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GALERKIN APPROXIMATION FOR INVERSE PROBLEMS FOR NONAUTONOMOUS NONLINEAR DISTRIBUTED SYSTEMS[1]

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ABSTRACT

We develop an abstract framework and convergence theory for Galerkin approximation for inverse problems involving the identification of nonautonomous nonlinear distributed parameter systems. We provide a set of relatively easily verified conditions which are sufficient to guarantee the existence of optimal solutions and their approximation by a sequence of solutions to a sequence of approximating finite dimensional identification problems. Our approach is based upon the theory of monotone operators in Banach spaces and is applicable to a reasonably broad class of nonlinear distributed systems. Operator theoretic and variational techniques are used to establish a fundamental convergence result. An example involving evolution systems with dynamics described by nonstationary quasi-linear elliptic operators along with some applications are presented and discussed.

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1. Introduction

In this paper we develop a general abstract approximation framework and convergence theory for Galerkin approximations for inverse problems involving nonautonomous nonlinear distributed parameter systems. We consider parameter estimation problems formulated as the minimization over a compact admissible parameter set, of a least-squares-like performance index subject to state constraints given by an inhomogeneous nonlinear distributed system. Our theory applies to systems whose dynamics can be described by nonstationary strongly maximal monotone operators defined on a reflexive Banach space which is densely and continuously embedded in a Hilbert space. This class of operators represents a nonlinear analog of the class of regularly dissipative, or abstract parabolic, linear operators. Our treatment relies heavily on the general theory for nonlinear evolution equations in Banach space given by Barbu in [9]. More specifically, we make extensive use of his Theorem III. 4.2 which serves as the basis for an existence, uniqueness, and regularity result that we require and for the fundamental convergence result we prove in section 3 below.

We employ standard Galerkin techniques to obtain a sequence of approximating identification problems, each of which involves finite dimensional state constraints. We demonstrate that if readily verifiable conditions on the system's dependence on the unknown parameters are satisfied, and the usual assumptions necessary for the convergence of Galerkin approximations hold, then solutions to the finite dimensionally constrained problems exist and, in some sense, approximate a solution to the original infinite dimensional identification problem.

Our treatment here and the approximation theory we develop generalizes and extends earlier results of ours and others in some significant ways. First, the variational approach that we use to establish our fundamental convergence result generalizes the techniques that have been widely used for this purpose elsewhere in the parameter identification literature (see, for example, [3],[5],[8]). However, in each of these instances, the variational arguments were given strictly in the context of particular inverse problems involving the identification of rather specialized linear distributed parameter systems. Second, the results given here extend the abstract approximation theories for autonomous linear system identification, and for quasi-autonomous nonlinear system identification developed in [4] and [7], respectively, to inverse problems for systems whose dynamics are described by temporarily inhomogeneous nonlinear operators. We note that although we have extended the theories developed in [4] and [7], the approach we have taken here in achieving this end is significantly different. Indeed, in [4] and [7] convergence is argued via an application of an abstract approximation result for evolution systems in Banach space such as the Trotter Kato Theorem (see [15]) or its nonlinear analogs (see, for example, [10] and [11]). While the approximation results given in [10] and [11] could be applied in the nonautonomous case, this would require that the resolvents of the time dependent system operators satisfy a Lipschitz-like

condition with respect to the time variable (see [7]). However, a condition on the resolvent is, in general, not easily verified for the infinite dimensional system and can be especially difficult to establish for the sequence of approximating finite dimensional systems. On the other hand, with the variational approach that we have taken here, a mildly restrictive, relatively easily verified, measurability condition on the temporarily varying system operator is the only assumption on the time dependence of the system that we require to establish existence of solutions and convergence.

We provide a brief outline of the remainder of the paper. In section 2 we define the class of nonlinear evolution systems and identification problems on which we shall be focusing our attention and we establish an existence, uniqueness, and regularity result for solutions to nonautonomous nonlinear distributed systems. In section 3 we develop our approximation theory and prove the fundamental convergence result. In section 4 we present an example and discuss some applications.

2. Inverse Problems for a Class of Nonautonomous Nonlinear Distributed Systems

Let $\mathcal{Q}$ be a metric space with Q, known as the admissible parameter set, a compact subset of $\mathcal{Q}$. Let Z, which we shall refer to as the observation space, be a normed linear space with norm $|\cdot|_Z$. Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and corresponding induced norm $|\cdot|$, and let V be a reflexive Banach space with norm $\|\cdot\|$. We assume that V is densely and continuously embedded in H. The latter assumption implies that there exists a constant $\mu > 0$ for which $|\varphi| \le \mu\|\varphi\|$, for all $\varphi \in V$. Let V^* be the space of continuous linear functionals defined on V and denote the usual dual space norm on V^* by $\|\cdot\|_*$. Identifying H with its dual, we have $V \subset H = H^* \subset V^*$ with H densely and continuously embedded in V^*. It follows that $\|\varphi\|_* \le \mu|\varphi|$, for all $\varphi \in H$ and that $\|\varphi\|_* \le \mu^2\|\varphi\|$, for all $\varphi \in V$. For $\varphi \in V^*$ and $\psi \in V$ we shall denote the duality pairing between φ and ψ by $\langle\varphi, \psi\rangle$. Of course when $\varphi \in H$, the pairing $\langle\varphi, \psi\rangle$ agrees with the usual H inner product of φ and ψ.

Let $T > 0$ be fixed, and for each $q \in Q$ and almost every $t \in [0,T]$ let A(t;q) be a hemicontinuous (i.e. $\text{w-}\lim_{h\to 0} A(t;q)(\varphi + h\psi) = A(t;q)\varphi$ for $\varphi, \psi \in V$) in general nonlinear, operator defined on all of V with range in V^*. In our discussions below, we shall require that the family of operators $A(t;q)$, $t \in [0,T]$, $q \in Q$, satisfy the following conditions.

(A) (Continuity) For each $\varphi \in V$ the map $q \to A(t;q)\varphi$ is continuous from $Q \subset \mathcal{Q}$ into V^* for almost every $t \in [0, T]$.

(B) (Equi-V-Monotonicity) There exist a constant ω and a positive constant α which do not depend upon $q \in Q$ or $t \in [0,T]$ such that

$$\langle A(t;q)\varphi - A(t;q)\psi,\ \varphi - \psi\rangle + \omega|\varphi - \psi|^2 \ge \alpha\|\varphi - \psi\|^p$$

for all $\varphi, \psi \in V$, almost every $t \in [0,T]$ and some p with $2 \le p < \infty$.

(C) (Equi-Boundedness) There exists a constant $\beta > 0$ which does not depend upon $q \in Q$ or $t \in [0, T]$ such that

$$\|A(t;q)\varphi\|_* \leq \beta(\|\varphi\|^{p-1} + 1)$$

for all $\varphi \in V$ and almost every $t \in [0, T]$.

(D) (Measurability) For each $q \in Q$ the function $A(t;q)u(t) : [0, T] \to V^*$ is strongly measurable for every $u \in L_p(0,T;V)$.

