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AN EFFICIENT MINDLIN FINITE STRIP PLATE ELEMENT BASED ON ASSUMED STRAIN DISTRIBUTION

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ABSTRACT

A simple two-node, linear, finite strip plate bending element based on Mindlin-Reissner plate theory for the analysis of very thin to thick bridges, plates, and axisymmetric shells is presented. The new transverse shear strains are assumed for constant distribution in the two-node linear strip. The important aspect is the choice of the points that relate the nodal displacements and rotations through the locking transverse shear strains. The element stiffness matrix is explicitly formulated for efficient computation and ease in computer implementation. Numerical results showing the efficiency and predictive capability of the element for analyzing plates with different supports, loading conditions, and a wide range of thicknesses are given. The results show no sign of the shear locking phenomenon.

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FINITE STRIP VERSUS FINITE ELEMENT

The finite strip method was first introduced by Cheung (1968a) for the analysis of elastic plates. It became well known because of its advantages over the conventional finite element method in the simplicity of the formulation and the reduction in size as well as bandwidth of the assembled stiffness matrix. The combined use of finite elements in one direction and Fourier-series expansions in another direction makes it simple and computationally efficient for analyzing a wide variety of structures (i.e., bridges, curved plates, sandwich plates, composite plates, and axisymmetric shells).

In the early stage classical Kirchhoff thin-plate theory, which does not account for shear deformation, was used (Cheung, 1968a and 1968b) and obviously restricted to "thin" situations only. Later the transverse shear effect based on Mindlin-Reissner plate theory (Mindlin, 1951) was included by Mawenya and Davies (1974), and this was applicable to modeling thin plates as well as moderately thick plates. However, despite its mathematical elegance overstiff numerical results, often called "shear locking" effect, were detected when using lower-order elements for analyzing thin and very thin structures. In this presentation the emphasis is on developing a simple, low-order element for general-purpose usage in thin and thick structures that will not produce the shear locking phenomenon.

FINITE ELEMENT METHOD (CONVENTIONAL)	FINITE STRIP METHOD
FORMULATION	
• TOTAL POTENTIAL ENERGY	• TOTAL POTENTIAL ENERGY
• MINDLIN-REISSNER PLATE THEORY	• MINDLIN-REISSNER PLATE THEORY
• DISPLACEMENT BASE	• DISPLACEMENT BASE
• INTERPOLATION IN ALL DIRECTIONS	• INTERPOLATION IN X-DIRECTION; FOURIER SERIES IN Y-DIRECTION
PERFORMANCE	
• ALMOST ALL TYPES OF BOUNDARY CONDITIONS	• LIMITED TO SOME PARTICULAR BOUNDARY CONDITION
• LARGE OVERALL MATRIX	• MUCH SMALLER OVERALL MATRIX
• LARGE BANDWIDTH	• MUCH SMALLER BANDWIDTH
• NORMALLY REQUIRES VERY FINE MESH FOR LOW-ORDER ELEMENT	• EXCELLENT PERFORMANCE FOR LOW-ORDER ELEMENT AND COARSE MESH
• LOCKING IN THIN SITUATION	• LOCKING IN THIN SITUATION

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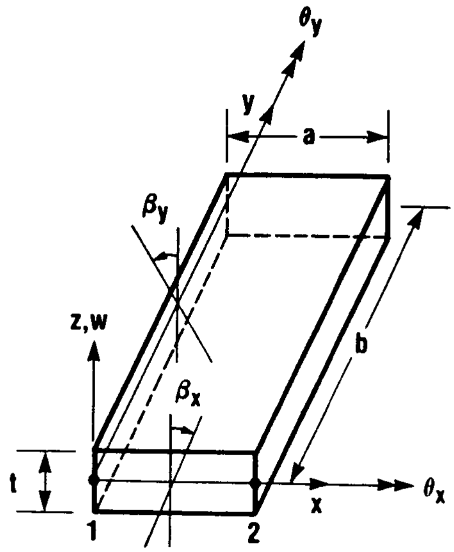
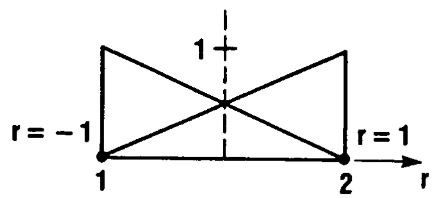
MINDLIN-REISSNER STRIP ELEMENT FORMULATION

