Method and Models for R-Curve Instability Calculations

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SUMMARY

This paper presents a simple method for performing elastic R-curve insta-
bility calculations. For a single material-structure combination, the calcula-
tions can be done on some pocket calculators. On microcomputers and larger,
it enables the development of a comprehensive program having libraries of driv-
ing force equations for different configurations and R-curve model equations
for different materials. The paper also presents several model equations for
fitting to experimental R-curve data, both linear elastic and elastoplastic.
The models are fit to data from the literature to demonstrate their viability.

INTRODUCTION

The R-curve is one of the most powerful concepts available to the fracture
analyst. It is probably the best phenomenological description of the mono-
tonic fracture process currently available. But it has not received the
acclaim and widespread use that it deserves. Perhaps the instability calcula-
tions are thought to be too involved or tedious. The literature contains very
little information on instability calculations and nothing recent. Creager
(ref. 1) presented a good graphical method. But it involves overlay transpar-
encies, is labor-intensive, and may lack precision. In a predictive round-
robin program (ref. 2), seven participants used the R-curve method. Only this
author used anything more elaborate than trial-and-error to perform the insta-
bility calculations.

The method I used is really quite simple. For a single material-structure
combination, the calculation can be done on some pocket calculators. On micro-
computers and larger, a comprehensive program having libraries of driving force
equations and R-curve model equations is possible.

This paper will describe the method of performing the instability calcula-
tion. It will also present several model equations for fitting to experimen-
tal R-curve data, both linear-elastic (K,G) and elastoplastic (J).

SYMBOLS

a	crack length
A,B,C,D,F,H	empirical coefficients in equations (3) to (7)
G	strain energy release rate
J	nonlinear crack parameter
K	stress intensity factor

L,M,N,P	empirical coefficients in equations (8) to (11)
W	width of specimen (or structure)
Y	stress intensity calibration factor
Δ	crack extension (effective or physical, as noted)
σ_f	flow Stress

Subscripts:

o	at initial (unloaded) condition
c	critical, at the instability point
R	related to the material's resistance

Specimen Notation:

M(T)	center-crack tension
CS	rectangular compact
CLWL	crack-line wedge loaded

INSTABILITY CALCULATION METHOD

The method was presented previously (ref. 3), but will be repeated here. The instability condition requires that both the magnitudes and the slopes of the crack driving force (G or K) curve and the material resistance curve (G_R) be equal at the point of instability. So we must solve two simultaneous equations. After writing the equations in a general form and doing a little algebra, we can write the instability condition as

$$0 = \frac{G_R(\Delta_c)}{G_R'(\Delta_c)} - \frac{a_0 + \Delta_c}{1 + 2\alpha_c} \tag{1}$$

where

$$G_R'(\Delta) = dG_R(\Delta)/d\Delta$$

and

$$\alpha = (a/Y)(dY/da)$$

For elastic R-curves Δ is the effective crack extension, a_0 is the initial crack length, Y is the stress intensity calibration factor, and the subscript (c) means "evaluated at the instability point." (If you prefer to work with the stress intensity factor K, substitute K/2K' for G/G'.) Now if we put in the expressions for G_R, G_R', and α then, for prescribed values of a_0 and specimen width W, Δ_c is the least positive root of equation (1). Any of several numerical methods will solve for the root, even on a programmable hand calculator. Once Δ_c is known, we can calculate G_c and the failure stress. The beauty of this equation is that the first term on the right side is a function of the material's R-curve only and the second is a function of the

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structural geometry only. Equations for the second term can be obtained from standard handbooks and will not be discussed here. Model R-curve equations to be used in the first term will be discussed later.

The effectiveness of this method is shown in appendix 11 of (ref. 2). In that analytical round-robin program, good predictions of fracture strengths were made for both simple and complex specimen geometries.

Unfortunately this simple method is not applicable to elastic-plastic (J_R) calculations. As discussed in Sections (2 and 6) of (ref. 4), the problem is much more complicated. Material properties are an integral part of the calculation of the driving force (J), and there appears to be no general way to separate them from the geometrical terms. However, the model equations for J_R curves to be discussed later may still prove useful.

