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ON THE SENSITIVITY OF COMPLEX,
INTERNALLY COUPLED SYSTEMS

Jaroslaw Sobieszczanski-Sobieski

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Jaroslaw Sobieszczanski-Sobieski*
NASA Langley Research Center, Hampton, Virginia

Abstract

A method is presented for computing sensitivity derivatives with respect to independent (input) variables for complex, internally coupled systems, while avoiding the cost and inaccuracy of finite differencing performed on the entire system analysis. The method entails two alternative algorithms: the first is based on the classical implicit function theorem formulated on residuals of governing equations, and the second develops the system sensitivity equations in a new form using the partial (local) sensitivity derivatives of the output with respect to the input for each part of the system. A few application examples are presented to illustrate the discussion. The method has a potential to answer the "what if" questions by presenting engineers with sensitivity information on design trade-offs to guide human judgment and formal optimization. In addition, the method is compatible with the modern technology of distributed computing as well as traditional division of design tasks among groups of specialists in the design process. The capability to quantify the effects of proposed design changes may provide the basis for a mathematical model of design.

Nomenclature

A_i the i-th CA
CA   contributing analysis, a "black box" transforming input into output data used in analysis of a system; usually associated with an engineering discipline, or a physical part of the system
c_o CPU time for computing system sensitivity derivatives by a one-step finite difference procedure involving repeated analysis of the entire system
c' CPU time for computing system sensitivity derivatives using GSE1
c'' CPU time for computing system sensitivity derivatives using GSE2
F   vector of functions forming the equations governing a physical phenomenon
f   functional relationship
GSE1   Global Sensitivity Equations based on partial derivatives of residuals
GSE2   Global Sensitivity Equations based on the partial derivatives of output with respect to input of each CA
H   number of input items received by a CA from other CA's
I   identity matrix
M   number of independent variables in a CA
m   number of unknown variables in a CA
N   number of the CA's in a system analysis.
System Analysis   one solution of Eq. 1 for all the output unknowns Y
X   vector of independent variables
Y   vector of dependent variables
Z   number of unknown variables in a CA

α,β,γ identifiers for CA's in a small system of three CA's, equivalent of A_1, A_2, A_3

Subscripts, superscripts, special markings:

i,j,k   subscripts identifying CA's, elements of vectors, and elements of matrices
o   superscript or subscript identifying an initial value, or a normalization denominator
overbar   linearized function
tilde   normalized, nondimensional quantity
arrow above character designates a vector

Other symbols used locally are identified where introduced.

Introduction

"What if" is the all important question that arises again and again in design. Indeed, it may be argued that the design process is not complete until all such pertinent questions have been asked, satisfactorily answered, and the answers translated into design changes toward a product as good as it can be made under a set of given restrictions. If the object being designed is a complex, coupled system, the "what if" questions are difficult to answer because, to borrow a phrase from (Ref. 1), "if you make any change to it there are likely to be many subtle consequences". A recent example from aerospace is the forebody shape in hypersonic aircraft whose change influences structures, aerodynamics, propulsion, and, ultimately, the performance.

Many "what if" questions cannot be quantified and engineering judgment is indispensable to answer them. However, in aerospace vehicle design a great deal of "what if" questions can be quantified either by assessing the effects of relatively large variations of the variables involved (a parametric study) or by considering very small, theoretically infinitesimal variations to calculate sensitivity derivatives.

The focus of this paper is on sensitivity analysis. While recent developments in numerical methods provided engineers with many useful techniques for disciplinary, or subsystem, sensitivity analysis, e.g., (Refs. 2, 3, 4), examination of literature, e.g., (Refs. 1, 5, 6) shows a void as far as the comparable methods applicable to entire systems are concerned. This paper's purpose is to address that void and to offer a system sensitivity analysis capable of answering the quantitative "what if" design questions. To that end, the paper presents a method for computing sensitivity derivatives with respect to independent (input) variables for complex, internally coupled systems, while avoiding the cost and inaccuracy of finite differencing performed on the entire system analysis. The method entails two alternative algorithms: the first is based on the classical implicit function theorem formulated on residuals of governing equations, and the second develops the system sensitivity equations in a new form using the partial

(local) sensitivity derivatives of the output with respect to the input for each part of the system. A few application examples are presented to illustrate the discussion.

Statement of the Problem

In this paper, a complex, internally coupled system is defined as physical object whose behavior is described by a vector Y obtainable as a solution of a set of simultaneous (coupled) equations which can be partitioned into subsets such as

$$\vec{\alpha}((X,Y_\beta,Y_\gamma),Y_\alpha) = 0$$

$$\vec{\beta}((X,Y_\alpha,Y_\gamma),Y_\beta) = 0 \qquad (1)$$

$$\vec{\gamma}((X,Y_\beta,Y_\alpha),Y_\gamma) = 0$$

Each of the system subsets represents a distinct, separate analysis that will be referred to as contributing analysis (CA), usually associated with a particular engineering discipline, or a distinct physical part (a subsystem) of the system, or both. Partitioning of the system analysis into separate but coupled CA's amounts to a system decomposition. The Operations Research literature calls such partitioning an aspect decomposition if the CA's correspond to disciplines, and an object decomposition if they correspond to physical subsystems (Ref. 7). In most engineering problems both types of decomposition are used simultaneously to break the large task into smaller ones. Mathematics developed in this paper applies equally to both types. All the mathematical discussion herein is based on three partitions because that is a number which is conveniently small, and yet sufficient to establish patterns that can easily be generalized to arbitrarily large number of partitions. Solving the entire set of equations will be referred to as the system analysis which can be written as $\vec{F}(Y,X) = 0$.

Each CA yields a solution in form of a vector Y_* (where subscript $*$ stands for α, β or γ, and identifies a subset of Y) listed last in the parentheses, given the input listed in the inner parentheses. The system is internally coupled because the input to one CA includes outputs from the other CAs - as shown by the arrows in Figs. 1. The coupled system is depicted in Fig. 2 by a directed graph representation (e.g., (Ref. 1)).

This paper's focus is on large scale applications in which at least some CA's are nonlinear and complex, so that the system analysis can only be done iteratively. Typically, a CA is carried out by a group of specialists, maybe at a separate subcontractor organization. This may be illustrated by an example of aircraft wing design incorporating nonlinear aerodynamics, structures, and active control (aspect decomposition), or substructuring (object decomposition).