A two-node, linear strip element is formulated for static analysis of plate structures. The midplane deflection and rotations are interpolated separately as the products of the sum of the Fourier series in the y-direction and the polynomial functions in the x-direction. The positive direction is shown below. Only the simply supported case is considered. The loads are also resolved into a sine series in the y-direction similar to the deflection. By using the orthogonality properties of the harmonic series (Cheung, 1976), the stiffness matrix is uncoupled. Note that the shear term of this stiffness matrix is derived by using the troublesome transverse shear strains. Next these strains will be replaced by the new assumed shear strains.





$$N_1 = \frac{1-r}{2}$$

$$N_2 = \frac{1+r}{2}$$

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$$w = \sum_{\ell=1}^{n} \sum_{i=1}^{2} N_i w_i^{\ell} \sin \frac{\ell \pi y}{b}$$

$$\beta_x = \sum_{\ell=1}^{n} \sum_{i=1}^{2} N_i \theta_{y_i}^{\ell} \sin \frac{\ell \pi y}{b}$$

$$\beta_y = \sum_{\ell=1}^{n} \sum_{i=1}^{2} N_i \theta_{x_i}^{\ell} \cos \frac{\ell \pi y}{b}$$

$$\epsilon_b = z\kappa$$

$$\kappa = \begin{bmatrix} \beta_{x,x} \\ -\beta_{y,y} \\ \beta_{x,y} - \beta_{y,x} \end{bmatrix} = \sum_{\ell=1}^{n} \sum_{i=1}^{2} [B_i^{\ell}]_b U_i^{\ell}$$

$$\epsilon_s = \begin{matrix} \partial w/\partial x + \beta_x \\ \partial w/\partial y - \beta_y \end{matrix} = \sum_{\ell=1}^{n} \sum_{i=1}^{2} [B_i^{\ell}]_s U_i^{\ell}$$

$$\sum_{\ell=1}^{n} \sum_{i,j} [K_{ij}^{\ell}] U_i^{\ell} = \sum_{\ell=1}^{n} R_i^{\ell}$$

$$\begin{bmatrix} [K^1] & & & \\ & [K^2] & & \\ & & \ddots & \\ & & & [K^n] \end{bmatrix} \begin{bmatrix} U^1 \\ U^2 \\ \vdots \\ U^n \end{bmatrix} = \begin{bmatrix} R^1 \\ R^2 \\ \vdots \\ R^n \end{bmatrix}$$

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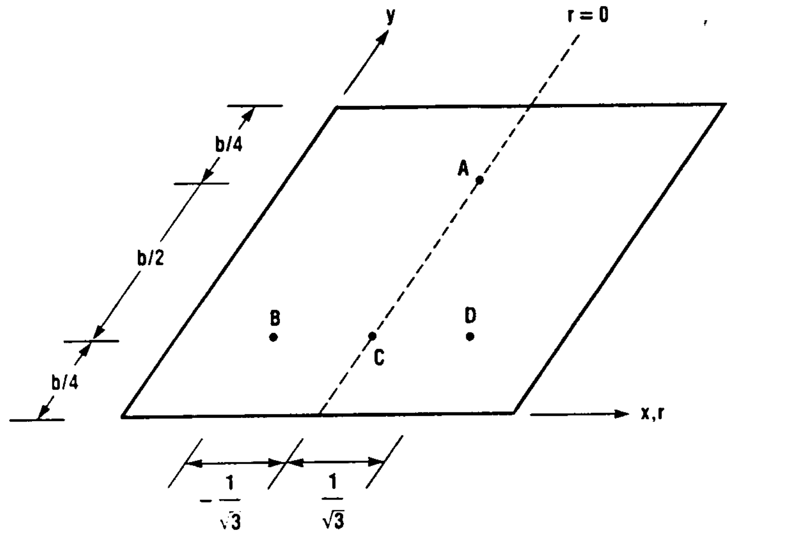
ASSUMED STRAIN DISTRIBUTIONS

The finite strip formulation has some drawbacks. The element locks when the structure is thin because it uses two-point Gauss quadrature integration for both bending and shear stiffness. The shear stiffness terms overwhelm the bending stiffness terms, and this leads to the overstiff element even though a very fine mesh is used. Although selective and reduced integration is a well-established approach, new assumed strain distributions are introduced to circumvent this locking phenomenon for transverse shear effects (MacNeal, 1982). Note that these assumed shear strains are evenly distributed across the cross sections and constrained to equal the troublesome shear strains at prespecified points. The choice of these points is of paramount importance in evaluating the predictive capability of the element even though the new assumed strains are an integral part of the overall performance.



