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PERIODICALLY CORRELATED PROCESSES

AND THEIR STATIONARY DILATIONS

by

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ABSTRACT

An explicit form for a stationary dilation for periodically correlated random processes is obtained. This is then used to give spectral conditions for a periodically correlated process to be non-deterministic, purely non-deterministic,minimal,and to have a positive angle between its past and future.

Key Words and Phrases: Periocically Correlated Processes, Harmonizable Processes, Stationary Dilation

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1. INTRODUCTION. Stationary stochastic processes have been extensively studied and the prediction theory of such processes is well developed and more or less complete. Although many facts are known for nonstationary processes, the theory of nonstationary processes needs more investigation. A wide class of nonstationary processes is that of harmonizable processes which were introduced by Loeve [15] and studied by several authors such as Cramer[3]. Rozanov [24], Abreu [1], Nieme [19], [20] and Miamee and Salehi [17]. A sequence X_n, $n \in Z$ in a Hilbert space H is called Harmonizable if its correlation function $\gamma(n,m) = (X_n, X_m)$ can be expressed as

$$(1) \qquad \gamma(n,m) = \int_0^{2\pi}\int_0^{2\pi} e^{-i(n\theta - m\Psi)} dF(\theta,\Psi),$$

where F is a complex-valued measure of bounded variation on the square $[0,2\pi] \times [0,2\pi]$. Harmonizalbe processes are natural generalizations of stationary processes. In fact, if the mass of the measure F is concentrated on the diagonal of that square, a harmonizable process reduces to a stationary one. Nevertheless, since the class of harmonizable processes is so broad, unlike the case of stationary processes, their studies are not as conclusive. Another class of nonstationary stochastic processes is that of periodically correlated processes: A sequence X_n in a Hilbert space H is called periodically correlated with period T if $\gamma(n,m) = \gamma(n + T, m + T)$, for all $m,n \in Z$, where $\gamma(n,m) = (X_n, X_m)$ is the correlation function of X_n. Periodically correlated processes have been recently studied by several authors including Gradyshev [6], [7] who gave the first mathematical treatment of these processes, Pourahmadi and Salehi [23], Ogura [21], Pagano [22], Hurd [11], [12], [13], and Miamee and Salehi [18]. Such processes have many applications [2], [5], [8].

For a periodically correlated process X_n, the behavior of its spectral distribution is quite clear now. In fact, (cf. [6]), its spectral distribution F in (1) is, in this case, concentrated on 2T-1 equidistant straight line segments parallel to the main diagonal of the square $[0,2\pi] \times [0,2\pi]$. For harmonizable processes, an important fact which was first observed by Abreu in [1] is the following theorem: *Given any harmonizable process* X_n *in a Hilbert space* H, *there exists a larger Hilbert space* $K \supseteq H$ *and a stationary process* $\underline{X}_n$ *in* K *such that*

$$X_n = P\,\underline{X}_n\ , \text{ for all } n \in Z,$$

where P *is the orthogonal projection of* K *onto* H. Any such stationary process $\underline{X}_n$ is called a stationary dilation for X_n. Abreu in [1] gives the spectral measure of one of these stationary dilations. However, that does not give a clear picture of what the connection between X_n and its stationary dilation $\underline{X}_n$ is, thus it would be useful if one could obtain an *explicit form* for a stationary dilation $\underline{X}_n$ in terms of X_n which would give a clear understanding of $\underline{X}_n$ in *both* the time and spectral domain.

In this note, we give a complete answer to this question for periodically correlated processes by giving an explicit closed form for a stationary dilation process $\underline{X}_n$ for X_n. This will express $\underline{X}_n$ in terms of the original process X_n in the time domain (Theorem 2.1) as well as in the spectral domain (Theorem 2.3). In section 3, we use these cruicial representations, established in section 2, to obtain spectral conditions for a harmonizable process to be nondeterministic, purely nondeterministic, minimal, and interpolable. In section 4, we obtain spectral conditions for a harmonizable process to have a positive angle between its past-present and future.