ELASTIC R-CURVE MODELS

To do these calculations we need an equation for the R-curve. While it would be nice if the equation had some physical significance, it's not absolutely necessary. Until a good theoretical analysis is available, all we need is a continuous equation that approximates reality, is readily differentiable, and fits the data. At present there is no single equation that fits all R-curve data. But there are several usable equations, so we have to pick the one that best fits the particular data. Let 'R' be a generic term to indicate either G_R or K_R.

Wang and McCabe (ref. 5) suggested fitting a polynomial to data in the region of the R-curve where you expect instability to occur. One could also describe the entire curve by a low-order (say, cubic) spline function. For either case we have

$$R_0 = A_0 + A_1 \Delta + A_2 \Delta^2 + A_3 \Delta^3 \tag{2}$$

(note: different polynomial coefficients are used for each spline segment.)

Broek and Vlieger (ref. 6) suggested a power curve of the form

$$R_1 = A \Delta^B \tag{3}$$

where $0 < B < 1$. This particular form works very well for the more ductile materials such as 2000-series aluminums and some stainless steels. It has two unique features. The slope is infinite at $\Delta = 0$, and there is no asymptote. In physical terms these imply there is no crack extension at small loads (which is tenable) and that a material's fracture toughness is limited only by the size of the specimen (which is not). So it may not be a good fundamental model, but it's quite good for curve fitting.

Lower-toughness materials are usually asymptotic to a plateau value' of toughness. Several models fit this description. Bluhm (ref. 7) proposed an exponential model,

$$R_2 = R_2^*[1 - \exp(-\Delta/C)] \tag{4}$$

This form has a finite slope at $\Delta = 0$ and is asymptotic to R_2^*. A somewhat similar model, which I will call the hyperbolic model, is

$$R_3 = R_3^* \Delta / (D + \Delta) \qquad (5)$$

This is actually an inverted and transposed rectangular hyperbola. Like the exponential model it has a finite slope at $\Delta = 0$ and is asymptotic to R_3^*. Trigonometry suggests two additional asymptotic forms. They are the arctangent function

$$R_4 = R_4^* (2/\pi) \arctan(\pi \Delta / 2F) \qquad (6)$$

and the hyperbolic tangent

$$R_5 = R_5^* \tanh(\Delta / H) \qquad (7)$$

The power model, equation (3), is simple enough that an illustration is not required. The four asymptotic models, equations (4) to (7), are shown schematically in figure 1(a). The first empirical coefficient (the asymptote) is the 'plateau value' of fracture toughness, the second is a characteristic value of crack extension. Note that in all cases the initial slope is equal to the first empirical coefficient divided by the second. That's why the second coefficient was placed in the denominator of the arguments in equations (4), (6), and (7) rather than in the numerator. Note also that equations (4) to (7) are linear in the first coefficient and nonlinear in the second. Most nonlinear regression routines require initial estimates of the coefficients. These estimates can easily be made from a plot of the raw data using figure 1(a) as a guideline. Furthermore, the fitted coefficients can easily be converted from one system of units to another.

Figure (1b) shows the asymptotic models, all with the same asymptote and initial slope. This helps to show the inherent characteristics of each. The hyperbolic tangent model (curve A) has the sharpest "knee," rising quickly to approach the asymptote. The knee of the exponential (B) is not as sharp but the curve still approaches the asymptote fairly quickly. The arctangent model (C) rises quickly at first but then more slowly. The hyperbolic model (D) has the slowest approach of all. The characteristics of these models should be kept in mind when attempting to fit them to data.

The low-order polynomial (or spline), equation (2), might well be the most accurate way to fit a single R-curve. But a comprehensive computer program must store the polynomial coefficients for each spline segment as well as the location of the knots. Interpolation for thickness or temperature effects could be a problem. The other R-curve models, however, only require that we store two constants and one equation code for each thickness and temperature. Interpolation for thickness or temperature should be much simpler.