The system solution Y is sensitive to the independent variables X present in the CA inputs. It is important to emphasize that the independent variables X may include not only the designer-decided inputs (design variables) but also other inputs external to the system, for example, loads,

heat flux, etc. In the most general case, all variables X may occur in the input to each CA, but in most practical applications only a subset of the vector X will enter the input of a particular CA.

One way to compute sensitivity derivatives of the solution Y with respect to the independent variables X is a finite difference technique depicted by a flowchart in Fig. 3 in its simplest, one-step-forward, version. It requires repetition of the system analysis for every perturbed X. This may be prohibitively costly, particularly if the system analysis is nonlinear and/or iterative. Even more importantly, it may be inaccurate to the point of producing meaningless results as the effect of small perturbations in X may drown in the noise of the iterative solution of the system (e.g., Ref. 8). Attempting to remedy this effect by increasing the perturbation magnitude may introduce significant error due to the analysis nonlinearity. Consequently, the perturbation range in which accuracy of finite differencing is acceptable becomes problem dependent and may not even exist.

Thus, the problem is how to calculate the sensitivity derivatives of the system solution Y with respect to the independent variables X without resorting to a finite difference operation involving the entire system analysis as in Fig. 3.

Solution

As mentioned in the Introduction, there are at least two ways of solving the system sensitivity problem. A residual-based solution will be introduced first, and an alternative using local output sensitivity will follow.

Residual-Based Solution

The implicit function theorem of functional analysis, e.g., Ref. 9 states that a set of governing equations

$$F(Y,X) = 0; \quad Y = f(X) \qquad (2)$$

has the following sensitivity equations

$$\left[\frac{\partial F}{\partial Y}\right]\left\{\frac{\partial Y}{\partial X_k}\right\} = -\left\{\frac{\partial F}{\partial X_k}\right\} \qquad (3)$$

The sensitivity equations are always simultaneous, linear, and algebraic, regardless of the mathematical nature (nonlinear, transcendental, etc.) of the governing equations of the system. In Eq. 3, the matrix of coefficients, m × m, is a Jacobian matrix of the partial derivatives with respect to dependent variables, and the right-hand-side vector contains the partial derivatives with respect to a particular independent variable. These partial derivatives are evaluated using the X and Y values which satisfy Eq. 2. In other words, solution of the governing equations, Eq. 2, is a prerequisite to forming and solving the sensitivity equations, Eq. 3.

The solution vector of Eq. 3 comprises the derivatives of the dependent variables with respect to a particular independent variable. It will be useful in further discussion to have noted at this point that Eq. 3 is based on residuals of Eq. 2,

i.e., a perturbation of one element in X alone would generate a vector of residuals of F replacing zero on the right hand side of the equation. Similarly, a perturbation of one element in Y alone would also generate a residual vector. Consequently, to maintain the right hand side at zero despite the perturbation of X, there must be a change of in Y subordinated to the change of X to make the residual vectors due to Y and X offset each other. Equation 3 merely states that to generate compensating residuals the rates of change of the residuals with respect to the dependent and independent variables must balance each other, taking into account the implicit dependence of Y on X. In other words, the total derivative with respect to X of the residuals of Eq. 2 must vanish.

The method for computing the terms given in Eq. 3 is problem-dependent. Obviously, an analytical differentiation is preferred but, if that is not possible, a finite difference technique may be applied. Since the finite difference technique in this application is used to calculate the partial derivatives of residuals, it requires only an evaluation of F(Y,X) for perturbations of its arguments instead of a solution of F(Y,X) = 0 for each perturbation. Thus, the finite difference operation performed on the entire system analysis as in Fig. 3 is eliminated.

When applied to the partitioned system in Eq. 1, the sensitivity equations 3 take on this form

$$
\begin{bmatrix}
\partial\alpha_i/\partial Y_{\alpha j} & \partial\alpha_i/\partial Y_{\beta j} & \partial\alpha_i/\partial Y_{\gamma j} \\
\partial\beta_i/\partial Y_{\alpha j} & \partial\beta_i/\partial Y_{\beta j} & \partial\beta_i/\partial Y_{\gamma j} \\
\partial\gamma_i/\partial Y_{\alpha j} & \partial\gamma_i/\partial Y_{\beta j} & \partial\gamma_i/\partial Y_{\gamma j}
\end{bmatrix}
\begin{Bmatrix}
\partial Y_{\alpha i}/\partial X_{*k} \\
\partial Y_{\beta i}/\partial X_{*k} \\
\partial Y_{\gamma i}/\partial X_{*k}
\end{Bmatrix}
$$

$$
= - \begin{Bmatrix}
0 \\
\partial_*/\partial X_k \\
0
\end{Bmatrix}
$$

(4)

referred to as the Global Sensitivity Equations 1 (GSE1). These equations contain as unknowns the sensitivity derivatives of the system solution Y (partitioned) with respect to an independent variable X (one at a time). Their matrix of coefficients is populated by the partial derivatives of the residuals of each CA with respect to the input that CA receives from the other CA's, and the right-hand-side vector is formed from the partial derivatives of the CA residuals with respect to the independent variable directly affecting that CA. For a general case of N CA's, the equations acquire a format shown in Appendix A.

Despite their potential cost and accuracy advantages, the use of the sensitivity equations 4 based on residuals may not be straightforward in engineering practice because existing disciplinary codes have usually no provisions to compute the residuals, and the residuals usually have no obvious physical meaning that would allow the user to judge validity of the numbers (An exception is

structural analysis where the residuals are unequilibrated loads). These reasons motivated derivation of a new form for the system sensitivity equations not predicated on the residuals.

Formulation Based on Sensitivities of Individual CA's

Residual-independent sensitivity equations may be derived in more than one way. The derivation shown below is based on linearization of the governing equations 1, an alternative derivation is shown in Appendix A.