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$$\gamma^{\ell}_{xz} = \frac{1}{2}\left(\gamma^{A}_{xz} + \gamma^{C}_{xz}\right)$$

$$\gamma^{\ell}_{yz} = \frac{1}{2}\left(\gamma^{B}_{yz} + \gamma^{D}_{yz}\right)$$

$$\gamma^{A}_{xz} = \frac{1}{2} S^{A}_{\ell}\left[-w^{\ell}_{1}/J + \theta^{\ell}_{y_1} + w^{\ell}_{2}/J + \theta^{\ell}_{y_2}\right]$$

$$\gamma^{C}_{xz} = \frac{1}{2} S^{C}_{\ell}\left[-w^{\ell}_{1}/J + \theta^{\ell}_{y_1} + w^{\ell}_{2}/J + \theta^{\ell}_{y_2}\right]$$

$S^{A}_{\ell} = \sin(3\ell\pi/4)$, $S^{C}_{\ell} = \sin(\ell\pi/4)$, AND $J = a/2$

$$\gamma^{B}_{yz} = C^{B}_{\ell}\left[(0.7887)(\ell\pi/b)w^{\ell}_{1} - (0.7887)\theta^{\ell}_{x_1} + (0.2113)(\ell\pi/b)w^{\ell}_{2} - (0.2113)\theta^{\ell}_{x_2}\right]$$

$$\gamma^{D}_{yz} = C^{D}_{\ell}\left[(0.2113)(\ell\pi/b)w^{\ell}_{1} - (0.2113)\theta^{\ell}_{x_1} + (0.7887)(\ell\pi/b)w^{\ell}_{2} - (0.7887)\theta^{\ell}_{x_2}\right]$$

$C^{B}_{\ell} = C^{D}_{\ell} = \cos(\ell\pi/4)$

$$[B^{\ell}]_s = \begin{bmatrix} -(S^{A}_{\ell} + S^{C}_{\ell})/4J & 0 & (S^{A}_{\ell} + S^{C}_{\ell})/4 & (S^{A}_{\ell} + S^{C}_{\ell})/4J & 0 & (S^{A}_{\ell} + S^{C}_{\ell})/4J \\ (\ell\pi/b)C^{B}_{\ell} & -C^{B}_{\ell}/2 & 0 & (\ell\pi/b)C^{D}_{\ell} & -C^{D}_{\ell}/2 & 0 \end{bmatrix}$$

