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OPTIMAL DISCRETE-TIME LQR PROBLEMS FOR PARABOLIC SYSTEMS
WITH UNBOUNDED INPUT - APPROXIMATION AND CONVERGENCE

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OPTIMAL DISCRETE-TIME LQR
PROBLEMS FOR PARABOLIC SYSTEMS
WITH UNBOUNDED INPUT -
APPROXIMATION AND CONVERGENCE[†]

by

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Abstract

An abstract approximation and convergence theory for the closed-loop solution of discrete-time linear-quadratic regulator problems for parabolic systems with unbounded input is developed. Under relatively mild stabilizability and detectability assumptions, functional analytic, operator theoretic techniques are used to demonstrate the norm convergence of Galerkin-based approximations to the optimal feedback control gains. The application of the general theory to a class of abstract boundary control systems is considered. Two examples, one involving the Neumann boundary control of a one dimensional heat equation, and the other, the vibration control of a cantilevered viscoelastic beam via shear input at the free end, are discussed.

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1. Introduction

In this paper we develop an abstract approximation framework for linear quadratic regulator (LQR) problems for infinite dimensional discrete-time parabolic systems with unbounded input. More specifically, we consider the application of the abstract approximation theory developed in [9] to the class of systems which are open-loop abstract parabolic and whose input operators have range in some space larger than the standard state space in which the problem is usually formulated. The theory we present here is a discrete-time analog of the results given in [3] for continuous-time parabolic systems. However, in contrast to the treatment in [3] which is restricted to the case of bounded input, we on the other hand are able to handle a relatively wide class of systems involving unbounded input. Our framework is applicable for example, to a variety of boundary control systems.

The abstract framework we develop here leads to a set of three relatively easily verified conditions (stabilizability, detectability, and strong convergence of the orthogonal projections corresponding to the approximating Galerkin subspaces) which when satisfied yield norm convergence of finite dimensional approximations to the optimal feedback gains. We employ a functional analytic, operator theoretic approach to obtain a relatively complete and reasonably general theory which is easily applied in practice to a wide class of important problems.

An outline of the remainder of the paper is as follows. In section 2 we briefly outline the abstract theory and approximation results for infinite dimensional discrete-time LQR problems developed in [9]. Section 3 is concerned with abstract parabolic control systems with unbounded input, their operator theoretic and discrete-time formulation, and the associated infinite dimensional optimal control problem. The approximation and convergence theories are discussed in section 4. In section 5 we consider abstract boundary control systems and present two examples; one involves the Neumann boundary control of a one dimensional heat equation, and the second is concerned with the vibration control of a cantilevered viscoelastic beam via a shear input at the free end. In section 6 we summarize our findings and make some concluding remarks. In an appendix we prove a discrete-time version of a continuous-time result due to Datko which is required in section 4.

2. The Discrete-Time LQR Problem-Feedback Solution and Approximation Theory

In this section we briefly outline and summarize the infinite dimensional discrete-time linear-quadratic theory developed in [17] and the approximation results from [9]. Let X be a Hilbert space with inner product $(\cdot, \cdot)_X$ and corresponding induced norm $\|\cdot\|_X$. We consider the optimal control problem given by

(P) Find $\bar{u} = \{\bar{u}_k\}_{k=0}^{\infty} \in \mathcal{L}_2(0, \infty; \mathbb{R}^m)$ which minimizes the quadratic performance index

$$(2.1) \quad J(u; x_0) = \sum_{k=0}^{\infty} (Qx_k, x_k)_X + u_k^T R u_k$$

subject to the linear discrete-time control system

$$(2.2) \quad x_{k+1} = T x_k + B u_k, \quad k = 0, 1, 2, \dots$$

$$(2.3) \quad x_0 \in X$$

where $T, Q \in \mathfrak{L}(X)$ with Q nonnegative self-adjoint, $B \in \mathfrak{L}(R^m, X)$, and R is an $m \times m$ positive definite symmetric matrix.

An input sequence $u = \{u_k\}_{k=0}^{\infty} \in \ell_2(0, \infty; R^m)$ is called admissible for the initial data x_0 if

$J(u; x_0) < \infty$. An operator $\Pi \in \mathfrak{L}(X)$ is called a solution to the algebraic Riccati equation corresponding to the plant defined by T, B, Q , and R if it satisfies

$$(2.4) \quad \Pi = T^*(\Pi - \Pi B(R + B^* \Pi B)^{-1} B^* \Pi) T + Q.$$

We have the following theorem concerning the solution of problem (\mathcal{P}) .

Theorem 2.1 There exists a nonnegative self-adjoint solution to the algebraic Riccati equation (2.4) if and only if there exists an admissible control for each $x_0 \in X$. If there exists an admissible control for each $x_0 \in X$, then the unique solution to problem (\mathcal{P}) is given in linear state feedback form by

$$\bar{u}_k = -\bar{F} \bar{x}_k, \quad k = 0, 1, 2, \dots$$

with the corresponding optimal trajectory $\bar{x} = \{\bar{x}_k\}_{k=0}^{\infty}$ given by

$$\bar{x}_{k+1} = \bar{S} \bar{x}_k, \quad k = 0, 1, 2, \dots$$

$$\bar{x}_0 = x_0$$

where $\bar{F} \in \mathfrak{L}(X, R^m)$ and $\bar{S} \in \mathfrak{L}(X)$ are given by

$$(2.5) \quad \bar{F} = (R + B^* \bar{\Pi} B)^{-1} B^* \bar{\Pi} T,$$

and

$$\bar{S} = T - B \bar{F},$$

and $\bar{\Pi} \in \mathfrak{L}(X)$ is the minimal nonnegative self-adjoint solution to (2.4). We have $\min J(u; x_0) = J(\bar{u}; x_0) = (\bar{\Pi} x_0, x_0)_X$. If, in addition, any admissible control drives the state x_k to zero

asymptotically as $k \rightarrow \infty$ (i.e. $\lim_{k \rightarrow \infty} \|x_k\|_X = 0$; this would be true for example, if $Q > 0$)

then the Riccati equation (2.4) admits a unique nonnegative self-adjoint solution $\bar{\Pi}$. If there exists an admissible control for each $x_0 \in X$, and $Q > 0$, then the spectral radius of \bar{S} is less than one and \bar{S} is uniformly exponentially stable. In particular if $Q \geq \delta > 0$ then $\|\bar{S}^k\| \leq M r^k$,

$k = 0, 1, 2, \dots$ with $M = \|\bar{\Pi}\| / \delta$ and $r = 1 - \delta / \|\bar{\Pi}\| < 1$.

In our discussion of the approximation theory below, we shall assume that the following hypothesis is satisfied.

(H1) There exists an admissible control for each $x_0 \in X$, and any admissible control drives the state to zero asymptotically.

We note that since \bar{F} given by (2.5) is an element in $\mathfrak{L}(X, R^m)$ we have

$$\bar{u}_k = -\bar{F}\bar{x}_k = -(\bar{f}, \bar{x}_k)_X, \quad k = 0, 1, 2, \dots$$

where $\bar{F} = (\bar{f}_1, \bar{f}_2, \dots, \bar{f}_m)^T$ with $\bar{f}_j \in X$, $j = 1, 2, \dots, m$, is referred to as the optimal functional feedback control gain.

The approximation theory in [9] is based upon the finite dimensional approximation of the space X and the operators T, B , and Q . For each $N = 1, 2, \dots$ let X^N be a finite dimensional subspace of X and let $P^N : X \rightarrow X^N$ be the corresponding orthogonal projection of X onto X^N . We take $T^N, Q^N \in \mathfrak{L}(X^N)$, $B^N \in \mathfrak{L}(R^m, X)$ and consider the following sequence of finite dimensional linear-quadratic regulator problems.

(P^N) Find $\bar{u}^N = \{\bar{u}_k^N\}_{k=0}^\infty \in \ell_2(0, \infty; R^m)$ which minimizes the quadratic performance index

$$J^N(u; x_0^N) = \sum_{k=0}^{\infty} (Q^N \bar{x}_k^N, \bar{x}_k^N)_X + u_k^T R u_k$$

subject to the linear discrete-time control system

$$(2.6) \quad \bar{x}_{k+1}^N = T^N \bar{x}_k^N + B^N u_k, \quad k = 0, 1, 2, \dots$$

$$(2.7) \quad \bar{x}_0^N = P^N x_0 \in X^N.$$

We require that the following hypothesis holds.

(H2) For each $N = 1, 2, \dots$, there exists an admissible control for every $x_0^N \in X^N$ and any admissible control drives the state \bar{x}_k^N to zero, asymptotically as $k \rightarrow \infty$.

It then follows from Theorem 2.1 that for each $N = 1, 2, \dots$ there exists a unique, nonnegative self-adjoint solution $\bar{\Pi}^N \in \mathfrak{L}(X^N)$ to the algebraic Riccati equation

$$\bar{\Pi}^N = (T^N)^* (\bar{\Pi}^N - \bar{\Pi}^N B^N (R + (B^N)^* \bar{\Pi}^N B^N)^{-1} (B^N)^* \bar{\Pi}^N) T^N + Q^N.$$

The unique solution to problem (\mathcal{P}^N) is given by

$$\bar{u}_k^N = -\bar{F}^N \bar{x}_k^N, \quad k = 0, 1, 2, \dots$$

with

$$\bar{x}_{k+1}^N = \bar{S}^N \bar{x}_k^N, \quad k = 0, 1, 2, \dots, \quad \bar{x}_0^N = P^N x_0,$$

where

$$\bar{F}^N = (R + (B^N)^* \bar{\Pi}^N B^N)^{-1} (B^N)^* \bar{\Pi}^N T^N$$

and $\bar{S}^N = T^N - B^N \bar{F}^N$. We have $\min J^N(u; x_0^N) = J^N(\bar{u}^N; x_0^N) = (\bar{\Pi}^N x_0^N, x_0^N)_X$, \bar{S}^N has spectral radius less than one and is uniformly exponentially stable. Since $\bar{F}^N \in \mathcal{L}(R^m, X^N)$ it follows that

$$\bar{u}_k^N = -(\bar{f}^N, \bar{x}_k^N)_X, \quad k = 0, 1, 2, \dots$$

where $\bar{f}^N = (\bar{f}_1^N, \dots, \bar{f}_m^N)^T$ with $\bar{f}_j^N \in X^N$, $j = 1, 2, \dots, m$.