Equation 1 relate each partition of Y to the X and the other partitions of Y so that from each equation:

$$
Y_\alpha = f_\alpha(X, Y_\beta, Y_\gamma)
$$

$$
Y_\beta = f_\beta(X, Y_\alpha, Y_\gamma)
$$

(5)

$$
Y_\gamma = f_\gamma(X, Y_\beta, Y_\alpha)
$$

These functions may be linearized in the neighborhood of the solution of Eq. 1 denoted $Y_{\alpha 0}$, $Y_{\beta 0}$, $Y_{\gamma 0}$ using a curtailed Taylor series. Using Y_α as an example:

$$
Y_\alpha = Y_{\alpha 0} + (\partial f_\alpha/\partial X)\Delta X + (\partial f_\alpha/\partial Y_\beta)\Delta Y_\beta
$$
$$
+ (\partial f_\alpha/\partial Y_\gamma)\Delta Y_\gamma ;
$$

(6)

By moving all terms to the left hand side, Eq. 6 are transformed into a linearized version of Eq. 1

$$
\bar{\alpha} = Y_\alpha - Y_{\alpha 0} - (\partial f_\alpha/\partial X)(X - X_0)
$$
$$
- (\partial f_\alpha/\partial Y_\beta)(Y_\beta - Y_{\beta 0})
$$
$$
+ - (\partial f_\alpha/\partial Y_\gamma)(Y_\gamma - Y_{\gamma 0}) = 0
$$

$$
\bar{\beta} = Y_\beta - Y_{\beta 0} - (\partial f_\beta/\partial X)(X - X_0)
$$
$$
- (\partial f_\beta/\partial Y_\alpha)(Y_\alpha - Y_{\alpha 0})
$$
$$
+ - (\partial f_\beta/\partial Y_\gamma)(Y_\gamma - Y_{\gamma 0}) = 0
$$

$$
\bar{\gamma} = Y_\gamma - Y_{\gamma 0} - (\partial f_\gamma/\partial X)(X - X_0)
$$
$$
- (\partial f_\gamma/\partial Y_\alpha)(Y_\alpha - Y_{\alpha 0})
$$
$$
+ - (\partial f_\gamma/\partial Y_\beta)(Y_\beta - Y_{\beta 0}) = 0
$$

(7)

Under some conditions discussed in Appendix A, Eq. 7 may have no solution because of singularity of their matrix of coefficients. Assuming that singularity conditions examined in the Appendix A do not occur, it is axiomatic that Eq. 7 and 1 have the same solutions Y and that these solutions have the same derivatives with respect to X, consequently, we may treat Eq. 7 as surrogate governing equations. The implicit function theorem may now be

applied to these equations just as it was applied to Eq. 1 by performing the differentiation shown in Eq. 3. This yields sensitivity equations in the form:

$$
\begin{bmatrix}
I & -\partial f_\alpha/\partial Y_\beta & -\partial f_\alpha/\partial Y_\gamma \\
\hline
-\partial f_\beta/\partial Y_\alpha & I & -\partial f_\beta/\partial Y_\gamma \\
\hline
-\partial f_\gamma/\partial Y_\alpha & -\partial f_\gamma/\partial Y_\beta & I
\end{bmatrix}
\begin{Bmatrix}
\partial Y_\alpha/\partial X_k \\
\hline
\partial Y_\beta/\partial X_k \\
\hline
\partial Y_\gamma/\partial X_k
\end{Bmatrix}
=
\begin{Bmatrix}
0 \\
\hline
\partial f_*/\partial X_k \\
\hline
0
\end{Bmatrix}
\tag{8}
$$

termed Global Sensitivity Equations 2. For a general case of N CA's, the equations acquire a format shown in Appendix A.

Equation 8, contains no residuals of the CA's. Instead, its matrix of coefficients is populated by the sensitivity derivatives of each CA output with respect to that CA's input, and its right hand side vector represents sensitivity of a CA's output with respect to the independent variable (one at a time) directly affecting that CA. As far as the complete system is concerned, these derivatives are partial (local) derivatives, while the solution of Eq. 8 yields the derivatives of the solution Y (partitioned) with respect to an independent variable (one at a time). By definition, the partial derivatives represent sensitivity of each isolated CA, and the derivatives of Y represent the system sensitivity with all the couplings (e.g., Figs. 1 and 2) fully accounted for.

Both GSE1 and GSE2, Eqs. 4 and 8, produce the same solution vector and both are equally exact because they are derived from a mathematical theorem without any simplifying assumptions or approximations. Similarity of mathematical characteristics and potential usage among the two sets of equations allows to limit the ensuing discussion to GSE2, occasionally using the notation GSEx to address both GSE1 and GSE2. This emphasis on GSE2 does not imply an unqualified recommendation of GSE2 over GSE1. The choice is up to the user and it depends on the factors already stated in the foregoing as motivations for the GSE2 development, and on the considerations of cost and benefits discussed later (supported by Appendix B).

Figure 4 illustrates details of the GSE2 structure for three CA's. The general pattern is easily extrapolated to larger number of CA's. The matrix of coefficients has identity submatrices on the diagonal. Each off-diagonal submatrix is a Jacobian matrix corresponding to a CA. For example, the Jacobian at the upper right corner in Fig. 4 contains partial sensitivity derivatives of every item of output from the α CA, a column m_α long, with respect to every one of m_γ input items the α CA receives from the γ CA, hence the m_γ

columns in the Jacobian. The corresponding Jacobian at the lower left corner comprises the partial derivatives of the output from the γ CA with respect to the input the γ CA receives from the α CA. In general case the two Jacobians are not symmetric. An example of the above two Jacobian matrices might be drawn from a case of an actively controlled flexible wing. Then, assuming the CA's α, β, and γ to be aerodynamics, structures, and active controls, respectively, the upper right Jacobian would include the partial sensitivity derivatives the aerodynamic pressure at selected locations on the wing to the control surface deflections. Correspondingly, the lower left Jacobian might comprise of the partial derivatives of the control surface deflections to the aerodynamic pressure coefficients. These derivatives derive from the control law that establishes a functional relationship between the control surface deflections and the aerodynamic pressure on the wing (sensed directly or indirectly).

On the right hand side, there is a vector of the partial sensitivity derivatives of the CA output with respect to a particular independent variable X. These partial derivatives are nonzero for those CA's that are directly influenced by that particular independent variable. Referring again to the above example of a flexible wing, if the variable X were, say, the planform aspect ratio, the nonzero elements of the right-hand-side vector would occur at the locations corresponding to the β and γ partitions, since only the aerodynamics and structures would be directly influenced.

The GSE2 matrix of coefficients depends only on the coupling among the CA's and not at all on the sensitivity to the design variables. The opposite is true for the right-hand-side vectors. Thus, the matrix can be formed and factored once, and the solutions for many design variables can be obtained by repeated back-substitutions of each right-hand-side vector. The coupling among the CA's is reflected in the topology of the GSE2 matrix as shown in a few examples in Fig. 5 and 6. If there are no couplings (#1), the matrix is an identity matrix and the derivatives of Y are equal directly to the partial derivatives on the right-hand-side. Each coupling link generates an off-diagonal Jacobian until the matrix becomes fully populated for a fully coupled system (#7). The coefficient matrix in the GSE2 exhibits a the same pattern.