For each $q \in Q$ let $u_0(q)$ be an element in H and let $f(\cdot\,;q)$ be a function in $L_{p'}(0,T; V^*)$ where p and p' are conjugate exponents (i.e. $1/p + 1/p' = 1$). We assume that the mappings $q \to u_0(q)$ and $q \to f(\cdot\,;q)$ are continuous from $Q \subset \mathcal{Q}$ into H and $L_{p'}(0,T; V^*)$ respectively. For each $z \in Z$, let $\Phi(\cdot\,;z)$ be a mapping defined on $L_p(0,T;V)$ with range in the nonnegative real numbers and which is continuous when restricted to one or the other of the two spaces $C(0,T;H)$ or $L_p(0,T;V)$ endowed with their respective usual topologies. We consider the following abstract parameter identification problem.

(ID) Given observations $z \in Z$, find parameters $\bar{q} \in Q$ which minimize the performance index

$$J(q) = \Phi(u(\cdot\,;q);z)$$

where $u(\cdot\,; q)$ is the solution to the initial value problem

$$\dot{u}(t) + A(t;q)u(t) = f(t;q), \qquad \text{a.e. } t \in [0, T] \tag{2.1}$$

$$u(0) = u_0(q) \tag{2.2}$$

corresponding to $q \in Q$.

The existence of a solution $\bar{q} \in Q$ to problem (ID) will follow as a consequence of the approximation theory and results to be presented in the next section. However, the notion of a solution to the abstract initial value problem (2.1), (2.2) for each $q \in Q$ must be made precise, and its existence, uniqueness, and regularity properties must be demonstrated. We have the following theorem.

Theorem 2.1 If condition (B) - (D) are satisfied, then for each $q \in Q$ there exists a unique function $u(\cdot\,; q)$ which is V^*-absolutely continuous on [0,T], and satisfies $u(\cdot\,;q) \in L_p(0,T;V) \cap C(0,T;H)$, $\dot{u}(\cdot\,;q) \in L_{p'}(0,T;V^*)$, the abstract differential equation (2.1) for almost every $t \in [0, T]$, and the initial condition (2.2).

Proof For each $q \in Q$ and almost every $t \in [0,T]$ define $A_\omega(t; q)$ and $f_\omega(t; q)$ by $A_\omega(t; q) = e^{-\omega t} A(t;q) e^{\omega t} + \omega I$ and $f_\omega(t; q) = e^{-\omega t} f(t; q)$. Then for almost every $t \in [0,T]$, $A_\omega(t;q)$ is a nonlinear hemicontinuous operator from V into V^* and $f_\omega(t;q) \in L_{p'}(0,T;V^*)$. Also, condition (D) implies that for every $u \in L_p(0,T;V)$, the function of t, $A_\omega(t;q)u(t) : [0, T] \to V^*$ is strongly measurable. Conditions (B) and (C), and straight forward calculations yield the existence of an $\alpha_\omega > 0$ and a $\beta_\omega > 0$ (both depending upon p) for which

$$< A_\omega(t;q)\varphi - A_\omega(t;q)\psi, \varphi - \psi> \geq \alpha_\omega \|\varphi - \psi\|^p ,$$

(i.e., that the operator $A_\omega(t;q)$ is strongly monotone) and

$$\| A_\omega(t;q)\varphi\|_* \leq \beta_\omega(\|\varphi\|^{p-1} + 1)$$

hold for all $\varphi,\psi \in V$ and almost every $t \in [0,T]$. Finally, for all $\varphi \in V$ we have

$$\begin{aligned}
&< A_\omega(t;q)\varphi, \varphi> = < A_\omega(t;q)\varphi - A_\omega(t;q)\theta, \varphi - \theta> + < A_\omega(t;q)\theta, \varphi> \\
&\geq \alpha_\omega\|\varphi\|^p - | < A_\omega(t;q)\theta, \varphi>| \\
&\geq \alpha_\omega\|\varphi\|^p - \| A_\omega(t;q)\theta\|_*\|\varphi\| \\
&\geq \alpha_\omega\|\varphi\|^p - \beta_\omega\|\varphi\| \\
&\geq \alpha_\omega \|\varphi\|^p - \frac{\beta_\omega^{p'}}{p'\varepsilon^{p'}} - \frac{\varepsilon^p}{p} \|\varphi\|^p \\
&= \gamma_\omega\|\varphi\|^p + \delta_\omega
\end{aligned}$$

where $\varepsilon > 0$ is chosen small enough so that $\gamma_\omega = \alpha_\omega - \varepsilon^p/p > 0$, $\delta_\omega = - \beta_\omega^{p'}/p'\varepsilon^{p'}$ and θ denotes the zero vector in V. Then an application of Theorem III. 4.2 in Barbu [9] yields the existence of a unique V^*-valued absolutely continous function $u_\omega(\cdot ;q)$ defined on [0, T] which satisfies $u_\omega(\cdot;q) \in L_p(0,T;V) \cap C(0,T;H)$, $\dot{u}_\omega(\cdot ;q) \in L_{p'}(0,T;V^*)$,

$$\dot{u}_\omega(t;q) + A_\omega(t;q)u_\omega(t;q) = f_\omega(t;q),$$

for almost every $t \in [0, T]$ and $u_\omega(0;q) = u_0(q)$. The conclusion of the theorem now follows immediately by setting $u(t;q) = e^{\omega t} u_\omega(t;q)$ for $t \in [0,T]$.

3. An Abstract Approximation Framework and Convergence Theory

For each n = 1,2,... let H_n be a finite dimensional subspace of H which is also contained in V. Let $P_n : H \to H_n$ denote the orthogonal projection of H onto H_n. We shall require that the

following convergence condition is satisfied by the subspaces H_n and the corresponding projections P_n.

(E) For each $\varphi \in V$, we have $\lim_{n\to\infty} \|P_n\varphi - \varphi\| = 0$.

Condition (E) and the Principle of Uniform Boundedness imply that there exists a constant $\nu > 0$, independent of $\varphi \in V$ and n, for which $\|P_n\varphi - \varphi\| \le \nu\|\varphi\|$. Note also that V densely and continuously embedded in H, $|P_n| = 1$, and condition (E) imply $\lim_{n\to\infty} |P_n\varphi - \varphi| = 0$ for all $\varphi \in H$.

For each n = 1,2,..., each $q \in Q$ and almost every $t \in [0, T]$, we define the operator $A_n(t; q) : H_n \to H_n$ to be the restriction of the operator $A(t;q)$ to H_n with the image in V^* of $\varphi_n \in H_n$, $A(t;q)\varphi_n$, considered to be a linear functional on H_n. Identifying H_n with its dual, for $\varphi_n \in H_n$ we obtain $A_n(t;q)\varphi_n = \psi_n$ where ψ_n is that element in H_n which satisfies $\langle A(t;q)\varphi_n, \chi_n\rangle = \langle\psi_n, \chi_n\rangle$ for all $\chi_n \in H_n$.

For each n = 1,2,..., and each $q \in Q$ we define $u_{0n}(q) \in H_n$ by $u_{0n}(q) = P_n u_0(q)$ and for almost every $t \in [0, T]$ we define $f_n(t;q)$ to be the restriction of $f(t;q) \in V^*$ to H_n. Note that the Riesz representation theorem implies that $f_n(\cdot\,;q) \in L_{p'}(0,T;H_n)$. We consider the following sequence of parameter identification problems.