The convergence theorem requires that the following hypothesis be satisfied.

(H3) For each $\phi \in X$, $P^N \phi \rightarrow \phi$, $T^N P^N \phi \rightarrow T\phi$, $(T^N)^* P^N \phi \rightarrow T^* \phi$, and $Q^N P^N \phi \rightarrow Q\phi$ as $N \rightarrow \infty$, and $B^N \rightarrow B$ in $\mathcal{L}(R^m, X)$ as $N \rightarrow \infty$.

Theorem 2.2 Assume that hypotheses (H1) - (H3) hold. Suppose further that there exist positive constants M_1 , M_2 , and r_2 , independent of N with $r_2 < 1$, for which

$$(2.8) \quad |\bar{\Pi}^N| \leq M_1$$

and

$$(2.9) \quad |(\bar{S}^N)^k| \leq M_2 r_2^k, \quad k = 0, 1, 2, \dots$$

Then for each $\phi \in X$, $\bar{\Pi}^N P^N \phi \rightarrow \bar{\Pi} \phi$ and $\bar{S}^N P^N \phi \rightarrow \bar{S} \phi$, as $N \rightarrow \infty$, $\bar{F}^N P^N \phi \rightarrow \bar{F}$ in $\mathcal{L}(X, R^m)$, $\bar{f}_j^N \rightarrow \bar{f}_j$ in X , $j = 1, 2, \dots, m$, and $\bar{u}_k^N \rightarrow \bar{u}_k$ in R^m and $\bar{x}_k^N \rightarrow \bar{x}_k$ in X , for each $k = 0, 1, 2, \dots$, as $N \rightarrow \infty$.

3. Abstract Parabolic Systems with Unbounded Input

Let H be a Hilbert space with inner product (\cdot, \cdot) and corresponding induced norm $|\cdot|$. Let V be another Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|\cdot\|$ and assume that V is densely and continuously embedded in H with $|\phi| \leq \mu \|\phi\|$ for $\phi \in V$. If we identify H with its dual, H^* , it then follows that $V \subset H = H^* \subset V^*$ with H densely and continuously embedded in

V^* , the dual of V , endowed with the usual operator norm $\|\cdot\|_*$. We have $\|\varphi\|_* \leq \mu|\varphi|$ for $\varphi \in H$ and $\|\varphi\|_* \leq \mu^2\|\varphi\|$ for $\varphi \in V$. We assume further that the embedding $V \subset H$ is compact.

Let $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{C}$ be a bounded, coercive, sesquilinear form on V . That is, there exist real constants $\alpha, \gamma > 0$ and β for which

$$\begin{aligned} \operatorname{Re} a(\varphi, \varphi) &\geq \alpha\|\varphi\|^2 - \beta|\varphi|^2, \quad \varphi \in V \\ |a(\varphi, \psi)| &\leq \gamma\|\varphi\|\|\psi\|, \quad \varphi, \psi \in V. \end{aligned}$$

The form $a(\cdot, \cdot)$ defines an operator $\mathcal{A} \in \mathfrak{L}(V, V^*)$ via

$$(\mathcal{A}\varphi)(\psi) = (\mathcal{A}\varphi, \psi) = -a(\varphi, \psi), \quad \varphi, \psi \in V$$

where (\cdot, \cdot) in the above definition denotes the natural extension of the H inner product to the duality pairing between V and V^* . If we define $\operatorname{Dom}(\mathcal{A}) = \{\varphi \in V : \mathcal{A}\varphi \in H\}$, then $\operatorname{Dom}(\mathcal{A}) = H$ and $\mathcal{A} : \operatorname{Dom}(\mathcal{A}) \subset H \rightarrow H$ is the infinitesimal generator of an analytic semigroup $\{\mathcal{T}(t) : t \geq 0\}$ of bounded linear operators on H with $|\mathcal{T}(t)| \leq e^{(\beta-\alpha/\mu)t}$, $t \geq 0$ (see [16]). In addition, it is shown in [16] that $\{\mathcal{T}(t) : t \geq 0\}$ can be extended to an analytic semigroup on V^* , and in [2] that it can be restricted to an analytic semigroup on V .

It is not difficult to argue (see [16]) that the H adjoint of $\mathcal{A}, \mathcal{A}^* : \operatorname{Dom}(\mathcal{A}^*) \subset H \rightarrow H$, is given by $\mathcal{A}^*\varphi = \psi$, where ψ is that element in H for which $-a(\theta, \varphi) = (\psi, \theta)$, for all $\theta \in V$, and $\operatorname{Dom}(\mathcal{A}^*)$ consists of all those elements $\varphi \in V$ for which such a $\psi \in H$ exists. The operator \mathcal{A}^* extends to an operator in $\mathfrak{L}(V, V^*)$ via

$$(\mathcal{A}^*\varphi)(\psi) = (\mathcal{A}^*\varphi, \psi) = -a(\psi, \varphi) \equiv -a^*(\varphi, \psi),$$

for $\varphi, \psi \in V$. It immediately follows that the sesquilinear form $a^*(\cdot, \cdot) : V \times V \rightarrow \mathbb{C}$ defined above satisfies

$$\begin{aligned} \operatorname{Re} a^*(\varphi, \varphi) &\geq \alpha\|\varphi\|^2 - \beta|\varphi|^2, \quad \varphi \in V \\ |a^*(\varphi, \psi)| &\leq \gamma\|\varphi\|\|\psi\|, \quad \varphi, \psi \in V, \end{aligned}$$

and consequently that \mathcal{A}^* is the infinitesimal generator of an analytic semigroup $\{\mathcal{T}^*(t) : t \geq 0\}$ on V, H , and V^* with $|\mathcal{T}^*(t)| \leq e^{(\beta-\alpha/\mu)t}$, $t \geq 0$. It also follows that $\mathcal{T}^*(t) = \mathcal{T}(t)^*$, the H adjoint of $\mathcal{T}(t)$, for all $t \geq 0$.

We consider the continuous time control system

$$(3.1) \quad \dot{x}(t) = \mathcal{A}x(t) + \mathfrak{B}u(t), \quad t > 0$$

$$(3.2) \quad x(0) = x_0$$

where $\mathcal{A} \in \mathfrak{L}(V, V^*)$ is as it was defined above, $\mathfrak{B} \in \mathfrak{L}(R^m, V^*)$, $u \in L_2(0, \infty; R^m)$, and

$x_0 \in H$. We note that the fact that \mathfrak{B} is assumed to have range in V^* rather than H indicates that our framework will be able to handle certain classes of unbounded input - for example, certain types of boundary control. This will become clearer when we discuss examples below. We shall be concerned with the so called mild solution to the initial value problem (3.1), (3.2). The mild solution to the system (3.1), (3.2) is the function $x \in L_2(0, t_f; V) \cap C(0, t_f; H) \cap H^1(0, t_f; V^*)$ for any $t_f > 0$ given by

$$(3.3) \quad x(t) = \mathcal{T}(t-s)x(s) + \int_s^t \mathcal{T}(t-\sigma)\mathfrak{B}u(\sigma)d\sigma, \quad 0 \leq s \leq t \leq T,$$

$$(3.4) \quad x(0) = x_0,$$

where the integral in (3.3) is interpreted as an integral in V^* .

To derive the discrete-time system of the form (2.2), (2.3) corresponding to the system (3.1), (3.2), we let $\tau > 0$ denote the length of the sampling interval, and consider piecewise constant (zero-order hold) controls of the form

$$u(t) = u_k, \quad t \in [k\tau, (k+1)\tau), \quad k = 0, 1, 2, \dots$$

where for each k , u_k is a constant vector in R^m . Defining $x_k = x(k\tau)$, $k = 0, 1, 2, \dots$, from (3.3), (3.4) we obtain

$$\begin{aligned} x_{k+1} &= Tx_k + Bu_k, \quad k = 0, 1, 2, \dots \\ x_0 &\in H \end{aligned}$$

where $T \in \mathfrak{L}(H)$ and $B \in \mathfrak{L}(R^m, H)$ are given by $T = \mathcal{T}(\tau)$ and $B = \int_0^\tau \mathcal{T}(t)\mathfrak{B}dt$, respectively.

In setting up the LQR problem, in the performance index (2.1) we assumed that $Q \in \mathfrak{L}(H)$ is nonnegative self-adjoint and that R is an $m \times m$ positive definite symmetric matrix. We make the following standing assumptions.

- (A) The pair $\{T, B\}$ is uniformly exponentially stabilizable. That is, there exists an operator $F \in \mathfrak{L}(H, R^m)$ for which the operator $S = T - BF$ is uniformly exponentially stable; i.e. there exist positive constants M and r with $r < 1$ for which $|S^k| \leq Mr^k$, $k = 0, 1, 2, \dots$
- (B) There exists a $\delta > 0$ for which $Q \geq \delta$ (i.e. $(Q\phi, \phi) \geq \delta|\phi|^2$, $\phi \in H$).

Lemma 3.1 If assumptions (A) and (B) hold then hypothesis (H1) is satisfied.

Proof Let $x_0 \in H$ be given and set $u_k = -Fx_k$, $k = 0, 1, 2, \dots$ where F is the operator in $\mathfrak{L}(H, R^m)$ guaranteed to exist by assumption (A). Then $x_k = S^k x_0$, $u_k = -FS^k x_0$, $k = 0, 1, 2, \dots$ where $S = T - BF \in \mathfrak{L}(H)$, and

$$J(u; x_0) = \sum_{k=0}^{\infty} (Qx_k, x_k) + u_k^T R u_k = \sum_{k=0}^{\infty} (QS^k x_0, S^k x_0) + (FS^k x_0)^T R FS^k x_0$$

$$\leq \sum_{k=0}^{\infty} |Q| |S^k x_0|^2 + |R| |F|^2 |S^k x_0|^2 \leq \{|Q| + |R| |F|^2\} M^2 \sum_{k=0}^{\infty} r^{2k} = \{|Q| + |R| |F|^2\} M^2 / 1 - r^2 < \infty.$$

Also, $Q \geq \delta > 0$ implies that any admissible control must drive the state x_k to zero asymptotically as $k \rightarrow \infty$, and the lemma is proved.

4. Galerkin Approximation and Convergence

For each $N = 1, 2, \dots$ let H^N be a finite dimensional subspace of H with $H^N \subset V$ for all N . Let $P^N : H \rightarrow H^N$ denote the corresponding orthogonal projection of H onto H^N . We make the following assumption concerning the approximation properties of the subspaces H^N .

