_i denotes the abscissa of Γ_i.

Chebyshev Gauss-Lobatto formula

$$(2.3)\qquad \int_{\Omega_i} g(x,y)\,\omega^i(x,y)dxdy \cong \sum_{k=0}^{N_x^i} \sum_{m=0}^{N_y} g(x_k^i,y_m)\, \omega_k^i\, \omega_m \quad ,$$

where

$$(2.4)\qquad \begin{cases} \omega^i(x,y) = [(x - x_{i-1})(x_i - x)]^{-1/2}\,(1 - y^2)^{-1/2} \quad , \\ \omega_k^i = \dfrac{a_i\pi}{2N_x^i} \ \text{for } k=1,\ldots,N_x^i - 1 \ ,\ \omega_k^i = \dfrac{a_i\pi}{4N_x^i} \ \text{for } k=0 \text{ and } k=N_x^i \quad , \\ \omega_m = \dfrac{\pi}{N_y} \ \text{for } m=1,\ldots,N_y - 1 \ ,\ \omega_m = \dfrac{\pi}{2N_y} \ \text{for } m=0 \text{ and } m=N_y \quad . \end{cases}$$

We can now introduce the *spectral collocation approximation* to the problem (1.1), which fits well the multidomain formulation (1.5)-(1.8). The spectral solution u_N verifies:

$$(2.5)\qquad u_{i,N} := u_{N|\Omega_i} \,\varepsilon\, P_N(\Omega_i) \ ,\ i=1,\ldots,M \quad ,$$

$$(2.6)\qquad Lu_{i,N} = f \qquad \text{in } C_i^{int} \quad ,\ i=1,\ldots,M \quad ,$$

$$(2.7)\qquad u_{i,N} = u_{i+1,N} \qquad \text{in } C^{\Gamma_i} \quad ,\ i=1,\ldots,M-1,$$

$$(2.8)\qquad u_{i,N} = 0 \qquad \text{in } C_i^b \quad ,\ i=1,\ldots,M \quad ,$$

$$(2.9)\qquad \frac{\partial u_{i,N}}{\partial x} - \frac{\partial u_{i+1,N}}{\partial x} = -\,(Lu_{i,N} - f)\,\omega_i^- - (Lu_{i+1,N} - f)\,\omega_i^+ \quad \text{in } C^{\Gamma_i} \quad ,\ i=1,\ldots,M-1 \quad .$$

Here we have set $\omega_i^- = \omega^i_{N^i_x}$ and $\omega_i^+ = \omega_0^{i+1}$ for i=1,...,M-1. We are assuming f to be a continuous function in order that (2.6) and (2.9) make sense. Note that at the interface collocation points, the jump of the normal derivative is not asked to vanish (as in (1.10)), but rather to balance a suitable linear combination of the residual of the equation from both sides. We observe, however, that the coefficients of such combination tend to zero if N^i_x tends to infinity. This follows from the definitions of the weights ω^i_k of the Gauss-Lobatto formula given in (2.2) and (2.4).

From a computational point of view, dealing with condition (2.9) rather than with the pure flux condition $\frac{\partial}{\partial x} u_{i,N} = \frac{\partial}{\partial x} u_{i+1,N}$ does not bring any appreciable extra work. On the other hand, a considerable advantage is achieved for the theoretical properties of the scheme, as is will be pointed out in the forthcom

Remark 2.1 The collocation scheme (2.5)-(2.9) has been first proposed by Funaro [13], who stressed its equivalence with a suitable variational approximation to (1.5)-(1.8) (see below). This equivalence allows one primarily to investigate the convergence properties, for N^i_x, $N_y \to \infty$, of the method.

The most relevant result (for the Legendre collocation) is the following (see [13] and [19]).

Setting $N_i = \min(N^i_x, N_y)$ and assuming that $u_i \in H^{\sigma_i}(\Omega_i)$ with $\sigma_i \geq 1$, for i=1,...,M, the following error estimate holds

$$
(2.10) \qquad \| u_i - u_{i,N} \|_{H^1(\Omega_i)} \leq C \sum_{j=1}^{M} N_j^{1-\sigma_j} \left(\| u_j \|_{H^{\sigma_j}(\Omega_j)} + \| f \|_{H^{\sigma_j - 1}(\Omega_j)} \right)
$$

where C is a positive constant independent of N_j and u_j, for j=1,...,M. In particular this estimate shows that if f = 0 then $u_{i,N} = 0$ for i = 1, ..., M, hence the discrete problem has a unique solution.

Another implication of (2.10) is that as N^i_x, $N_y \to \infty$ the jumps in the normal derivatives $\frac{\partial}{\partial x} u_{i,N} - \frac{\partial}{\partial x} u_{i+1,N}$ at the interface points vanish with spectral rate, hence so does the right hand side of (2.9). In [13] it is also shown by numerical experiments that using (2.9) rather than the pure flux condition does not reduce the accuracy of the results. ◊

We now introduce an iteration-by-subdomains procedure for the solution of the above collocation problem. Let us suppose, here and in the sequel, that M is an odd number; the case of M even can be studied analogously. We set $\Gamma = \cup \Gamma_i$, and define

$$P^{o}_{N}(\Gamma) := \{p : \Gamma \to R : p_i := p_{|\Gamma_i} \text{ is a polynomial of degree} \le N_y \text{ on } \Gamma_i,$$
$$\text{vanishing at the endpoints of } \Gamma_i\}.$$

For a given $g^0 \in P^0_N(\Gamma)$ we look for a sequence $u^n_{i,N}, n \ge 1$, satisfying: $u^n_{i,N} \in P_N(\Omega_i)$, $i=1,\dots,M$, and:

<u>for i odd</u>:

(2.11) $\quad Lu^n_{i,N} = f \qquad \text{in } C_i^{int}$,

(2.12) $\quad u^n_{i,N} = g^{n-1} \quad \text{in } C^{\Gamma_{i-1}} \cup C^{\Gamma_i}$ for $i \ne 1, M$, $u^n_{1,N} = g^{n-1}$ in C^{Γ_1}, $u^n_{M,N} = g^{n-1}$ in $C^{\Gamma_{M-1}}$,

(2.13) $\quad u^n_{i,N} = 0 \qquad \text{in } C^b_i$;

<u>for i even</u>:

(2.14) $\quad Lu^n_{i,N} = f \qquad \text{in } C_i^{int}$,

(2.15) $$\frac{\partial u^n_{i,N}}{\partial x} - (Lu^n_{i,N} - f)\,\omega^+_{i-1} = \frac{\partial u^n_{i-1,N}}{\partial x} + (Lu^n_{i-1,N} - f)\,\omega^-_{i-1} \qquad \text{in } C^{\Gamma_{i-1}} ,$$

(2.16) $$\frac{\partial u^n_{i,N}}{\partial x} + (Lu^n_{i,N} - f)\,\omega^-_{i} = \frac{\partial u^n_{i+1,N}}{\partial x} - (Lu^n_{i+1} - f)\,\omega^+_{i} \qquad \text{in } C^{\Gamma_i} ,$$

(2.17) $\quad u^n_{i,N} = 0 \qquad \text{in } C^b_i$

Then set:

(2.18) $$g^n = \begin{cases} \theta_n u^n_{i-1,N} + (1-\theta_n)\,g^{n-1} & \text{on } C^{\Gamma_{i-1}} \quad \text{for } i=3,\dots,M \quad , \text{ i odd} \\ \theta_n u^n_{i+1,N} + (1-\theta_n)\,g^{n-1} & \text{on } C^{\Gamma_i} \quad \text{for } i=1,\dots,M-2 \quad , \text{ i odd} \end{cases}$$

and restart from (2.11) with n+1 instead of n. In (2.18), θ_n is a positive relaxation parameter; how to choose it will be dealt with in the remark 5.4.

Following this method, at each step one solves the M/2+1 *independent Dirichlet problems* (2.11)-(2.13); then, after computing the (independent) right-hand sides of (2.15)-(2.16), one has to solve the M/2 *independent mixed problems* (2.14)-(2.17). The *degree of parallelism* of this algorithm is therefore M/2.

In the simple situation of a <u>decomposition by two subdomains</u>, the above method simplifies to:

$$(2.19)\qquad Lu^n_{1,N} = f \qquad \text{in } C_1^{int},$$

$$(2.20)\qquad u^n_{1,N} = g^{n-1} \qquad \text{in } C_1^{\Gamma},$$

$$(2.21)\qquad u^n_{1,N} = 0 \qquad \text{in } C^b_1,$$

$$(2.22)\qquad Lu^n_{2,N} = f \qquad \text{in } C_2^{int}$$

$$(2.23)\qquad \frac{\partial u^n_{2,N}}{\partial x} - (Lu^n_{2,N} - f)\,\omega_1^+ = \frac{\partial u^n_{1,N}}{\partial x} + (Lu^n_{1,N} - f)\,\omega_1^- \qquad \text{in } C^{\Gamma}$$

$$(2.24)\qquad u^n_{2,N} = 0 \qquad \text{in } C^b_2,$$

and $g^n = \theta_n u^n_{2,N} + (1 - \theta_n)\, g^{n-1}$ on C^{Γ}. Here Γ is the interface between Ω_1 and Ω_2.