(ID_n) Given observations $z \in Z$, find parameters $\bar{q}_n \in Q$ which minimize the performance index

$$J_n(q) = \Phi(u_n(\cdot\,;q);z) \tag{3.1}$$

where $u_n(\cdot\,;q)$ is the solution to the initial value problem in H_n

$$\dot{u}_n(t) + A_n(t;q)u_n(t) = f_n(t;q), \text{ a.e. } t \in [0,T] \tag{3.2}$$

$$u_n(0) = u_{0n}(q) \tag{3.3}$$

corresponding to $q \in Q$.

The initial value problem (3.2), (3.3) in H_n is the standard Galerkin approximation to the initial value problem (2.1), (2.2). With appropriate minor modifications, Theorem 2.1 yields for each n = 1,2,... and each $q \in Q$ the existence of a unique absolutely continuous function $u_n(\cdot\,;q)$: $[0,T] \to H_n$ which satisfies the finite dimensional ordinary differential equation (3.2) for almost every $t \in [0,T]$, and the initial conditions (3.3), with $\dot{u}_n(\cdot\,;q) \in L_{p'}(0,T;H_n)$.

We would like to demonstrate that if the conditions (A) - (E) above are satisfied, then a solution to the inverse problem (ID_n) exists for each n = 1,2,..., and that these solutions imply the existence of, and in some sense approximate, a solution to problem (ID). In Theorem 3.1 below

we show that the mapping $q \to u_n(\cdot\,;q)$ is continuous from $Q \subset \mathcal{Q}$ into both $C(0,T;H)$ and $L_p(0,T;V)$ and that $\lim_{n\to\infty} u_n(\cdot\,;q_n) = u(\cdot\,;q_0)$ in both $C(0,T;H)$ and $L_p(0,T;V)$ whenever $\{q_n\}$ is a sequence in Q with $\lim_{n\to\infty} q_n = q_0$. Theorem 3.1 is in fact sufficient to conclude that the desired existence and approximation results obtain. Indeed, the continuous dependence of $u_n(\cdot\,;q)$ on q implies that for each $n = 1,2,\ldots, J_n$ given by (3.1) is continuous on Q. Since Q was assumed to be a compact subset of the metric space $\mathcal{Q}$, the existence of a solution $\bar{q}_n$ to problem (ID_n) follows immediately. Now $\{\bar{q}_n\} \subset Q$ and Q compact imply the existence of a convergent subsequence $\{\bar{q}_{n_j}\}$ of the sequence $\{\bar{q}_n\}$. If $\lim_{j\to\infty} \bar{q}_{n_j} = \bar{q}$, then $\bar{q} \in Q$ and

$$
\begin{aligned}
J(\bar{q}) = \Phi(u(\cdot\,;\bar{q});z) &= \Phi(\lim_{j\to\infty} u_{n_j}(\cdot\,;\bar{q}_{n_j});z) \\
&= \lim_{j\to\infty} \Phi(u_{n_j}(\cdot\,;\bar{q}_{n_j});z) = \lim_{j\to\infty} J_{n_j}(\bar{q}_{n_j}) \\
&\le \lim_{j\to\infty} J_{n_j}(q) = \lim_{j\to\infty} \Phi(u_{n_j}(\cdot\,;q);z) \\
&= \Phi(\lim_{j\to\infty} u_{n_j}(\cdot\,;q);z) = \Phi(u(\cdot\,;q);z) \\
&= J(q)
\end{aligned}
$$

for every $q \in Q$. Consequently $\bar{q}$ is a solution to problem (ID). We note that the limit of any convergent subsequence of $\{\bar{q}_n\}$ is a solution to problem (ID). When problem (ID) admits a unique solution $\bar{q}$, the sequence $\{\bar{q}_n\}$ itself is convergent and its limit is $\bar{q}$.

Theorem 3.1 If conditions (A) - (E) are satisfied, then

(i) $\lim_{n\to\infty} u_n(\cdot\,;q_n) = u(\cdot\,;q_0)$ in $C(0,T;H)$ and $L_p(0,T;V)$ whenever $\{q_n\}$ is a sequence in Q with $\lim_{n\to\infty} q_n = q_0$, and

(ii) for each fixed $n = 1,2,\ldots, \lim_{m\to\infty} u_n(\cdot\,;q_m) = u_n(\cdot\,;q_0)$ in $C(0,T;H)$ and $L_p(0,T;V)$ whenever $\{q_m\}$ is a sequence in Q with $\lim_{m\to\infty} q_m = q_0$.

Proof We shall prove (i) only; the proof of (ii) is analogous. We establish (i) with an argument in the spirit of those in [3], [8] set in an abstract framework similar to the one used by Barbu to prove his Theorem III. 4.2. in [9]. Fix $t \in [0,T]$ and define the Hilbert space $\mathcal{H}$ by $\mathcal{H} = L_2(0,t;H)$ together with the inner product

$$(x,y) = \int_0^t \langle x(s),y(s)\rangle ds$$

and corresponding induced norm $|x|_{\mathcal{H}} = \sqrt{(x,x)}$. Define the reflexive Banach space $\mathcal{V}$ by $\mathcal{V} = L_p(0,t;V)$ with norm

$$\|x\|_{\mathcal{V}} = \left(\int_0^t \|x(s)\|^p ds \right)^{1/p}$$

Then $\mathcal{V}^* = L_{p'}(0,t;V^*)$, the dual norm is given by

$$\|x\|_{\mathcal{V}^*} = \left(\int_0^t \|x(s)\|_*^{p'} ds \right)^{1/p'}$$

and the dense and continuous embedding $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$ holds.

For each $q \in Q$, define the operator $\mathcal{B}(q) : \text{Dom}(\mathcal{B}(q)) \subset \mathcal{V} \to \mathcal{V}^*$ by $\mathcal{B}(q)v = \dot{v}$, for $v \in \text{Dom}(\mathcal{B}(q)) = \{u \in \mathcal{V} : \dot{u} \in \mathcal{V}^*, u(0) = u_0(q)\}$ where the derivative in the above definition is in a generalized or distributional sense (see Lions, [13]). It is shown in [9] that the operator $\mathcal{B}(q)$ is maximal monotone on $\mathcal{V} \times \mathcal{V}^*$ and that $\text{Dom}(\mathcal{B}(q)) \subset C(0,t;H)$. For each $q \in Q$ define the operator $\mathcal{A}(q) : \mathcal{V} \to \mathcal{V}^*$ by

$$(\mathcal{A}(q)x)(s) = A(s;q)x(s), \qquad x \in \mathcal{V}, \text{ a.e. } s \in [0,t].$$