(C) For each $\phi \in V$, $\lim_{N \rightarrow \infty} \|P^N \phi - \phi\| = 0$.

Note that assumption (C) implies that $P^N \phi \rightarrow \phi$ in H as $N \rightarrow \infty$ for each $\phi \in H$ and that

$\|P^N \phi - \phi\| \leq v \|\phi\|$ for all $\phi \in V$ and some $v > 0$ which does not depend upon ϕ or N .

We use a standard Galerkin approach to define the operators $\mathcal{A}^N \in \mathfrak{L}(H^N)$. For $\phi^N \in H^N$, Let $\mathcal{A}^N \phi^N = \psi^N$ where ψ^N is the unique element in H^N guaranteed to exist by the Riesz Representation Theorem which satisfies $-a(\phi^N, \theta^N) = (\psi^N, \theta^N)$ for all $\theta^N \in H^N$. If we set $\mathcal{T}^N(t) = \exp(\mathcal{A}^N t)$, $t \geq 0$, then $\{\mathcal{T}^N(t) : t \geq 0\}$ is a semigroup of bounded linear operators on H^N with $|\mathcal{T}^N(t)| \leq e^{(\beta - \frac{\alpha}{\mu})t}$, $t \geq 0$, $N = 1, 2, \dots$. The adjoint of \mathcal{A}^N , $(\mathcal{A}^N)^* \in \mathfrak{L}(H^N)$, is given by $(\mathcal{A}^N)^* \phi^N = \psi^N$ where $\psi^N \in H^N$ satisfies $-a^*(\phi^N, \theta^N) = (\psi^N, \theta^N)$, $\theta^N \in H^N$. It follows that $\mathcal{T}^N(t)^* = \exp((\mathcal{A}^N)^* t)$, $t \geq 0$.

Lemma 4.1 If assumption (C) holds, then for each $\phi \in H$ we have $\mathcal{T}^N(t) P^N \phi \rightarrow \mathcal{T}(t) \phi$, and $\mathcal{T}^N(t)^* P^N \phi \rightarrow \mathcal{T}(t)^* \phi$, as $N \rightarrow \infty$, uniformly in t for t in bounded subintervals of $[0, \infty)$.

Proof The result will follow from the Trotter-Kato semigroup approximation theorem (see [12]) once we have shown that $(\mathcal{A}^N - \lambda)^{-1} P^N \phi \rightarrow (\mathcal{A} - \lambda)^{-1} \phi$ and $((\mathcal{A}^N)^* - \lambda)^{-1} P^N \phi \rightarrow (\mathcal{A}^* - \lambda)^{-1} \phi$ as $N \rightarrow \infty$ for each $\phi \in H$ and some $\lambda \geq \beta$. We argue that the first convergence holds only; the proof that the second holds as well is completely analogous.

Let $\phi \in H$ and set $\psi^N = (\mathcal{A}^N - \lambda)^{-1} P^N \phi$ and $\psi = (\mathcal{A} - \lambda)^{-1} \phi$ for some fixed $\lambda \geq \beta$. (Note that $\lambda \geq \beta$ implies $\lambda \in \rho(\mathcal{A}) \cap \rho(\mathcal{A}^N)$ for all $N = 1, 2, \dots$). Then

$$\begin{aligned}
\alpha \|\psi^N - P^N \psi\|^2 &\leq \operatorname{Re} a(\psi^N - P^N \psi, \psi^N - P^N \psi) + |\beta| \|\psi^N - P^N \psi\|^2 \\
&= -\operatorname{Re}((\mathcal{A}^N - \lambda)(\psi^N - P^N \psi), \psi^N - P^N \psi) + (\beta - \lambda) \|\psi^N - P^N \psi\|^2 \\
&\leq -\operatorname{Re}(P^N \phi, \psi^N - P^N \psi) + \operatorname{Re}((\mathcal{A} - \lambda)(P^N \psi - \psi), \psi^N - P^N \psi) + \operatorname{Re}((\mathcal{A} - \lambda)\psi, \psi^N - P^N \psi) \\
&= \operatorname{Re}(\phi - P^N \phi, \psi^N - P^N \psi) - \operatorname{Re} a(P^N \psi - \psi, \psi^N - P^N \psi) - \lambda \operatorname{Re}(P^N \psi - \psi, \psi^N - P^N \psi) \\
&\leq |\phi - P^N \phi| \|\psi^N - P^N \psi\| + \gamma \|P^N \psi - \psi\| \|\psi^N - P^N \psi\| + \lambda \|P^N \psi - \psi\| \|\psi^N - P^N \psi\| \\
&\leq \frac{1}{4\varepsilon} \|\phi - P^N \phi\|^2 + \varepsilon \|\psi^N - P^N \psi\|^2 + \frac{\gamma}{4\varepsilon} \|P^N \psi - \psi\|^2 \\
&\quad + \gamma \varepsilon \|\psi^N - P^N \psi\|^2 + \frac{\lambda^2}{4\varepsilon} \|P^N \psi - \psi\|^2 + \varepsilon \|\psi^N - P^N \psi\|^2
\end{aligned}$$

for any $\varepsilon > 0$. Recalling that $|\theta| \leq \mu \|\theta\|$ for $\theta \in V$, and choosing $\varepsilon > 0$ so that $\alpha - \varepsilon(2\mu^2 + \gamma) = \alpha/4$, we obtain the estimate

$$\|\psi^N - P^N \psi\|^2 \leq (\alpha\varepsilon)^{-1} \{ \|\phi - P^N \phi\|^2 + \gamma \|P^N \psi - \psi\|^2 + \lambda \|P^N \psi - \psi\|^2 \},$$

the right hand side of which tends to zero as $N \rightarrow \infty$ by assumption (C). It follows that

$$\|\psi^N - \psi\| \leq \|\psi^N - P^N \psi\| + \|P^N \psi - \psi\| \leq \mu \|\psi^N - P^N \psi\| + \|P^N \psi - \psi\|$$

which tends to zero as $N \rightarrow \infty$ by the previous estimate and assumption (C), and the lemma is proved.

Setting $T^N = \mathfrak{T}^N(\tau) \in \mathfrak{L}(H^N)$, Lemma 4.1 implies $T^N P^N \phi \rightarrow T\phi$ and $(T^N)^* P^N \phi \rightarrow T^* \phi$ as $N \rightarrow \infty$ for each $\phi \in H$.

Define $\mathfrak{B}^N \in \mathfrak{L}(R^m, H^N)$ by $\mathfrak{B}^N v = \psi^N$, where for $v \in R^m$, ψ^N is that element in H^N which satisfies (again by the Riesz Representation Theorem) $(\mathfrak{B}^N v)(\theta^N) = (\psi^N, \theta^N)$, for all $\theta^N \in H^N$. Define $B^N \in \mathfrak{L}(R^m, H^N)$ by $B^N = \int_0^\tau \mathfrak{T}^N(t) \mathfrak{B}^N dt$.

Lemma 4.2 If assumption (C) holds, then $B^N \rightarrow B$ in $\mathfrak{L}(R^m, H)$ as $N \rightarrow \infty$.

Proof We assume $\beta > 0$. If $\beta \leq 0$, then β can be taken equal to zero in the argument that

follows. For $t \geq 0$, $v \in R^m$, and $N = 1, 2, \dots$ set $z(t) = \int_0^t \mathcal{T}(s) \mathfrak{B} v ds$ and $z^N(t) = \int_0^t \mathcal{T}^N(s) \mathfrak{B}^N v ds$.

Then $z, z^N \in L_2(0, \tau; V) \cap C(0, \tau; H) \cap H^1(0, \tau; V^*)$,

$$\dot{z}(t) = \mathcal{A}z(t) + \mathfrak{B}v, \quad t > 0, \quad z(0) = 0$$

$$\dot{z}^N(t) = \mathcal{A}^N z^N(t) + \mathfrak{B}^N v, \quad t > 0, \quad z^N(0) = 0,$$

and $Bv = z(\tau)$ and $B^N v = z^N(\tau)$. It follows that

$$\begin{aligned} & \alpha \int_0^\tau e^{-4\beta t} \|z^N(t) - z(t)\|^2 dt \\ & \leq \int_0^\tau e^{-4\beta t} \operatorname{Re} a(z^N(t) - z(t), z^N(t) - z(t)) + e^{-4\beta t} \beta \|z^N(t) - z(t)\|^2 dt \\ & = \int_0^\tau e^{-4\beta t} \operatorname{Re}(\mathcal{A}z^N(t) - \mathcal{A}z(t), z^N(t) - P^N z(t)) + \beta e^{-4\beta t} \|z^N(t) - z(t)\|^2 dt \\ & \quad - \int_0^\tau e^{-4\beta t} \operatorname{Re}(\mathcal{A}z^N(t) - \mathcal{A}z(t), P^N z(t) - z(t)) dt \\ & = \int_0^\tau e^{-4\beta t} \operatorname{Re}(\dot{z}^N(t) - \dot{z}(t), z^N(t) - P^N z(t)) + \beta e^{-4\beta t} \|z^N(t) - z(t)\|^2 dt \\ & \quad + \int_0^\tau e^{-4\beta t} \operatorname{Re}(\mathcal{A}z(t) - \mathcal{A}z^N(t), P^N z(t) - z(t)) dt \end{aligned}$$

$$\begin{aligned}
&= \int_0^\tau -e^{-4\beta t} \operatorname{Re}(\dot{z}^N(t) - \frac{d}{dt} \{P^N z(t)\}, z^N(t) - P^N z(t)) + \beta e^{-4\beta t} |z^N(t) - z(t)|^2 dt \\
&\quad + \int_0^\tau e^{-4\beta t} \operatorname{Re}(\mathcal{A}z(t) - \mathcal{A}z^N(t), P^N z(t) - z(t)) dt
\end{aligned}$$

where in the last equality above we have used the fact that the definition of generalized derivative implies that

$$\int_0^\tau \left(\frac{d}{dt} \{P^N z(t)\} - \dot{z}(t), \theta^N(t) \right) dt = 0$$