There is a coincidence of form between the matrix of coefficients in GSEx and the so-called equation precedence matrix (or N-Square Matrix) used in Operations Research literature (e.g., (Ref. 1, p. 87)) to analyze internal couplings in systems. Namely, each non-zero, off-diagonal Jacobian in the GSEx matrix of coefficients corresponds to a non-zero element in the N-Square matrix for the same system.

As a matter of a particular interest to a structural engineer, one may observe that if F in Eq. 2 represents structural load-deflection equations, then the Jacobians in Eq. 8 correspond to substructures or, ultimately, individual finite elements.

In some applications it may be convenient (for instance, when the X variables are measured in

4

different units) to have all terms in GSE2 dimensionless. A nondimensional version of the GSE2 is given in Appendix A.

Examples

Since both the GSE1 and GSE2 (Eqs. 4 and 8, respectively) are rigorously derived from a fundamental theorem they do not need numerical verification. However, a few examples are provided for the usage of GSE2 to support the discussion of costs and benefits that will follow.

A simple example of a 2D airfoil in airflow is shown in Fig. 7. The airfoil is supported by two linear springs attached to a ramp whose angle of inclination ψ is an independent variable. The elastic degrees of freedom allowed are only the pitch and the plunge. The lift coefficient is assumed to be a nonlinear function of the angle of attack illustrated in Fig. 8 and defined in Table 1 - Aerodynamics. The function is set up deliberately as a transcendental function to admit only an iterative system analysis. The angle of attack θ depends on the ramp angle (design variable ψ and the airfoil elastic support pitch angle ϕ.

The airfoil on springs is an aerodynamic-structure system abstracted as a directed graph in Fig. 9. All the equations that constitute the Aerodynamic and Structures CA's in the graph are listed in Table 1 which also shows the problem notation and its correspondence to the generic notation used in the paper, and the numerical data for the example. The purpose of the example is to show computation of the derivatives of the system solution output - the lift L and the elastic pitch angle ϕ - by means of the GSE2 and to compare the results with those from a finite difference technique.

The system solution was found iteratively and is listed in Table 2 for arbitrary ψ value of .05 rad. Next, the sensitivity derivatives of L and ψ with respect to the angle ψ were obtained by the finite differences procedure at the system level illustrated in Fig. 3 which required repetition of the iterative solution for the angle ψ incremented by .0025 rad to .0525 rad. These derivatives are shown in Table 2 and provide reference for comparison with the same derivatives computed using the GSE2. Incidentally, the derivative of L is greater than the partial derivative due to the elastic effect. The GSE2 and the numerical values of the partial derivatives that enter these equations are also given in Table 2 (these equations are also shown in a dimensionless format in Appendix A). The partial derivatives were obtained by the same, simple, one-step-forward, finite difference procedure referred to above but applied separately to Aerodynamics and Structures CA's. Finally, Table 2 presents the GSE2 solution that agrees with the finite difference results obtained at the system level.

The second example shows how the GSE2 equations for a system are made up of the partial derivatives for the system CA's. The system is a flexible wing with an active control intended to reduce the root bending moment. The system directed graph and the coupling information are shown in Fig. 10 - upper part. The bottom part of the figure illustrates the make-up of the GSE2.

Dimensions of the arrays entering the GSE2 depend on the number of the individual pieces of data (coupling channel bandwidth, referred to as bandwidth, for short) communicated from one CA to another. These dimensions have a strong impact on the computational cost of the method as shown in the next section and in Appendix B, therefore, it is important to keep the bandwidths as small as possible. In this example, the Structures-Active Control channel does not need to transmit more than a few strain gage readings. Similarly, the Aerodynamics-Active Control channel transmits only a few dynamic pressure sensor indications (or only the Mach number and the angle of attack value from which the pressures may be inferred) and one, or two, control surface deflection angles. In contrast, the information moving along the Aerodynamic-Structures channel may include hundreds of the dynamic pressure values for discrete locations on the wing, if a panel-based CFD code is used, and thousands of the nodal point displacements output from a finite element code. It is evident, that this channel will require attention to reduce its bandwidth. Such reduction may be achieved by representing deformations and loads by a relatively small number of generalized coordinates and corresponding generalized forces based on modal analysis, following the practice well established in aeroelasticity analysis.

Another example of the use of the GSE2 for a system with active control is described in (Ref. 8).

Costs and Benefits

By using GSEx (Eqs. 4 or 8), the cost of repetitive system analysis required by finite difference procedure (Fig. 3) is eliminated, but the cost of generating the input into these equations and solving them is added. Using the CPU time as a simplified measure of the computational cost, Appendix B shows, under a set of assumptions, that the cost for the finite difference procedure of Fig. 3 increases with the square of the number of CA's in the system. On the other hand, the cost of generating the input into GSEx under the same set of assumptions is proportional to the product of the number of CA's and the bandwidth. These relations suggest that there is a limit on the finite difference procedure's applicability to large systems, and show the importance of the bandwidth to the cost of the GSEx. As far as the accuracy of GSEx is concerned, it depends on the conditioning of its matrix of coefficients (see Appendix A) but is not affected by the system dimensionality.

The principal qualitative advantage of the sensitivity analysis based on the GSE2 is that it allows to treat the system as decomposed into a set of "black boxes" coupled by a well-defined sets of data. Each black box may, then, be subjected to its own sensitivity analysis performed by specialists intimately familiar with the specifics. The specialists may use any means for the partial sensitivity analysis available such as: finite difference procedures, historical statistical data, approximate methods, or even judgmental assessment. It should also be stressed that they may also draw on the disciplinary, quasi-analytical sensitivity analysis algorithms that are now undergoing an intensive development (Ref. 6). They may even obtain

the sensitivity data experimentally. In general, the approach divides the labor and thus creates opportunity for concurrent data processing in the contemporary distributed computing environment, and supports a broad workfront in the engineering organization.

Another benefit from the "black box" approach is that the GSEx are inherently recursive, in the sense that each of the system's "black boxes" (the CA's) may be a complex system within itself. If so, its sensitivity analysis may be carried out as described herein, to produce the sensitivity derivatives that will be treated as the sensitivity partial derivatives in the GSEx of the parent system.

Regarding of the choice of GSE2 vs. GSE1, if the computational cost was the only factor (see Appendix B) GSE1 would be recommended over GSE2. However, the nonavailability of the residuals in existing disciplinary codes, and difficulties with physical interpretation of the residuals clearly favor the GSE2 format. Furthermore, the disciplinary sensitivity analyses are formulated to yield data compatible with input to GSE2 but not GSE1. These considerations may be overridden in the future by new disciplinary code developments (availability of the residual options) encouraged by the strong cost advantage of the GSE1. For now, the choice is judgmental.

