PARALLEL COMPUTER METHODS FOR EIGENVALUE EXTRACTION

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ABSTRACT

This paper presents a new numerical algorithm for the solution of large-order eigenproblems typically encountered in linear elastic finite element systems. The architecture of parallel processing is used in the algorithm to achieve increased speed and efficiency of calculations. The algorithm is based on the frontal technique for the solution of linear simultaneous equations and the modified subspace eigenanalysis method for the solution of the eigenproblem. Assembly, elimination, and back-substitution of degrees-of-freedom are performed concurrently by using a number of fronts. All fronts converge to and diverge from a predefined global front during elimination and back-substitution, respectively. In the meantime, reduction of the stiffness and mass matrices required by the modified subspace method can be completed during the convergence/divergence cycle, and an estimate of the required eigenpairs can be obtained. Successive cycles of convergence and divergence are repeated until the desired accuracy of calculations is achieved. The advantages of this new algorithm in parallel computer architecture are discussed.

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GENERALIZED EIGENPROBLEM

- New parallel algorithm for the solution of large-scale eigenproblems in finite element applications

- Assumptions

  (1) Linear elastic finite element models

  (2) n lower order eigenpairs are required, i.e., $\omega_1^2 \leq \omega_2^2 \leq \ldots \omega_n^2$

  (3) [K] is positive-definite

  (4) [M] is semipositive definite

$[K][\Phi] = [M][\Phi][\Omega]$

N – DEGREES OF FREEDOM

REQUIRED n EIGENPAIRS, $n \leq N$

[K] POSITIVE-DEFINITE

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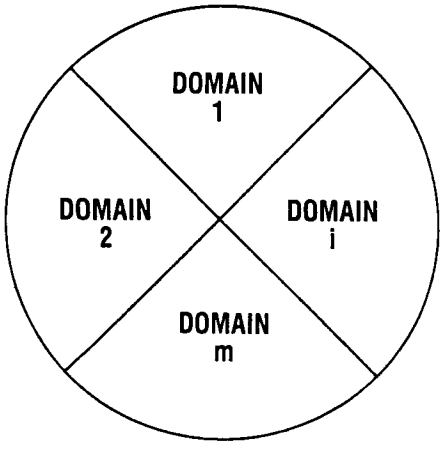
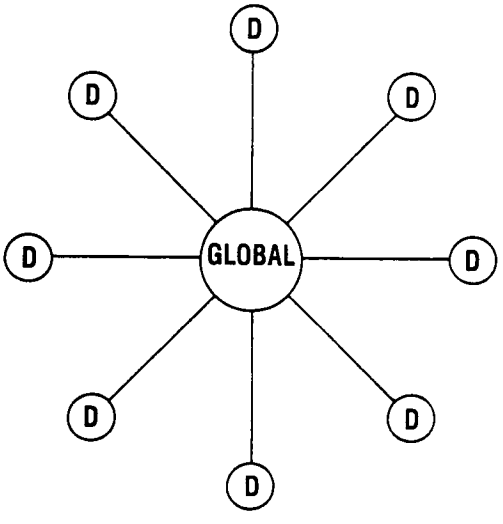
PARALLEL ARCHITECTURE

- Consider a parallel computer with (m+1) processors (tasks).
- Designate the first processor as a global processor (task).
- Designate the remaining m-processors as domain processors (tasks).
- A finite element model can be divided into a number of domains equal to m.
- A star architecture (or tree) is the first to be investigated.

FINITE ELEMENT MODEL SUBDIVIDED INTO m DOMAINS



STAR ARCHITECTURE



DOMAIN

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MAJOR COMPUTATIONAL TASKS

Three major steps of large computational requirements

(1) Creation of element stiffness and mass matrices

(2) Extraction of a set of eigenpairs

(3) Solution of a set of simultaneous linear equations

The merits of selecting the modified subspace method for step 2 and the frontal solution for step 3 are discussed in the next viewgraphs.

$$[K]\,[\Phi] = [M]\,[\Phi]\,[\Omega]$$

(1) CREATION OF K^e AND M^e

(2) EIGENSOLUTION (MODIFIED SUBSPACE)

(3) EQUATION SOLVER (FRONTAL SOLUTION)

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MODIFIED SUBSPACE METHOD

The Modified Subspace Method iterates simultaneously for a subset of eigenpairs $[\phi,\omega^2]$ of the generalized eigenproblem:

(1) Let $[V]_1$ be n starting eigenvectors.