for all $\theta^N \in L_2(0, \tau; V)$. Continuing, we find that

$$\begin{aligned}
&\alpha \int_0^\tau e^{-4\beta t} \|z^N(t) - z(t)\|^2 dt \\
&\leq \int_0^\tau -e^{-4\beta t} \operatorname{Re}(\dot{z}^N(t) - \frac{d}{dt} \{P^N z(t)\}, z^N(t) - P^N z(t)) + 2\beta e^{-4\beta t} |z^N(t) - P^N z(t)|^2 dt \\
&\quad + 2\beta \int_0^\tau e^{-4\beta t} |P^N z(t) - z(t)|^2 dt + \int_0^\tau e^{-4\beta t} \operatorname{Re}(\mathcal{A}z(t) - \mathcal{A}z^N(t), P^N z(t) - z(t)) dt \\
&= \int_0^\tau -\frac{1}{2} \frac{d}{dt} e^{-4\beta t} |z^N(t) - P^N z(t)|^2 dt + 2\beta \int_0^\tau e^{-4\beta t} |P^N z(t) - z(t)|^2 dt \\
&\quad + \int_0^\tau e^{-4\beta t} \operatorname{Re}(\mathcal{A}z(t) - \mathcal{A}z^N(t), P^N z(t) - z(t)) dt \\
&\leq -\frac{1}{2} e^{-4\beta \tau} |z^N(\tau) - P^N z(\tau)|^2 + 2\beta \int_0^\tau |P^N z(t) - z(t)|^2 dt \\
&\quad + \alpha \int_0^\tau \|z(t) - z^N(t)\|^2 dt + \frac{\gamma^2}{4\alpha} \int_0^\tau \|P^N z(t) - z(t)\|^2 dt,
\end{aligned}$$

and hence that

$$|z^N(\tau) - P^N z(\tau)|^2 \leq 4\beta e^{4\beta\tau} \int_0^\tau |P^N z(t) - z(t)|^2 dt + \frac{\gamma^2 e^{4\beta\tau}}{2\alpha} \int_0^\tau \|P^N z(t) - z(t)\|^2 dt.$$

Assumption (C), the remark immediately following it, the fact that $z \in L_2(0, \tau; V) \cap C(0, \tau; H)$, and the dominated convergence theorem therefore imply that $\|B^N v - P^N B v\| \rightarrow 0$ as $N \rightarrow \infty$ for each $v \in R^m$. The triangle inequality, assumption (C) and the finite dimensionality of R^m then immediately yield the desired result.

If we define $Q^N \in \mathfrak{L}(H^N)$ by $Q^N = P^N Q$ then Q^N is nonnegative (in fact positive, by assumption (B)), self-adjoint, and by assumption (C) satisfies $Q^N P^N \phi \rightarrow Q\phi$ as $N \rightarrow \infty$ for each $\phi \in H$. This together with Lemmas 4.1 and 4.2 yield the following result.

Lemma 4.3 For H^N , P^N , T^N , B^N , and Q^N as defined above, assumption (C) implies that hypothesis (H3) is satisfied.

To verify that hypothesis (H2) and the conditions of the convergence theorem, Theorem 2.2, are satisfied we shall require the fact that the operators $B^N : R^m \rightarrow H^N$ are uniformly bounded in $\mathfrak{L}(R^m, V)$. Toward this end, we note that it can be argued (see [2]) that $\|\mathcal{T}^N(t)\| = \|\mathcal{T}^N(t)\| \leq C e^{\beta t}$, $t \geq 0$ where C is a positive constant which does not depend upon N .

Lemma 4.4 If assumption (C) holds then there exists a constant $L_1 > 0$ which does not depend on N for which $\|B^N v\| \leq L_1 \|v\|$, $v \in R^m$; that is the operators B^N are uniformly bounded in $\mathfrak{L}(R^m, V)$.

Proof For $v \in R^m$ we have

$$\begin{aligned} \alpha \|B^N v\|^2 &\leq \operatorname{Re} a(B^N v, B^N v) + \beta \|B^N v\|^2 \\ &= -\operatorname{Re} \left(\mathcal{A}^N \int_0^\tau \mathcal{T}^N(t) \mathfrak{B}^N v dt, B^N v \right) + \beta (B^N v, B^N v) \\ &= \operatorname{Re} (\mathfrak{B}^N v - \mathcal{T}^N(\tau) \mathfrak{B}^N v, B^N v) + \beta (B^N v, B^N v) \end{aligned}$$

$$\begin{aligned}
&= \operatorname{Re}(\mathfrak{B}v, B^N v) - \operatorname{Re}(\mathfrak{B}v, \mathfrak{T}^N(\tau)^* B^N v) + \beta(B^N v, B^N v) \\
&\leq \|\mathfrak{B}v\|_* \|B^N v\| + Ce^{\beta\tau} \|\mathfrak{B}v\|_* \|B^N v\| + |\beta| \|B^N v\|_* \|B^N v\| \\
&\leq \|\mathfrak{B}^*\| \{1 + Ce^{\beta\tau}\} |v| \|B^N v\| + |\beta|\mu \|B^N v\| \|B^N v\|.
\end{aligned}$$

Thus

$$\|B^N v\| \leq (\|\mathfrak{B}^*\| / \alpha) \{1 + Ce^{\beta\tau}\} |v| + (|\beta|\mu / \alpha) \|B^N v\|,$$

which together with Lemma 4.3 yields the desired result.

Lemma 4.5 Suppose that assumptions (A) and (C) hold and let $F \in \mathfrak{L}(H, R^m)$ be the operator in assumption (A) which uniformly exponentially stabilizes the pair $\{T, B\}$. Then for all N sufficiently large, F uniformly exponentially stabilizes the pairs $\{T^N, B^N\}$. That is, the pairs $\{T^N, B^N\}$ are uniformly exponentially stabilizable, uniformly in N for all N sufficiently large.

Proof Let $S^N = T^N - B^N F$ and let $S = T - B F$. Then Lemma 4.3 implies that $S^N P^N \varphi \rightarrow S \varphi$ and $(S^N)^* P^N \varphi \rightarrow S^* \varphi$ as $N \rightarrow \infty$ for each $\varphi \in H$. It follows that there exists a constant $K_1 > 0$ for which $\|S^N\| = \|(S^N)^*\| \leq K_1$ and $\|S\| = \|S^*\| \leq K_1$, and consequently that the spectra of the operators S, S^*, S^N , and $(S^N)^*$ are contained in the closed disc $\{z : |z| \leq K_1\}$ in the complex plane. Let $\eta > K_1$. Then $\eta \in \rho(S) \cap \rho(S^N)$ and since $S^N P^N$ converges strongly to S as $N \rightarrow \infty$, and the embedding $V \subset H$ is compact, it follows (see [1]) that there exists a constant $K_2 > 0$ which does not depend on N for which $\|(\eta - S^N)^{-1} P^N\| \leq K_2$. Therefore for $\varphi \in H$ we have

$$\begin{aligned}
&\|(\eta - S^N)^{-1} P^N \varphi - (\eta - S)^{-1} \varphi\| \\
&\leq \|(\eta - S^N)^{-1} P^N \varphi - P^N (\eta - S)^{-1} \varphi\| + \|(P^N - I)(\eta - S)^{-1} \varphi\| \\
&= \|(\eta - S^N)^{-1} P^N (S^N P^N - S)(\eta - S)^{-1} \varphi\| + \|(P^N - I)(\eta - S)^{-1} \varphi\| \\
&\leq K_2 \|(S^N P^N - S)(\eta - S)^{-1} \varphi\| + \|(P^N - I)(\eta - S)^{-1} \varphi\|
\end{aligned}$$

which by Lemma 4.3 tends to zero as $N \rightarrow \infty$. Analogously it can be shown that

$$(\eta - (S^N)^*)^{-1} P^N \varphi \rightarrow (\eta - S^*)^{-1} \varphi \text{ as } N \rightarrow \infty \text{ for each } \varphi \in H \text{ and } \eta > K_1.$$

We claim next that for some positive integer N_0 we have

$$(4.1) \quad \bigcap_{N=N_0}^{\infty} \sigma(S^N) \subset \{z : |z| \leq \varepsilon\}$$

for some $\varepsilon < 1$. Suppose not. Then there exists a sequence of positive integers $\{N_j\}_{j=1}^{\infty}$

with $N_j \rightarrow \infty$ as $j \rightarrow \infty$ and sequences of complex numbers $\{\lambda^{N_j}\}_{j=1}^{\infty}$ and elements in H^{N_j} ,

$\{\varphi^{N_j}\}_{j=1}^{\infty}$ for which $[\lambda^{N_j}, \varphi^{N_j}]$ is an eigenvalue / eigenvector pair for the operator S^{N_j} with

$|\lambda^{N_j}| > 1 - 1/j$ and $|\varphi^{N_j}| = 1$. Now $|\lambda^{N_j}| \leq K_1$. Therefore $\{\lambda^{N_j}\}_{j=1}^{\infty}$ must admit a convergent

subsequence $\{\lambda^{N_{j_k}}\}_{k=1}^{\infty}$ with limit $\hat{\lambda}$ satisfying $|\hat{\lambda}| \geq 1$. For convenience we re-index and

say $\lim_{N \rightarrow \infty} \lambda^N = \hat{\lambda}$. We claim that $\hat{\lambda}$ is an eigenvalue of S . Indeed, for $\eta > K_1$ we have

$$(\eta - \lambda^N) \varphi^N = (\eta - S^N) \varphi^N, \text{ or}$$

$$(4.2) \quad \varphi^N = (\eta - S^N)^{-1} (\eta - \lambda^N) \varphi^N.$$

The estimate (see [16]) $\|\mathcal{T}^N(t) \psi^N\| \leq \frac{L_2}{\sqrt{t}} \|\psi^N\|$, $t > 0$, $\psi^N \in H^N$, for some positive constant

L_2 which does not depend on N , and Lemma 4.4 imply

$$|\lambda^N| \|\varphi^N\| = \|\lambda^N \varphi^N\| = \|S^N \varphi^N\| \leq \|\mathcal{T}^N(\tau) \varphi^N\| + \|B^N F \varphi^N\|$$

$$\leq \frac{L_2}{\sqrt{\tau}} \|\varphi^N\| + L_1 \|F\| \|\varphi^N\| = \frac{L_2}{\sqrt{\tau}} + L_1 \|F\|.$$

Recalling that $\lambda^N \rightarrow \hat{\lambda}$ and $|\hat{\lambda}| \geq 1$, it follows that the sequence $\{\varphi^N\}$ lies in a bounded subset

of V . The assumption that the embedding $V \subset H$ is compact then implies that $\{\varphi^N\}$ admits an

H -convergent subsequence $\{\varphi^{N_j}\}$ with limit $\hat{\varphi} \in H$. Once again we re-index and assume that $\lim_{N \rightarrow \infty} \|\varphi^N - \hat{\varphi}\| = 0$. Then for any $\psi \in H$, (4.2) yields

$$((\eta - \lambda^N) \varphi^N, (\eta - (S^N)^*)^{-1} P^N \psi) = (\varphi^N, P^N \psi) = (\varphi^N, \psi).$$

Taking the limit as $N \rightarrow \infty$ we obtain

$$((\eta - \hat{\lambda})\hat{\phi}, (\eta - S^*)^{-1}\psi) = (\hat{\phi}, \psi),$$

or $S\hat{\phi} = \hat{\lambda}\hat{\phi}$; i.e. $[\hat{\lambda}, \hat{\phi}]$ is an eigenvalue/eigenvector pair for the operator S . But this is a contradiction since $|\hat{\lambda}| \geq 1$ and $S = T - BF$ is assumed to be uniformly exponentially stable. It follows that (4.1) must hold for some positive integer N_0 sufficiently large.