Usage in Design

Systematic procedure for generating the system sensitivity data in a design process using the GSE2 may be organized in a way shown by a Chapin-format (Ref. 10) flowchart in Fig. 11. It begins with the system analysis for a given X. The partial sensitivity derivative computations follow in each CA independently for the given X and given Y (the latter obtained from the system analysis). The partial derivative calculations may be carried out concurrently. The final operation is an assembly and solution of the GSE2. The usage of GSE1 is similar. The results of a system sensitivity analysis may be used to identify the "design drivers", and to determine design modifications toward improvement, or they can be input into a formal optimization procedure. Since, in general case, the system solution and its sensitivity analysis have to be updated after moving away from the previously solved design, the flowchart procedure has to be iterated as the design process advances.

The GSEx may also be used to assess the coupling strength between any two CA's. This can be done by computing derivatives of the GSEx solution to the elements of the GSEx matrix in a manner shown below for the GSE2 used as an example. The GSE2 may be written as

$$[A] \left\{ \frac{\partial Y}{\partial X_k} \right\} = \{RHS\}; \tag{9}$$

where A and RHS are the matrix of coefficients and the right hand side vector, respectively, defined in Eq. 8. Since the equations are linear, the derivatives of their solution with respect to the elements of the matrix A may be obtained by substituting Eq. 9 for F in Eq. 2 and, then, writing the corresponding sensitivity equations using the differentiation pattern of Eq. 3

$$[A] \frac{\partial}{\partial A_{ij}} \left\{ \frac{\partial Y}{\partial X_k} \right\} = -\left[\frac{\partial A}{\partial A_{ij}} \right] \left\{ \frac{\partial Y}{\partial X_k} \right\}; \tag{10}$$

In this set of equations the matrix $\partial A/\partial A_{ij}$ is all empty except unity at the location occupied by the element A_{ij} in the matrix A. The vector of the partial derivatives of Y with respect to X_k is available from the solution of Eq. 9, and the derivatives of the RHS in Eq. 9 with respect to A_{ij} are null so they do not appear in Eq. 10. Consequently, the unknown derivatives $\partial(\partial Y/\partial X_k)/\partial A_{ij}$ may be obtained by backsubstitution of the new right hand side vector over the matrix A decomposed once in the solution of Eq. 9 and saved.

These derivatives measure the influence of the partial sensitivity derivatives A_{ij} on the sensitivity of the system with respect to X and may be adopted as indicators of the strength of the couplings among the parts of the system. A full survey of the coupling strengths would require solution of Eq. 9 and 10 for each combination of A_{ij} and X_k. In case of Y and X expressed in nonhomogeneous physical units, a dimensionless form shown in Appendix A for Eq. 8 would have to be used to obtain the coupling strength indicators that could be compared with each other. The comparison would be useful to identify relatively weak couplings that might be dropped from the system's mathematical model. Thus, the coupling strength indicators may augment the system analyst's judgment in searching for a compromise between the system model simplicity and its predictive accuracy.

Conclusions

The paper addresses problem of a sensitivity of a complex, internally coupled system behavior (response) to changes in independent variables. It is assumed that the system analysis is made up of self-contained analyses, corresponding to disciplines and/or physical subsystems, which exchange input/output data with each other. The problem is solved by formulating rigorous sensitivity equations - the global sensitivity equations - derived in a form based on the residuals of the governing equations, and in a new form that does not rely on the residuals. The latter is judged to be more useful to engineers than the former. It allows evaluation of system sensitivity to independent variables on the basis of the partial derivative information obtained locally within each contributing engineering discipline, or within each physical subsystem analysis, consistent with the decomposition of the design process among the specialty groups, and compatible with the technology of distributed computing. The equations eliminate the need for costly and potentially inaccurate finite differencing performed on the entire system analysis, and are capable of accepting experimentally obtained sensitivity data. Their computational cost advantage over the reference finite difference procedure increases with the number of self-contained analyses into which the system analysis can be partitioned, and is reduced proportionally to the volume of the coupling information.

Derivatives of the solution of the sensitivity equations with respect to their own coefficients may be useful as indicators of the strength of the

couplings among the parts of the system. Ranking these indicators by their magnitudes may identify weak couplings that might be eliminated from the system's model to make it simpler without significant loss of its predictive accuracy.

The global sensitivity equations in either form are offered as a tool to support the design process by contributing the system sensitivity information as an aid for human judgment and/or for use in formal optimization. Inasmuch as the global sensitivity equations quantitatively answer, with the first order of the accuracy, the "what if" questions underlying the design process, they may be regarded as a first order mathematical model of that process.

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Appendix A

Alternative Derivation, Generalization to a Case of N CA's, and a Dimensionless Format for the Global Sensitivity Equations 2.

Alternative Derivation

An alternative derivation begins with Eq. 5, differentiated with respect to one particular independent variable, say, X_k. Using the chain rule, the derivatives of Y are

$$\partial Y_\alpha/\partial X_k = (\partial f_\alpha/\partial X_k) + (\partial f_\alpha/\partial Y_\beta)(\partial Y_\beta/\partial X_k)$$
$$+ (\partial f_\alpha/\partial Y_\gamma)(\partial Y_\gamma/\partial X_k)$$

$$\partial Y_\beta/\partial X_k = (\partial f_\beta/\partial X_k) + (\partial f_\beta/\partial Y_\alpha)(\partial Y_\alpha/\partial X_k)$$
$$+ (\partial f_\beta/\partial Y_\gamma)(\partial Y_\gamma/\partial X_k)$$

$$\partial Y_\gamma/\partial X_k = (\partial f_\gamma/\partial X_k) + (\partial f_\gamma/\partial Y_\alpha)(\partial Y_\alpha/\partial X_k)$$
$$+ (\partial f_\gamma/\partial Y_\beta)(\partial Y_\beta/\partial X_k) \tag{A1}$$

Collecting the given and unknown terms, and rearranging, yields the GSE2 in the format of Eq. 8.

Possible Singularity of the Matrix of Coefficients in Eq. 7 and GSEx.