Using the properties of the operators $A(s;q)$, it is not difficult to argue that $\mathcal{A}(q)$ is hemicontinuous and satisfies

$$(3.4) \qquad (\mathcal{A}(q)x - \mathcal{A}(q)y,\ x - y) + \omega |x - y|_{\mathcal{H}}^2 \geq \alpha \|x - y\|_{\mathcal{V}}^p$$

for all $x,y \in \mathcal{V}$. In light of our proof of Theorem 2.1, it is clear that we may, without loss of generality take ω in condition (B) and (3.4) above equal to zero. We shall do this in our discussions below. It follows (see [9]) that the operator $\mathcal{T}(q) : \text{Dom}(\mathcal{B}(q)) \subset \mathcal{V} \to \mathcal{V}^*$ given by $\mathcal{T}(q) = \mathcal{B}(q) + \mathcal{A}(q)$ is maximal monotone on $\mathcal{V} \times \mathcal{V}^*$ and that $\mathcal{R}(\mathcal{T}(q)) = \mathcal{V}^*$. Consequently, the operator $\mathcal{T}(q)^{-1} : \mathcal{V}^* \to \text{Dom}(\mathcal{B}(q))$ is well defined. Henceforth denoting $u(\cdot\,;q_0)$ and $f(\cdot\,;q_0)$ by u and f respectively, we have $f \in \mathcal{V}^*$, $u \in \text{Dom}(\mathcal{B}(q_0))$ and $u = \mathcal{T}(q_0)^{-1}f$.

For each $n = 1,2,\ldots$ let V_n denote the linear space H_n endowed with the V-topology. That is, V_n is H_n considered as a subspace of V rather than H. Let $\mathcal{H}_n = L_2(0,t;H_n)$ and $\mathcal{V}_n = L_p(0,t;V_n)$. Then, since H_n is finite dimensional, V_n^* is H_n endowed with the V^*-topology and $\mathcal{V}_n^* = L_{p'}(0,t;V_n^*)$. Define the operators $\mathcal{B}_n(q)$: $\text{Dom}(\mathcal{B}_n(q)) \subset \mathcal{V}_n \to \mathcal{V}_n^*$, $\mathcal{A}_n(q)$: $\mathcal{V}_n \to \mathcal{V}_n^*$ and $\mathcal{T}_n(q) : \text{Dom}(\mathcal{B}_n(q)) \subset \mathcal{V}_n \to \mathcal{V}_n^*$ by

$$\mathcal{B}_n(q)x_n = \dot{x}_n, \qquad x_n \in \text{Dom}(\mathcal{B}_n(q)) = \{v_n \in \mathcal{V}_n : \dot{v}_n \in \mathcal{V}_n^*,\ v_n(0) = P_n u_0(q)\},$$

$$(\mathcal{A}_n(q)x_n)(s) = A_n(s;q)x_n(s), \qquad x_n \in \mathcal{V}_n, \text{ a.e. } s \in [0,t],$$

and

$$\mathcal{T}_n(q) = \mathcal{B}_n(q) + \mathcal{A}_n(q),$$

respectively. As was the case above we have that $\mathcal{B}_n(q)$ is maximal monotone, $\mathcal{A}_n(q)$ (actually, $\mathcal{A}_n(q) + \omega I$) is strongly monotone on $\mathcal{V}_n \times \mathcal{V}_n^*$ and $\mathcal{R}(\mathcal{T}_n(q)) = \mathcal{V}_n^*$. Denoting $u_n(\cdot\,;q_n)$ by u_n and $f_n(\cdot\,;q_n)$ by f_n we have that $f_n \in \mathcal{V}_n^*$ and that $u_n = \mathcal{T}_n(q_n)^{-1} f_n$ with $u_n \in \mathrm{Dom}\,(\mathcal{B}_n(q_n))$.

Now for $x \in \mathcal{V}$ and $q \in Q$, condition (C) implies

$$\begin{aligned}
(3.5)\qquad \|\mathcal{A}(q)x\|_{\mathcal{V}^*} &= \Big(\int_0^t \|A(s;q)x(s)\|_*^{p'} ds\Big)^{1/p'} \\
&\le \Big(\int_0^t \beta^{p'} (\|x(s)\|^{p-1} + 1)^{p'} ds\Big)^{1/p'} \\
&\le \beta\Big\{\Big(\int_0^t \|x(s)\|^{(p-1)p'} ds\Big)^{1/p'} + \Big(\int_0^t ds\Big)^{1/p'}\Big\} \\
&\le \beta\{\|x\|_{\mathcal{V}}^{p-1} + T^{1/p'}\} \\
&\le \tilde{\beta}\{\|x\|_{\mathcal{V}}^{p-1} + 1\}.
\end{aligned}$$

This estimate, which also holds for $\mathcal{A}_n(q)$, in turn implies that there exists a constant $M > 0$, independent of $t \in [0, T]$ and $n = 1,2,\ldots$ for which $\|u_n\|_{\mathcal{V}} \le M$, $n = 1,2,\ldots$. Indeed, using θ to denote the zero vector in $\mathcal{V}$ and noting that $(\mathcal{A}_n(q)w_n, v_n) = (\mathcal{A}(q)w_n, v_n)$ for all w_n, v_n in $\mathcal{V}_n$, (3.5) can be used to argue

$$\begin{aligned}
\alpha\|u_n\|_{\mathcal{V}}^p &\le (\mathcal{A}(q_n)u_n - \mathcal{A}(q_n)\theta,\, u_n) \\
&= (\mathcal{T}_n(q_n)u_n, u_n) - (\mathcal{B}_n(q_n)u_n, u_n) - (\mathcal{A}(q_n)\theta, u_n) \\
&= (f_n, u_n) - \frac{1}{2}\int_0^t \frac{d}{ds}|u_n(s)|^2 ds - (\mathcal{A}(q_n)\theta, u_n) \\
&\le \|f_n\|_{\mathcal{V}^*}\|u_n\|_{\mathcal{V}} + \frac{1}{2}|P_n u_0(q_n)|^2 - \frac{1}{2}|u_n(t;q_n)|^2 + \|\mathcal{A}(q_n)\theta\|_{\mathcal{V}^*}\|u_n\|_{\mathcal{V}} \\
&\le \|f_n - f\|_{\mathcal{V}^*}\|u_n\|_{\mathcal{V}} + \|f\|_{\mathcal{V}^*}\|u_n\|_{\mathcal{V}} + \frac{1}{2}|u_0(q_n) - u_0(q_0)|^2 + \frac{1}{2}|u_0(q_0)|^2 + \tilde{\beta}\|u_n\|_{\mathcal{V}}.
\end{aligned}$$

The continuous dependence assumptions on $f(\cdot\,; q)$ and $u_0(q)$, $\lim_{n\to\infty} q_n = q_0$, and the application of the familiar inequality

$$ab \le \frac{\varepsilon^p a^p}{p} + \frac{b^{p'}}{\varepsilon^{p'} p'},\quad a, b \ge 0,\quad \varepsilon > 0$$

with ε chosen sufficiently small, yields the desired uniform bound.

For $x \in \mathcal{H}$, define $x^n \in \mathcal{H}_n$ by $x^n(s) = P_n x(s)$, a.e. $s \in [0, t]$. Then for $x \in \mathrm{Dom}(\mathcal{B}(q))$, using the definition of generalized derivative, it is not difficult to argue that $x^n \in \mathrm{Dom}(\mathcal{B}_n(q))$ and that $(\mathcal{B}_n(q)x^n - \mathcal{B}(q)x, v_n) = 0$ for all $v_n \in \mathcal{V}_n$.