Now for $N \geq N_0$, let $x^N \in H^N$ and set $x_k^N = (S^N)^k x^N$. If we denote the z -transform of the sequence $\{x_k^N\}_{k=0}^\infty$ by $x^N(z)$, then $x^N(z) = (I - z^{-1}S^N)^{-1}x^N$, and

$$x_k^N = \frac{1}{2\pi i} \int (z - S^N)^{-1} x^N z^k dz$$

where the integral in the above expression is around a closed contour in the complex plane which contains the spectrum of the operator S^N in its interior. If we choose the contour to be the circle

$z = re^{i\theta}$, $0 \leq \theta \leq 2\pi$ with $\varepsilon < r < 1$, we obtain $|x_k^N| \leq K_2 r^{k+1} |x^N|$, $k = 0, 1, 2, \dots$, or

$$|(S^N)^k| \leq M r^k, \quad k = 0, 1, 2, \dots$$

with $M = K_2 r$ and the proof is complete.

Theorem 4.1 If assumptions (A), (B), and (C) hold, then the conditions of Theorem 2.2 are satisfied and therefore the convergence results stated as the conclusions of that theorem are valid for the class of problems considered in section 3.

Proof. Lemma 3.1 implies that hypothesis (H1) is satisfied and Lemma 4.3 implies that hypothesis (H3) holds. Assumption (B) and the definition of Q^N yields $(Q^N \phi^N, \phi^N) = (P^N Q \phi^N, \phi^N) = (Q \phi^N, \phi^N) \geq \delta |\phi^N|^2$, $\phi^N \in H^N$. This together with Lemma 4.5 and the same arguments used to prove Lemma 3.1 imply that hypothesis (H2) is satisfied. Thus we need only to verify (2.8) and (2.9). Toward this end we note that

$$(4.3) \quad |\bar{\Pi}^N| = \sup \{(\bar{\Pi}^N \phi^N, \phi^N) : \phi^N \in H^N, |\phi^N| = 1\} = \sup_{|\phi^N|=1} J^N(\bar{u}^N; \phi^N)$$

where \bar{u}^N is the optimal control corresponding to the initial data ϕ^N . For $\phi^N \in H^N$ with

$|\phi^N| = 1$ define $u^N = \{u_k^N\}_{k=0}^\infty$ by $u_k^N = -F x_k^N$, $k = 0, 1, 2, \dots$ where $F \in \mathfrak{L}(H, R^m)$ is the

operator from assumption (A) and $\{x_k^N\}_{k=0}^\infty$ is given by (2.6), (2.7) with $x_0 = \phi^N$. Then for $N \geq N_0$ with $S^N = T - B^N F$, Lemma 4.5 implies

$$\begin{aligned}
J^N(\bar{u}^N; \varphi^N) &\leq J^N(u^N; \varphi^N) = \sum_{k=0}^{\infty} (Q^N(S^N)^k \varphi^N, (S^N)^k \varphi^N) + (F(S^N)^k \varphi^N)^T R F(S^N)^k \varphi^N \\
&\leq \sum_{k=0}^{\infty} |Q^N| |(S^N)^k \varphi^N|^2 + |R| |F|^2 |(S^N)^k \varphi^N|^2 \leq \{|Q| + |R| |F|^2\} M^2 |\varphi^N|^2 \sum_{k=0}^{\infty} r^{2k} \\
&= \{|Q| + |R| |F|^2\} M^2 / (1 - r^2) \equiv M_1
\end{aligned}$$

which together with (4.3) establishes (2.8).

Now $|\bar{\Pi}^N|$ uniformly bounded implies that the operators $\bar{S}^N = T^N - B^N \bar{F}^N$ are uniformly bounded. Thus

$$(4.4) \quad |(\bar{S}^N)^k| \leq s^k, \quad k = 0, 1, 2, \dots$$

for some positive constant s which does not depend on N . Also, for $\varphi^N \in H^N$

$$(4.5) \quad \sum_{k=0}^{\infty} (Q^N (\bar{S}^N)^k \varphi^N, (\bar{S}^N)^k \varphi^N) \leq (\bar{\Pi}^N \varphi^N, \varphi^N) \leq M_1 |\varphi^N|^2.$$

Since $Q^N \geq \delta > 0$, (4.4) and (4.5) together with a discrete-time version of a result due to Datko [5] (see the Appendix) establishes (2.9) for some $M_2, r_2 > 0$ with $r_2 < 1$ and the theorem is proved.

Finally we note that once a basis has been selected for the finite dimensional subspace H^N , the matrix representations for the operators T^N, B^N and Q^N , which are required to solve the approximating optimal control problem (\mathfrak{P}^N) , are easily computed. Indeed, if $\{\varphi_j^N\}_{j=1}^{K^N}$

denotes a basis for H^N , then $[\mathcal{A}^N] = -(\varphi_i^N, \varphi_j^N)^{-1} a(\varphi_k^N, \varphi_\ell^N)^T$ with $[T^N] = \exp(\tau[\mathcal{A}^N])$.

Similarly we find $[\mathfrak{B}^N] = (\varphi_i^N, \varphi_j^N)^{-1} (\mathfrak{B} e_k, \varphi_\ell^N)^T$ where e_k denotes the k^{th} standard unit vector in \mathbb{R}^m , $[B^N] = \int_0^\tau \exp(t[\mathcal{A}^N]) [\mathfrak{B}^N] dt$, and $[Q^N] = (\varphi_i^N, \varphi_j^N)^{-1} (Q \varphi_k^N, \varphi_\ell^N)$. A more

complete and detailed treatment of the computational and implementational aspects of solving the finite dimensional approximating linear-quadratic regulator problems (\mathfrak{P}^N) can be found in [9].

5. Examples

We describe a generic class of boundary control systems and some specific examples to which the abstract approximation framework and theory we have developed above applies. Let H be a

Hilbert space with inner product (\cdot, \cdot) and corresponding induced norm $\|\cdot\|$, and let W be another Hilbert space with inner product and norm $[\cdot, \cdot]$ and $\|\cdot\|_W$ respectively. Let $\langle \cdot, \cdot \rangle$ be another inner product defined on $W \times W$ and let the corresponding induced norm be denoted by $\|\cdot\|$. We assume that $W \subset H$ and that there exist positive constants μ and ρ for which $\|\varphi\| \leq \mu \|\varphi\|_W \leq \rho \|\varphi\|$, for all $\varphi \in W$.

We consider the generic boundary control system given by

$$(5.1) \quad \dot{x}(t) = \Delta x(t), \quad t > 0$$

$$(5.2) \quad \Gamma x(t) = u(t), \quad t > 0$$

$$(5.3) \quad x(0) = x_0$$

where $x(t) \in H$, $u(t) \in \mathbb{R}^m$, $x_0 \in H$, $\Delta \in \mathcal{L}(W, H)$ and $\Gamma \in \mathcal{L}(W, \mathbb{R}^m)$ is assumed to be surjective. Let W_0 denote the null space of Γ , $\mathcal{N}(\Gamma)$. That is, $W_0 = \{\varphi \in W : \Gamma \varphi = 0\}$ and define V to be the completion of W_0 with respect to the norm $\|\cdot\|$. It follows that V is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding induced norm $\|\cdot\|$. We shall require the following assumptions:

- (1) V is dense in H
- (2) The embedding $V \subset H$ is compact
- (3) $W \subseteq V$.

Then choosing H as our pivot space, we have $V \subset H \subset V^*$ with the embeddings dense and continuous.

Define the bounded sesquilinear form $\sigma(\cdot, \cdot) : W \times H \rightarrow \mathbb{C}$ by $\sigma(\varphi, \psi) = -(\Delta \varphi, \psi)$ for $\varphi \in W$ and $\psi \in H$, and assume that there exist real constants α, β, γ with $\alpha, \gamma > 0$, for which

$$(4) \quad \operatorname{Re} \sigma(\varphi, \varphi) \geq \alpha \|\varphi\|^2 - \beta |\varphi|^2, \quad \varphi \in W_0$$

$$(5) \quad |\sigma(\varphi, \psi)| \leq \gamma \|\varphi\| \|\psi\|, \quad \varphi, \psi \in W_0.$$

Define the sesquilinear form $a(\cdot, \cdot) : W_0 \times W_0 \rightarrow \mathbb{C}$ to be the restriction of the form $\sigma(\cdot, \cdot)$ to

$W_0 \times W_0$. It follows from density and continuity (i.e. assumption (5) above) that $a(\cdot, \cdot)$ admits a unique extension to a bounded coercive form on $V \times V$. Thus assumptions (4) and (5) continue to hold with $\sigma(\cdot, \cdot)$ replaced by $a(\cdot, \cdot)$ and the space W_0 replaced by V , and consequently we may define an operator $\mathcal{A} \in \mathfrak{L}(V, V^*)$ via

$$(\mathcal{A}\varphi)(\psi) = (\mathcal{A}\varphi, \psi) = -a(\varphi, \psi), \quad \varphi, \psi \in V.$$

If we restrict \mathcal{A} to $\text{Dom}(\mathcal{A}) = \{\varphi \in V : \mathcal{A}\varphi \in H\}$ then $\mathcal{A} : \text{Dom}(\mathcal{A}) \subset H \rightarrow H$ is densely defined and is the infinitesimal generator of an analytic semigroup $\{\mathcal{T}(t) : t \geq 0\}$ of bounded linear operators on H with $|\mathcal{T}(t)| \leq e^{(\beta - \alpha/\mu)t}$, $t \geq 0$. Also, $\{\mathcal{T}(t) : t \geq 0\}$ admits an extension and a restriction to an analytic semigroup of bounded linear operators on V^* and V , respectively.