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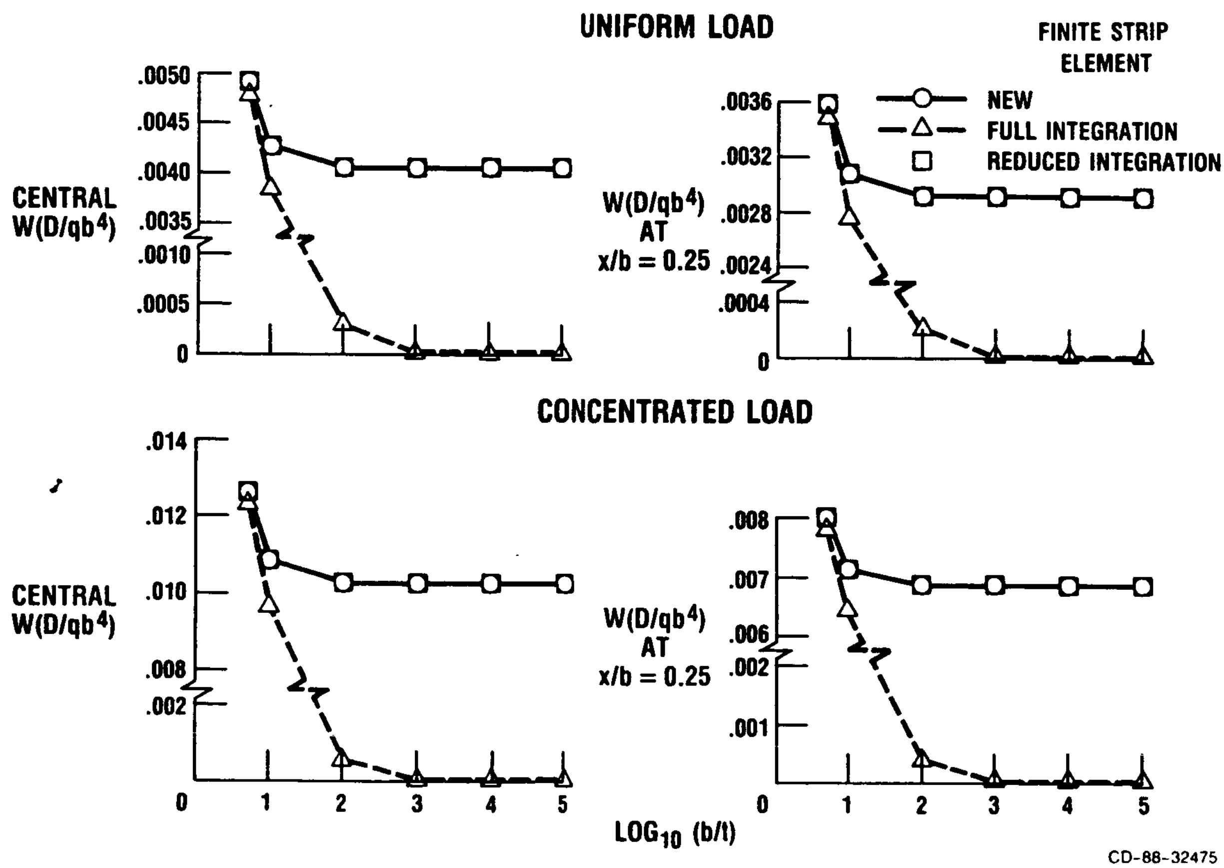
CASE STUDIES - SHEAR LOCKING INVESTIGATION

The new finite strip element has been implemented into the finite element computer program FEAP (Zienkiewicz, 1982) with relative ease. The subroutines written in Fortran 77 for the formulation of the element stiffness matrix consist of approximately 200 lines. Results of numerical benchmark problems are presented to evaluate the performance of the element in different aspects: mesh size and harmonic term convergence characteristics, shear locking phenomenon as the thickness decreases, and shear force and bending moment prediction. The "full" numerical integration is employed.

A benchmark problem is used to detect the shear locking effects. Using eight strip elements with four nonzero harmonic terms and a Poisson's ratio ν of 0.3, a simply supported square plate subjected to two loading conditions, uniform and concentrated loads, is investigated for a range of aspect (span to thickness) ratios from 5 to 10^5. The resulting central deflections of the plate are normalized by the classical thin-plate solution (Timoshenko and Wionowsky-Krieyer, 1959) and plotted below for both uniform and concentrated loads. For the entire range of aspect ratios shear locking was not detected. Note that the range of aspect ratios investigated varies from relatively thick to very thin situations; therefore this strip element would be useful for a wide range of applications, including the analysis of bridges, curved plates, and axisymmetric shells.



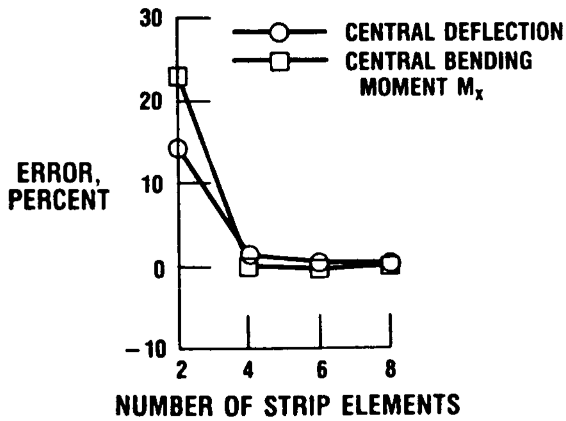
CASE STUDIES - CONVERGENCE OF MESH SIZE

To assess the convergence characteristics predicted by the new element, we consider here again the analysis of a simply supported, uniformly loaded square plate with aspect ratio equal to 100 and $\nu = 0.3$. For mesh convergence the central deflections and central bending moments in the x- and y-directions are tabulated for the four meshes as well as for the exact solution. The number of available degrees of freedom for each mesh is shown as well. This gives a more realistic view of the numerical computation cost. Graphical representation is presented by plotting the percentage error of the central deflections and the central bending moment in the x-direction versus mesh size. The rate of convergence is rapid for both displacement and moment, with no sign of shear locking. As illustrated below, satisfactory convergence is reached by using four strips.