When a solution of a set of nonlinear equations, corresponding to Eq. 2, exists by virtue of intersection as illustrated for an example of two functions in a two-variable space in Fig. 12(a), then, the same solution coordinates are defined by a set of corresponding linearized equations, corresponding to Eq. 7, and represented by the dashed tangents in the figure. However, in case of the solution existing by virtue of tangency, as in

Fig. 12(b), the corresponding tangents overlap. Hence, their equations, Eq. 7, become singular and have no solution. Therefore, they no longer can be used as a substitute for the original, nonlinear equations, or as a basis for deriving linear sensitivity equations, since the matrix of coefficients in these equations will also be singular. It will be so because Eq. 7 may be written as

$$[A(X)]\{Y(X)\} = \{RHS(X)\} \tag{A2}$$

and the corresponding sensitivity equations obtained either from Eq. 4 or Eq. 8 are

$$[A(X)]\{\partial Y(X)/\partial X\} = -[\partial A(X)/\partial X]\{Y\}$$
$$+ \{\partial RHS(X)/\partial X\} ; \tag{A3}$$

Since Eq. A2 and A3 share the same matrix of coefficients, its singularity affects both sets of equations.

One may add that the tangency-type solution to Eq. 2 depicted in Fig. 12b has also a drawback of rendering the coordinates of the solution point S ill-defined. As shown in Fig. 13, due to error in numerical definition of the tangent functions, the coordinates of the point of tangency fall into broad intervals of uncertainty. This may also occur for the intersection type-type solution, if the intersection angle is small, then the matrix of coefficients A(X) may be ill-conditioned, although non-singular. Occurrence of these cases in design usually indicates that the design analysis is ill-posed and should be reformulated.

The GSEx matrix of coefficients may also be singular, if the system is physically unstable, e.g., a wing divergence. However, such instability would normally manifest itself at the prerequisite stage of the system analysis (solution of Eq. 1), so it is not expected to become a problem in the sensitivity analysis.

GSE1 and GSE2 Generalized to a Case of N CA's.

A simple extrapolation of the pattern from the case of three CA's discussed in conjunction with Eq. 1, 4, and 8, leads to Eq. 1, GSE1, and GSE2, respectively, taking on the form of Eq. A4, A5, A6:

$$A_i((X,\ldots Y_j\ldots Y_l\ldots),Y_i) = 0 \quad \begin{array}{l} i = 1{\rightarrow}N \\ j \ne i; l \ne i \end{array} \tag{A4}$$

$$[B] \cdot \{\partial Y/\partial X_k\} = -\{\partial A_i/\partial X_k\} \tag{A5}$$

$$B_{ij} = \partial A_i/\partial Y_j$$

$$[B] \cdot \{\partial Y/\partial X_k\} = \{\partial f_i/\partial X_k\} \tag{A6}$$

$$B_{ij} = -\partial f_i/\partial Y_j; \quad B_{ii} = I$$

Dimensionless Form

To transform the GSE2 from the form of Eq. 8 to a dimensionless form, we normalize the variables Y and X in Eq. 1 to unity by dividing them by their initial values (if any of them is zero, a suitable nonzero value is used instead). The normalization yields, showing two partial derivatives as examples

$$(\partial \tilde{Y}_{\alpha i}/\partial \tilde{Y}_{\beta j}) = (\partial Y_{\alpha i}/\partial Y_{\beta j})(Y^0_{\beta j}/Y^0_{\alpha i}) \tag{A7}$$

$$(\partial \tilde{Y}_{\alpha i}/\partial \tilde{X}_k) = (\partial Y_{\alpha i}/\partial X_k)(X^0_k/Y^0_{\alpha i}) \tag{A8}$$

This normalization introduced into Eqs. 5, 6, and 7 leads to a dimensionless form for Eq. 8. For instance, the partition β of Eq. 8 becomes

$$\left[-\frac{\partial f_\beta}{\partial Y_\alpha} 1_{\beta\alpha ij} \; \middle| \; I \; \middle| \; -\frac{\partial f_\beta}{\partial Y_\gamma} 1_{\beta\gamma ij} \; \middle| \; \frac{\partial Y_\beta}{\partial X_k} 1_{\beta xk} \; \middle| \; \frac{\partial f_\beta}{\partial X_k} 1_{\beta xk} \right] \tag{A9}$$

where:

$$1_{\beta\alpha ij} = Y^0_{\alpha i}/Y_{\beta j}; \quad 1_{\beta\gamma ij} = Y^0_{\gamma i}/Y_{\beta j}; \quad 1_{\beta xk} = X^0_k/Y^0_{\beta j}$$

Referring to the example defined in Tables 1 and 2, the GSE2 shown in Table 2 transform to the following dimensionless form:

$$Y_\alpha \equiv L; \quad Y_\beta \equiv \phi; \quad L = 502.2995 \text{ N};$$

$$\phi = .0175805 \text{ rd}$$

$$X_k = \psi; \quad \psi = .05 \text{ rd}$$

$$\begin{bmatrix} 1 & -.343 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \partial \tilde{L}/\partial \tilde{\psi} \\ \partial \tilde{\phi}/\partial \tilde{\psi} \end{Bmatrix} = \begin{Bmatrix} .343 \\ 0 \end{Bmatrix} \tag{A10}$$

in which the great numerical disparity of the terms visible in the dimensional form of the equations in Table 2 is avoided which is one benefit from the dimensionless form.

Appendix B

Computational costs of the GSEx-based sensitivity analysis and the reference finite difference procedure measured by the CPU times are influenced by a very large number of the problem-dependent and computer type-dependent (hardware and software) variables, but a reasonable estimates can be made if a number of simplifying assumptions are introduced. The assumptions used here are:

1. there are N CA's, each having the same CPU time c_1 for one solution.

2. complete solution of the system is iterative; it requires p_1 iterations, and Nc_1 CPU time in each iteration.

3. repeated solution of the system for a small perturbation of a design variable requires $p_2 < p_1$ iterations.

4. each CA is directly influenced by M independent variables so that the total number of independent variables is MN.

5. there are Z unknown variables Y in each CA.

6. each CA receives H input variables from the remainder of the system.

7. partial sensitivity derivatives of a CA are computed by finite differences, and each CA solution repeated for a small perturbation of input requires $c_2 < c_1$ of CPU time.

8. computation of residuals for a CA requires c_4 CPU time.

9. solution of the GSE2 and GSE1 with multiple right hand sides takes c_3 and c_5 CPU times, respectively. These times are expected to be relatively small, due to the use of the parallel and vector processing technology.