ELASTIC-PLASTIC R-CURVE MODELS

The four asymptotic models for elastic R-curves can be modified to represent elastic-plastic J_R curves by replacing $[R^*]$ with $[R^* + T\Delta]$ and renaming the coefficients. Then equations (4) to (7) become

$$R_6 = (R_6^* + T_6 \Delta)[1 - \exp(-\Delta/L)] \qquad (8)$$

$$R_7 = (R_7^* + T_7 \Delta)\Delta / (M + \Delta) \qquad (9)$$

$$R_8 = (R_8^* + T_8\Delta)(2/\pi) \arctan(\pi \Delta/2N) \tag{10}$$

$$R_9 = (R_9^* + T_9\Delta) \tanh(\Delta/P) \tag{11}$$

These four models are shown schematically in figure 2. Here the first coefficient R_n^* is a reference toughness value, similar to J_{Ic}. The second coefficient, T, is analogous to the tearing modulus. The initial slope is equal to the first coefficient, R^*, divided by the third (L, M, N, or P). Using figure 2 as a guideline, all three coefficients can be estimated from a plot of the raw data. However, the construction is a bit more elaborate than for elastic R-curves. As before, the fitted coefficients may easily be converted from one system of units to another.

At this point we have a choice. We can prescribe the initial slope as twice the flow stress, which is a customary assumption. Then only the parameters R^* and T need to be determined empirically. Equations (8) to (11) then become

$$R_{6A} = (R_{6A}^* + T_{6A}\Delta) [1 - \exp(-2\sigma_f\Delta/R_{6A}^*)] \tag{8a}$$

$$R_{7A} = (R_{7A}^* + T_{7A}\Delta)[\Delta/(R_{7A}^*/2\sigma_f) + \Delta] \tag{9a}$$

$$R_{8A} = (R_{8A}^* + T_{8A}\Delta)(2/\pi) \arctan(\pi\sigma_f\Delta/R_{8A}^*) \tag{10a}$$

$$R_{9A} = (R_{9A}^* + T_{9A}\Delta) \tanh(2\sigma_f\Delta/R_{9A}^*) \tag{11a}$$

On first thought, prescribing the flow stress appears attractive. However, we still really have three parameters in the equation. We are merely fixing one of them. In multiparameter curve fitting, this can sometimes result in poor fits. Remember, too, that the flow stress is not readily measured and is only defined by custom. But both approaches are worthy of investigation. The coefficients of these elastic-plastic equations can also be estimated from a plot of the raw data (see fig. 2) and readily converted from one system of units to another.

APPLICATIONS TO DATA

To use this method and these models, we must first obtain the model equation coefficients by fitting to experimental data. This can be done by nonlinear regression analysis. The data to follow were all fitted using the program MARQFIT (ref. 8) on a microcomputer. If the data used were not available in tabular form, published figures were enlarged xerographically and digitized. Fitted values of the coefficients for the data sets presented here are given in tables I and II. In the discussion to follow, the models of equations (8) to (11) will be referred to as 'modified' models and those of equations (8a) to (11a) as 'special' models.

Elastic R-curves

Figure 3 shows the power model, equation (3), fit to data for 2014-T6 aluminum center-crack specimens tested at 77 K (ref. 9). Figure 4 shows the exponential model, equation (4), fit to data for 7475-T761 aluminum CLWL specimens

(ref. 5). Figure 5 shows the hyperbolic model, equation (5), fit to data for 7075-T651 aluminum CLWL specimens (ref. 2). Figure 6 shows the arctangent model, equation (6), fit to data for 2024-T3 aluminum CLWL specimens (ref. 5). (Tabular data for figures 4 and 6 were obtained by private communication from the second author.) Figure 7 shows the hyperbolic tangent model, equation (7), fit to the same data as figure 4.

These figures all show good fits, but these are not necessarily the best fits that could be obtained. All possible combinations of model equations and data sets cannot be shown for space reasons. The combinations that are presented were selected to show that each model is a viable one. That is, there is at least one data set that each model can describe well.

Elastic-Plastic R-curves

The following figures were developed using equations (8) to (11) as given. That is, the initial slope (flow stress) was considered a third fitting parameter. Figure 8 shows the modified exponential model, equation (8), fit to data for A106C steel compact specimens at 135 °C (ref. 10). Figure 9 shows the modified hyperbolic model, equation (9), fit to data for 2024-T351 CLWL specimens (ref. 2). Figure 10 shows the modified arctangent model, equation (10), fit to data for A533B steel compact specimens at 149 °C (ref. 11). Figure 11 shows the modified hyperbolic tangent model, equation (11), fit to data for A106B steel compact specimens (ref. 12). Again the figures presented were chosen to show that each model is a viable one.