2. EXPLICIT FORM OF A STATIONARY DILATION FOR PERIODICALLY CORRELATED PROCESSES.

In this section, after introducing some preliminary results concerning periodically correlated processes, we give a closed form expression for a stationary dilation process $\underline{X}_n$ for a given periodically correlated process X_n. This expression gives $\underline{X}_n$ in terms of X_n in the time domain (Theorem 2.1). Then we find the spectral measure of $\underline{X}_n$, and thereby give an explicit expression for this stationary dilation in the spectral domain (Theorem 2.3).

Let X_n be a periodically correlated process with period T. Then for each τ, the function $R(n,\tau)$ defined by

$$R(n,\tau) = \gamma(n+\tau,n) = (X_{n+\tau}, X_n)$$

is periodic in n with period T. Since $R(n,\tau)$ is periodic in n, one can write

$$R(n,\tau) = \sum_{k=0}^{T-1} R_k(\tau) \exp\left(\frac{2\pi ikn}{T}\right). \tag{2}$$

For convenience, we extend the definition of these $R_k(\tau)$, k=0, 1, 2, ..., T-1, to all integers by $R_k(\tau) = R_{k+T}(\tau)$. It is shown [6] that each $R_k(\tau)$ has a representation of the form

$$R_k(\tau) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\tau\lambda} \, dF_k(\lambda), \tag{3}$$

where each $F_k(.)$ is a complex valued measure on $[0, 2\pi]$. It is also shown in [6] that

$$R(n,\tau) = \frac{1}{4\pi^2} \int_0^{2\pi}\int_0^{2\pi} e^{-i(n+\tau)\theta + in\Psi} \, dF(\theta,\Psi) \tag{4}$$

or

$$\gamma(n,m) = \frac{1}{4\pi^2} \int_0^{2\pi}\int^{2\pi} e^{-i(m\theta - n\Psi)} \, dF(\theta,\Psi), \tag{5}$$

where the spectral measure $F(.,.)$ is given by

$$F(A,B) = \sum_{k=-T+1}^{T-1} \int_{A\cap\left(B - \frac{2\pi k}{T}\right)} dF_k(\lambda), \tag{6}$$

and B - a is the set of all b - a with b ϵ B. This shows that periodically correlated processes do satisfy (1) with a spectral measure which is concentrated on $2T - 1$ straight line segments $\theta - \Psi = 2\pi k/T$, $k = -T + 1, \ldots, T - 1$ contained in the square $[0,2\pi] \times [0,2\pi]$.

Representations (5) and (6), which are of the form (1), in particular show that any periodically correlated process X_n is harmonizable and hence, by Abreu's result mentioned in section 1, has a stationary dilation. The next theorem gives our explicit form for one such dilation which is probably the most natural one in this case. But first, we introduce the direct sum H^T of T copies of H. H^T consists of all vectors $\underline{X} = (X^1, X^2, \ldots, X^T)$ with $X^i \in H$, $i = 1, 2, \ldots, n$. We endow H^T with the Euclidian inner product. For $\underline{X} = (X^1, X^2, \ldots, X^T)$ and $\underline{Y} = (Y^1, Y^2, \ldots, Y^T)$ in H^T, we define their inner product $((\underline{X}, \underline{Y}))$ to be

$$((\underline{X}, \underline{Y})) = \sum_{i=1}^{T} (X^i, Y^i),$$

and hence the norm to be $|||\ \underline{X}\ ||| = \sum_{i=1}^{T} |X^i|^2$.

<u>2.1 THEOREM</u>. Let the sequence X_n in the Hilbert space H be a periodically correlated process with period T. Then the process $\underline{X}_n = (X_n, \ldots, X_{n+T-1})$ in $K = H^T$ is stationary and

$$X_n = P\,\underline{X}_n,$$

where P is the projection of K onto its first coordinate (which is certainly an orthogonal projection).