Recall that Γ was assumed to be surjective and let $\Gamma^+ \in \mathfrak{L}(R^m, W)$ denote a fixed but arbitrary right inverse of Γ . That is $\Gamma\Gamma^+u = u$ for $u \in R^m$. Define $\mathfrak{B} \in \mathfrak{L}(V, V^*)$ by

$$(5.4) \quad (\mathfrak{B}u)(\varphi) = (\mathfrak{B}u, \varphi) = a(\Gamma^+u, \varphi) - \sigma(\Gamma^+u, \varphi),$$

for $u \in R^m$ and $\varphi \in V$. We note that the operator \mathfrak{B} is indeed well defined. For if Γ_1^+ and Γ_2^+ are two distinct right inverses of Γ and we let \mathfrak{B}_1 and \mathfrak{B}_2 denote the corresponding operators defined as in (5.4), then for any $u \in R^m$ and $\varphi \in V$ we have $\Gamma_1^+u - \Gamma_2^+u = (\Gamma_1^+ - \Gamma_2^+)u \in W_0$ and

$$\begin{aligned} (\mathfrak{B}_1u)(\varphi) - (\mathfrak{B}_2u)(\varphi) &= a((\Gamma_1^+ - \Gamma_2^+)u, \varphi) - \sigma((\Gamma_1^+ - \Gamma_2^+)u, \varphi) \\ &= \sigma((\Gamma_1^+ - \Gamma_2^+)u, \varphi) - \sigma((\Gamma_1^+ - \Gamma_2^+)u, \varphi) = 0, \end{aligned}$$

or $\mathfrak{B}_1 = \mathfrak{B}_2$.

Following Curtain and Salmon [4], for any $t_f > 0$ we call $x \in L_2(0, t_f; V) \cap C(0, t_f; H) \cap H^1(0, t_f; V^*)$ the weak solution to the boundary control system (5.1) - (5.3) if it is the unique mild solution to the initial value problem

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t) + \mathfrak{B}u(t), \quad t > 0 \\ x(0) &= x_0. \end{aligned}$$

That is, if

$$x(t) = \mathcal{T}(t-s)x(s) + \int_s^t \mathcal{T}(t-\sigma)\mathfrak{B}u(\sigma)d\sigma, \quad 0 \leq s \leq t \leq t_f.$$

$$x(s) = x_0.$$

For the discrete-time system, with τ denoting the length of the sampling interval, we have

$$T = \mathcal{T}(\tau) \in \mathfrak{L}(H) \text{ and } B = \int_0^\tau \mathcal{T}(t) \mathfrak{B} dt \in \mathfrak{L}(R^m, H). \text{ From (5.4) together with the definitions}$$

of the forms $\sigma(\cdot, \cdot)$ and $a(\cdot, \cdot)$, we obtain

$$\begin{aligned} Bu &= \int_0^\tau \mathcal{T}(t) \mathfrak{B} u dt = \int_0^\tau \mathcal{T}(t) (\Delta - \mathcal{A}) \Gamma^+ u dt = \int_0^\tau \mathcal{T}(t) \Delta \Gamma^+ u dt - \int_0^\tau \frac{d}{dt} \mathcal{T}(t) \Gamma^+ u dt \\ &= (I - \mathcal{T}(\tau)) \Gamma^+ u + \int_0^\tau \mathcal{T}(t) \Delta \Gamma^+ u dt. \end{aligned}$$

Similarly, for the approximating input operators B^N we find

$$B^N u = (I - \mathcal{T}^N(\tau)) P^N \Gamma^+ u + \int_0^\tau \mathcal{T}^N(t) P^N \Delta \Gamma^+ u dt.$$

If Γ^+ is chosen so that $\mathfrak{R}(\Gamma^+) \subset \mathfrak{N}(\Delta)$, the discrete-time input operators B and B^N take on the particularly simple forms $B = (I - \mathcal{T}(\tau)) \Gamma^+ = (I - T) \Gamma^+$ and $B^N = (I - \mathcal{T}^N(\tau)) P^N \Gamma^+ = (I - T^N) P^N \Gamma^+$.

We discuss two specific examples of boundary control systems of the general form that we have just described, and outline the arguments necessary to verify that assumptions (A) - (C) are satisfied.

Example 5.1 We consider a system whose dynamics are described by the one dimensional heat equation

$$(5.5) \quad \frac{\partial w}{\partial t}(t, \eta) = a_0 \frac{\partial^2 w}{\partial \eta^2}(t, \eta), \quad t > 0, \quad 0 < \eta < 1$$

where $a_0 > 0$, with a homogeneous Dirichlet boundary condition at $\eta = 0$,

$$(5.6) \quad w(t, 0) = 0, \quad t > 0$$

and Neumann boundary control at $\eta = 1$,

$$(5.7) \quad a_0 \frac{\partial w}{\partial \eta}(t, 1) = u(t), \quad t > 0.$$

The initial conditions are assumed to be of the form

$$(5.8) \quad w(0, \eta) = w_0(\eta), \quad 0 < \eta < 1$$

where $w_0 \in L_2(0,1)$ is given.

To put the system (5.5) - (5.8) in the form of (5.1) - (5.3) we take $H = L_2(0,1)$, and $W = H^2(0,1) \cap H_L^1(0,1)$ where $H_L^1(0,1) = \{\varphi \in H^1(0,1) : \varphi(0) = 0\}$. The inner products (\cdot, \cdot) , $\langle \cdot, \cdot \rangle$, and $[\cdot, \cdot]$ are taken to be

$$\begin{aligned} (\varphi, \psi) &= \int_0^1 \varphi \psi, & \varphi, \psi \in H \\ \langle \varphi, \psi \rangle &= \int_0^1 D\varphi D\psi, & \varphi, \psi \in W \\ [\varphi, \psi] &= \int_0^1 \varphi \psi + \int_0^1 D\varphi D\psi + \int_0^1 D^2\varphi D^2\psi, & \varphi, \psi \in W. \end{aligned}$$

The operators $\Delta \in \mathfrak{L}(W, H)$ and $\Gamma \in \mathfrak{L}(W, \mathbb{R}^1)$ are given by $\Delta\varphi = a_0 D^2\varphi$ and $\Gamma\varphi = a_0 D\varphi(1)$, for

$\varphi \in W$. We then have $W_0 = \mathcal{N}(\Gamma) = \{\varphi \in H^2(0,1) : \varphi(0) = D\varphi(1) = 0\}$, $\sigma(\cdot, \cdot) : W \times H \rightarrow \mathbb{C}$

given by $\sigma(\varphi, \psi) = -(a_0 D^2\varphi, \psi)$, $\varphi \in W$, $\psi \in H$, $V = H_L^1(0,1)$, $V^* \subset H^{-1}(0,1)$, and $a(\cdot, \cdot) :$

$V \times V \rightarrow \mathbb{C}$ given by $a(\varphi, \psi) = (a_0 D\varphi, D\psi)$, $\varphi, \psi \in V$. The operator $\mathcal{A} \in \mathfrak{L}(V, V^*)$ takes the form

$(\mathcal{A}\varphi)(\psi) = -(a_0 D\varphi, D\psi)$ for $\varphi, \psi \in V$. It follows that $\mathcal{A} : \text{Dom}(\mathcal{A}) \subset H \rightarrow H$ is given by $\mathcal{A}\varphi =$

$a_0 D^2\varphi$ for $\varphi \in \text{Dom}(\mathcal{A}) = W_0$ and is self-adjoint. We have

$$a(\varphi, \varphi) \geq a_0 \|\varphi\|^2, \quad \varphi \in V$$

$$|a(\varphi, \psi)| \leq a_0 \|\varphi\| \|\psi\|, \quad \varphi, \psi \in V,$$

and therefore that $(\mathcal{A}\varphi, \varphi) \leq -a_0 \|\varphi\|^2$ for $\varphi \in \text{Dom}(\mathcal{A})$. Thus \mathcal{A} is the infinitesimal generator of

an analytic semigroup $\{\mathcal{T}(t) : t \geq 0\}$ of bounded, self-adjoint linear operators on H (and V and V^*) which is uniformly exponentially stable; that is

$$(5.9) \quad |\mathcal{T}(t)| \leq e^{-a_0 t}, \quad t \geq 0.$$

We take $\Gamma^+ \in \mathfrak{L}(\mathbb{R}^1, W)$ to be $(\Gamma^+ u)(\eta) = (u/a_0)\eta$, $0 \leq \eta \leq 1$ and note that with this choice

of Γ^+ we have $\mathfrak{R}(\Gamma^+) \subset \eta(\Delta)$. We note further that W is indeed a subset of V and thus

$\mathfrak{B} \in \mathfrak{L}(R^1, V^*)$ as defined in (5.4) is given by $(\mathfrak{B}u)(\varphi) = u\varphi(1)$, for $u \in R^1$ and $\varphi \in V$. The discrete-time input operator $B \in \mathfrak{L}(R^1, V^*)$ will take the form $Bu = (I - \mathfrak{T}(\tau)) \Gamma^+ u$. We have $x(t) = w(t, \cdot)$, $t \geq 0$ and $x_0 = w_0$.

The space V is densely, continuously, and compactly embedded in H , it is clear from (5.9) that assumption (A) is satisfied and if, for example, we choose $Q \in \mathfrak{L}(H)$ to be $Q\varphi = q\varphi$ where $q \in L_\infty(0,1)$ is such that $q(\eta) > 0$, a.e. $\eta \in [0,1]$, then assumption (B) will be satisfied as well. With regard to approximation, a linear spline based scheme is one for which assumption (C) can be shown to hold. For each $N = 1, 2, \dots$ let $\{\varphi_j^N\}_{j=0}^N$ denote the usual $N+1$ linear B-splines (i.e. "hat" functions) defined on $[0,1]$ with respect to the uniform mesh $\{0, 1/N, 2/N, \dots, 1\}$. Let H^N be the N -dimensional subspace of $V = H_L^1(0,1)$ given by $H^N = \text{span } \{\varphi_j^N\}_{j=1}^N$, and let $P^N: H \rightarrow H^N$ denote the corresponding orthogonal projection of $L_2(0,1)$ onto H^N . Using the approximation properties of interpolatory splines (see [15]), the Schmidt inequality (see [16]), and the variational properties of the orthogonal projection, it is not difficult to argue that $\lim_{N \rightarrow \infty} \|P^N \varphi - \varphi\| = 0$ for each $\varphi \in H_L^1(0,1)$ and consequently that assumption (C) is satisfied. We note that other commonly used finite element methods for the heat equation can be shown to lead to approximation schemes for which assumption (C) holds, including for example, modal and spectral methods.