Elastic-Plastic R-curves (flow stress prescribed)

Figures 12 to 15 were developed using equations (8a) to (11a). The flow stress was taken as the mean of the reported yield and ultimate strengths. Figure 12 shows the special exponential model, equation (8a), fit to the same data as figure 7. Figure 13 shows the special hyperbolic model fit to data for 5456-H117 aluminum compact specimens (ref. 13). Figure 14 shows the special arctangent model fit to the same data as figure 9. Figure 15 shows the special hyperbolic tangent model fit to the same data as figure 10.

DISCUSSION

All possible combinations of model equations and data sets cannot be shown for space reasons. The combinations that are presented were chosen to show that each model is a viable one. And indeed, each model is a good fit to at least one data set.

For elastic R-curves, the eye is usually able to judge whether a power model or an asymptotic model is appropriate. But there is no obvious way to predict which asymptotic model will be the best fit. For the elastic-plastic case, there also is no way to tell in advance whether the flow stress can be specified or whether it should be a parameter to be fitted. However, for 4 of the 6 sets of J_R data, the best fit overall was obtained when the initial slope (flow stress) was fitted rather than prescribed.

CONCLUSIONS

In summary, the instability calculation method is simple and effective. The R-curve model equations presented here are all viable, and at least one of them should fit almost any data. The empirical coefficients are easily estimated from a plot of the raw data and converted from one system of units to another. In combination, the method and the models make powerful tools for fracture analysis.

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TABLE I. - ELASTIC R-CURVES

Figure	Equation	Fitted equation
3	3	$K_R = 43.92\Delta^{0.2175}$
4	4	$K_R = 161.3[1 - \exp(-\Delta/10.43)]$
5	5	$K_R = 50.98\Delta/(0.9718 + \Delta)$
6	6	$K_R = 155.0(2/\pi) \arctan[(\pi/2)(\Delta/15.06)]$
7	7	$K_R = 158.6 \tanh(\Delta/13.76)$

TABLE II. - ELASTIC-PLASTIC R-CURVES

Figure	Equation	Fitted equation
8	8	$J_R = (315.1 + 129.6\Delta)[1 - \exp(-\Delta/0.3300)]$
9	9	$J_R = (357.7 + 3.600\Delta)\Delta/(5.172 + \Delta)$
10	10	$J_R = (373.5 + 130.7\Delta)(2/\pi) \arctan[(\pi/2)(\Delta/0.3258)]$
11	11	$J_R = (502.7 + 177.6\Delta) \tanh(\Delta/0.6318)$
12	8a	$J_R = (347.6 + 122.6\Delta)[1 - \exp(-\Delta/0.4237)]$
13	9a	$J_R = (38.87 + 13.00\Delta)\Delta/(0.06645 + \Delta)$
14	10a	$J_R = (396.7 + 124.2\Delta)(2/\pi) \arctan(\pi/2)(\Delta/0.2493)]$
15	11a	$J_R = (553.5 + 165.2\Delta) \tanh(\Delta/0.8763)$
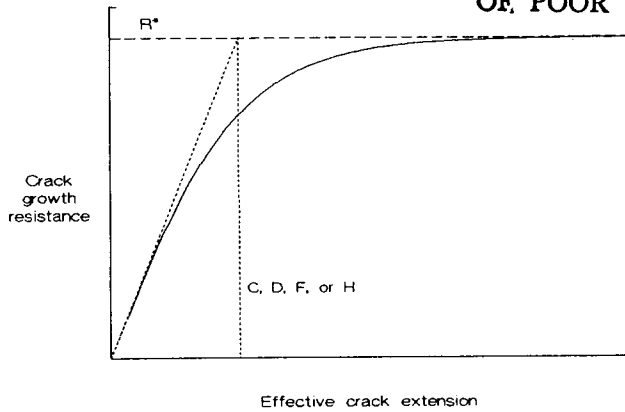
FIGURE 1(a). - SCHEMATIC REPRESENTATION OF ELASTIC-
ASYMPTOTIC R-CURVES (C, D, F, AND H ARE COEFFI-
CIENTS IN EQS. (4) TO (7)).
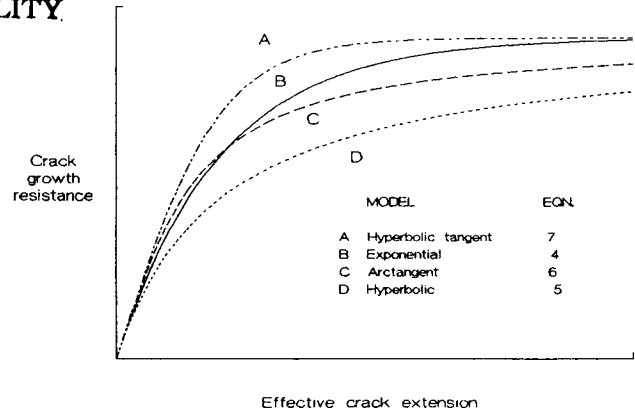
FIGURE 1(b). - ELASTIC ASYMPTOTIC R-CURVES HAVING THE
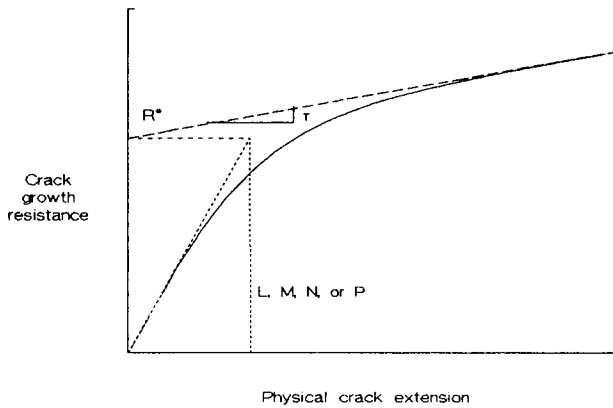SAME ASYMPTOTE AND INITIAL SLOPE.