Remark 2.2 (More general differential operators) The multidomain approach (2.5)-(2.9), and the corresponding iteration-by-subdomain procedure (2.11)-(2.18), apply also to the more general differential operator:

$$(2.25)\qquad Lu := \frac{\partial}{\partial x}\left(\alpha_{11}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(\alpha_{12}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\alpha_{21}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\alpha_{22}\frac{\partial u}{\partial y}\right) + \alpha_0 u$$

with α_{ij} symmetric, uniformly negative definite, bounded, and smooth on Ω. In such a case, the operator L in Ω_i should be replaced by its *pseudo-spectral* realisation

$$(2.26)\qquad L_{i,N}u := \frac{\partial}{\partial x}\left[I^i_N\left(\alpha_{11}\frac{\partial u}{\partial x} + \alpha_{12}\frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[I^i_N\left(\alpha_{21}\frac{\partial u}{\partial x} + \alpha_{22}\frac{\partial u}{\partial y}\right)\right] + \alpha_0 u\,.$$

Here I^i_N v denotes the polynomial of $P_N(\Omega_i)$ which interpolates v at the collocation nodes of C_i (see [6], Ch. 2). Moreover, the conormal derivative $\alpha_{11}\partial/\partial x + \alpha_{12}\partial/\partial y$ should be used instead of the normal derivative $\partial/\partial x$ at the subdomain interfaces.

In particular, we remind that this situation occurs whenever we consider the Laplace equation within a domain Ω subdivided into subdomains with curved boundaries. The mapping of the physical subdomain Ω_i into the computational one $D=(-1,1)^2$ introduces a transformed operator which has the form (2.25).◊

Remark 2.3 The iteration procedure for two subdomains (2.19)-(2.24) has been introduced in [15] for Chebyshev collocation methods, and subsequently applied in [21] to finite element approximations. The convergence analysis was carried out in [15] for the Helmholtz operator using a separation of variable argument. In section 4, we apply a proof technique based on a suitable extension theorem, similar to the one presented in [21] for finite elements. The higher generality of this technique allows us to deal with the case of several subdomains.◊

3 VARIATIONAL FORM OF THE COLLOCATION MULTIDOMAIN PROBLEM

In this section we give a variational formulation of the collocation multidomain problem (2.5)-(2.9), and, accordingly, of the iterative procedure (2.11)-(2.18). The new formulation is more suited for the convergence analysis that will be carried out in section 4.

We confine ourselves to the case of Legendre collocation points, where we will make use of the following notation. For functions continuous in Ω_i we define a discrete inner product, approximating that of $L^2(\Omega_i)$, as follows:

$$(3.1) \qquad (w,z)_{i,N} := \sum_{k=0}^{N_x^i} \sum_{m=0}^{N_y} (wz)\,(x_k^i, y_m)\,\omega_k^i\,\omega_m \quad , \ i=1,\dots,M$$

Due to the exactness of the quadrature formula (2.1) for polynomials $g \in P_{2N-1}$, we have

$$(3.2) \qquad (w,z)_{i,N} = \int_{\Omega_i} wz\,dx \qquad \text{if } wz \in P_{2N-1}(\Omega_i) \quad , \ i=1,\dots,M$$

We also define the discrete bilinear form

$$(3.3) \qquad a_{i,N}(w,z) := \left(\frac{\partial w}{\partial x}, \frac{\partial z}{\partial x}\right)_{i,N} + \left(\frac{\partial w}{\partial y}, \frac{\partial z}{\partial y}\right)_{i,N} + (\alpha_0 w,z)_{i,N} \qquad i=1,\dots,M$$

For any $\varphi \in P^\circ_N(\Gamma)$, we denote with $\overline{\varphi} \in P_N(\Omega)$ any piecewise polynomial extension of φ to Ω which is determined by the values of φ_{i-1} and φ_i solely. $\overline{\varphi}$ can be, e.g., either the discrete harmonic extension, or the interpolant extension of φ. We can now prove the following equivalence result.

Proposition 3.1 The collocation multidomain problem (2.5) - (2.9) is equivalent to look for u_N such that $u_{i,N} := u_{N|\Omega_i} \in P_N(\Omega_i)$, $i=1,\dots,M$ and satisfies

$$(3.4) \qquad a_{i,N}(u_{i,N}, v) = (f,v)_{i,N} \qquad \forall\, v \in P^\circ_N(\Omega_i) \quad , \ i=1,\dots,M \ ,$$

$$(3.5) \qquad u_{i,N} = 0 \qquad \text{on } \partial\Omega_i \cap \partial\Omega \quad , \ i=1,\dots,M \ ,$$

$$(3.6) \qquad u_{i,N} = u_{i+1,N} \qquad \text{on } \Gamma_i \quad , \quad i=1,\dots,M-1 \quad ,$$

$$(3.7) \qquad \sum_{\substack{i=1 \\ i \text{ even}}}^{M} a_{i,N}(u_{i,N}, \overline{\varphi}) = -\sum_{\substack{j=1 \\ i \text{ odd}}}^{M} a_{j,N}(u_{j,N}, \overline{\varphi}) + \sum_{k=1}^{M} (f,\overline{\varphi})_{k,N} \quad \forall \varphi \in P_N^0(\Gamma) \quad .$$

Proof. This proof generalizes the proof of an analogous equivalence statement given in [13] for a decomposition by two subdomains.

We start proving the following discrete integration by parts formula:

$$(3.8) \qquad a_{i,N}(w,z) = (Lw,z)_{i,N} + \sum_{m=0}^{N_y} \left[\left(\frac{\partial w}{\partial x} z\right)(x_i, y_m) - \left(\frac{\partial w}{\partial x} z\right)(x_{i-1}, y_m) \right] \omega_m$$

for $i=1,\dots,M$, $\forall\, w, z \in P_N(\Omega_i)$ such that $w=z=0$ on $\partial\Omega_i \cap \partial\Omega$.

Indeed, by (3.3) and (3.1) we obtain, using the property (3.2):

$$a_{i,N}(w,z) = \sum_{m=0}^{N_y} \omega_m \int_{x_{i-1}}^{x_i} \left(\frac{\partial w}{\partial x}\frac{\partial z}{\partial x}\right)(x,y_m)\, dx + \sum_{k=0}^{N_x^i} \omega_k^i \int_{-1}^{1} \left(\frac{\partial w}{\partial y}\frac{\partial z}{\partial y}\right)(x_k^i, y)\, dy + (\alpha_0\, w, z)_{i,N}$$

$$= -\sum_{m=0}^{N_y} \omega_m \left\{ \int_{x_{i-1}}^{x_i} \left(\frac{\partial^2 w}{\partial x^2} z\right)(x,y_m)\, dx - \left(\frac{\partial w}{\partial x} z\right)(x_i,y_m) + \left(\frac{\partial w}{\partial x} z\right)(x_{i-1}, y_m) \right\}$$

$$-\sum_{k=0}^{N_x^i} \omega_k^i \int_{-1}^{1} \left(\frac{\partial^2 w}{\partial y^2} z\right)(x_k^i, y)\, dy + (\alpha_0\, w, z)_{i,N}$$

Now we get (3.8) using again (3.2). Let now u_N be the solution to (2.5)-(2.9). Equations (2.6) give:

$$(3.9) \qquad (Lu_{i,N}, v)_{i,N} = (f,v)_{i,N} \qquad \forall\, v \in P^{\circ}_N(\Omega_i) \quad , \quad i=1,\dots,M$$

This yields (3.4) by virtue of (3.8).

We verify now (3.7). For this, let i be an odd integer between 1 and M, and let φ be any function of $P^{\circ}_N(\Gamma)$. In Ω_i one has $\overline{\varphi} = \psi^{(i-1)} + \psi^{(i)}$, where $\psi^{(j)}$ is a polynomial extension of the function which coincides with φ on C^{Γ_j} and vanishes on C^{Γ_l}, $l \neq j$. In particular, we can set $\psi^{(j)} = 0$ on C_l^{int} if $l \neq j, j+1$. Using (3.8) we have

$$
\begin{aligned}
a_{i,N}(u_{i,N}, \psi^{(i)}) &= (Lu_{i,N}, \psi^{(i)})_{i,N} + \sum_{m=0}^{N_y} \omega_m \left[\frac{\partial u_{i,N}}{\partial x}_{|x=x_i} \varphi_i \right](y_m) = \text{(using (2.6))} \\
&\sum_{m=0}^{N_y} \omega_m \left[\bar{\omega}_i^- (Lu_{i,N} - f)_{|x=x_i} + \frac{\partial u_{i,N}}{\partial x}_{|x=x_i} \right](y_m)\, \varphi_i(y_m) + (f, \psi^{(i)})_{i,N} = \text{(using (2.9))} \\
&- \sum_{m=0}^{N_y} \omega_m \left[\omega_i^+ (Lu_{i+1,N} - f)_{|x=x_i} - \frac{\partial u_{i+1,N}}{\partial x}_{|x=x_i} \right](y_m)\, \varphi_i(y_m) + (f, \psi^{(i)})_{i,N} = \text{(using (2.6))} \\
&- (Lu_{i+1,N}, \psi^{(i)})_{i+1,N} + \sum_{m=0}^{N_y} \omega_m \left[\frac{\partial u_{i+1,N}}{\partial x}_{|x=x_i} \varphi_i \right](y_m) + (f, \psi^{(i)})_{i,N} + (f, \psi^{(i)})_{i+1,N} \quad .
\end{aligned}
$$

Then, using (3.8) in Ω_{i+1} we find

$$a_{i,N}(u_{i,N}, \psi^{(i)}) = -a_{i+1,N}(u_{i+1,N}, \psi^{(i)}) + (f, \psi^{(i)})_{i,N} + (f, \psi^{(i)})_{i+1,N} \quad .$$

In a similar way we find that:

$$a_{i,N}(u_{i,N}, \psi^{(i-1)}) = -a_{i-1,N}(u_{i-1,N}, \psi^{(i-1)}) + (f, \psi^{(i-1)})_{i,N} + (f, \psi^{(i-1)})_{i-1,N} \quad .$$

By summation of the last two equations it follows:

$$
\begin{aligned}
a_{i,N}(u_{i,N}, \bar{\varphi}) &= -a_{i-1,N}(u_{i-1,N}, \psi^{(i-1)}) - a_{i+1,N}(u_{i+1,N}, \psi^{(i)}) \\
&\quad + (f, \bar{\varphi})_{i,N} + (f, \psi^{(i-1)})_{i-1,N} + (f, \psi^{(i)})_{i+1,N}
\end{aligned}
\tag{3.10}
$$

It is now an easy task to see that (3.7) can be obtained by summing the relations (3.10) on all odd i between 1 and M.