<u>Proof</u>. To see that $\underline{X}_n$ is stationary, we write

$$((\underline{X}_n, \underline{X}_m)) = \sum_{j=0}^{T-1} (X_{n+j}, X_{m+j})$$

$$= (X_n, X_m) + \sum_{j=1}^{T-1} (X_{n+j}, X_{m+j})$$

Since X_n is periodically correlated with Period T, we have $(X_m, X_n) = (X_{m+T}, X_{n+T})$ so we can write

$$((\underline{X}_n, \underline{X}_m)) = (X_{n+T}, X_{m+T}) + \sum_{j=1}^{T-1} (X_{n+j}, X_{m+j})$$
$$= \sum_{j=1}^{T} (X_{n+j}, X_{m+j}) = \sum_{j=0}^{T-1} (X_{n+1+j}, X_{m+1+j})$$
$$= ((\underline{X}_{n+1}, \underline{X}_{m+1})).$$

The rest of the theorem is clear.

In particular, this theorem shows that given any periodically correlated process X_n with period T and values in H, then $\underline{X}_n = (X_n, \ldots, X_{n+T-1})$ is a stationary dilation for it which takes values in $K = H^T$.

2.2 REMARK. One can easily see that the stationarity of $\underline{X}_n$ is equivalent to the fact that X_n is periodically correlated.

The next Theorem which gives the spectral measure of this particular stationary dilation of X_n is crucial for any spectral analysis of our process X_n via its stationary dilation.

2.3 THEOREM. If X_n is a periodically correlated process with period T and $\underline{X}_n$ is its stationary dilation as introduced in Theorem 2.1, then the spectral measure of $\underline{X}_n$ is T times $dF_0(\lambda)$ (cf. (3) where F_0 is the diagonal part of the measure dF of X_n).

PROOF. Using (2) and (3) we can write

$$((\underline{X}_n, \underline{X}_0)) = \sum_{j=0}^{T-1} (X_{n+j}, X_j) = \sum_{j=0}^{T-1} R(j, n)$$
$$= \sum_{j=0}^{T-1} \sum_{k=0}^{T-1} R_k(n) \exp\left(\frac{2\pi ikj}{T}\right)$$
$$= \frac{1}{2\pi} \sum_{k=0}^{T-1} \sum_{j=0}^{T-1} \exp\left(\frac{2\pi ikj}{T}\right) \int_0^{2\pi} e^{-in\lambda}\, dF_k(\lambda)$$
$$= \frac{1}{2\pi} \sum_{k=0}^{T-1} \int_0^{2\pi} e^{-in\lambda} \sum_{j=0}^{T-1} \exp\left(\frac{2\pi ikj}{T}\right) dF_k(\lambda).$$

But one can easily check that

$$\sum_{j=0}^{T-1} \exp\left(\frac{2\pi ikj}{T}\right) = \begin{cases} 1 & , \text{ if } k=0 \\ 0 & , \text{ otherwise.} \end{cases}$$

Taking this into consideration, we can continue to write

$$((\underline{X}_n, \underline{X}_0)) = \frac{T}{2\pi} \int_0^{2\pi} e^{-in\lambda}\, dF_0(\lambda),$$

which completes the proof.

3. MINIMALITY, INTERPOLABILITY, DETERMINISM AND PURELY NONDETERMINISM.

In this section, we establish spectral conditions for a periodically correlated process to be minimal, interpolable, deterministic, or purely nondeterministic.

Given any process X_n (not necessarily stationary), we define $H_x(+\infty) = \overline{SP}\{X_k : k \in Z\}$, $H_x(n) = \overline{SP}\{X_k : k \le n\}$, $H_x(-\infty) = \bigcap_n H_x(n)$ and $H'_x(n) = \overline{SP}\{X_k : k \ne n\}$. The process X_n is called <u>deterministic</u> if $H_x(-\infty) = H_x(n)$ for all n, <u>minimal</u> if $H'_x(n) \ne H_x(+\infty)$ for some $n \in Z$, and <u>purely nondeterministic</u> if $H_x(-\infty) = 0$.

<u>3.1 THEOREM</u>. Suppose the periodically correlated process with period T is purely nondeterministic. Then F_0 must be a.c. with respect to Lebesque measure and

$$\int_0^{2\pi} \log f_0(\lambda)\, d\lambda > -\infty,$$

where f_0 is the density of F_0.