MODEL		EQN.
A	Hyperbolic tangent	7
B	Exponential	4
C	Arctangent	6
D	Hyperbolic	5

FIGURE 2. - SCHEMATIC REPRESENTATION OF ELASTIC-
PLASTIC R-CURVES (L, M, N, OR P ARE COEFFICIENTS
IN EQS. (8) TO (11)).
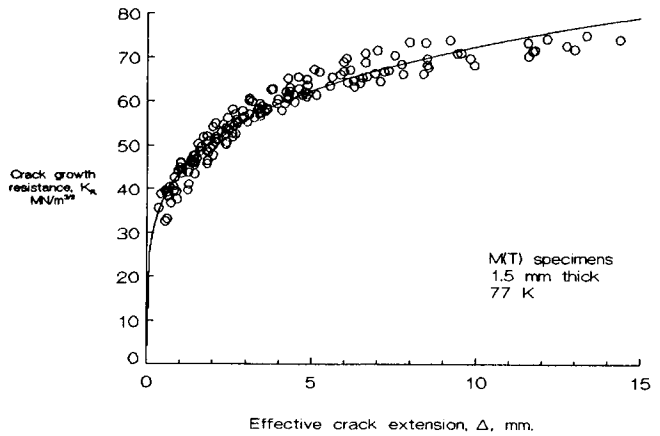
FIGURE 3. - POWER MODEL (EQ. (3)) FIT TO DATA FOR
2014-T6 ALUMINUM (REF. 9).
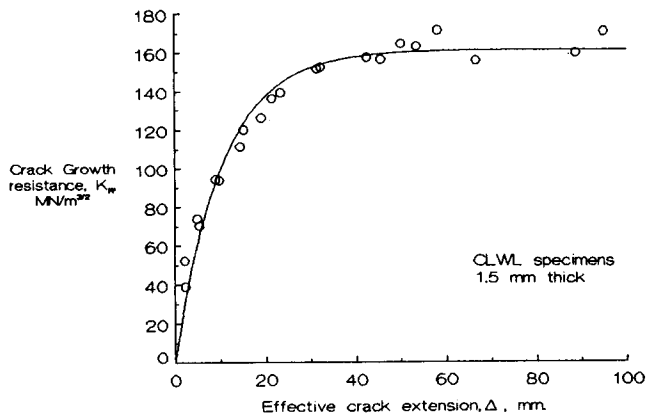
FIGURE 4. - EXPONENTIAL MODEL (EQ. (4)) FIT TO
DATA FOR 7475-T761 ALUMINUM (REF. 5).
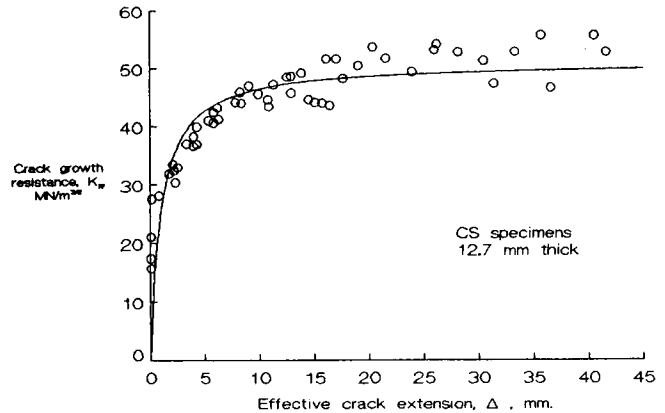
FIGURE 5. - HYPERBOLIC MODEL (EQ. (5)) FIT TO DATA
FOR 7075-T651 ALUMINUM (REF. 2).