$b/t = 100$

DEGREES OF FREEDOM	NUMBER OF STRIP ELEMENTS	CENTRAL DEFLECTION	CENTRAL BENDING MOMENTS*	
			M_x	M_y
5	2	0.00348	0.03691	0.03865
11	4	.00401	.04785	.04699
17	6	.00404	.04799	.04753
23	8	.00405	.04794	.04766
EXACT		0.00406	0.0479	0.0479
MULTIPLIER		qb^4/D	qb^2	qb^2

*LINEARLY EXTRAPOLATED FROM GAUSS POINTS.



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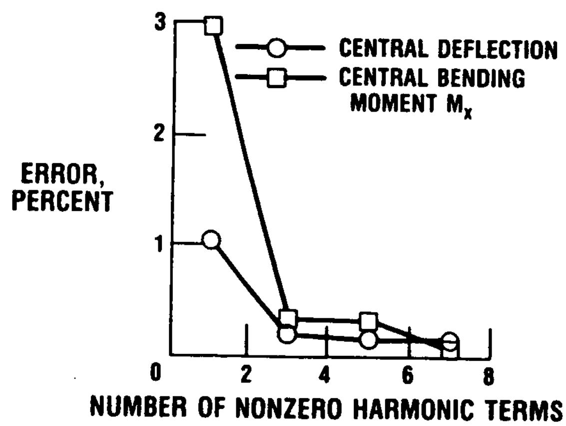
CASE STUDIES - CONVERGENCE OF HARMONIC TERM

For the convergence of the harmonic term an eight-strip element mesh is used. The numerical results, tabulated with four nonzero harmonic terms, result in fairly good convergence once again. Errors are less than 0.4 percent when the third nonzero harmonic term is specified for both quantities. Note that this element is a low-order, two-node strip. The numerical computation is minimal but the convergence rate is relatively high.

b/t = 100; EIGHT STRIP ELEMENTS

HARMONIC TERM	CENTRAL DEFLECTION	CENTRAL BENDING MOMENTS*	
		M_x	M_y
$\ell = 1$	−0.004101	−0.04930	−0.05162
$\ell = 3$	.000051	.00158	.00461
$\ell = 5$	−.000004	−.00032	−.00103
$\ell = 7$	.000001	.00011	.00038
SUM	−.004054	−.04794	−.04766
EXACT	−0.00406	−0.0479	−0.0479
MULTIPLIER	qb^4/D	qb^2	qb^2

*LINEARLY EXTRAPOLATED FROM GAUSS POINTS.