It then follows that

$$
\begin{aligned}
\alpha\|u_n - u\|_{\mathcal{V}}^p &\le (\mathcal{A}(q_n)u_n - \mathcal{A}(q_n)u, u_n - u) \\
&= (\mathcal{A}(q_n)u_n - \mathcal{A}(q_0)u, u_n - u^n) + (\mathcal{A}(q_n)u_n - \mathcal{A}(q_0)u, u^n - u) \\
&\quad + (\mathcal{A}(q_0)u - \mathcal{A}(q_n)u, u_n - u) \\
&= (\mathcal{T}_n(q_n)u_n - \mathcal{T}(q_0)u, u_n - u^n) - (\mathcal{B}_n(q_n)u_n - \mathcal{B}(q_0)u, u_n - u^n) \\
&\quad + (\mathcal{A}(q_n)u_n - \mathcal{A}(q_0)u, u^n - u) + (\mathcal{A}(q_0)u - \mathcal{A}(q_n)u, u_n - u) \\
&= (f_n - f, u_n - u^n) - (\mathcal{B}_n(q_n)u_n - \mathcal{B}_n(q_0)u^n, u_n - u^n) \\
&\quad + (\mathcal{A}(q_n)u_n - \mathcal{A}(q_0)u, u^n - u) + (\mathcal{A}(q_0)u - \mathcal{A}(q_n)u, u_n - u) \\
&\le \|f_n - f\|_{\mathcal{V}^*} \|u_n - u^n\|_{\mathcal{V}} - \frac{1}{2}\int_0^t \frac{d}{ds}|u_n(s) - u^n(s)|^2 ds \\
&\quad + \|\mathcal{A}(q_n)u_n - \mathcal{A}(q_0)u\|_{\mathcal{V}^*} \|u^n - u\|_{\mathcal{V}} + \|\mathcal{A}(q_0)u - \mathcal{A}(q_n)u\|_{\mathcal{V}^*} \|u_n - u\|_{\mathcal{V}}.
\end{aligned}
$$

Condition (E), the uniform bound on $\|u_n\|_{\mathcal{V}}$, (3.5), and the final estimate above imply

$$
\begin{aligned}
\alpha\|u_n - u\|_{\mathcal{V}}^p &\le \{M + (\nu + 1)\|u\|_{\mathcal{V}}\} \|f_n - f\|_{\mathcal{V}^*} \\
&\quad + \frac{1}{2}|u_0(q_n) - u_0(q_0)|^2 - \frac{1}{2}|u_n(t;q_n) - P_n u(t;q_0)|^2 \\
&\quad + \tilde{\beta}(M^{p-1} + \|u\|_{\mathcal{V}}^{p-1} + 2)\|u^n - u\|_{\mathcal{V}} + \frac{1}{p'\varepsilon^{p'}} \|\mathcal{A}(q_0)u - \mathcal{A}(q_n)u\|_{\mathcal{V}^*}^{p'} + \frac{\varepsilon^p}{p}\|u_n - u\|_{\mathcal{V}}^p.
\end{aligned}
$$

With $\varepsilon > 0$ chosen sufficiently small, we obtain

$$
\begin{aligned}
(3.6)\quad &\frac{1}{2}|u_n(t\,;q_n) - P_n u(t;q_0)|^2 + (\alpha - \frac{\varepsilon^p}{p})\|u_n - u\|_{\mathcal{V}}^p \\
&\le \{M + (\nu + 1)\|u\|_{\mathcal{V}}\} \|f_n - f\|_{\mathcal{V}^*} + \frac{1}{2}|u_0(q_n) - u_0(q_0)|^2 \\
&\quad + \tilde{\beta}(M^{p-1} + \|u\|_{\mathcal{V}}^{p-1} + 2)\|u^n - u\|_{\mathcal{V}} + \frac{1}{p'\varepsilon^{p'}} \|\mathcal{A}(q_0)u - \mathcal{A}(q_n)u\|_{\mathcal{V}^*}^{p'}.
\end{aligned}
$$

Conditions (A), (C), and (E), and the continuous dependence assumption on $u_0(q)$ and $f(\cdot\,;q)$ imply that the right hand side of the estimate (3.6) tends to zero as $n \to \infty$. Consequently the left hand side tends to zero as well. Moreover, by replacing any t dependence on the right hand side with T we find $\lim_{n\to\infty} u_n(\cdot\,;q_n) = u(\cdot\,;q_0)$ in $L_p(0,T;V)$ and $\lim_{n\to\infty} \{u_n(\cdot\,;q_n) - u^n(\cdot\,;q_0)\} = 0$ in $C(0,T;H)$. However

$$|u_n(t;q_n) - u(t;q_0)| \le |u_n(t;q_n) - u^n(t;q_0)| + |P_n u(t;q_0) - u(t;q_0)|$$

for $t \in [0, T]$. Therefore, condition (E) and $u(\cdot\,; q_0) \in C(0,T;H)$ imply that $\lim_{n\to\infty} u_n(\cdot\,;q_n) = u(\cdot\,;q_0)$ in $C(0,T;H)$ and the theorem is proved.

Remark In practice, it is frequently the case that the parameter space $\mathcal{Q}$ and the admissible parameter set Q are infinite dimensional with elements consisting of spatially and/or temporally varying functions. If this is the case, to actually solve the approximating identification problem (ID_n), in addition to making the state discretization, the admissible parameter set Q must be discretized as well. Briefly, this can be carried out as follows. For each m = 1,2,... let $I^m : Q \subset \mathcal{Q} \to \mathcal{Q}$ be a continuous map with finite dimensional range and the property that $\lim_{m\to\infty} I^m(q) = q$, uniformly in q for $q \in Q$. We set $Q^m = I^m(Q)$ (note that for each m, Q^m is a compact subset of $\mathcal{Q}$) and consider the doubly indexed sequence of approximating identification problems (ID_n^m) where (ID_n^m) is the problem (ID_n) with Q replaced by Q^m. Each of the problems admits a solution $\bar{q}_n^m \in Q^m$ and it can be argued that there exists a subsequence $\{\bar{q}_{n_k}^{m_j}\} \subset \{\bar{q}_n^m\}$ for which $\lim_{j,k\to\infty} \bar{q}_{n_k}^{m_j} = \bar{q}$, with $\bar{q}$ a solution to problem (ID). (A more detailed discussion of the double discretization procedure outlined above can be found in [5].) Once bases for H_n and the range of I^m have been chosen, the problem (ID_n^m) becomes one involving the minimization of a continuous functional over a closed and bounded subset of Euclidean space subject to finite dimensional (ODE) constraints. The resulting optimization problem is typically solved using an iterative search procedure, a variety of which have been implemented as a part of any one of a number of standard readily available software packages (for example, IMSL, MINPACK, etc.) (see [3]).

4. An Example and Applications

We present an example of a class of temporally inhomogeneous nonlinear operators and corresponding nonautonomous nonlinear evolution systems to which the general theory developed above applies. We also briefly outline a sampling of application areas which give rise to inverse problems involving both linear and nonlinear nonautonomous distributed systems of the type

discussed in our example. We treat here the relevant theory only; implementation questions will be discussed and numerical findings are and will be reported on elsewhere (see e.g. [1], [8]).