Under this assumption, the cost of the finite difference procedure is a sum of two terms: a reference system analysis, and the system analysis repeated for small perturbation of each of the independent variables. The two terms form, respectively, the following expression:

$$c_0 = Nc_1p_1 + Nc_1p_2MN$$

$$= c_1p_1(N + (p_2/p_1)MN^2 \tag{B1}$$

For the GSE2-based sensitivity analysis the cost is a sum of the cost of one system analysis, the cost of solution of each of the N CA's repeated for a small perturbation of its (H + M) inputs to compute the partial derivatives, and the cost of solving the GSE2. These three cost contributions are represented by the respective terms in the following expression:

$$c'' = Nc_1p_1 + c_2N(H + M) + c_3 \tag{B2}$$

Finally, for the GSE1-based sensitivity analysis the cost is a sum of the cost of one system analysis, the cost of computing the residuals of each CA for small perturbations of each of: its unknown Y variables, each of the coupling Y variables, and each of the X variables influencing that CA. The total cost includes also the cost of solving the GSE1. These three cost contributions are represented by the respective terms in the expression:

$$c' = N_1p_1 + c_4(Z + H + M) + c_5 \tag{B3}$$

The above equations reveal that the finite difference procedure cost may be tending toward overwhelming values for large number of CA's because of the presence of the N^2 term in Eq. B1. On the other hand, the cost of the GSEx-based sensitivity analysis does not depend on N^2 but is proportional to H and the cost of solution of the GSEx. This

suggests that its cost advantage over the finite
difference procedure will increase with the size of
the system measured by N, provided that magnitude
of the coupling bandwidth H is judiciously kept
under control, and the full advantage is taken of
the progress in computing technology to keep c_3 and
c_5 as low as possible.

It is apparent from Eq. B2 and B3 that the
GSE1 cost has a potential of being much smaller
than that of GSE2 because evaluation of the
residuals for a CA takes much less time than
computation of the partial derivatives of its
output with respect to its input. Therefore, one
may expect $c_4 \ll c_2$ - a reasonable estimate would
be a one, or two, orders of magnitude difference.

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TABLE 1. DEFINITION OF EXAMPLE 1

NOTATION AND DATA

- Notation: b - span

    C - chord

    $\bar{z}_1 \equiv z_1/C$; $\bar{z}_2 \equiv z_2/C$; $\bar{a} = a/C$

    $\bar{h}_1 \equiv \bar{a} - \bar{z}_1$; $\bar{h}_2 \equiv \bar{z}_2 - \bar{a}$

    $p \equiv \bar{h}_1/\bar{h}_2$

    $S = B \cdot C$

    $Y_\alpha = \{L, c, \bar{a}\}$ $X_\alpha = \{S, u, r, \theta_0, c, \bar{a}, \Psi\}$

    $L = f_\alpha(\Psi, \phi)$; $\phi = f_\beta(L)$;

    $Y_\beta = \{\phi\}$ $X_\beta = \{\bar{z}_1, \bar{z}_2, k_1, k_2\}$

- Data: B = 100 cm, C = 10 cm, $\bar{z}_1 = .2$; $\bar{z}_2 = .7$;

    $k_1 = 4000$ N/cm; $k_2 = 2000$ N/cm

    $\bar{a} = .25$; $q = 1$N/cm^2;

COUPLING DATA MOVING DOWN	AERODYNAMIC CA.		COUPLING DATA MOVING UP
	$L = q \cdot S \cdot c_L$ $c_L = u\theta + r(1$ $- \cos((\pi/2)(\theta/\theta_0)))$		#
L	STRUCTURAL CA.		ϕ
	$R_1 = L/(1 + p)$ $R_2 = Lp/(1 + p)$ $d_1 = R_1/k_1$ $d_2 = R_2/k_2$ $\phi = (d_1 - d_2)/(C \cdot (\bar{z}_2 - \bar{z}_1))$		

TABLE 2. SENSITIVITY ANALYSIS OF EXAMPLE 1.

SYSTEM SOLUTION
$L = 502.3$ N; $\phi = .0176$ rd
DERIVATIVES WITH RESPECT TO Ψ BY FINITE DIFFERENCES
$\Delta\Psi = .0025$ $\dfrac{\partial L}{\partial \Psi} = 14925.16$ N/rd; $\dfrac{\partial\phi}{\partial\Psi} = .5221287$ rd/rd
GSE2
Symbolically: $\begin{bmatrix} I & -\partial f_\alpha/\partial\phi \\ -\partial f_\beta/\partial L & I \end{bmatrix} \begin{Bmatrix} \partial L/\partial\Psi \\ \partial\phi/\partial\Psi \end{Bmatrix} = \begin{Bmatrix} \partial f_\alpha/\partial\Psi \\ 0 \end{Bmatrix}$ Numerically: $\begin{bmatrix} 1 & -9805.105 \\ -.3510^{-4} & 1 \end{bmatrix} \begin{Bmatrix} \partial L/\partial\Psi \\ \partial\phi/\partial\Psi \end{Bmatrix} = \begin{Bmatrix} 9805.105 \\ 0 \end{Bmatrix}$ *
DERIVATIVES WITH RESPECT TO Ψ FROM GSE2
$\dfrac{\partial L}{\partial\Psi} = 14928.12$ N/rd; $\dfrac{\partial\phi}{\partial\Psi} = .5224841$ rd/rd
*Numerical values of $\partial f_\alpha/\partial\phi$ and $\partial f_\alpha/\partial\Psi$ are equal because of relation marked # in Table 1

Fig. 1 Function vectors (a) forming a set of coupled equations (b).
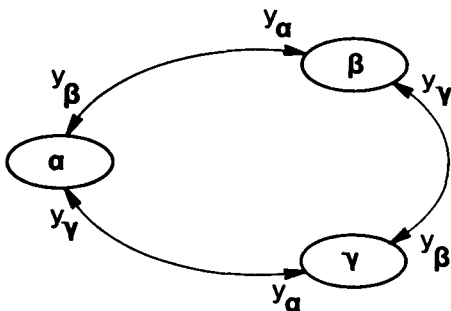
Fig. 2 Directed graph representation of the system shown in Fig. 1.

Fig. 3 Finite difference procedure involving the system analysis.

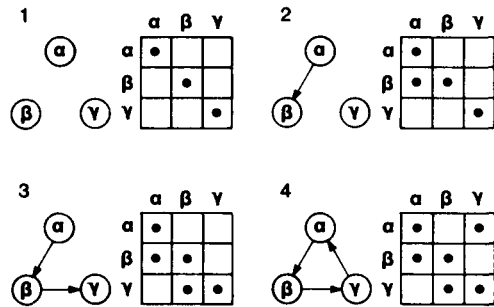

Fig. 4 Anatomy of the GSE2.