(2) Operate on each $[V]_\ell$ as follows:

$$[V]^*_{\ell+1} = [K]^{-1}[M][V]_\ell = [K]^{-1}[B]_\ell, \quad \ell = 1,2,3, \ldots$$

(3) Modify $[V]^*_{\ell+1}$ to increase convergence rate by one third on average

$$[V]^*_{\ell+1} \leftarrow [V]^*_{\ell+1} - \beta_\ell [V]_\ell$$

where $\beta_\ell = 0$ for $\ell = 1$ and $\ell > 11$

$$\beta_\ell = 0.5\,(1+r_{\ell-1})/\omega_n^2 \quad 1 < \ell \leq 11$$

$r_{\ell-1}$ are the interval points of the 11th order Labatoo rule [-1, 1]

(4) Project K and M onto the required subspace.

(5) Solve the auxiliary eigenproblem to obtain $[Q]_{\ell+1}$ and $[\Omega]_{\ell+1}$.

(6) An improved set of eigenvectors $[V]_{\ell+1}$ can be obtained.

(7) Test for convergence on ω_n^2. Repeat steps 2 to 6 until desired accuracy is achieved.

$$[K]^*_{\ell+1} = \sum_e [V]^{*T}_{\ell+1}[K][V]^*_{\ell+1}$$

$$[M]^*_{\ell+1} = \sum_e [V]^{*T}_{\ell+1}[M][V]^*_{\ell+1}$$

AUXILIARY EIGENPROBLEM

$$[K]^*_{\ell+1}[Q]_{\ell+1} = [M]^*_{\ell+1}[Q]_{\ell+1}[\Omega]_{\ell+1}$$

IMPROVED EIGENVECTORS

$$[V]^e_{\ell+1} = [V]^{*e}_{\ell+1}[Q]_{\ell+1}$$

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RATE OF CONVERGENCE

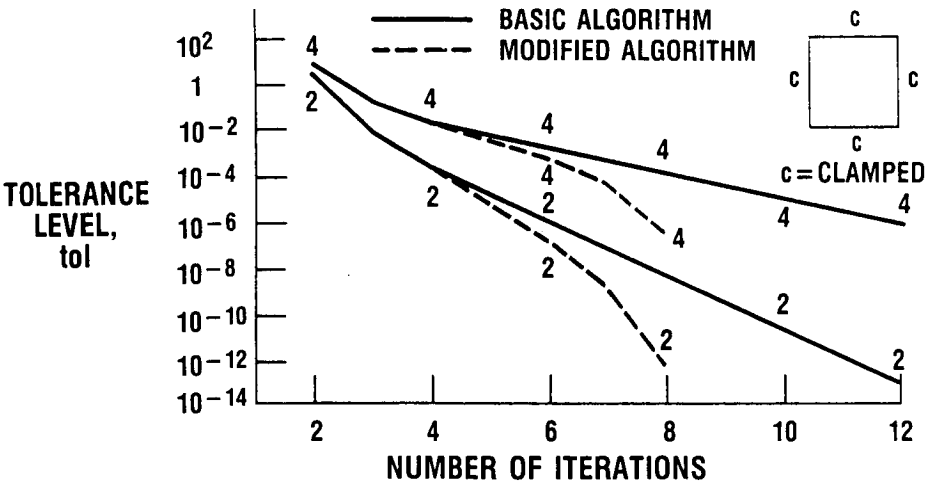
Rate of convergence of the modified subspace is 33 percent faster on average compared to the classical subspace method.

The figure shows typical behavior.

Most computations are performed on an element-by-element basis.