9
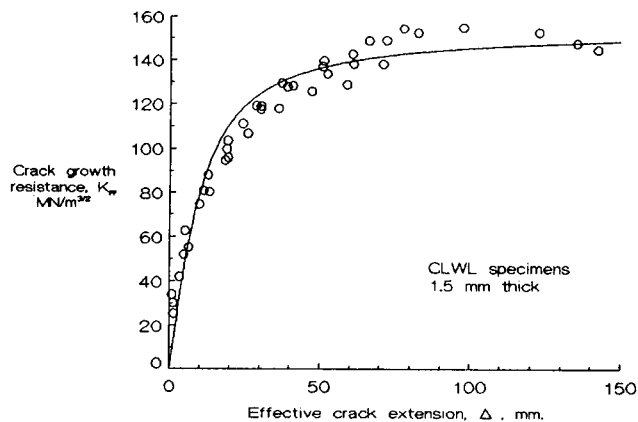
FIGURE 6. - ARCTANGENT MODEL (EQ. (6)) FIT TO
DATA FOR 2024-T3 ALUMINUM (REF. 5).
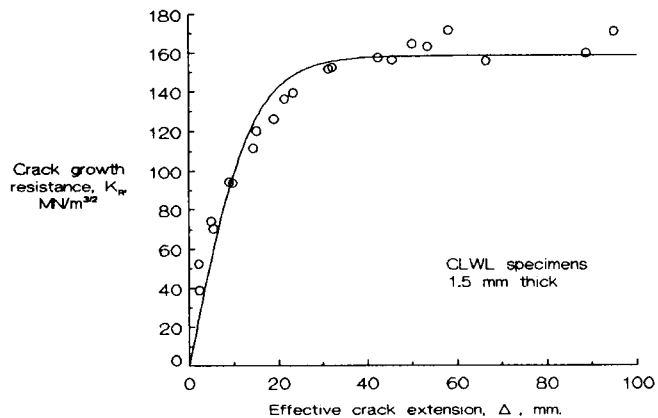
FIGURE 7. - HYPERBOLIC TANGENT MODEL (EQ. (7))
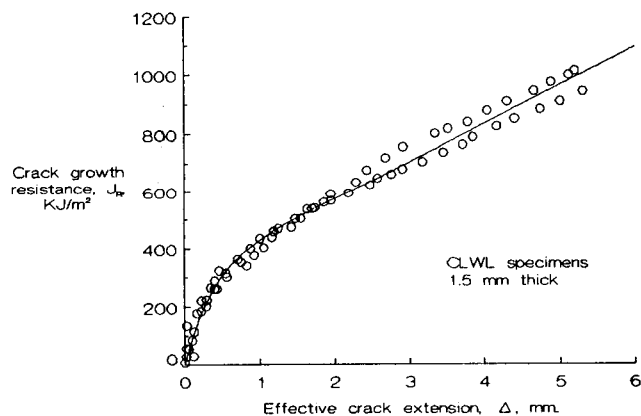FIT TO DATA FOR 7475-T761 ALUMINUM (REF. 5).