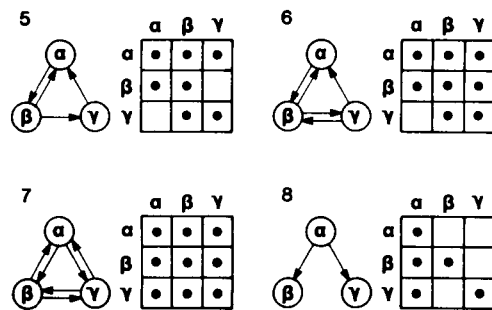Fig. 5 System couplings reflected in the GSE2 matrix.

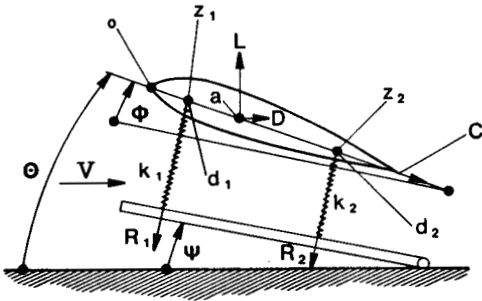Fig. 6 More examples of system couplings and their reflection in the GSE2 matrix.

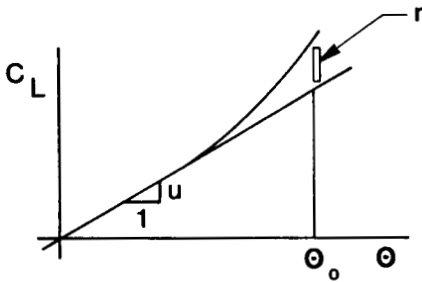Fig. 7 Example 1: a simple aerodynamic-structures system

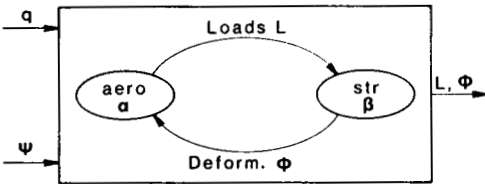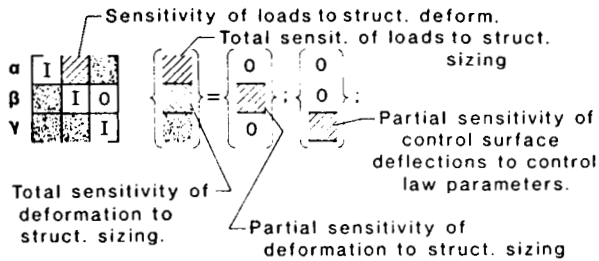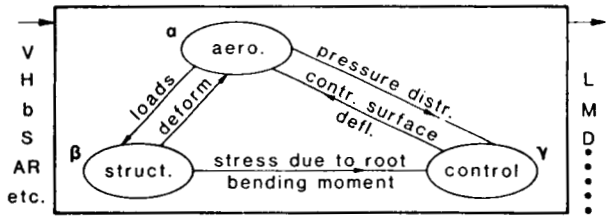Fig. 8 Nonlinear relationship C_L vs. angle of attack in Example 1.

Fig. 9 The system from Fig. 7 abstracted as a "black box" with a directed graph showing internal coupling.

Wing system=aerodynamics + structures + active controls

- Sensitivity of loads to struct. deform.
- Total sensit. of loads to struct. sizing
- Partial sensitivity of control surface deflections to control law parameters.
- Partial sensitivity of deformation to struct. sizing
- Total sensitivity of deformation to struct. sizing.

• Lift $L = \int_s$ loads dS: $dL/d(str.\ siz.) = \int_s d(loads)/d(str.\ siz.)dS$:

Fig. 10 Example of a flexible wing: a system comprising aerodynamics, structures, and active control.
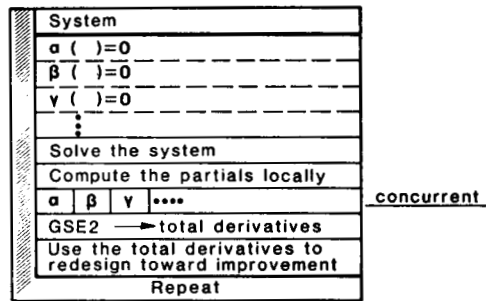

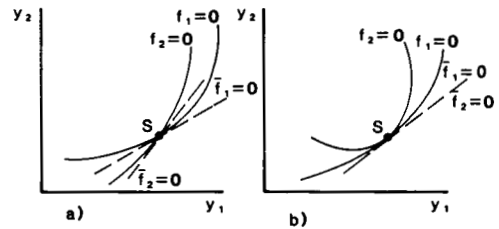Fig. 11 Sensitivity procedure in design process.

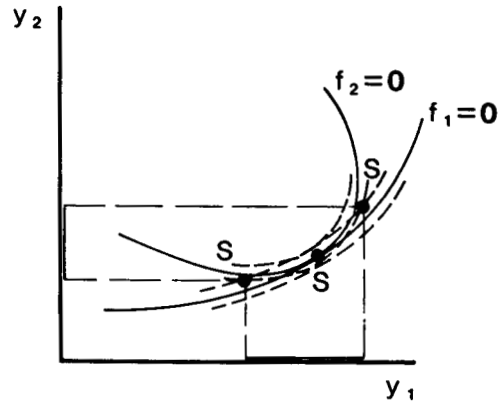Figure 12.- Constrained minimum defined by intersection (a) and tangency (b).

Figure 13.- Ill-defined constrained minimum.

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16. Abstract

A method is presented for computing sensitivity derivatives with respect to independent (input) variables for complex, internally coupled systems, while avoiding the cost and inaccuracy of finite differencing performed on the entire system analysis. The method entails two alternative algorithms: the first is based on the classical implicit function theorem formulated on residuals of governing equations, and the second develops the system sensitivity equations in a new form using the partial (local) sensitivity derivatives of the output with respect to the input for each part of the system. A few application examples are presented to illustrate the discussion. The method has a potential to answer the "what if" questions by presenting engineers with sensitivity information on design trade-offs to guide human judgment and formal optimization. In addition, the method is compatible with the modern technology of distributed computing as well as traditional division of design tasks among groups of specialists in the design process. The capability to quantify the effects of proposed design changes may provide the basis for a methematical model of design.

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