Let $T > 0$ be fixed, let Ω be a bounded region in R^{ℓ} with smooth boundary $\partial\Omega$, and for $\alpha = (\alpha_1,\alpha_2,...,\alpha_{\ell})$ a multi-index of nonnegative integers let $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell}$. We denote the α^{th} order generalized, or distributional, derivative of a measurable function u defined on Ω by $D^{\alpha}u$; that is

$$D^{\alpha}u(x) = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_{\ell}}}{\partial x_{\ell}^{\alpha_{\ell}}} u(x), \qquad \text{a.e. } x \in \Omega$$

where $\alpha = (\alpha_1, \alpha_2,..., \alpha_{\ell})$. Let m be a fixed nonnegative integer, set $N = \sum_{j=0}^{m} \ell^j$, and for u measurable on Ω, let δu denote the N-vector valued function whose components are $D^{\alpha}u$ for all multi-indices α with $0 \le |\alpha| \le m$. Define the metric space $\mathcal{Q}$ by

$$\mathcal{Q} = \bigtimes_{0\le|\alpha|,|\beta|\le m} L_{\infty}((0,T) \times \Omega \times R^N)$$

and let Q be a compact subset of $\mathcal{Q}$ with the properties given in (1) and (2) below.

The vector valued function $q = \{q_{\alpha,\beta}\}$, $0 \le |\alpha|, |\beta| \le m$, is an element in Q if

(1) The real valued mapping $\zeta \to q_{\alpha,\beta}(t,x,\zeta)$ defined on R^N is continuous for almost every $(t,x) \in [0, T] \times \Omega$ and all multi-indices α and β with $0 \le |\alpha|, |\beta| \le m$.

(2) There exists a positive constant λ which does not depend on $q \in Q$ for which

$$\sum_{0\le|\alpha|,|\beta|\le m} (q_{\alpha,\beta}(t,x,\zeta)\,\zeta_{\beta} - q_{\alpha,\beta}(t,x,\eta)\eta_{\beta})(\zeta_{\alpha} - \eta_{\alpha}) \ge \lambda \sum_{0\le|\alpha|\le m} |\zeta_{\alpha} - \eta_{\alpha}|^2$$

for almost every $(t, x) \in [0,T] \times \Omega$ and all $\zeta, \eta \in R^N$.

We take $H = L_2(\Omega)$ and let V be any closed linear subspace of $H^m(\Omega)$ which contains $H_0^m(\Omega)$. Then $V^* \subset H^{-m}(\Omega)$, and for each $q = \{q_{\alpha,\beta}\} \in Q$ and almost every $t \in [0,T]$ define the operator $A(t;q) : V \to V^*$ by

$$(4.1) \qquad \langle A(t;q)u,v\rangle = \sum_{0\le|\alpha|,|\beta|\le m} \int_{\Omega} q_{\alpha,\beta}(t,x,\delta u(x))D^{\beta}u(x)D^{\alpha}v(x)dx$$

for u,v ε V. The notation $\langle\cdot,\cdot\rangle$ appearing in the definition (4.1) above will be used to denote both the usual inner product on $H = L_2(\Omega)$ and the duality pairing between V and V^*. The operator $A(t;q): V \to V^*$ is the distributional form of the formal differential operator

$$A(t;q)u(x) = \sum_{0\le|\alpha|,|\beta|\le m} (-1)^{|\alpha|} D^{\alpha} q_{\alpha,\beta}(t,x,\delta u(x)) D^{\beta} u(x)$$

for almost every $(t,x) \varepsilon [0,T] \times \Omega$, and is said to be of quasi-linear elliptic type. It is not difficult to show that (1), (2) above imply that for each $q \varepsilon Q$ the operator A(t;q) given by (4.1) is hemicontinuous from V into V^* and that conditions (A) - (D) are satisfied. We note that since V is separable, weak and strong measurability are equivalent (see [12]). Consequently condition (D) can be verified by showing that the real valued mapping $t \to \langle A(t;q)u,v\rangle$, $u, v \varepsilon V$, is measurable in the usual sense on [0, T]. When V is chosen as either $H_0^m(\Omega)$ or $H^m(\Omega)$, the abstract evolution equation (2.1) with $A(t;q)\ V \to V^*$ given by (4.1) corresponds to the quasi-linear parabolic partial differential equation

$$\frac{\partial u}{\partial t}(t,x) + \sum_{0\le|\alpha|,|\beta|\le m} (-1)^{|\alpha|} D^{\alpha} q_{\alpha,\beta}(t,x,\delta u(t,x)) D^{\beta} u(t,x) = f(t,x;q), \quad \text{a.e. } x \varepsilon \Omega,\ t > 0$$

together with the standard homogeneous Dirichlet or Neumann boundary conditions, respectively. We illustrate the applicability of the general theory which we have developed above with a brief description (along with appropriate references to more detailed treatments) of some inverse problems of the form of problem (ID) which have arisen in practice and have been documented elsewhere in the literature.

4.1 Size Structured Population Dynamics

We consider an application of the Fokker Planck theory to model size structured population dynamics (see [20]). With the spatial domain representing the size distribution of the species in question, and u(t,x) denoting its population density at time t and size x, the assumption that growth or aging is a Markov transition process and an argument based upon a Brownian motion paradigm (see [14]) leads to the equation

$$\frac{\partial u}{\partial t}(t,x) + \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} \frac{\partial^j}{\partial x^j} (M_j(t,x)u(t,x)) = 0$$

where for $j = 1,2,...$

$$M_j(t,x) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} (\xi - x)^j \varphi(t,x; t + \Delta t, \xi) d\xi.$$

The kernel $\varphi(t,x, t + \Delta t, \xi)$ is the probability density function for the transition from size x at time t to size ξ at time $t + \Delta t$, and the functions M_j can be interpreted as the moments of the time rate of increase in size or as the time rate of change of the moments of the growth process. If we make the usual assumption that $M_j(t,x) \approx 0$, $j \geq 3$, and set $q_1(t,x) = M_1(t,x)$ and $q_2(t,x) = \frac{1}{2} M_2(t,x)$, we obtain the Fokker Planck equation

$$(4.2) \quad \frac{\partial u}{\partial t}(t,x) - \frac{\partial^2}{\partial x^2}(q_2(t,x)u(t,x)) + \frac{\partial}{\partial x}(q_1(t,x)u(t,x)) + q_0(t,x)u(t,x) = 0, \quad x_0 < x < x_1, \quad t > 0$$

where we have included a mortality term, $q_0 u$, and taken x_0 and x_1 to be respectively the minimum and maximum sizes of individuals in the population.