We shall now prove the converse, namely that the solution of (3.4)-(3.7) is also solution of (2.5)-(2.9). To this end, for each i=1,...,M let P denote a point of C_i. The discrete characteristic function associated with P is the polynomial of $P_N(\Omega_i)$ which equals 1 at P and vanishes at the remaining points of C_i. Equations (2.6) are obtained from (3.4) taking φ to be either of the discrete characteristic functions associated with either point of C_i^{int}. Similarly, (2.9) follows from (3.7) taking as φ a function of $P_{N_y}(\Gamma_i)$ which equals 1 at a point of C^{Γ_i}, and vanishes at all the other points of C^{Γ_i}, then working as done for proving (3.10). Clearly, the relations (2.7) and (2.8) follow from (3.5) and (3.6).◊

In view of the Proposition 3.1, it is now clear that the iterative procedure (2.11)-(2.18) admits the following equivalent variational formulation:

For all i odd (i=1,...,M) solve

(3.11) $$a_{i,N}(u^n_{i,N}, v) = (f, v)_{i,N} \qquad \forall\, v \in P^\circ_N(\Omega_i) \ , \quad i=1,\dots,M \ ,$$

(3.12) $$u^n_{i,N} = 0 \qquad \text{on } \partial\Omega \cap \partial\Omega_i \ , \quad i=1,\dots,M \ ,$$

(3.13) $$u^n_{i,N} = \begin{cases} \theta_{n-1}\, u^{n-1}_{i-1,N} + (1-\theta_{n-1})\, u^{n-1}_{i,N} & \text{on } \Gamma_{i-1} \ , \quad i=3,\dots,M-1 \\ \theta_{n-1}\, u^{n-1}_{i+1,N} + (1-\theta_{n-1})\, u^{n-1}_{i,N} & \text{on } \Gamma_i \ , \quad i=1,\dots,M-2 \ , \end{cases}$$

then for all i even, (i=2,...,M-1) solve

(3.14) $$a_{i,N}(u^n_{i,N}, v) = (f, v)_{i,N} \qquad \forall\, v \in P^\circ_N(\Omega_i) \ ,$$

(3.15) $$u^n_{i,N} = 0 \qquad \text{on } \partial\Omega \cap \partial\Omega_i \ ,$$

(3.16) $$a_{i,N}(u^n_{i,N}, \bar\varphi) = -a_{i-1,N}(u^n_{i-1,N}, \bar\varphi) + (f,\bar\varphi)_{i,N} + (f,\bar\varphi)_{i-1,N} \quad \forall\, \varphi \in P^\circ_N(\Gamma) : \ \varphi_{|\Gamma_j} \equiv 0 \ \text{if } j \neq i-1,$$

(3.17) $$a_{i,N}(u^n_{i,N}, \bar\varphi) = -a_{i+1,N}(u^n_{i+1,N}, \bar\varphi) + (f,\bar\varphi)_{i,N} + (f,\bar\varphi)_{i+1,N} \quad \forall\, \varphi \in P^0_N(\Gamma) : \varphi_{|\Gamma_j} \equiv 0 \ \text{if } j \neq i$$

For each i odd, (3.11)-(3.13) is a Dirichlet problem, while, for i even, (3.14)-(3.17) is a mixed type problem. The above variational formulation of the multidomain collocation iterations (2.11)-(2.18) will be used in the next section to investigate the convergence properties of this method.

Remark 3.1 If Ω is a plurirectangle, partitioned into subrectangles Ω_i without internal cross points (as in fig.0.1.a), the above multidomain problem and the relative iteration procedure (2.11)-(2.18) can still be formulated. In such case each vertex of the decomposition lies on the boundary of Ω, hence the value of each iterate $u^n_{i,N}$ is set to zero (the prescribed boundary data) there. ◊

Remark 3.2 For one dimensional multidomain methods using the Chebyshev collocation points an equivalent variational formulation is still available (see [14]). However, the bilinear form $a_{i,N}(\cdot,\cdot)$ that holds such an equivalence is much more involved than the Legendre one (3.3). (Actually it must take into account the Chebyshev weight functions which blow up at each subdomain interface). The generalization of this equivalence to two dimensional problems is not yet available. Thus the convergence analysis we carry out in the next section will deal with Legendre collocation points only.◊

4 CONVERGENCE ANALYSIS

In this section we confine ourselves to the case of the Legendre collocation. We prove that the sequence $\{u^n_{i,N}\}$, solution to (2.11)-(2.18) (or, equivalently to (3.11)-(3.17)) converges, as $n \to \infty$, to $u_{i,N}$ and that the rate of convergence is independent of N.

For any $\varphi \in P^\circ_N(\Gamma)$ we recall that φ on Γ_i is denoted by φ_i, i=1,...,M-1; then we define $R_{i,N}\,\varphi \in P_N(\Omega_i)$ through the equations:

(4.1) $$a_{i,N}(R_{i,N}\,\varphi, v) = 0 \qquad \forall\, v \in P^\circ_N(\Omega_i),$$

(4.2) $$R_{i,N}\,\varphi = \varphi_i \qquad \text{on } \Gamma_i,$$

(4.3) $$R_{i,N}\,\varphi = \varphi_{i-1} \qquad \text{on } \Gamma_{i-1},$$

(4.4) $$R_{i,N}\,\varphi = 0 \qquad \text{on } \partial\Omega \cap \partial\Omega_i .$$

If i=1 or i=M the conditions (4.3) and (4.2) respectivey must be dropped. In view of the analogy between this problem and (1.4), $R_{i,N}\,\varphi$ is referred as the *discrete harmonic extension* of φ to Ω_i. We now define the following norms.

(4.5) $$\|\varphi\|_{odd} := \left\{ \sum_{\substack{i=1 \\ i \text{ odd}}}^{M} [a_{i,N}(R_{i,N}\,\varphi, R_{i,N}\,\varphi)] \right\}^{1/2}$$

(4.6) $$\|\varphi\|_{even} := \left\{ \sum_{\substack{i=1 \\ i \text{ even}}}^{M} [a_{i,N}(R_{i,N}\,\varphi, R_{i,N}\,\varphi)] \right\}^{1/2}$$

In the next lemma we state that the norms (4.5) and (4.6) are uniformly equivalent on $P^\circ_N(\Gamma)$. This is an extension type theorem for spectral collocation approximations similar to the one stated in [3] and [21] for finite elements. Its proof is essentialy based on a result due to Bernardi and Maday [2].

Theorem 4.1 There exist two positive constants σ and τ independent of N such that

(4.7) $$\|\varphi\|^2_{odd} \le \sigma\, \|\varphi\|^2_{even} \quad \text{and} \quad \|\varphi\|^2_{even} \le \tau\, \|\varphi\|^2_{odd}, \quad \text{for any } \varphi \in P^0_N(\Gamma).$$

with

$$\text{(4.8)} \qquad \begin{cases} \sigma = \sigma_0\, \rho(M)\ ,\ \tau = \tau_0\, \rho(M)\ ,\ \text{and } \rho(M) = (1 + H^{-1}) \max(1, H^{-1})\ , \\ H = \min\,\{\text{meas}\,(\Omega_i)\ ,\ 1 \le i \le M\} \end{cases}$$

Proof Let us fix i such that $2 \le i \le M-1$. For any $\varphi \in P^{\circ}_N(\Gamma)$ let $Q_{i,N}\,\varphi \in P_N(\Omega_i)$ be the solution to the collocation problem:

$$\begin{cases} -\Delta\, Q_{i,N}\,\varphi = 0 & \text{in } C_i^{int}\ , \\ Q_{i,N}\,\varphi = \varphi_i & \text{on } \Gamma_i\ , \\ Q_{i,N}\,\varphi = \varphi_{i-1} & \text{on } \Gamma_{i-1}\ , \\ Q_{i,N}\,\varphi = 0 & \text{on } \partial\Omega \cap \partial\Omega_i\ . \end{cases}$$

It is proven in [2] that (for $\Omega_i = (-1, 1)^2$)

$$\text{(4.9)} \qquad \| Q_{i,N}\,\varphi \|_{H^1(\Omega_i)} \le C_1 \left[\| \varphi_i \|^2_{H^{1/2}_{00}(\Gamma_i)} + \| \varphi_{i-1} \|^2_{H^{1/2}_{00}(\Gamma_{i-1})} \right]^{1/2}$$