FIGURE 8. - MODIFIED EXPONENTIAL MODEL (EQ. (8))
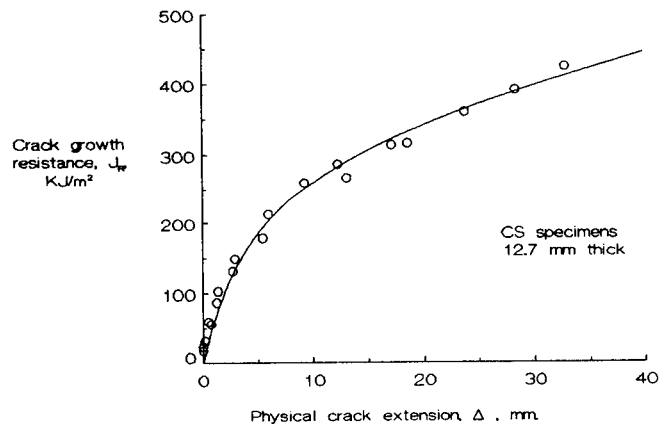FIT TO DATA FOR A106C STEEL (REF. 10).

FIGURE 9. - MODIFIED HYPERBOLIC MODEL (EQ. (9))
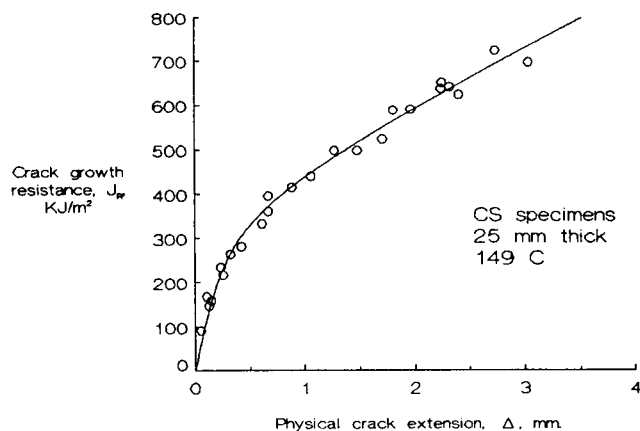FIT TO DATA FOR 2024-T351 ALUMINUM (REF. 2).

FIGURE 10. - MODIFIED ARCTANGENT MODEL
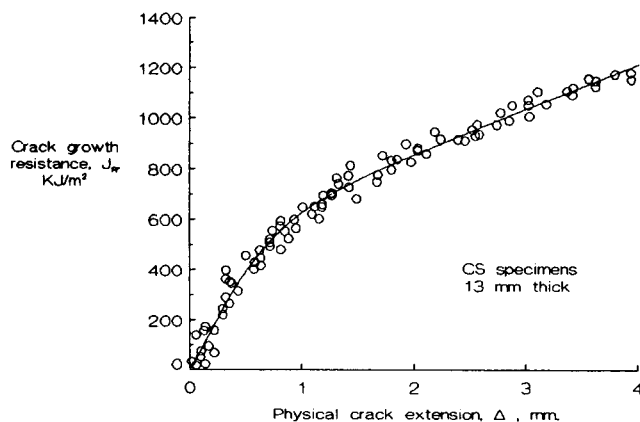(EQ. (10)) FIT TO DATA FOR A533B STEEL
(REF. 11).

FIGURE 11. - MODIFIED HYPERBOLIC TANGENT MODEL
(EQ. (11)) FIT TO DATA FOR A106B STEEL
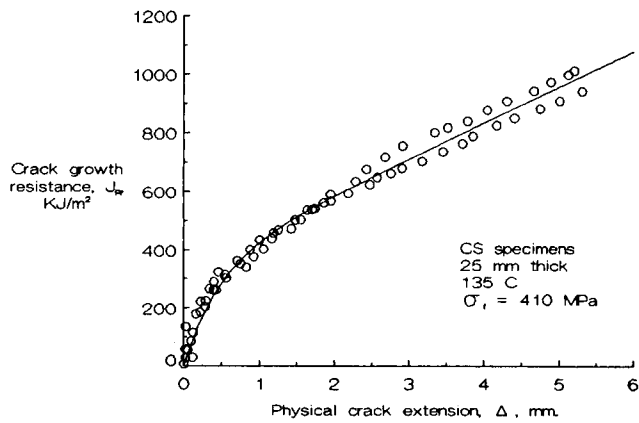(REF. 12).