Along with the linear partial differential equation (4.2), appropriate boundary conditions have to be specified. We shall impose a renewal type boundary condition at the minimum size,

$$(4.3) \quad \{q_1(t,x)u(t,x) - \frac{\partial}{\partial x}(q_2(t,x)u(t,x))\}_{x=x_0} = \int_{x_0}^{x_1} q_3(t,x)u(t,x)dx, \quad t > 0 \ ,$$

and a zero flux condition (i.e. individuals can not grow beyond the maximum size)

$$(4.4) \quad \{q_1(t,x)u(t,x) - \frac{\partial}{\partial x}(q_2(t,x)u(t,x))\}_{x=x_1} = 0, \quad t > 0$$

at the maximum size. (A more detailed discussion of the modeling considerations implicit in this choice of boundary conditions can be found in [2] and [6].) We assume that the size dependent population distribution at time t = 0 is given by $u_0(x)$, $x_0 \leq x \leq x_1$ and set

$$(4.5) \quad u(0,x) = u_0(x), \quad x_0 \leq x \leq x_1 \ .$$

In order to apply the model (4.2) - (4.5) effectively, one must be able to determine the moments q_1 and q_2, the mortality rate q_0 and the fecundity kernel q_3 with some degree of accuracy, and consistently with biological principles and experimental observation. In problems of practical interest to population ecologists, one can expect only limited success in determining these parameters directly from knowledge of the growth process. An alternative is to formulate an inverse problem wherein the parameters q_0, q_1, q_2 and q_3 would be determined from observations $z(t_j,x)$, $x_0 \leq x \leq x_1$ of the population density at times t_j .

Let $\mathcal{Q} = L_\infty([0,T] \times [x_0,x_1]) \times L_\infty([0,T] \times [x_0,x_1]) \times L_\infty(0,T; W^{1,\infty}(x_0,x_1)) \times L_\infty([0,T] \times [x_0,x_1])$ and let Q be a compact subset of $\mathcal{Q}$ with the property that $q = (q_0, q_1, q_2, q_3) \in Q$ if and only if $q_2(t,x) \geq \nu > 0$ for almost every $t \in [0, T]$ and every $x \in [x_0, x_1]$. Let $H = L_2(x_0,x_1)$ with $\langle \cdot,\cdot \rangle$ denoting the usual L_2 inner product on H and set $V = H^1(x_0,x_1)$. For each $q = (q_0, q_1, q_2, q_3) \in Q$ and almost every $t \in [0, T]$ define the operator $A(t;q) : V \to V^*$ by

$$(4.6) \quad \langle A(t; q)\varphi,\psi \rangle = - \langle q_1(t ; \cdot)\varphi - D(q_2(t; \cdot)\varphi), D\psi \rangle - \psi(x_0) \int_{x_0}^{x_1} q_3(t,x)\varphi(x)dx + \langle q_3(t ; \cdot)\varphi,\psi \rangle$$

for $\varphi, \psi \in V$. Relatively straight forward arguments can be used to show that conditions (A) - (D) are satisfied with $p = 2$ (see [1]). If we define the least squares performance index

$$J(q) = \Phi(u(\cdot ;q); z) = \sum_{i=1}^{m} \int_{x_0}^{x_1} (u(t_i,x;q) - z(t_i,x))^2 dx$$

where for each $q \in Q$ $u(\cdot ;q) \in C(0,T;H)$ is the unique solution to the initial value problem (2.1), (2.2) with A(t;q) given by (4.6) and $f = 0$, and $z = (z(t_1, \cdot),...,z(t_m, \cdot)) \in Z = \overset{m}{\underset{1}{\times}} L_2(x_0,x_1)$, then, with an appropriate choice of the Galerkin subspaces H_n (see [1] and below), our theory applies.

4.2 Biological Mixing in Sea Sediment Cores

The modeling of biological mixing in lake and sea sediment cores yields a second example of a temporally inhomogeneous linear distributed system with an associated inverse problem. Lake and deep sea sediment core samples play an important role in geophysical research by providing a record of the geological, oceanographic, climatic, and biological history of the earth. Unfortunately however, the stratigraphic records contained in these core samples are frequently corrupted by the mixing activities of benthic organisms near the sea-floor-sea interface. Through the use of tracers from dated events (e.g. radioactive fallout, volcanic eruptions, etc.) data can be obtained from which the identification of an appropriate model for the mixing process is possible. Note that with an appropriate model, the deconvolution of the corrupted signal becomes a possibility.

One approach to modeling the mixing process is based upon the assumption that mixing occurs only to a fixed depth ℓ from the seafloor/sea interface. Under this assumption, the mixed layer in the sediment is modeled as a one dimensional moving diffusion chamber. The linear partial differential equation

$$(4.7) \quad \frac{\partial u}{\partial t}(t,x) - \frac{\partial}{\partial x}\{q_1(x)\frac{\partial}{\partial x}u(t,x)\} + q_2(t)\frac{\partial}{\partial x}u(t,x) = 0, \quad 0 \leq x \leq \ell \ , \quad t > 0$$

results where u(t,x) denotes the concentration of tracer at time t and depth x in the mixed layer, q_1

is a depth dependent diffusion coefficient (it is hypothesized that mixing is most intense at the seafloor/sea interface and then decreases with depth), and q_2 is a temporally varying sedimentation rate. If we assume that the concentration of tracer in the mixed layer at time $t = 0$ is given by $u_0(x)$, $0 \le x \le \ell$, and that no other sources of tracer exist, we are led to the boundary and initial conditions given by

$$(4.8) \quad \left\{ - q_1(x) \frac{\partial u}{\partial x}(t,x) + q_2(t)u(t,x) \right\}_{x=0} = 0, \quad t > 0$$

$$(4.9) \quad \left\{ - q_1(x) \frac{\partial u}{\partial x}(t,x) \right\}_{x=\ell} = 0, \quad t > 0$$

$$(4.10) \quad u(0,x) = u_0(x), \quad 0 \le x \le \ell.$$

The boundary condition (4.8) states that there is no tracer flux through the top of the mixed layer while the boundary condition (4.9) implies that the tracer at depth ℓ comes to rest in the stationary sediment layers below the mixed layer. (That is, as more sediment is deposited, since mixing is assumed to occur only to a fixed depth, the diffusion chamber moves up, leaving the tracer at the bottom behind.)

The data for an associated inverse problem will come from analysis of the core sample. Our observations, $z(t)$, $t > 0$, therefore, are the concentration of tracer at a depth $x = \ell$ in the mixed layer over some fixed time period, $[0, T]$. For brevity, we assume that u_0 and ℓ are known, or can be determined either experimentally or from the data, and consider the inverse problem of identifying the functional parameters q_1 and q_2. (Other inverse problems associated with the model given by (4.7) - (4.10), and a more detailed discussion of the modeling process itself can be found in [8].)

Let $\mathcal{Q} = L_\infty(0,\ell) \times L_\infty(0,T)$ and let Q be a compact subset of $\mathcal{Q}$ with the property that $q = (q_1, q_2) \in Q$ if and only if $q_1(x) \ge \nu > 0$ for almost every $x \in [0,\ell]$. We take $H = L_2(0,\ell)$ endowed with the standard L_2 inner product, $\langle \cdot,\cdot \rangle$, set $V = H^1(0,\ell)$, and for $q = (q_1, q_2) \in Q$ and almost every $t \in [0,T]$ we define $A(t; q) : V \to V^*$ by

$$(4.11) \quad \langle A(t;q)\varphi,\psi \rangle = \langle q_1 D\varphi, D\psi \rangle - q_2(t)\langle \varphi, D\psi \rangle + q_2(t)\varphi(\ell)\psi(\ell),$$

for $\varphi,\psi \in V$.