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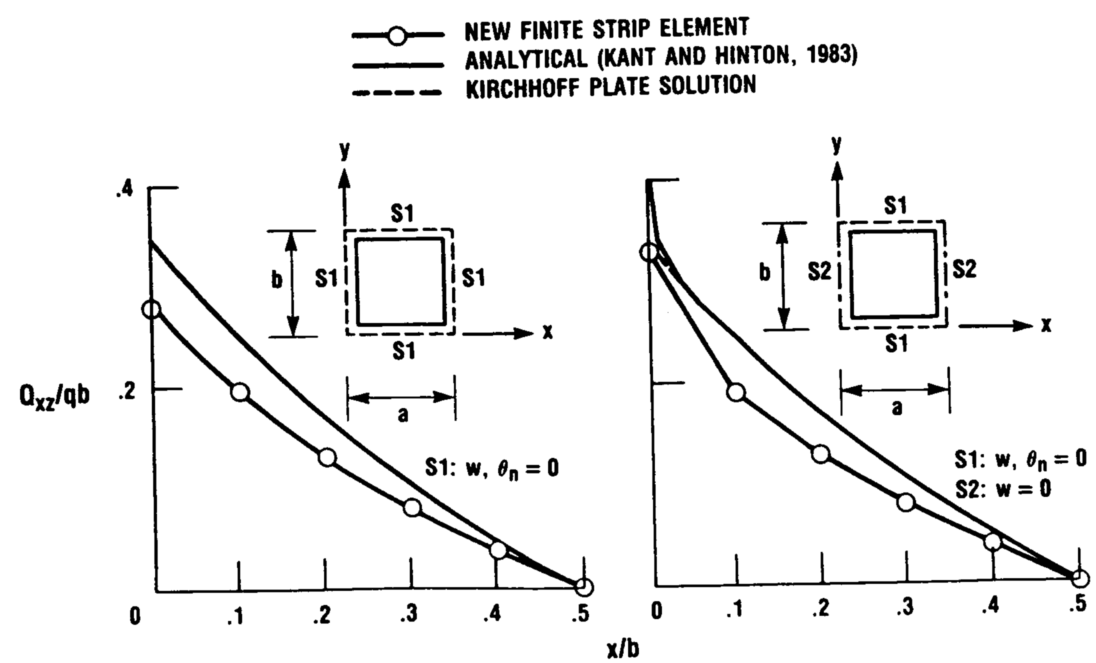
For a low-order element the standard finite strip formulation based on Mindlin-Reissner plate theory is well recognized to predict accurate displacements and fairly good bending moments when using the selective and reduced integration technique (Onate and Suarez, 1983b). However, the shear force predictions are poor and rarely found in the finite strip literature even though they are desperately needed in designing structures such as bridges and slabs. Therefore the shear force predictive capability of this new finite strip element is presented here. Because of the limited number of cases of the analytical solution, only four cases for shear forces and one case for bending moment are compared.

The uniformly loaded square plates involving a variety of support conditions in the x-direction are investigated. In order to capture the steep gradient of the dependent variables near the plate edge, a rather fine mesh is used in the analysis with the new strip element. The resulting variations of shear forces and bending moments across the center of the plate in the various cases are plotted along with analytical solutions by Kant and Hinton (1983) and by Kirchhoff plate theory. These analytical solutions based on Mindlin plate theory assume transverse displacement and sectional rotations similar to those for a standard finite strip element. Kant and Hinton (1983) claimed that the analytical results compared favorably well with the finite strip method.

The results of shear force and bending moment, shown with the pertinent data and boundary conditions, are in good agreement near the center of the plates. In the regions further away from the center the differences start to magnify and are average at 15 percent near the edge of the plate. However, the curves for both solutions seem to follow the same pattern. Note that this new strip element is only a simple, two-node linear element and its predictive capabilities are shown to exceed its expectation.