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FIGURE 12. - SPECIAL EXPONENTIAL MODEL (EQ. (8a))
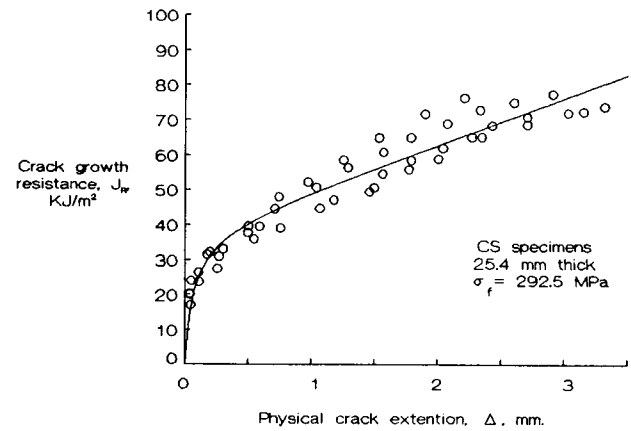FIT TO DATA FOR A106C STEEL (REF. 10).

FIGURE 13. - SPECIAL HYPERBOLIC MODEL (EQ. (9a))
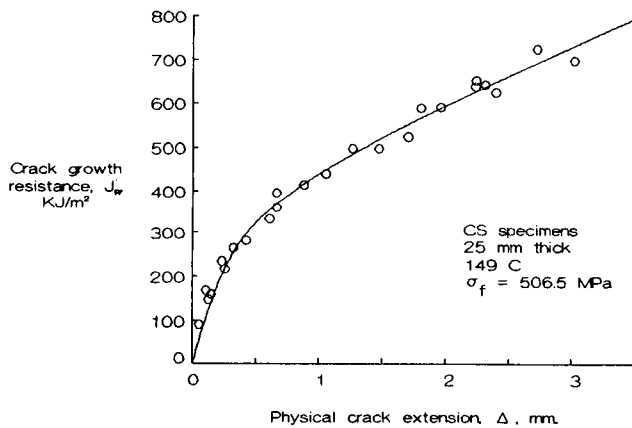FIT TO DATA FOR 5456-H117 ALUMINUM (REF. 13).

FIGURE 14. - SPECIAL ARCTANGENT MODEL (EQ. (10a))
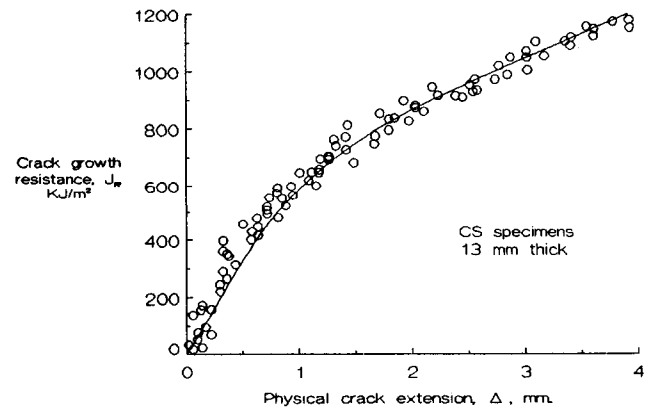FIT TO DATA FOR A533B STEEL (REF. 11).

FIGURE 15. - SPECIAL HYPERBOLIC TANGENT MODEL
(EQ. (11a)) FIT TO DATA FOR A533B STEEL
(REF. 11).

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16. Abstract

This paper presents a simple method for performing elastic R-curve instability calculations. For a single material-structure combination, the calculations can be done on some pocket calculators. On microcomputers and larger, it enables the development of a comprehensive program having libraries of driving force equations for different configurations and R-curve model equations for different materials. The paper also presents several model equations for fitting to experimental R-curve data, both linear elastic and elastoplastic. The models are fit to data from the literature to demonstrate their viability.

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