For the system which has just been discussed, the optimal functional feedback control gain takes the form of a function $\bar{f} \in L_2(0,1)$ with the optimal control given by

$$\bar{u}_k = - \int_0^1 \bar{f}(\eta) \bar{w}(k\tau, \eta) d\eta, \quad k = 0, 1, \dots$$

The approximating optimal gain is a function $\bar{f}^N \in H^N$ with

$$\bar{u}_k^N = - \int_0^1 \bar{f}^N(\eta) \bar{w}^N(k\tau, \eta) d\eta, \quad k = 0, 1, 2, \dots$$

where \bar{w}^N is the system state which results from the input $\bar{u}^N = \{\bar{u}_k^N\}_{k=0}^\infty$. Our theory yields that $\lim_{N \rightarrow \infty} \|\bar{f}^N - \bar{f}\| = 0$ and numerical studies that we have carried out and reported on elsewhere (see, for example, [7], [9], [10], [13]) substantiate our theoretical findings.

Example 5.2 In this example we consider the control of the transverse vibration of a cantilevered Euler-Bernoulli beam with Kelvin-Voigt viscoelastic damping and shear boundary control at the free end. The dynamics of such a system are described by

$$(5.10) \quad \frac{\partial^2 w}{\partial t^2}(t, \eta) + c_0 \frac{\partial^4}{\partial \eta^4} \frac{\partial w}{\partial t}(t, \eta) + a_0 \frac{\partial^4 w}{\partial \eta^4}(t, \eta) = 0, \quad t > 0, \quad 0 < \eta < 1$$

$$(5.11) \quad w(t, 0) = \frac{\partial w}{\partial \eta}(t, 0) = 0, \quad t > 0$$

$$(5.12) \quad c_0 \frac{\partial^2}{\partial \eta^2} \frac{\partial w}{\partial t}(t, 1) + a_0 \frac{\partial^2 w}{\partial \eta^2}(t, 1) = 0, \quad t > 0$$

$$(5.13) \quad -c_0 \frac{\partial^3}{\partial \eta^3} \frac{\partial w}{\partial t}(t, 1) - a_0 \frac{\partial^3 w}{\partial \eta^3}(t, 1) = u(t), \quad t > 0$$

$$(5.14) \quad w(0, \eta) = w_0(\eta), \quad \frac{\partial w}{\partial t}(0, \eta) = w_1(\eta), \quad 0 < \eta < 1,$$

where $a_0, c_0 > 0$, and $w_0 \in H_L^2(0, 1) = \{\varphi \in H^2(0, 1) : \varphi(0) = D\varphi(0) = 0\}$, and $w_1 \in L_2(0, 1)$ are given.

To put the boundary control system (5.10) - (5.14) in the form of (5.1) - (5.3), we let $H = H_L^2(0, 1) \times L_2(0, 1)$ with inner product

$$((\varphi_1, \psi_1), (\varphi_2, \psi_2)) = a_0 \int_0^1 D^2 \varphi_1 D^2 \varphi_2 + \int_0^1 \psi_1 \psi_2,$$

and let $W = \{(\varphi, \psi) \in H : c_0 \psi + a_0 \varphi \in H^4(0, 1), \psi(0) = D\psi(0) = 0, c_0 D^2 \psi(1) + a_0 D^2 \varphi(1) = 0\}$ endowed with the inner product $[\cdot, \cdot]$ taken to be the standard inner product on $H^4(0, 1)$. We define the inner product $\langle \cdot, \cdot \rangle$ on W by

$$\langle (\varphi_1, \psi_1), (\varphi_2, \psi_2) \rangle = a_0 \int_0^1 D^2 \varphi_1 D^2 \varphi_2 + \int_0^1 D^2 \psi_1 D^2 \psi_2.$$

The operators $\Delta \in \mathfrak{L}(W, H)$ and $\Gamma \in \mathfrak{L}(W, \mathbb{R}^1)$ are given by $\Delta(\varphi, \psi) = (\psi, -c_0 D^4 \psi - a_0 D^4 \varphi)$ and $\Gamma(\varphi, \psi) = -c_0 D^3 \psi(1) - a_0 D^3 \varphi(1)$, for $(\varphi, \psi) \in W$. It follows that $W_0 = \{(\varphi, \psi) \in W : -$

$c_0 D^3 \psi(1) - a_0 D^3 \varphi(1) = 0$ and that $V = H_L^2(0,1) \times H_L^2(0,1)$. The form $\sigma(\cdot, \cdot) : W \times H \rightarrow \mathbb{C}$ is given by

$$\sigma((\varphi, \psi), (\theta, \chi)) = -a_0 \int_0^1 D^2 \psi D^2 \theta + a_0 \int_0^1 D^4 \varphi \chi + c_0 \int_0^1 D^4 \psi \chi$$

for $(\varphi, \psi) \in W$ and $(\theta, \chi) \in H$, and the form $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{C}$ is given by

$$a((\varphi_1, \psi_1), (\varphi_2, \psi_2)) = -a_0 \int_0^1 D^2 \psi_1 D^2 \varphi_2 + a_0 \int_0^1 D^2 \varphi_1 D^2 \psi_2 + c_0 \int_0^1 D^2 \psi_1 D^2 \psi_2,$$

for $(\varphi_i, \psi_i) \in V$, $i = 1, 2$.

It is not difficult to show that

$$\begin{aligned} a((\varphi, \psi), (\varphi, \psi)) &\geq c_0 \|(\varphi, \psi)\|^2 - c_0 |(\varphi, \psi)|^2 \\ |a((\varphi, \psi), (\theta, \chi))| &\leq \sqrt{2 \max(1, c_0)} \|(\varphi, \psi)\| \|(\theta, \chi)\| \end{aligned}$$

for $(\varphi, \psi), (\theta, \chi) \in V$. With the operator $\mathcal{A} \in \mathfrak{L}(V, V^*)$ defined by

$$(\mathcal{A}(\varphi, \psi))(\theta, \chi) = -a((\varphi, \psi), (\theta, \chi))$$

for $(\varphi, \psi), (\theta, \chi) \in V$, we have $\mathcal{A} : \text{Dom}(\mathcal{A}) \subset H \rightarrow H$ given by $\mathcal{A}(\varphi, \psi) = \Delta(\varphi, \psi)$ for $(\varphi, \psi) \in \text{Dom}(\mathcal{A}) = W_0$. The operator $\mathcal{A} : \text{Dom}(\mathcal{A}) \subset H \rightarrow H$ is densely defined and the infinitesimal generator of an analytic semigroup of contractions $\{\mathcal{T}(t) : t \geq 0\}$ on H . Moreover, it can be shown (see [6]) that the semigroup $\{\mathcal{T}(t) : t \geq 0\}$ is in fact uniformly exponentially stable.

Let φ_0 be the cubic polynomial which satisfies the interpolatory conditions

$\varphi_0(0) = D\varphi_0(0) = D^2\varphi_0(1) = 0, D^3\varphi_0(1) = -1/a_0$. We choose $\Gamma^+ \in \mathfrak{L}(R^1, W)$ as $\Gamma^+ u = (\varphi_0 u, 0)$, $u \in R^1$. Once again we note that $\mathfrak{R}(\Gamma^+) \subset \eta(\Delta)$, and, since $W \subset V$, the operator \mathfrak{B} as defined by (5.4) is an element in $\mathfrak{L}(R^1, V^*)$ with $(\mathfrak{B}u)(\varphi, \psi) = u\psi(1)$, $u \in R^1, (\varphi, \psi) \in V$. The discrete-time state transition operator is given by $T = \mathcal{T}(\tau) \in \mathfrak{L}(H)$ and the discrete-time input operator is given by $B = (I - T)\Gamma^+ \in \mathfrak{L}(R^1, H)$.

The uniform exponential stability of the open-loop semigroup $\{\mathcal{T}(t) : t \geq 0\}$ implies that assumption (A) is satisfied. In flexible structure control problems such as the one we have just described, the state penalization operator Q in the performance index is frequently chosen to be the

identity (i.e. the energy of the system is to be driven to zero). Thus when this is the case, we have that assumption (B) is also satisfied.

For an approximation scheme, for each $N = 1, 2, \dots$ we let $\{B_j^N\}_{j=1}^{N+1}$ denote the standard cubic B-spline functions defined on the interval $[0, 1]$ with respect to the uniform mesh $\{0, 1/N, 2/N, \dots, 1\}$. Let $\{\beta_j^N\}_{j=1}^{N+1}$ denote the modified cubic B-splines which satisfy $\beta_j^N(0) = D\beta_j^N(0) = 0$, $j = 1, 2, \dots, N+1$ (that is, $B_1^N = B_0^N - 2B_1^N - 2B_{-1}^N$, $\beta_j^N = B_j^N$, $j = 2, 3, \dots, N+1$), and set $\hat{\beta}_j^N = (\beta_j^N, 0)$, $j = 1, 2, \dots, N+1$, $\hat{\beta}_{N+1+j}^N = (0, \beta_j^N)$, $j = 1, 2, \dots, N+1$. Let $H^N = \text{span} \{\hat{\beta}_j^N\}_{j=1}^{2N+2} \subset V$ and let $P^N : H \rightarrow H^N$ denote the orthogonal projection of H onto H^N . Once again elementary properties of spline functions (see [15]) and standard techniques from the theory of approximation (see [14]) can be used to argue that $\lim_{N \rightarrow \infty} \|P^N(\varphi, \psi) - (\varphi, \psi)\| = 0$ for each $(\varphi, \psi) \in V$, and consequently that assumption (C) is satisfied.