$y/b = 0.5$; $b/t = 50$; $\nu = 0.3$; UNIFORM LOAD; FOUR NONZERO HARMONIC TERMS



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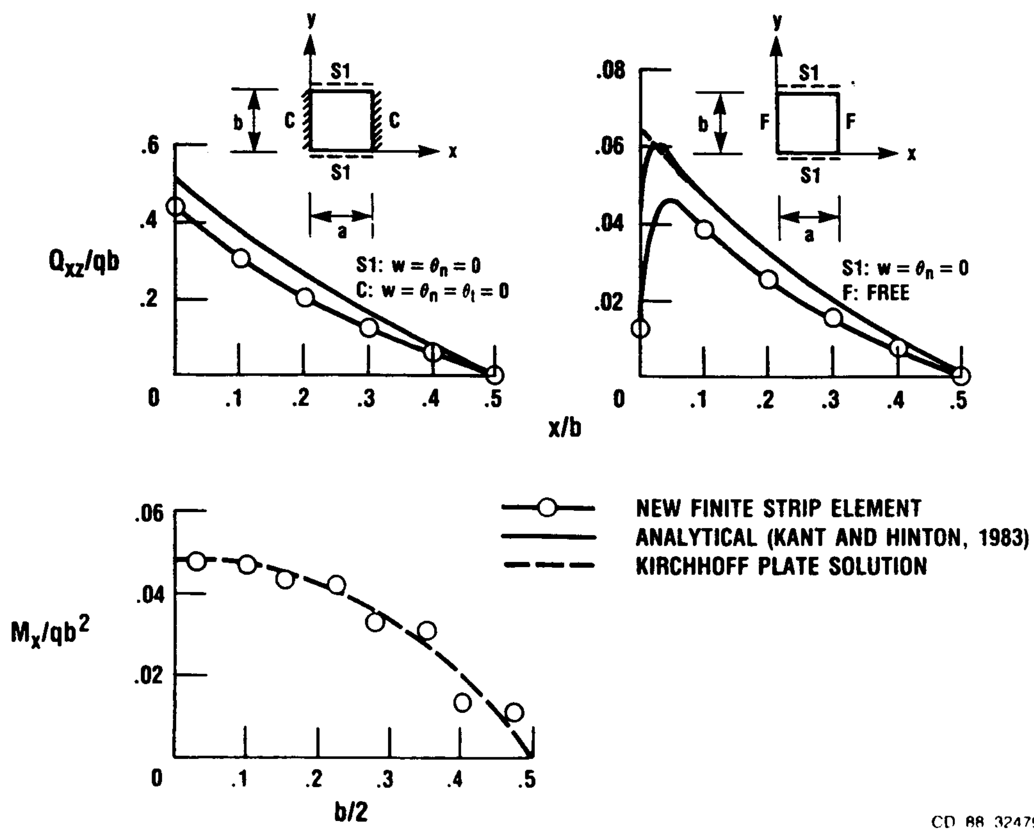
y/b = 0.5; b/t = 50; ν = 0.3; UNIFORM LOAD; FOUR NONZERO HARMONIC TERMS



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PROPERTIES OF NEW FINITE STRIP ELEMENT

A two-node linear strip element based on Mindlin-Reissner plate theory is presented for the static analysis of bending plates. The new shear strain distributions are assumed and connected to the standard shear strains at the preselected points. These points are chosen by following the guideline of removing the shear locking phenomenon without the need for the "reduced integration" technique. Because of the uncoupling nature of the finite strip method the element stiffness matrix can be explicitly formulated for efficient computations and computer implementation. On the basis of the results obtained, the following properties can be stated:

- **SIMPLE AND RELIABLE**
- **COMPUTATIONALLY EFFICIENT**
- **EASY FOR COMPUTER IMPLEMENTATION**
- **GOOD CONVERGENCE CHARACTERISTICS**
- **NO SHEAR LOCKING EFFECT FOR THIN SITUATION**
- **FAIRLY ACCURATE MOMENT AND SHEAR FORCE PREDICTIONS**

$D_1 = Et^3/12(1-\nu^2)$ $\quad D_2 = Et^3\nu/12(1-\nu^2)$ $\quad C = \cos^2(\ell\pi/4)$ $\quad F = \ell\pi/b$

$D_3 = Et^3/24(1+\nu)$ $\quad D_4 = Etk/2(1+\nu)$ $\quad H = 0.25\left(\sin 3\ell\frac{\pi}{4} + \sin \ell\frac{\pi}{4}\right)$

WHERE E IS YOUNG'S MODULUS AND ν IS POISSON'S RATIO

$$[K]' = \begin{bmatrix} D_4\left(4\frac{b}{a}H^2 + a\frac{b}{4}F^2C\right) & -D_4a\frac{b}{4}\bullet FC & -2bH^2D_4 & -D_4\left(4\frac{b}{a}H^2 + a\frac{b}{4}F^2C\right) & -D_4a\frac{b}{4}\bullet FC & -2bH^2D_4 \\ & \frac{b}{2}\left(D_1F^2\frac{a}{3} + \frac{D_3}{a} + D_4\frac{a}{2}C\right) & \frac{b}{4}F(D_3-D_2) & -D_4a\frac{b}{4}\bullet FC & \frac{b}{2}\left(D_1F^2\frac{a}{6} - \frac{D_3}{a} + D_4\frac{a}{2}C\right) & \frac{b}{4}F(D_2+D_3) \\ & & b\left(\frac{D_1}{2a} + D_3F^2\frac{a}{6} + D_4aH^2\right) & 2D_4bH^2 & -\frac{b}{4}F(D_2+D_3) & b\left(-\frac{D_1}{2a} + D_3F^2\frac{a}{12} + D_4aH^2\right) \\ & & & D_4\left(4\frac{b}{a}H^2 + a\frac{b}{4}F^2C\right) & -D_4a\frac{b}{4}\bullet FC & 2bH^2D_4 \\ & & & & \frac{b}{2}\left(D_1F^2\frac{a}{3} + \frac{D_3}{a} + D_4\frac{a}{2}C\right) & \frac{b}{4}F(D_2-D_3) \\ & \text{SYMMETRIC} & & & & b\left(\frac{D_1}{2a} + D_3F^2\frac{a}{6} + D_4aH^2\right) \end{bmatrix}$$