For a general rectangle Ω_i, it is easy to see that C_1 is proportional to $1 + \text{meas}\,(\Omega_i)^{-1}$

Setting $v = R_{i,N}\,\varphi - Q_{i,N}\,\varphi$ in (4.1) we obtain

$$\text{(4.10)} \qquad a_{i,N}(R_{i,N}\,\varphi - Q_{i,N}\,\varphi\, ,\, R_{i,N}\,\varphi - Q_{i,N}\,\varphi) = -a_{i,N}(Q_{i,N}\,\varphi,\, R_{i,N}\,\varphi - Q_{i,N}\,\varphi)\ .$$

On the other hand we recall that (see [12] and [6, Ch.11])

$$\text{(4.11)} \qquad C_2 \| w \|^2_{H^1(\Omega_i)} \le a_{i,N}(w,w) \qquad \forall\, w \in P_N(\Omega_i)$$

and

$$\text{(4.12)} \qquad | a_{i,N}(w,z) | \le C_3 \| w \|_{H^1(\Omega_i)} \| z \|_{H^1(\Omega_i)} \qquad \forall\, w, z \in P_N(\Omega)$$

where C_2 and C_3 are positive constants independent of N. Hence, from (4.9) and (4.10) we deduce

$$\text{(4.13)} \qquad a_{i,N}(R_{i,N}\,\varphi\, ,\, R_{i,N}\,\varphi) \le C_4 \left[\| \varphi_i \|^2_{H^{1/2}_{00}(\Gamma_i)} + \| \varphi_{i-1} \|^2_{H^{1/2}_{00}(\Gamma_{i-1})} \right]$$

In order to estimate the right hand side of (4.13) we use now the continuity of the trace operator from $H^1(\Omega_i)$ to $H^{1/2}(\partial\Omega_i)$ (see [18])

$$a_{i,N}(R_{i,N}\varphi, R_{i,N}\varphi) \le C_5 \left[\| R_{i+1,N}\varphi \|^2_{H^1(\Omega_{i+1})} + \| R_{i-1,N}\varphi \|^2_{H^1(\Omega_{i-1})} \right]$$

where C_5 is proportional to $\rho(M)$

Using again (4.11) we also get

$$[a_{i,N}(R_{i,N}\varphi, R_{i,N}\varphi)]^2 \le C_6 \left([a_{i+1,N}(R_{i+1,N}\varphi, R_{i+1,N}\varphi)]^2 + [a_{i-1,N}(R_{i-1,N}\varphi, R_{i-1,N}\varphi)]^2 \right). \tag{4.14}$$

For i=1 or i=M working in the same way we have

$$\begin{cases} [a_{1,N}(R_{1,N}\varphi, R_{1,N}\varphi)]^2 \le C_7 [a_{2,N}(R_{2,N}\varphi, R_{2,N}\varphi)]^2 , \\ [a_{M,N}(R_{M,N}\varphi, R_{M,N}\varphi)]^2 \le C_8 [a_{M-1,N}(R_{M-1,N}\varphi, R_{M-1,N}\varphi)]^2 . \end{cases} \tag{4.15}$$

Finally summing up (4.14) and (4.15) for i odd and for i even we deduce the desired result (4.7).◊

In order to prove that the solution $u^n_{i,N}$ of (3.11)-(3.18) converges, as $n \to \infty$, to the one $u_{i,N}$ of (3.4)-(3.7), we look for the operator T_θ such that

$$u^{n+1}_{i,N} - u_{i,N} = T_\theta(u^n_{i,N} - u_{i,N}) \qquad \text{on } \Gamma \text{ , i even .} \tag{4.16}$$

The operator T_θ can be defined as follows. For any $\varphi \in P^\circ_N(\Gamma)$ let $w_{i,N} \in P_N(\Omega_i)$, i=1,...,M, i <u>even</u>, be defined through:

$$a_{i,N}(w_{i,N}, v) = 0 \qquad \forall\, v \in P^\circ_N(\Omega_i) \tag{4.17}$$

$$\sum_{\substack{i=1 \\ i \text{ even}}}^{M} a_{i,N}(w_{i,N}, R_{i,N}\psi) = -\sum_{\substack{i=1 \\ i \text{ odd}}}^{M} a_{i,N}(R_{i,N}\varphi, R_{i,N}\psi) \qquad \forall\, \psi \in P^0_N(\Gamma) \tag{4.18}$$

$$w_{i,N} = 0 \qquad \text{on } \partial\Omega \cap \partial\Omega_i \tag{4.19}$$

Then we define

$$T\varphi := w_{i,N|\Gamma_i} \qquad \text{for i=1,...,M-1 ,} \tag{4.20}$$

and

$$T_\theta\varphi := \theta T\varphi + (1-\theta)\varphi \quad , \quad \forall\, \varphi \in P^\circ_N(\Gamma) \ . \tag{4.21}$$

<u>Remark 4.1</u> Due to (4.1) and (4.17) one could substitute $R_{i,N}\psi$ in (4.18) with any other extension of ψ in Ω_i, without affecting the definition of $w_{i,N}$.◊

<u>Remark 4.2</u> The operator T can be interpreted as follows (suppose $M > 2$). For a given $\varphi \in P^{\circ}_N(\Gamma)$ and an even number i with $2 \leq i \leq M-1$ one can find $R_{i-1,N}\,\varphi$ and $R_{i+1,N}\,\varphi$ by solving the Dirichlet problems (4.1)-(4.4). Then compute the left flux at the boundary side Γ_{i-1} and the right flux and the boundary side Γ_i as follows

$$D_{i-1,m} = \frac{\partial}{\partial x} R_{i-1,N}\,\varphi\,(x_N^{i-1}, y_m) + LR_{i-1}\,\varphi\,(x_N^{i-1}, y_m)\,\omega_N$$

$$D_{i+1,m} = -\frac{\partial}{\partial x} R_{i+1,N}\,\varphi\,(x_0^{i+1}, y_m) + LR_{i+1}\,\varphi\,(x_0^{i+1}, y_m)\,\omega_0$$

for m=1,..., N-1. The functions $w_{i,N}$ are now obtained by solving the mixed problem

$$\begin{cases} w_{i,N} \in P_N(\Omega_i) \\ Lw_{i,N} = 0 \quad \text{in } C_i^{int} \\ \dfrac{\partial w_{i,N}}{\partial x}(x_N^i, y_m) + Lw_{i,N}(x_N^i, y_m)\,\omega^i_{N_x^i} = D_{i+1,m} \quad \text{for } m=1, \cdot\,, N-1 \\ \dfrac{\partial w_{i,N}}{\partial x}(x_0^i, y_m) - Lw_{i,N}(x_0^i, y_m)\,\omega_0^i = D_{i-1,m} \quad \text{for } m=1, \cdot\,, N-1 \\ w_{i,N} = 0 \quad \text{on } C_i^b \quad . \end{cases}$$

Finally $T\varphi$ on Γ_{i-1} and Γ_i is obtained by computing $w_{i,N}$ on Γ_{i-1} and Γ_i.

The operator which associates to the values of $D_{i-1,m}$ and $D_{i+1,m}$ the values of $w_{i,N}$ on Γ_{i-1} and Γ_i is a discrete Poincaré-Steklov operator (see [1]).

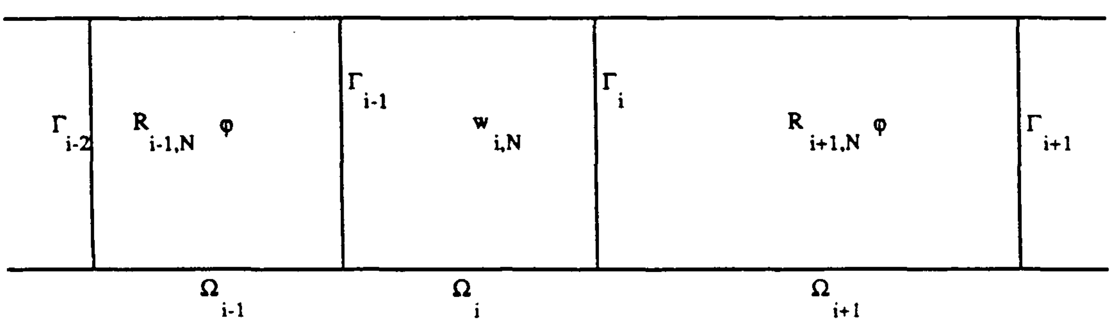


FIG.4.1 Computation of T φ

In the following theorem we prove that for suitable values of θ, T_θ is a contraction. This, together with (4.16), will imply the convergence of $u^n_{i,N}$ to $u_{i,N}$ for $n \to \infty$.