The optimal functional feedback control gains are of the form $\bar{f} = (\bar{f}_1, \bar{f}_2) \in H$ (i.e. $\bar{f}_1 \in H_L^2(0, 1)$, $\bar{f}_2 \in L_2(0, 1)$) with

$$\bar{u}_k = -a_0 \int_0^1 D^2 \bar{f}_1(\eta) D^2 \bar{w}(k\tau, \eta) d\eta - \int_0^1 \bar{f}_2(\eta) \dot{\bar{w}}(k\tau, \eta) d\eta, \quad k = 0, 1, 2, \dots$$

(note that in this example we have $x(t) = (w(t, \cdot), \dot{w}(t, \cdot))$, $t > 0$). The approximating functional feedback control gains are of the form $\bar{f}^N = (\bar{f}_1^N, \bar{f}_2^N) \in H^N$, and our theory yields $\lim_{N \rightarrow \infty} \bar{f}_1^N = \bar{f}_1$ in H^2 and $\lim_{N \rightarrow \infty} \bar{f}_2^N = \bar{f}_2$ in L_2 . Numerical studies for a flexible structure example such as the one we have described above were reported on in [8] and [13].

Finally we note that both of the examples that were treated above admit generalization to higher dimensions with the approximation subspaces H^N being formed via tensor products of one dimensional elements. Also, it is worth noting that not every control system involving unbounded input that we might formulate will conform to our general framework. For example, if we replace (5.7) with Dirichlet boundary control, such a system results. In this case we have that W is not a subset of V and consequently the continuous-time input operator \mathfrak{B} will have range in some space larger than V^* . More specifically $\mathfrak{R}(\mathfrak{B})$ will be contained in the dual of $\text{Dom}(\mathcal{A}^*)$ (see [4], [8], [10], [13]), and the range of the discrete-time input operator, $\mathfrak{R}(\mathfrak{B})$ will be a subset of W rather than V . While it is in fact possible to apply the theory outlined in section 2 to this particular example (see [9]), the question as to whether or not the results in section 3 and 4 can be extended so as to be able to handle this more general class of systems with unbounded input, at present, remains open.

6. Summary and Concluding Remarks

We have developed a rather complete abstract approximation framework for discrete-time LQR problems for parabolic systems with unbounded input. Our results are in some sense analogous to those found in [3] for continuous time systems but also represent an extension in that the theory developed here can handle at least some, but certainly not all, forms of unbounded (continuous-time) input. Requiring only that (1) the infinite dimensional open-loop discrete-time system be uniformly exponentially stabilizable, (2) that the state penalization operator in the quadratic performance index be positive definite, and (3) that the usual Galerkin hypothesis on the finite dimensional approximation subspaces (i.e. the strong V-norm convergence of the corresponding orthogonal projections) be satisfied, we are able to apply the theory developed in [9] to obtain norm convergence of the approximating optimal feedback gains.

In addition to the question raised at the end of the previous section, other related open problems remain. For example, in regard to the condition (1) above, one might ask if the uniform exponential stabilizability of the open-loop continuous-time system implies uniform exponential stabilizability of the corresponding discrete-time system. In a finite dimensional setting, this notion is treated in a paper by Hautus [11]. It is shown there that the answer to this question is yes for all but a finite number of sampling interval lengths τ . The extension of this result to the most general of infinite dimensional systems is not immediately clear, and is worthy of some consideration.

The extension of our theory to the LQG problem for infinite dimensional discrete-time systems with both unbounded input and output in the spirit of the treatment of this problem in [10] should also be looked at. In particular, since the semigroup $\{\mathcal{T}(t) : t \geq 0\}$ is analytic, the discrete-time input operator B is in fact an element in $\mathfrak{L}(R^m, V)$. Therefore, if the output operator is bounded on V , one might attempt to develop a general theory wherein the problem is formulated in the state space V rather than H . However, without additional structure, for example, the operator \mathcal{A} self-adjoint, such results may be difficult to obtain.

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Appendix: A Discrete-Time Version of a Result of Datko's

Let X be a Banach space with norm $\|\cdot\|_X$ and suppose $S \in \mathcal{B}(X)$. Suppose further that there exist positive constants M_1, M_2 and r_1 with $r_1 > 1$ for which $\|S^k\| \leq M_1 r_1^k$, $k = 0, 1, 2, \dots$ and

$$(A.1) \quad \sum_{k=0}^{\infty} \|S^k x\|_X \leq M_2 \|x\|_X^2, \quad x \in X.$$

Then there exist positive constants M and r with $r < 1$ such that $\|S^k\| \leq M r^k$, $k = 0, 1, 2, \dots$.

Proof. Following Datko's proof in the continuous time case we first show that there exists an $M_3 > 0$ such that $\|S^k x\|_X \leq M_3 \|x\|_X$, $k = 0, 1, 2, \dots$, $x \in X$. Suppose not. Then there exist sequences $\{(k_n, \bar{k}_n)\}$ and $\{x_n\}$ with $0 \leq k_n \leq \bar{k}_n < \infty$ and $\|x_n\| = 1$ such that

$$(A.2) \quad M_1 n \leq \|S^{\bar{k}_n - k_n} x_n\|_X \leq M_1 r_1^{\bar{k}_n - k_n}.$$

Then for $k \in [k_n, \bar{k}_n]$ an integer, we have

$$M_1 n \leq \|S^{\bar{k}_n - k} S^{k - k_n} x_n\|_X \leq M_1 r_1^{\bar{k}_n - k} \|S^{k - k_n} x_n\|_X$$

which implies

$$n r_1^{-(\bar{k}_n - k)} \leq \|S^{k - k_n} x_n\|_X.$$

Squaring both sides of the last inequality and summing from $k = k_n$ to $k = \bar{k}_n - 1$, we obtain

$$\sum_{k=k_n}^{\bar{k}_n - 1} n^2 r_1^{-2(\bar{k}_n - k)} \leq \sum_{k=k_n}^{\bar{k}_n - 1} \|S^{k - k_n} x_n\|_X^2.$$

But

$$\sum_{k=k_n}^{\bar{k}_n - 1} \|S^{k - k_n} x_n\|_X^2 \leq \sum_{k=k_n}^{\infty} \|S^{k - k_n} x_n\|_X^2 \leq M_2 \|x_n\|_X^2 = M_2$$

and

$$\sum_{k=k_n}^{\bar{k}_n-1} n^2 r_1^{-2(\bar{k}_n-k)} = n^2 r_1^{-2\bar{k}_n} \sum_{k=k_n}^{\bar{k}_n-1} r_1^{2k} = \frac{n^2}{r_1^2 - 1} (1 - r_1^{-2(\bar{k}_n-k_n)}) .$$

Thus

$$\frac{n^2}{r_1^2 - 1} (1 - r_1^{-2(\bar{k}_n-k_n)}) \leq M_2 .$$

Now (A.2) implies $n \leq r_1^{\bar{k}_n-k_n}$, or $\bar{k}_n - k_n \geq \log_{r_1} n$. Therefore for some n_0 sufficiently large we have $1 - r_1^{-2(\bar{k}_n-k_n)} > \frac{1}{2}$ for all $n \geq n_0$. Thus for all $n \geq n_0$ we have $n^2/2(r_1^2 - 1) \leq M_2$, which is a contradiction.

We next argue that given any $\varepsilon > 0$ with $\sqrt{M_2} \varepsilon < 1$ there exist a $K \in \mathbb{Z}^+$ with $0 \leq K \leq 1/\varepsilon^2$ for which $|S^K x|_X \leq \sqrt{M_2} \varepsilon |x|_X$ for all $x \in X$. The bound (A.1) implies that for each $x \neq 0$ there exists a $K = K(x, \varepsilon)$ such that

$$|S^k x|_X \leq \sqrt{M_2} \varepsilon |x|_X \text{ and } \sqrt{M_2} \varepsilon |x|_X < |S^k x|_X, \quad 0 \leq k \leq K.$$

This implies that

$$M_2 \varepsilon^2 |x|_X^2 K \leq \sum_{k=0}^{K-1} |S^k x|_X^2 \leq M_2 |x|_X^2$$

which in turn implies that $\varepsilon^2 K \leq 1$ or $K = K(\varepsilon) \leq 1/\varepsilon^2$. Thus K may be chosen independently of $x \in X$ and the claim is established.

Now let $\delta > 0$ be given and choose $\varepsilon > 0$ so small that $\sqrt{M_2} \varepsilon < 1$ and $M_3 \sqrt{M_2} \varepsilon < \delta$.

Then for $k \geq K(\varepsilon)$ we have $k = K + \ell$, $\ell \geq 0$ and

$$|S^k x|_X = |S^{K+\ell} x|_X = |S^\ell S^K x|_X \leq M_3 |S^K x|_X \leq M_3 \sqrt{M_2} \varepsilon |x|_X < \delta |x|_X$$

for any $x \in X$. Therefore S is uniformly asymptotically stable.

Since S is uniformly asymptotically stable there exists a $K > 0$, an integer, such that $|S^k| \leq \frac{1}{2}$, $k \geq K$. Now any $k = 0, 1, 2, \dots$ can be written as $k = nK + m$ where n is a natural number and m is an integer such that $0 \leq m < K$. Then for $k = 0, 1, 2, \dots$

$$|S^k| = |S^{nK+m}| \leq |S^m| |S^{nK}| \leq |S^m| |S^K|^n \leq M_1 r_1^m (1/2)^n = M_1 e^{m \ln r_1} e^{-n \ln 2}$$

$$= M_1 e^{(m \ln r_1 + (m/K) \ln 2)} e^{-(k/K) \ln 2} \leq M_1 e^{K \ln r_1 + \ln 2} e^{-(k/K) \ln 2}$$

$$= 2M_1 r_1^K \left(\frac{1}{2}\right)^{\frac{k}{K}} = M r^k$$

where $M = 2M_1 r_1^K$ and $r = 1/\sqrt[k]{2} < 1$.

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16. Abstract An abstract approximation and convergence theory for the closed-loop solution of discrete-time linear-quadratic regulator problems for parabolic systems with unbounded input is developed. Under relatively mild stabilizability and detectability assumptions, functional analytic, operator theoretic techniques are used to demonstrate the norm convergence of Galerkin-based approximations to the optimal feedback control gains. The application of the general theory to a class of abstract boundary control systems is considered. Two examples, one involving the Neumann boundary control of a one dimensional heat equation, and the other, the vibration control of a cantilevered viscoelastic beam via shear input at the free end, are discussed.					
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