<u>PROOF</u>. Let $\underline{X}_n$ be the stationary dilation of X_n given in Theorem 2.1. Then it is not hard to see that

$$H_{\underline{x}}(n) \subseteq H_x(n) \oplus \ldots \oplus H_x(n+T-1), \text{ for all } n \in Z.$$

From this one immediately gets

$$H_{\underline{x}}(-\infty) \subseteq H_x(-\infty) \oplus \ldots \oplus H_x(-\infty).$$

Since X_n is assumed to be purely nondeterministic and $H_x(-\infty) = 0$. We conclude that $H_{\underline{x}}(-\infty) = 0$, which means $\underline{X}_n$ is purely nondeterministic. Now the theorem follows from a well known result for stationary processes [4] [14] which says that our stationary process $\underline{X}_n$ is purely nondeterministic if and only if its spectral measure (see Theorem 2.3) TF_0 is a.c. and its spectral measure Tf_0 satisfies $\int_0^{2\pi} \log(Tf_0(\lambda))d\lambda > -\infty$, which is the same as saying F_0 in a.c. and $\int_0^{2\pi} \log f_0 > -\infty$.

3.2 THEOREM. Let X_n be a periodically correlated process with period T. For X_n to be deterministic, it suffices to have $\int_0^{2\pi} \log \frac{dF_0}{d\lambda} d\lambda = -\infty$, where $\frac{dF_0}{d\lambda}$ denotes the Radon-Nykodixm derivative of the absolutely continuous part of F_0.

PROOF. If $\int_0^{2\pi} \log \frac{dF_0}{d\theta} d\lambda = -\infty$, then $\int_0^{2\pi} \log\left(\frac{TdF_0}{d\lambda}\right) d\lambda = -\infty$. This together with Theorem 2.3 and the well known characterization of deterministic stationary processes [4],[14] implies that the stationary dilation $\underline{X}_n$ of X_n is deterministic. But since $X_n = P\underline{X}_n$, one can easily see that this forces X_n to be deterministic.

The above proof is similar to that of a corresponding proposition in [1] regarding harmonizable processes.

Proofs of the following two theorems are similar to those of Theorems 3.1 and 3.2 above.In fact, they follow from Theorems 2.1 and 2.3 by recalling the corresponding well knowns results for stationary processes (cf. [4], [14], [25] in exactly the same way. Hence we omit the proofs,

3.2 THEOREM. Let X_n be a periodically correlated process which is minimal. Then F_0 is a.c. and its density $f_0(\lambda)$ is a.e. invertible with

$$\int_0^{2\pi} \frac{d\lambda}{f_0(\lambda)} < \infty.$$

3.4 THEOREM. The periodically correlated process X_n is interpolable if

$$\int_0^{2\pi} \frac{|p(e^{i\lambda})|^2}{\frac{dF_0}{d\lambda}} \, d\lambda = \infty$$

for every non-zero trigonometric polynomial P.

4. POSITIVITY OF THE ANGLE BETWEEN PAST-PRESENT AND FUTURE. In this section, we study the angle between past-present and future for a periodically correlated process and, using our results of section 2, we get some necessary spectral conditions for this angle to be positive. The positivity of this angle is closely tied with the set $\{X_n\}$ forming a Shoulder basis for its time domain $H_x(+\infty)$. (For this and other applications of this concept one can see [9], [10] and [16]).

For a given process X_n, besides the subspaces we introduced at the beginning of section 3, we need to define $F(n) = \overline{SP}\{X_k : n < k\}$. As a measure of the angle between two spaces M and N (in a Hilbert space) it is customary to consider the quantity (cf.[9])

$$P(M,N) = \sup_{X,Y \neq 0} \left\{ \frac{|(X,Y)|}{||X|| \cdot ||Y||} : X \in M,\ Y \in N. \right\}$$

It is clear that $P(M, N) \leq 1$ and if it is strictly less than 1, it is said that the angle between M and N is positive. If, in particular, M is taken to be $H_x(n)$ and N is taken to be $F_x(n)$, then we say that the angle between past-present and future of X_n at time n is positive if $\rho_x(n) = \rho(H_x(n), F_x(n)) < 1$. If X_n is stationary, then $\rho(n)$ is a constant function of n, and if X_n is periodically correlated with period T, then $\rho(n)$ is periodic with period T. The stationary process X_n is said to have a positive angle between its past and future if $\rho(n) < 1$ for some and hence every n. A periodically correlated process X_n has a positive angle between its past and future if $\rho(n) < 1$ for every $n = 0,1,\ldots T-1$ and hence every n. Positivity of the angle between past and future is some kind of regularity which is stronger than the ones often used in prediction theory, namely nondeterminism, purely nondeterminism, minimality, etc. (cf. [9] and [16]).