BASIC SUBSPACE $\leq \dfrac{\omega_i^2}{\omega_{n+1}^2}$

MODIFIED SUBSPACE $\leq \dfrac{\omega_i^2}{\omega_{n+1}^2} \quad \dfrac{1-\beta_\ell \omega_{n+1}^2}{1-\beta_\ell \omega_i^2}$



CONVERGENCE OF λ_2 AND λ_4 FOR SQUARE PLATE CD-88-31672

FRONTAL SOLUTION

(1) Gauss elimination technique

(2) Underlying philosophy based on processing of elements one by one

(3) Simultaneous assembly and elimination of variables

(4) The optimum frontal width at most equal to the optimum bandwidth

(5) Numbering of nodes - no impact on optimality; numbering of elements - important to minimize the frontal width

(6) More efficient for solid elements and elements with midside nodes

(7) Requires a prefront to determine last appearance of each node

(8) Lends itself to parallel solutions

WITHIN EACH DOMAIN

$$k_{ij} \leftarrow k_{ij} - \sum \left[\frac{k_{is} k_{sj}}{k_{ss}} \right]$$

$$b_{iq} \leftarrow b_{iq} - \sum \left[\frac{k_{is} b_{sq}}{k_{ss}} \right]$$

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DOMAIN PROCESSING

Multitasking on the Cray supercomputer provides tools for implementing the frontal solution concurrently within a number of domains. Assembly and elimination for the i-th domain is assigned to a task (processor). Simultaneous assembly and elimination of variables within the domains is performed in parallel tasks until the domain fronts reach their respective global fronts. However, it is instructive to analyze the set of simultaneous equations for the i-th domain assuming that the domain stiffness matrix and right-hand sides are fully assembled before Gauss elimination is performed.

For domain i

$$[K][V]^{*}_{\ell+1} = [B]_{\ell} \qquad \text{at iteration } \ell$$

Elimination gives

$$U_d V^{*}_d + K_d V^{*}_F = B_d$$

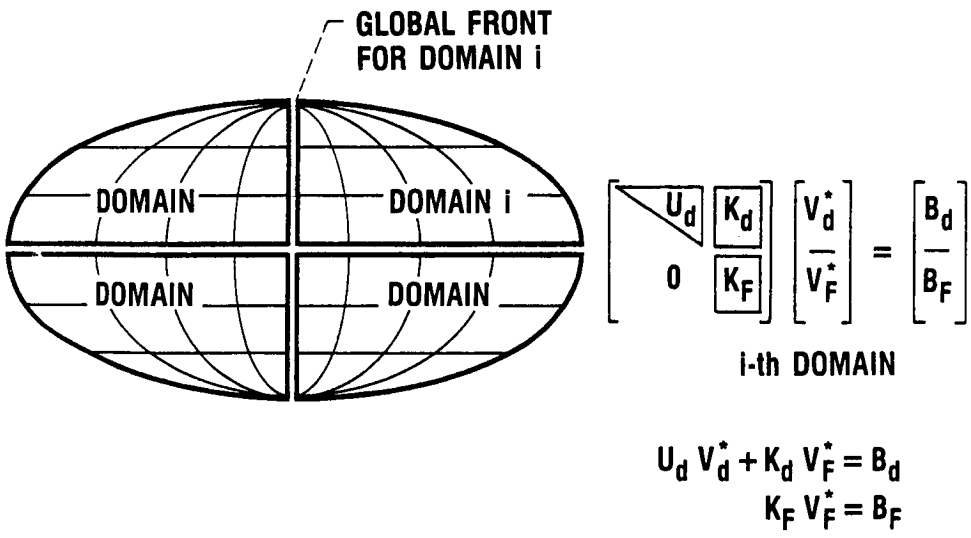
$$K_F V^{*}_F = B_F$$

where U_d upper Δ matrix for domain i

V^{*}_d variables within domain i

V^{*}_F variables along global front of domain i

B_d and B_F are right-hand sides for domain variables and global fronts, respectively.