<u>Theorem 4.2</u> <u>For each</u> $M \geq 2$ <u>there exists</u> $\theta^* > 0$ <u>such that for any</u> $N > 0$ <u>the following holds</u>

(4.22) $$\forall\, \theta \in (0, \theta^*)\ \exists\, k(\theta) < 1 :\ \| T_\theta \varphi \|_{odd} \leq k(\theta)\, \| \varphi \|_{odd} \qquad \forall\, \varphi \in P^\circ_N(\Gamma)$$

<u>Moreover there exist</u> θ', θ'' <u>and</u> κ <u>with</u> $0 < \theta' < \theta'' < \theta^*$ <u>and</u> $\kappa < 1$ <u>such that for all</u> $N > 0$

(4.23) $$\forall\, \theta \in [\theta', \theta''] \qquad k(\theta) \leq \kappa < 1 \ .$$

<u>Proof</u> From (4.5) and (4.21) we have, using the symmetry of the forms $a_{i,N}(\cdot,\cdot)$

(4.24) $$\| T_\theta \varphi \|^2_{odd} = \theta^2 \| T\varphi \|^2_{odd} + 2\theta(1-\theta) \sum_{\substack{i=1 \\ i\ odd}}^{M} a_{i,N}(R_{i,N}\varphi, R_{i,N}T\varphi) + (1-\theta)^2 \| \varphi \|^2_{odd} \ .$$

Moreover, using (4.17) and (4.18), we have

(4.25) $$\sum_{\substack{i=1 \\ i\ odd}}^{M} a_{i,N}(R_{i,N}\varphi, R_{i,N}T\varphi) = -\sum_{\substack{i=1 \\ i\ even}}^{M} a_{i,N}(w_{i,N}, R_{i,N}T\varphi) = -\sum_{\substack{i=1 \\ i\ even}}^{M} a_{i,N}(R_{i,N}T\varphi, R_{i,N}T\varphi) \ ,$$

hence, from theorem 4.1 and (4.24) we deduce

(4.26) $$\| T_\theta \varphi \|^2_{odd} \leq \theta^2 \sigma \| T\varphi \|^2_{even} + (1-\theta)^2 \| \varphi \|^2_{odd} - 2\theta(1-\theta) \| T\varphi \|^2_{even} \ .$$

From (4.25), the Cauchy-Schwarz inequality and (4.7) we also have

(4.27) $$\| T\varphi \|_{even} \leq \sqrt{\sigma}\ \| \varphi \|_{odd} \ .$$

In order to find a lower bound for $\| T\varphi \|_{even}$ we remark that from (4.17)-(4.18) we have the fundamental relation

$$\sum_{\substack{i=1\\ i\ \mathrm{odd}}}^{M} a_{i,N}(R_{i,N}\varphi, R_{i,N}\varphi) = -\sum_{\substack{i=1\\ i\ \mathrm{even}}}^{M} a_{i,N}(w_{i,N}, R_{i,N}\varphi) = -\sum_{\substack{i=1\\ i\ \mathrm{even}}}^{M} a_{i,N}(R_{i,N}T\varphi, R_{i,N}\varphi) \quad ,$$

Using the Cauchy-Schwarz inequality, the property (4.7) and the definitions (4.5) and (4.6) we deduce:

$$\frac{1}{\sqrt{\tau}} \| \varphi \|_{odd} \le \| T\varphi \|_{even} \quad . \tag{4.28}$$

From (4.26),(4.27),(4.28) it follows

$$\| T_\theta \varphi \|^2_{odd} \le \left(\theta^2 \sigma^2 + (1-\theta)^2 - \frac{2\theta\,(1-\theta)}{\tau} \right) \| \varphi \|^2_{odd} \quad .$$

If we define

$$k(\theta) := \left[\frac{\theta^2 (\sigma^2 \tau + \tau + 2) - 2\theta\,(\tau + 1) + \tau}{\tau} \right]^{1/2}$$

We can readily see that (4.22) holds and that

$$k(\theta) < 1 \quad \text{iff} \quad 0<\theta<\theta^* = \min\left(1 \, , \frac{2\,(\tau+1)}{\sigma^2 \tau + \tau + 2} \right) .$$

Then (4.23) follows from the continuity of $k(\theta)$.◊

We can now derive the following convergence result.

Corollary 4.1 Let $u_{i,N}$, $i=1,\dots, M$, be the solution of (3.4)-(3.7) and let $u^n_{i,N}$, for $n \ge 1$ and $i=1,\dots , M$, be the solution of (3.11)-(3.18). If $\theta_n \in [\theta', \theta'']$ we have

$$\| (u^{n+1}_{i,N} - u_{i,N})_{|\Gamma} \|_{odd} \le \kappa^{n+1} \| (u^{0}_{i,N} - u_{i,N})_{|\Gamma} \|_{odd} \quad , \tag{4.29}$$

moreover

$$\left[\sum_{i=1}^{M} \| u^{n+1}_{i,N} - u_{i,N} \|^2_{H^1(\Omega_i)} \right]^{1/2} \le C\,\kappa^{n+1} \quad , \tag{4.30}$$

where C is a constant dependent on g^0 but independent of both N and M.

Proof The estimate (4.29) follows using repeatidly (4.16) and theorem 4.1. Due to the theorem 4.1 and to (4.11), the estimate (4.30) can be deduced from (4.29).◊

Remark 4.3 In view of (4.8), if H is much smaller than one, then σ and τ grow like H^{-2}. If θ_{opt}

is the value of θ that minimizes $\kappa(\theta)$, one has $\theta_{opt} = \dfrac{\tau+1}{\sigma^2\tau+\tau+2}$ and $\kappa(\theta_{opt})$ behaves like $(1-CH^2)^{1/2}$, where C is a positive constant. ◊

Remark 4.4 The proof of theorem 3.1 and corollary 3.1 can also be applied to the case of a plurirectangle Ω partitioned into rectangles Ω_i without internal cross points (see fig.0.1 and remark 3.1).◊

5 MATRIX INTERPRETATION OF THE ITERATION-BY-SUBDOMAIN METHOD

To solve the multidomain problem (2.5)-(2.9) we have proposed, in section 2, an iterative method based on a sequence of differential problems to be solved in each subdomain. An (a priori) different point of view consists of applying the influence (or capacitance) matrix method for solving (2.5)-(2.9). This approach, in the framework of spectral approximations, has been extensively pursued by Peyret and his Coworkers (see e.g. [17], [24]) and by other authors (see [20],[22]). The influence matrix coincides with the Schur complement of the matrix of the system (2.5)-(2.9) with respect to the interface variables (see e.g. [9]). It is precisely the matrix of the system of the interface unknowns and it is derived from the global system by block Gaussian elimination (see e.g. [3]). In this section we show that the iteration-by-subdomain method introduced in section 2 is equivalent to a preconditioned Richardson iterative method for the resolution of the influence system. In particular we will give the precise structure of the preconditioner for the influence matrix.

We proceed now to construct the influence matrix associated with the problem (2.5)-(2.9).

Let $\{\eta_i \in P^\circ_N(\Gamma_i),\ i=1,\dots,M-1\}$ be M-1 given polynomials. Correspondingly, for each $i=2,\dots,M$, denote by $v_{i,N}$ the solution to the problem

$$(5.1)\qquad \begin{cases} v_{i,N} \in P_N(\Omega_i) & \\ Lv_{i,N} = 0 & \text{in } C_i^{int} \\ v_{i,N} = \eta_{i-1} & \text{in } C^{\Gamma_{i-1}} \\ v_{i,N} = 0 & \text{in } C^{\Gamma_i} \\ v_{i,N} = 0 & \text{in } C_i^b \end{cases}$$

Moreover, for each $i=1,\dots,M-1$ we let $z_{i,N}$ be such that

$$(5.2)\qquad \begin{cases} z_{i,N} \in P_N(\Omega_i) & \\ Lz_{i,N} = 0 & \text{in } C_i^{int} \\ z_{i,N} = 0 & \text{in } C^{\Gamma_{i-1}} \\ z_{i,N} = \eta_i & \text{in } C^{\Gamma_i} \\ z_{i,N} = 0 & \text{in } C_i^b \end{cases}$$

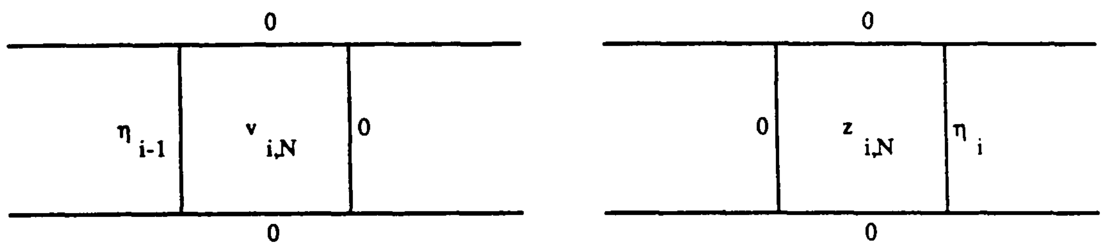


FIG.5.1 Boundary conditions for $v_{i,N}$ and $z_{i,N}$

Let us denote with η_i the vector of the values attained by the polynomial η_i at the collocation points C^{Γ_i} of Γ_i. Then we define S^i_1, S^i_2, S^i_3, S^i_4 to be the square matrices of order N-1 such that

$$(5.3)\qquad (S^i_1\,\eta_i)_m := \frac{\partial}{\partial x} z_{i,N}(x^i_N, y_m) + Lz_{i,N}(x^i_N, y_m)\,\omega^-_i$$

$$(5.4)\qquad (S^i_2\,\eta_{i-1})_m := -\frac{\partial}{\partial x} v_{i,N}(x^i_0, y_m) + Lv_{i,N}(x^i_0, y_m)\,\omega^+_{i-1}$$

$$(5.5)\qquad (S^i_3\,\eta_i)_m := -\frac{\partial}{\partial x} z_{i,N}(x^i_0, y_m) + Lz_{i,N}(x^i_0, y_m)\,\omega^+_{i-1}$$

$$(5.6)\qquad (S^i_4\,\eta_{i-1})_m := \frac{\partial}{\partial x} v_{i,N}(x^i_N, y_m) + Lv_{i,N}(x^i_N, y_m)\,\omega^-_i$$

for m=1,...,N-1. S^i_1 is defined for i=1,...,M-1, S^i_2 for i=2,...,M and S^i_3, S^i_4 for i=2,...,M-1. We shall call for brevity *interface flux* on Γ_i of a function w defined on Ω_i the quantity

$$\frac{\partial}{\partial x} w(x^i_N, y_m) + Lw(x^i_N, y_m)\,\omega^-_i \qquad\qquad \text{for } m=1,\ldots,N-1 \quad .$$

Then the matrix S^i_1 represents the operator which associates to η_i on Γ_i the interface flux on Γ_i of the left discrete harmonic extension of η_i on Ω_i. S^i_2, S^i_3, S^i_4 can be interpreted analogously.