The following theorem connects the positivity of a periodically correlated process X_n with its stationary dilation $\underline{X}_n$ introduced in Theorem 2.1.

4.1 THEOREM. Let X_n be a periodically correlated process whose angle between past and future is positive. Then, the angle between past and future of its stationary dilation $\underline{X}_n$ given in Theorem 2.1 is also positive.

PROOF. Take a finite linear combination $\alpha = \sum_{k\le 0} a_k \underline{X}_k$ in $H_{\underline{x}}(0)$ and another one $\beta = \sum_{j\ge 1} b_j \underline{X}_j$ in $F_{\underline{x}}(1)$. Then we can write

$$|((\alpha,\beta))| = |((\sum_{k\le 0} a_k \underline{X}_k, \sum_{j\ge 1} b_j \underline{X}_j))|$$

$$= |\sum_{i=0}^{T-1} (\sum_{k\le 0} a_k X_{k+i}, \sum_{j\ge 0} b_j X_{j+i})|$$

$$\le \sum_{i=0}^{T-1} |(\sum_{k\le 0} a_k X_{k+i}, \sum_{j\ge 0} b_j X_{j+i})|$$

$$\le \sum_{i=0}^{T-1} \rho_x(i) \, |\sum_{k\le 0} a_k X_{k+i}| \, |\sum_{j\ge 1} b_j X_{j+i}|$$

$$< \rho_x \sum_{i=0}^{T-1} |\sum_{k\le 0} a_k X_{k+i}| |\sum_{j\ge 1} b_0 X_{j+i}|,$$

where $\rho_x = \max \{\rho_x(i): i=0,1,2,\dots,T-1\}$. We can further write

$$|((\alpha,\beta))| \le \rho_x \left(\sum_{i=0}^{T-1} |\sum_{k\le 0} a_k X_{k+i}|^2\right)^{\frac{1}{2}} \left(\sum_{i=0}^{T-1} |\sum_{j\ge 1} b_j X_{j+i}|^2\right)^{\frac{1}{2}}.$$

So we have

$$|((\alpha,B))| < \rho_x ||| \alpha ||| \, ||| \beta |||,$$

which implies that $\rho_{\underline{x}} \le \rho_x$. Now since we have assumed that $\rho_x < 1$, we conclude that $\rho_{\underline{x}} < 1$, which completes the proof.

From the well known characterizations for positivity of the angle for stationary processes [10], [9], it follows that the process $\underline{X}_n$ has a positive angle if and only if F_0 is a.c. with a density f_0 which either satisfies the Muckenhoupt condition [10]

$$\text{(7)} \qquad \operatorname{Sup}\left(\frac{1}{|I|}\int_I f_0(\lambda)\, d\lambda\right)\left(\frac{1}{|I|}\int_I \frac{d\lambda}{f_0(\lambda)}\right) < \infty,$$

where Sup is taken over all intervals $I \subseteq [0,2\pi]$, or is of the Helson -Szego type [9]

$$\text{(8)} \qquad f_0 = e^{u+\tilde{v}},$$

where u is a bounded real function and $\tilde{v}$ is the conjugate of a real function v which is bounded and satisfies the condition $\| v \|_\infty < \frac{\pi}{2}$.

Putting the facts just mentioned along with our Theorem 4.1, we get the following.

4.2 THEOREM. Let X_n be a periodically correlated process. For X_n to have a positive angle between its past and future, it is necessary that F_0 be a.c. and f_0 satisfies either the Muckenhoupt condition (7) or is of Helson-Szegö type (8).

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