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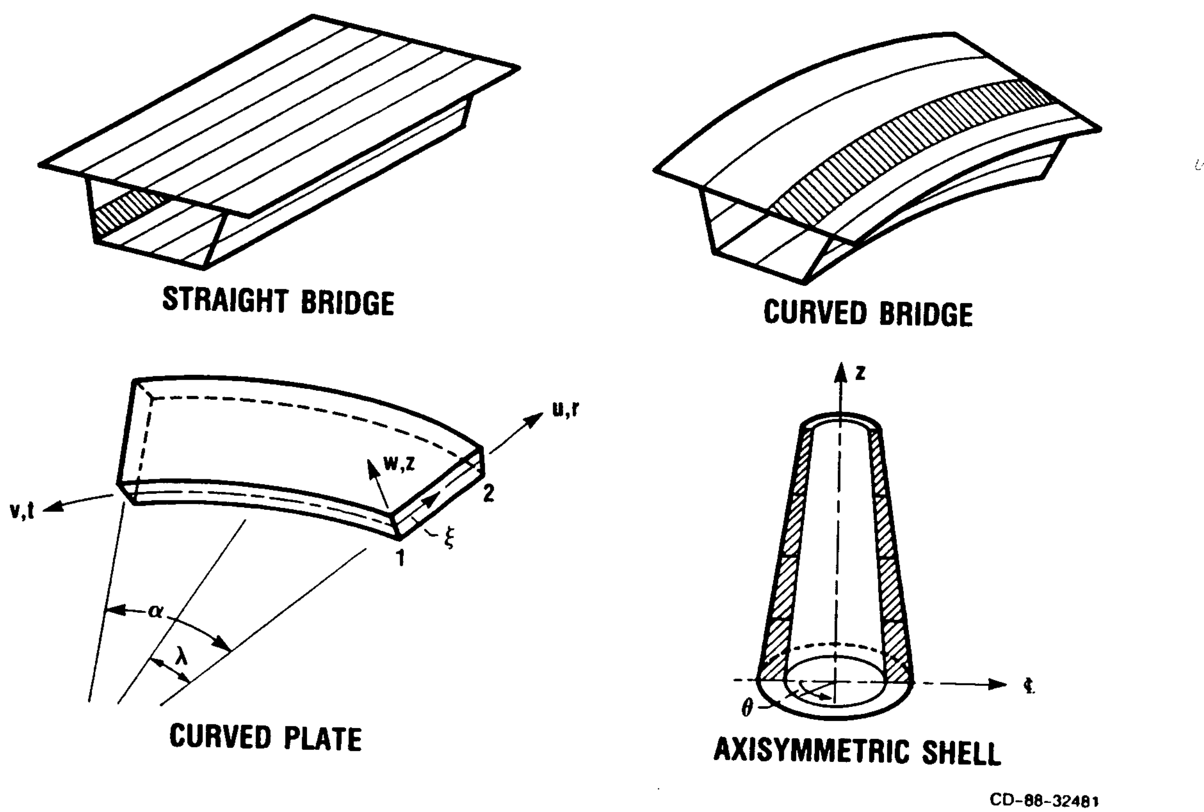
APPLICATIONS AND FUTURE DEVELOPMENT

Because of the simplicity of its formulation the application of the finite strip method to the analysis of bridges, curved plates, and axisymmetric shells is straightforward. Onate and Suarez (1983a) demonstrated this in detail. Since the element stiffness matrix can be explicitly formulated, it is very convenient for practical engineers to implement this element into existing conventional finite element computer programs such as NFAP, developed by Chang (1987). NFAP is available to general users here at the NASA Lewis Research Center. Future research will involve extending the concept of this strip element for geometrically nonlinear analysis.



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