Using arguments similar to those given in [8], it is not difficult to show that conditions (A) - (D) are satisfied with $p = 2$. The appropriate performance index is given by

$$(4.12) \quad J(q) = \Phi(u(\cdot\,;q);z) = \int_0^T (u(t,\ell;q) - z(t))^2 dt$$

where $u(\cdot,q) \in L_2(0, T;V)$ is the unique solution to the initial value problem (2.1), (2.2) with

A(t;q) given by (4.11) and f = 0, and $z \in Z \equiv L_2(0,T)$. Note that $\Phi(\cdot\,;z)$ given by (4.12) is continuous on $L_2(0, T;V)$. Once again, with an appropriate choice of the H_n, our theory applies.

4.3 Nonlinear Heat Conduction

A well known model for nonlinear heat conduction or mass transfer provides still another opportunity for the application of our framework. The example we present now can be considered as a special case of the general class of nonlinear systems which were discussed earlier in this section. However, for simplicity in notation we define directly the relevent operators and spaces in this particular example. Let Ω be a bounded region in R^{ℓ} with smooth boundary, let $\mathcal{Q} = L_\infty((0,T) \times \Omega \times R^{\ell})$ and let Q be a compact subset of $\mathcal{Q}$ with the property that $q \in Q$ if and only if

(1) The mapping $\zeta \to q(t,x,\zeta)$ is C^1 for almost every $(t,x) \in [0,T] \times \Omega$, and
(2) There exists a constant $\lambda > 0$ which does not depend upon $q \in Q$ for which

$$(4.13) \qquad \theta_i \nabla_\zeta q(t,x,\zeta)\,|_{\zeta=\theta} \cdot (\zeta - \eta) + q(t,x,\theta)(\zeta_i - \eta_i) \geq \lambda(\zeta_i - \eta_i), \quad i = 1,2,\ldots,\ell$$

for almost every $(t,x) \in [0, T] \times \Omega$ and all $\theta, \zeta, \eta \in R^{\ell}$. (Note that when $\ell = 1$, the function $q(t,x,\zeta) = q(\zeta) = 1 - .5e^{-\zeta^2}$ satisfies (4.13).) Then for $q \in Q$, consider the nonautonomous, nonlinear model for heat conduction or mass transfer given by (see [17], [18])

$$(4.14) \qquad \frac{\partial u}{\partial t}(t,x) - \nabla \cdot \{q(t,x,\nabla u(t,x))\nabla u(t,x)\} = 0, \quad x \in \Omega, \quad t > 0$$

together with appropriate boundary conditions.

We take $H = L_2(\Omega)$ and take V to be an appropriately chosen closed subspace of $H^1(\Omega)$ which contains $H_0^1(\Omega)$. The precise choice of V of course depends upon the boundary conditions which accompany (4.14). For example with Dirichlet boundary conditions, we take $V = H_0^1(\Omega)$ and when Neumann boundary conditions have been specified we take $V = H^1(\Omega)$. For each $q \in Q$ and almost every $t \in [0, T]$, we define the operator $A(t;q) : V \to V^*$ by

$$\langle A(t; q)\varphi, \psi \rangle = \int_\Omega q(t,x,\nabla\varphi(x))\nabla\varphi(x)\nabla\psi(x)dx$$

for $\varphi, \psi \in V$. Once again, the hemicontinuity of the operator A(t ;q) and that conditions (A) - (D) are satisfied are not difficult to argue.

Finally, we point out that it is usually not difficult to choose Galerkin subspaces H_n and corresponding orthogonal projections P_n which satisfy condition (E). It is frequently the case that choosing the H_n as the span of appropriately modified Hermite or polynomial spline functions

(calling upon tensor products in higher dimensions) will suffice. For example, in one dimension, we suppose that $H = L_2(0,1)$ and $V = H^1(0,1)$ endowed with the usual inner products and norms. For each $n = 1,2,..$ let $\{\varphi_j^n\}_{j=0}^n$ denote the usual linear spline (or "hat") functions (see [16]) defined with respect to the uniform mesh $\{0, 1/n, 2/n,...,1\}$ on the interval $[0,1]$. Set $H_n = \text{span}\ \{\varphi_j^n\}_{j=0}^n$ and let P_n denote the orthogonal projection of H onto the $n + 1$ dimensional subspace H_n with respect to the L_2 inner product. For $I_n : V \to H_n$ the interpolation operator defined by $(I_n\varphi)(j/n) = \varphi(j/n)$, $j = 0,1,2,...,n$, for $\varphi \in V$, one can obtain (see [19]) the estimates

$$|D^i(I_n\varphi - \varphi)| \le k_{ij} n^{i-j} |D^j\varphi|, \quad \varphi \in H^j(0,1),$$

$i = 0,...,j$, $j = 1,2$, where the k_{ij} are positive constants which do not depend on n or φ. It then follows that

$$|P_n\varphi - \varphi| \le |I_n\varphi - \varphi| \le k_1 n^{-1} \|\varphi\|,$$

for $\varphi \in V$, and by using the Schmidt inequality (see [16]) that

$$\begin{aligned} (4.15) \quad \|P_n\varphi - \varphi\|^2 &= |P_n\varphi - \varphi|^2 + |D(P_n\varphi - \varphi)|^2 \\ &\le k_{01}^2 n^{-2} \|\varphi\|^2 + \tilde{k} n^2 |P_n\varphi - I_n\varphi|^2 + 2|D(I_n\varphi - \varphi)|^2 \\ &\le \nu^2 \|\varphi\|^2 \end{aligned}$$

where ν is a positive constant independent of n and $\varphi \in V$. Similar arguments can be used to show that $\lim_{n\to\infty} \|P_n\varphi - \varphi\| = 0$, for $\varphi \in H^2(0, 1)$. This, together with density and (4.15), establishes condition (E). These ideas extend to higher order splines, other sets of boundary conditions (i.e. other choices for V), and higher dimensions via tensor products.

Schemes based on Galerkin spline approximations as outlined in the previous paragraph can be used in a number of important areas of application. In [1], [2], [8] the reader will find computational results involving linear nonautonomous systems for which the theory in this paper provides a sound theoretical foundation. In addition, we are currently carrying out computations for several nonlinear examples including heat conduction models such as the one outlined in section 4.3. These findings will be reported in a subsequent paper on numerical and implementation aspects of the ideas.

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16. Abstract

We develop an abstract framework and convergence theory for Galerkin approximation for inverse problems involving the identification of nonautonomous nonlinear distributed parameter systems. We provide a set of relatively easily verified conditions which are sufficient to guarantee the existence of optimal solutions and their approximation by a sequence of solutions to a sequence of approximating finite dimensional identification problems. Our approach is based upon the theory of monotone operators in Banach spaces and is applicable to a reasonably broad class of nonlinear distributed systems. Operator theoretic and variational techniques are used to establish a fundamental convergence result. An example involving evolution systems with dynamics described by nonstationary quasi-linear elliptic operators along with some applications are presented and discussed.

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