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GLOBAL FRONTS

$$K_{FF} = \sum^{m} K_F$$

$$B_{FF} = \sum^{m} B_F$$

where m = total numbers of domains

$$K_{FF}\ V_{FF} = B_{FF}$$

Solution for V_{FF} is then obtained by using the frontal solution on the global fronts. Since V_{FF} is a superset of all V_F^*, back-substitution for V_d^* follows concurrently within the domains.

$$K_{FF} = \sum^{m} K_F$$

$$B_{FF} = \sum^{m} B_F$$

WHERE m = TOTAL NUMBER OF DOMAINS

$$K_{FF} V_{FF} = B_{FF}$$

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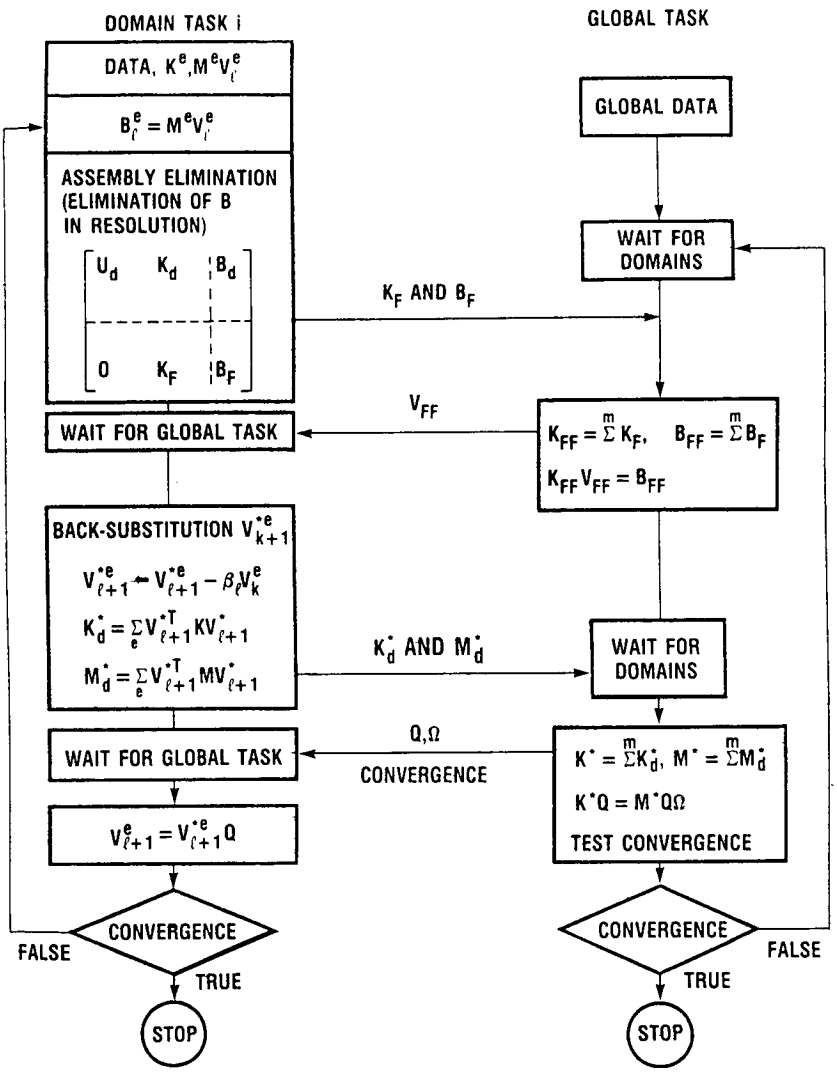
IMPLEMENTATION

Successful implementation of the new parallel algorithm depends on

(1) Maximizing the efficiency of communication links between the global task and the domains

(2) Minimizing sequential computational steps

(3) Multithreaded I/O



CONCLUSIONS

Parallel solution method for eigenvalue extraction for linear elastic finite element models has been successfully implemented on the Cray supercomputer by using the multitasking environment. Preliminary results are encouraging and extensive testing of the new algorithm is currently progressing. The new algorithm enhances the speed-up of similar sequential solution methods. Both the frontal method for the solution of the set of simultaneous equations arising in finite element problems and the modified subspace method for the solution of a subset of eigenpairs offer a new algorithm which has been efficiently parallelized. The parallel tasks are associated with recognizable finite element domains rather than dissected blocks of abstract equations. Moreover, the complexity of data management and data flow normally associated with parallel solution methods is avoided in this new algorithm.

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