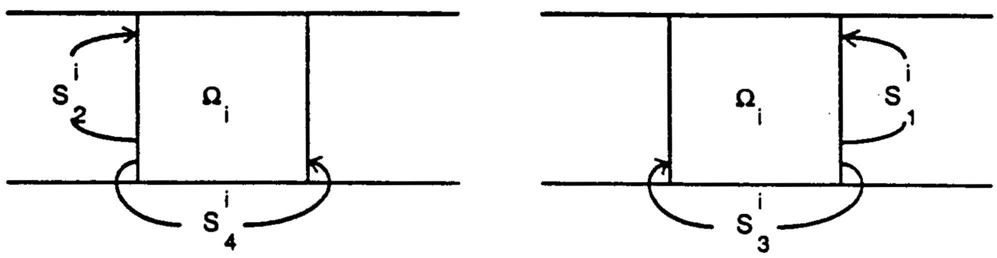


FIG.5.2 Interpretation of S^i_1, S^i_2, S^i_3, S^i_4.

We now define the block tridiagonal matrix S, of order (M-1) (N-1), as follows.

$$
(5.7)\quad S=\begin{bmatrix}
S^1_1+S^2_2 & S^2_3 & & & O \\
S^2_4 & S^2_1+S^3_2 & S^3_3 & & \\
 & & & & \\
 & & S^{M-2}_4 & S^{M-2}_1+S^{M-1}_2 & S^{M-1}_3 \\
O & & & S^{M-1}_4 & S^{M-1}_1+S^{M}_2
\end{bmatrix}
$$

We claim that the matrix S in (5.7) is actually the influence (or capacitance) matrix of the system (2.5)-(2.9). As a matter of fact, let us solve the homogeneous collocation problems

$$
(5.8)\quad \begin{cases}
\tilde{u}_{i,N} \in P_N(\Omega_i) & \\
L\tilde{u}_{i,N} = f & \text{in } C_i^{int} \\
\tilde{u}_{i,N} = 0 & \text{in } C^{\Gamma_i} \cup C_i^b
\end{cases}
$$

for $i=1,\dots,M$ and compute the vector $\mathbf{b} = [b_1,\dots,b_i,\dots,b_{M-1}]^T$ defined by

$$(b_i)_m = \frac{\partial}{\partial x}\tilde{u}_{i,N}(x_N^i, y_m) + (L\tilde{u}_{i,N} - f)(x_N^i, y_m)\,\omega_i^- - \frac{\partial}{\partial x}\tilde{u}_{i+1,N}(x_0^{i+1}, y_m) + \tag{5.9}$$
$$(L\tilde{u}_{i+1,N} - f)(x_0^{i+1}, y_m)\,\omega_i^+ \quad , \quad \text{for } m=1,\dots,N-1,\ i=1,\dots,M-1 \quad .$$

Then define the vector $\underline{\xi} = [\xi_1,\dots,\xi_i,\dots,\xi_{M-1}]^T$ as the solution of the linear system

$$S\underline{\xi} = \mathbf{b} \quad . \tag{5.10}$$

It is now readily seen (using (5.1)-(5.10)) that the solution of (2.5)-(2.9) is given by

$$\begin{cases} u_{i,N} = \tilde{u}_{i,N} + w_{i,N} + z_{i,N} & \text{for } i=2,\dots,M-1 \\ u_{1,N} = \tilde{u}_{1,N} + z_{1,N} \\ u_{M,N} = \tilde{u}_{M,N} + w_{M,N} \end{cases} \tag{5.11}$$

where $w_{i,N}$ and $z_{i,N}$ are obtained from (5.1) and (5.2) taking $\eta_i = \xi_i$. Thus, the matrix S defined in (5.7) is actually the influence matrix associated with the problem (2.5)-(2.9).

Remark 5.1 It can be proved, using repeatedly the integration by parts, that in the case of the Legendre collocation the matrix S is symmetric. On the other hand S is not symmetric if the Chebyshev collocation points are used. S is again non symmetric in the case of both Legendre or Chebyshev collocation if the pure flux condition $\frac{\partial u_{i,N}}{\partial x} = \frac{\partial u_{i+1,N}}{\partial x}$ on Γ_i is used instead of (2.9).◊

The relations (5.8)-(5.11) suggest a possible algorithm, based on the influence matrix system, for the solution of the multidomain problem (2.5)-(2.9). The crucial step of this algorithm is the use of an efficient method for the solution of the system (5.10). To this end, preconditioned iterative methods (which do not require the explicit construction of the matrix S but simply the calculation of the product of S time a given vector) are currently used (see e.g. [3], [4], [5], [7], [8],[11],[16]). In this section we show that our iteration by subdomain method amounts to the use of the Richardson iterative method for (5.10) preconditioned with a proper matrix P (see (5.15) below).

Let us recall that if P is any non singular matrix of order (M-1) (N-1), the Richardson iterative method for solving (5.10) preconditioned with P is:

$$P(\underline{\xi}^{n+1} - \underline{\xi}^n) = \theta_n(\mathbf{b} - S\underline{\xi}^n) \quad . \tag{5.12}$$

The iteration matrix at the step n is

$$(1-\theta_n)I + \theta_n(I - P^{-1}S) \quad . \tag{5.13}$$

Let us define the matrix P as follows (we assume, e.g., M odd).

$$(5.14)\quad P = \begin{bmatrix} S_2^2 & S_3^2 & & & & & \\ S_4^2 & S_1^2 & & & & & \\ & & S_2^4 & S_3^4 & & & \\ & & S_4^4 & S_1^4 & & & \\ & & & & \ddots & & \\ & & & & & S_2^{M-1} & S_3^{M-1} \\ & & & & & S_4^{M-1} & S_1^{M-1} \end{bmatrix}$$

We denote with T_θ and T the matrices associated with the operators defined in (4.20) and (4.21) and bearing the same name.

Proposition 5.1 The following identity holds

$$(5.15)\qquad T_\theta = (1-\theta)\, I + \theta\, (I - P^{-1} S) \quad .$$

Proof In view of (4.20) and (4.21) it is sufficient to prove that $T = I - P^{-1} S$ or equivalently that $PT = P-S$. Recalling (5.3)-(5.6) the action of P consists of solving a sequence of Dirichlet problems in Ω_i for i even, and then to compute the interface fluxes on Γ. Recalling the interpretation of the operator T given in the remark 4.2, it is readily seen that applying the matrix PT to the values of a polynomial on C^Γ amounts to solve a sequence of Dirichlet problems in Ω_i for i odd with such boundary conditions, and then to compute the interface fluxes on Γ of the obtained solution.

On the other hand the matrix P-S is given by

$$P\text{-}S = \begin{bmatrix} -S_1^1 & & & \bigcirc \\ & -S_2^3 & S_3^3 & \\ & S^3 & -S_1^3 & \\ & & & \\ \bigcirc & & & -S_2^M \end{bmatrix}$$

From (5.3)-(5.6) we deduce that the matrix P-S is associated with the solution of Dirichlet problems in Ω_i for i odd and to the computation of the interface fluxes on Γ. This shows that PT=P-S. ◊

Remark 5.2 (The case of two subdomains) If M is even it is possible to give an analogous interpretation of the iteration-by-subdomain method. In the particular case in which M=2 we have

(5.16) $$S = S^1{}_1 + S^2{}_2 \quad \text{and} \quad P = S^2{}_2 \quad ,$$

and therefore (5.15) becomes $T_\theta = (1 - \theta)\, I - \theta\, (S^2{}_2)^{-1}\, S^1{}_1)$. If Ω_1 and Ω_2 have the same measure and $L = -\Delta + \alpha I$, with $\alpha \in R^+$, then $S^2{}_2 = S^1{}_1$, hence taking $\theta_n = 1/2$, $\forall n$, yields exact convergence in two iterations (this result was already found in [15]). ◊

Remark 5.3 (Condition number of the preconditioned matrix) Let us consider the case of the Legendre collocation with M=2. Recalling the definitions of $S^1{}_1$ and $S^2{}_2$ (see (5.3) (5.4)) and using the discrete integration by parts formula (3.8) we deduce from (5.16) that

(5.17) $$\forall\, \varphi, \psi \in P_N^0(\Gamma_i) \quad (S\, \underline{\varphi}, \underline{\psi})_{N,\Gamma_i} = a_{1,N}\, (R_{1,N}\, \varphi, R_{1,N}\, \psi) + a_{2,N}\, (R_{2,N}\, \varphi, R_{2,N}\, \psi)\, .$$

where φ is the vector such that $\varphi_m = \varphi(y_m)$ m=0,...,N_y, and

(5.18) $$(S\, \underline{\varphi}, \underline{\psi})_{N,\Gamma_1} := \sum_{m=0}^{N_y} \sum_{k=0}^{N_y} S_{mk}\, \varphi_k\, \psi_m\, \omega_m$$

Similarly the preconditioner $P = S^2{}_2$ satisfies

$$\forall\, \varphi, \psi \in P_N^0(\Gamma_1) \qquad (P\underline{\varphi}, \underline{\psi})_{N,\Gamma_1} = a_{2,N}(R_{2,N}\,\varphi, R_{2,N}\,\psi)\ . \tag{5.19}$$

From (5.18),(5.19) and theorem 4.1 we obtain the fundamental relation

$$\forall\, \varphi \in P_N^0(\Gamma_1) \qquad (P\,\underline{\varphi}, \underline{\varphi})_{N,\Gamma_1} \le (S\underline{\varphi}, \underline{\varphi})_{N,\Gamma_1} \le \tau\,(P\underline{\varphi}, \underline{\varphi})_{N,\Gamma_1} \tag{5.20}$$

where τ is a constant *independent of N*. This ensures that the eigenvalues of the matrix $P^{-1}S$ are real and positive. Moreover the ratio $\lambda_{max}/\lambda_{min}$ between the maximum and the minimum eigenvalue of $P^{-1}S$ is bounded by τ.◊

Remark 5.4 (Choice of the relaxation parameter θ_n). The interpretation of the iterative method(2.11)-(2.18) as a Richardson scheme suggests some convenient choices of the parameter θ_n appearing in (2.18). When information is available on the extreme eigenvalues of the preconditioned influence matrix the optimal parameter $\theta = \dfrac{1}{\lambda_{max} + \lambda_{min}}$ can be used for all n (see e.g.[7]).
Otherwise a dynamical choice inspired by a "minimal residual" strategy can be used (see [1] for the case of multidomain finite element approximation).
Another strategy is suggested by the proof theorem 4.2. The minimal value of the contraction constant $k(\theta)$ is achieved for $\theta_{opt} = (\tau + 1/\sigma^2\tau + \tau + 2)$. In [21] the values of θ_n are choosen so that $\sup\{\theta_n\} = \theta_{opt}$ and no extra computational work is needed to obtain θ_n. The same strategy can be applied to the case of spectral collocation approximations here discussed.◊

6 OTHER PRECONDITIONERS

Based on the interpretation of the preconditioned Richardson method for solving the influence system (5.10), one can formulate other iteration-by-subdomain procedures by simply taking a preconditioner P in (5.12) different than the one in (5.14). For instance if we take

$$P = \begin{bmatrix} S_2^2 & & & \bigcirc \\ & S_2^3 & & \\ & & \ddots & \\ \bigcirc & & & S_2^M \end{bmatrix} \tag{6.1}$$

then the iteration (5.12) yields the following iteration by subdomain procedure. (We identify, for simplicity of notation, the vector ξ^n with the polynomial $\xi^n \in P^o_N(\Gamma)$ whose values at the collocation points are the components of ξ^n).

Given ξ^o, look for $U^n_i \in P_N(\Omega_i)$ for i=1,...,M-1 such that for any $n \geq 1$

$$(6.2) \qquad \begin{cases} LU^n_i = f & \text{in } C^{int}_i \ , \\ U^n_i = \xi^{n-1}_i & \text{in } C^{\Gamma_i} \ , \quad U^n_i = 0 \ \text{in } C^b_i \text{ and, if } i \geq 2 \ , \\ U^n_i = \xi^{n-1}_{i-1} & \text{in } C^{\Gamma_{i-1}} \ . \end{cases}$$

Then, for i=2,...,M look for $V^n_i \in P_N(\Omega_i)$ such that

$$(6.3) \qquad \begin{cases} LV^n_i = f \quad \text{in } C^{int}_i \\ \frac{\partial}{\partial x} V^n_i - (LV^n_i - f)\, \omega^+_{i-1} = \frac{\partial}{\partial x} U^n_{i-1} + (LU^n_{i-1} - f)\, \omega^-_{i-1} \quad \text{in } C^{\Gamma_{i-1}} \\ V^n_i = \xi^n_i \ \text{in } C^{\Gamma_i} \ , \quad V^n_i = 0 \ \text{in } C^b_i \ \text{if } i < M, \text{ and } V^n_M = 0 \text{ in } C^b_M \ . \end{cases}$$

Finally $\underline{\xi}^n$ is defined by

$$(6.4) \qquad \xi^n_i = \theta_n V^n_{i+1} + (1 - \theta_n)\, \xi^{n-1}_i \quad \text{on } C^{\Gamma_i} \ \text{for } i=1,...,M-1$$

Once $\underline{\xi}^{n-1}$ is available, one can compute U^n_i, i=2,...,M by solving the M-1 independent problems (6.2). Then the M-1 independent systems (6.3) can be solved simultaneously. At each step of this algorithm one has to solve 2(M-1) differential problems, instead of M as prescribed by the method (2.11)-(2.18), however in this case the degree of parallelism of the algorithm is M-1, instead of M/2.

The choice of the preconditioner

$$(6.5)\quad P = \begin{bmatrix} S_1^1 & & & \\ & S_1^2 & & \\ & & \ddots & \\ & & & S_1^{M-1} \end{bmatrix}$$

gives rise to an algorithm analogous to (6.2)-(6.4). In this case the Dirichlet problems are solved in Ω^i for i=2,...,M, and the mixed Dirichlet-Neumann problems in Ω_i for i=1,...,M-1.
Another preconditioner which corresponds to an iteration-by-subdomain technique is

$$(6.6)\quad P = \begin{bmatrix} S_1^1 & & & & O \\ S_4^2 & S_1^2 & & & \\ & & \ddots & & \\ & & & & \\ O & & & S_4^{M-1} & S_1^{M-1} \end{bmatrix}$$

The differential interpretation of the preconditioned Richardson iteration is now the following. First one solves M-1 independent Dirichlet problems in Ω_i, i=2,...,M, with ξ_i^{n-1} as boundary data. Then computes the interface flux ϕ_{i-1} on Γ_{i-1} of the solution in Ω_i. Finally one solves sequentially M-1 mixed problems going from Ω_1 to Ω_{M-1}. For the mixed problem in Ω_i one enforces the flux ϕ_i in Γ_i, while on Γ_{i-1} the value of the solution coming from Ω_{i-1} is enforced as Dirichlet data. We note that these mixed problems are not solvable in parallel.

An analogous algorithm is obtained by using the upper triangular preconditioner

$$
P = \begin{bmatrix}
S_2^2 & S_3^2 & & & \\
& S_2^3 & S_3^3 & & \\
& & \ddots & \ddots & \\
& & & S_2^{M-1} & S_3^{M-1} \\
& & & & S_2^M
\end{bmatrix}
$$

The convergence analysis for these preconditioners is under investigation.

Notice that in the iteration by subdomain methods we have considered so far the subdomains can be handled in parallel whenever the preconditioner P has a block diagonal structure. The degree of parallelism is precisely the number of diagonal blocks.

7 DECOMPOSITION OF Ω WITH INTERNAL CROSS POINTS

In this section we consider the case of decompositions of Ω with internal cross points (see fig.0.1 b).

For simplicity we consider the case of a square Ω partitioned into four squares Ω_i, $i=1,\ldots,4$ (see fig.6.1). The general case can be studied combining the results of this case with those of the previous sections.

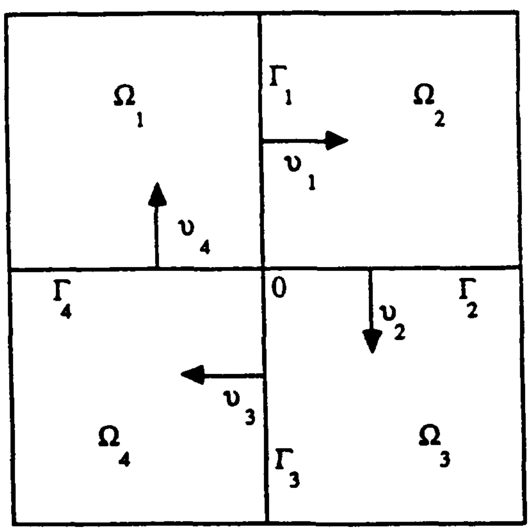


FIG.7.1 The decomposition of Ω.

We denote by Φ the space of the functions which are the traces on $\Gamma := \bigcup_{j=1}^{4} \Gamma_j$ of the functions of $H^1_0(\Omega)$. The variational multidomain formulation of problem (1.1) is given by (1.5)-(1.8) provided u_1 is identified with u_5.

The collocation points in Ω_i, i=1,..., 4, are defined in section 2. We recall that here C^{Γ_i} is the set of the internal collocation points of the interval Γ_i, i=1,...,4. The cross point **0** is the unique collocation point common to all the subdomains. For simplicity we look for a spectral solution which is of equal degree (N) in the four subdomains. Hence the weights defined in (2.2) and (2.4) are such that $\omega^i_k = \omega_k$, $\forall$ k = 0,...,N, moreover we have $\omega_{N_y} = \omega_0$, and we denote this value with ω.

Let us consider the following multidomain spectral approximation of problem (1.5)-(1.8). For i=1,...,4 we look for $u_{i,N} \in P_N(\Omega_i)$ such that for i=1, ... ,4 (we identify $u_{5,N}$ with $u_{1,N}$)

$$(7.1) \qquad Lu_{i,N} = f \qquad \text{in } C_i^{int}$$

$$(7.2) \qquad u_{i,N} = u_{i+1,N} \quad \text{in } C^{\Gamma_i}$$

$$(7.3) \qquad u_{i,N} = 0 \qquad \text{in } C_i^b$$

$$(7.4) \qquad \frac{\partial u_{i,N}}{\partial \upsilon_i} - \frac{\partial u_{i+1,N}}{\partial \upsilon_i} = -(Lu_{i,N} - f)\omega - (Lu_{i+1,N} - f)\omega \ \text{ in } C^{\Gamma_i} \quad .$$

$$(7.5) \qquad \left(\frac{\partial u_{1,N}}{\partial x} - \frac{\partial u_{1,N}}{\partial y}\right)(0) - \left(\frac{\partial u_{2,N}}{\partial x} + \frac{\partial u_{2,N}}{\partial y}\right)(0) - \left(\frac{\partial u_{3,N}}{\partial x} - \frac{\partial u_{3,N}}{\partial y}\right)(0) + \left(\frac{\partial u_{4,N}}{\partial x} + \frac{\partial u_{4,N}}{\partial y}\right)(0)$$
$$= -\sum_{i=1}^{4} (Lu_{i,N} - f)\,(0)\,\omega \quad .$$

The υ_i's are the outward normal directions to Γ_i, ordered clockwise as indicated in fig.6.1.

The conditions (7.4) are analogous to the ones in (2.9). The "cross point condition" (7.5) gives a relation between the four polynomials $u_{i,N}$, i=1, - ,4. In the spirit of the method (2.11)-(2.18) it is possible to formulate an iterative method for solving (7.1)-(7.5). The idea is to solve, at each iteration, two Dirichlet problems in Ω_1 and Ω_3, and two mixed problems in Ω_2 and Ω_4 as follows.

For i=1,...,4 let $g^0_i \in P_N(\Gamma_i)$ be such that $g^0_i = 0$ on $\partial\Omega$ and $g^0_1 = g^0_2 = g^0_3 = g^0_4$ in **0**; we look for a sequence $u^n_{i,N}$, $n \geq 1$, satisfying: $u^n_{i,N} \in P_N(\Omega_i)$, i=1,...,4 and

<u>in Ω_1 and Ω_3:</u>

$$(7.6) \quad Lu^n_{1,N} = f \text{ in } C_1^{int},\ u^n_{1,N} = g_1^{n-1} \text{ in } C^{\Gamma_1},\ u^n_{1,N} = g_4^{n-1} \text{ in } C^{\Gamma_4},\ u^n_{1,N} = g_1^{n-1} \text{ in } 0,\ u^n_{1,N} = 0 \text{ in } C_1^b$$

(7.7) $Lu^n_{3,N} = f$ in C_3^{int}, $u^n_{3,N} = g_2^{n-1}$ in C^{Γ_2}, $u^n_{3,N} = g_3^{n-1}$ in C^{Γ_3}, $u^n_{3,N} = g_1^{n-1}$ in 0, $u^n_{1,N} = 0$ in C_3^b

<u>in Ω_2 and Ω_4:</u>

(7.8) $Lu^n_{2,N} = f \quad \text{in } C_2^{int} \quad , \quad u^n_{2,N} = 0 \quad \text{in } C_2^b$

(7.9) $$-\frac{\partial u^n_{2,N}}{\partial \upsilon_1} + (Lu^n_{2,N} - f)\omega = -\frac{\partial u^n_{1,N}}{\partial \upsilon_1} - (Lu^n_{1,N} - f)\omega \quad \text{in } C^{\Gamma_1}$$

(7.10) $$-\frac{\partial u^n_{2,N}}{\partial \upsilon_2} - (Lu^n_{2,N} - f)\omega = -\frac{\partial u^n_{3,N}}{\partial \upsilon_2} + (Lu^n_3 - f)\omega \quad \text{in } C^{\Gamma_2}$$

(7.11) $Lu^n_{4,N} = f \quad \text{in } C_4^{int} \quad , \quad u^n_{4,N} = 0 \quad \text{in } C_4^b$

(7.12) $$\frac{\partial u^n_{4,N}}{\partial \upsilon_3} - (Lu^n_{4,N} - f)\omega = -\frac{\partial u^n_{3,N}}{\partial \upsilon_3} + (Lu^n_{3,N} - f)\omega \quad \text{in } C^{\Gamma_3}$$

(7.13) $$\frac{\partial u^n_{4,N}}{\partial \upsilon_4} + (Lu^n_{4,N} - f)\omega = -\frac{\partial u^n_{1,N}}{\partial \upsilon_4} - (Lu^n_{1,N} - f)\omega \quad \text{in } C^{\Gamma_4}$$

(7.14) $$-\left(\frac{\partial u^n_{2,N}}{\partial x} + \frac{\partial u^n_{2,N}}{\partial y}\right)(0) + \left(\frac{\partial u^n_{4,N}}{\partial x} + \frac{\partial u^n_{4,N}}{\partial y}\right)(0) + (Lu^n_{2,N} - f)(0)\omega + (Lu^n_{4,N} - f)(0)\omega =$$
$$= -\left(\frac{\partial u^n_{1,N}}{\partial x} - \frac{\partial u^n_{1,N}}{\partial y}\right)(0) + \left(\frac{\partial u^n_{3,N}}{\partial x} - \frac{\partial u^n_{3,N}}{\partial y}\right)(0) - (Lu^n_{1,N} - f)(0)\omega - (Lu^n_{3,N} - f)(0)\omega$$

(7.15) $u^n_{2,N} = u^n_{4,N} \qquad \text{in } \mathbf{0}$.

The polynomial g^n_i are defined recursively as follows:

(7.16) $$\begin{cases} g_1^n = \theta_n u^n_{2,N} + (1-\theta_n)\, g_1^{n-1} & \text{in } C^{\Gamma_1} \\ g_2^n = \theta_n u^n_{2,N} + (1-\theta_n)\, g_2^{n-1} & \text{in } C^{\Gamma_2} \\ g_3^n = \theta_n u^n_{4,N} + (1-\theta_n)\, g_3^{n-1} & \text{in } C^{\Gamma_3} \\ g_4^n = \theta_n u^n_{4,N} + (1-\theta_n)\, g_4^{n-1} & \text{in } C^{\Gamma_4} \\ g_i^n = \theta_n u^n_{2,N} + (1-\theta_n)\, g_i^{n-1} & \text{for } i=1,\dots,4 \text{ in } \mathbf{0} \,. \end{cases}$$

Note that problems in Ω_1 and Ω_3 are independent, while those in Ω_2 and Ω_4 are coupled through the two cross point conditions (7.14) and (7.15).

If the Legendre collocation points are used, the problem (7.1)-(7.5) can be written in a variational form which coincide with (3.4)-(3.7) provided M=4, $u_{5,N}$ is identified with $u_{1,N}$, and $P^{\circ}_N(\Gamma)$ is substituted with the space

$$\Phi_N := \{\varphi : \bigcup_{i=1}^{4} \Gamma_i \to R : \varphi_{|\Gamma_i} \in P_N(\Gamma_i),\ i=1,\dots,4,\ \varphi \text{ is continuous in } 0,\ \varphi=0 \text{ in } \partial\Omega \cap (\cup \Gamma_i)\}.$$

We skip the proof which is analogous to the one of proposition 3.1. With this variational formulation we can now follow the guideline of section 4 to prove the convergence, as $n \to \infty$, of the iterative scheme (7.6)-(7.16). In this case however the extension theorem 4.1 takes a different form. Precisely (4.7) holds now with two constants σ and τ which depend on N as follows

$$\sigma = \sigma_0 \log N, \quad \tau = \tau_0 \log N .$$

A consequence of this is that the constant θ^* of theorem 4.2, as well as the error reduction factor $k(\theta)$, for $\theta \in (0, \theta^*)$, depend now on N. The convergence result (4.29) is still true, but κ tends to one as $N \to \infty$ with a logarithmical speed. The precise form of this result and a detailed proof of it can be found in [25].

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National Aeronautics and Space Administration

Report Documentation Page

1. Report No. NASA CR-181620 ICASE Report No. 88-11	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle DOMAIN DECOMPOSITION PRECONDITIONERS FOR THE SPECTRAL COLLOCATION METHOD		5. Report Date January 1988
		6. Performing Organization Code
7. Author(s) Alfio Quarteroni and Giovanni Sacchi-Landriani		8. Performing Organization Report No. 88-11
		10. Work Unit No. 505-90-21-01
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665-5225		11. Contract or Grant No. NAS1-18107
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Langley Research Center Hampton, VA 23665-5225		13. Type of Report and Period Covered Contractor Report
		14. Sponsoring Agency Code

15. Supplementary Notes

Langley Technical Monitor: Richard W. Barnwell

Submitted to Journal of Scientific Computing

Final Report

16. Abstract

We propose and analyze several block iteration preconditioners for the solution of elliptic problems by spectral collocation methods in a region partitioned into several rectangles. It is shown that convergence is achieved with a rate which does not depend on the polynomial degree of the spectral solution. The iterative methods here presented can be effectively implemented on multiprocessor systems due to their high degree of parallelism.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
domain decomposition, spectral collocation methods, parallel calculations, approximation of elliptic equations	64 - Numerical Analysis Unclassified - unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of pages	22. Price
Unclassified	Unclassified	37	A03

NASA FORM 1626 OCT 86

NASA-Langley, 1988