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VIBRATION FREQUENCIES OF TAPERED BARS WITH NONCLASSICAL BOUNDARY CONDITIONS

A report to

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Loads and Structural Dynamics
NASA Johnson Space Center
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July, 1988 by

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INTRODUCTION

In the interim report of October 1986 the goals for this research were revised and clarified. These goals are restated in the next section of this report, page 2, along with an evaluation of the accomplishment of the goal.

All of the cases of the truncated-cone beams that were originally proposed to be solved have been solved. A summary of these solutions is shown in this report under the section, problems solved, page 4. In addition, some cases of beams with unequal tapers have been solved and are discussed in the same section.

Since this research is continuing, the problems in progress section of this report, page 28, discusses problems under research at present. The final sections of this report present an up-to-date status of the budget, personnel, and bibliography for this project.

GOALS FOR THIS RESEARCH

Listed below are the goals for this research as revised in the interim report of October 1986.

Goal 1: Bibliography

In the process of this research we have begun to accumulate a comprehensive bibliography on transverse vibrations of uniform and tapered bars. There has been extensive publication of research on the solutions of eigenfrequencies and eigenfunctions of transversely freely vibrating bars using the Bernoulli-Euler equation. One of the goals will be to review and discuss a comprehensive bibliography of the research. This should be useful for researchers and designers.

Evaluation of Accomplishment: The bibliography at the end of this report is a comprehensive list of publications of transversely freely vibrating bars using the Bernoulli-Euler equation which has been compiled during our research. This bibliography contains two presentations made on the research in this project, references numbers 10 and 35. A survey paper on this subject is only in the beginning stages.

Goal 2: New Solutions

The main objective of the original proposal was to solve some of the previously unsolved problems on truncated-cone tapered bars with nonclassical boundary conditions. It must be realized that this is an open-ended goal, since the "unsolved problems" is not a defined list.

Evaluation of Accomplishment: All of the cases that were originally perceived have been solved. In addition some cases of bars with unequal tapers have been solved. Thus there has been even more progress on this goal than originally planned.

Goal 3: Monograph

Since there has been extensive research in this area it would be desirable to have the results presented in a form useable for researchers and designers. The results of research of these problems has been presented in many different ways. The purpose here will be to present the results in a form that will allow the most expeditious use. Some study of the results obtained by this research, in addition to study of past publications should result in accomplishment of this goal. At the conclusion of this project, it is expected that all these results will be accumulated in a comprehensive monograph, similar to that by Gorman (1975).

Evaluation of Accomplishment: Much of the data for this monograph has been accumulated, either from publications in the bibliography or from the research accomplished in this project. The outline, some descriptions, tables, and graphs have been developed for the monograph. The work under progress will also eventually be included in this monograph.

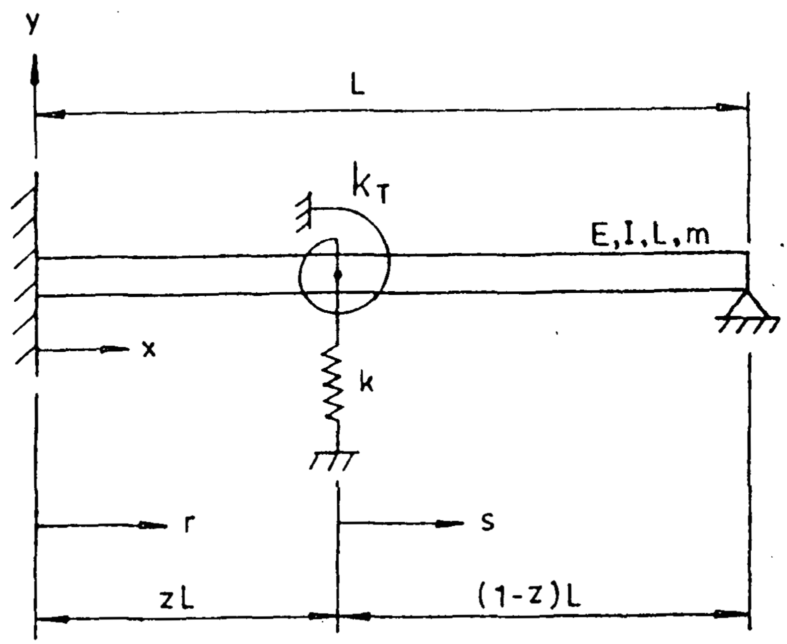
PROBLEMS SOLVED

This is a tabulation of the problems solved for this project through December 1987.

I. Uniform Beam with Constraining Springs at Various Locations Along the Length and Various End Conditions

The cases presented here have not been found in the literature. Although the beams are not tapered, the main purpose for solving these was to initiate the Rayleigh-Ritz method of solution with a simple case. The Rayleigh-Ritz method has been used in these cases as a check, and this will also be done for the tapered beams.

A. Uniform Cantilever with Simple Support at Right End.



Right end simply supported

For this beam the variables are shown in the figure above. To make the results more widely applicable, dimensionless spring constants were used:

$$\bar{k} = \frac{kL^3}{EI} \quad , \qquad \bar{k}_t = \frac{k_t L}{EI}$$

With the natural frequencies of the system being ω and the following substitutions made for simplification,

$$\beta^4 = \frac{\omega^2 m}{EI}$$

$a=\beta L$, $b=az=\beta Lz$,

$s=\sin(b)=\sin(\beta Lz)$

$c=\cos(b)=\cos(\beta Lz)$

$sh=\sinh(b)=\sinh(\beta Lz)$

$ch=\cosh(b)=\cosh(\beta Lz)$

$\bar{s}=\sin(a-b)=\sin(\beta L-\beta Lz)$

$\bar{c}=\cos(a-b)=\cos(\beta L-\beta Lz)$

$\bar{t}=\tan(a-b)=\tan(\beta L-\beta Lz)$

$\overline{sh}=\sinh(a-b)=\sinh(\beta L-\beta Lz)$

$\overline{ch}=\cosh(a-b)=\cosh(\beta L-\beta Lz)$

$\overline{th}=\tanh(a-b)=\tanh(\beta L-\beta Lz)$

the characteristic equation is:

$$\left\{ \frac{[-(c)(\bar{c})+(s)(\bar{s})+(ch)(\overline{ch})+(sh)(\overline{sh})]+(\frac{\overline{k_t}}{2\beta L})[-(s)(\bar{c})+(s)(\overline{ch})}{[\ (c)(\bar{s})+(s)(\bar{c})-(ch)(\overline{sh})-(sh)(\overline{ch})]+(\frac{\overline{k_t}}{2\beta L})[-(c)(\bar{c})+(c)(\overline{ch})} \right.$$

$$\left. \frac{-(\bar{c})(sh)+(sh)(\overline{ch})]+\frac{\bar{k}}{2\beta^3 L^3}[-(c)(\bar{s})+(c)(\overline{sh})+(\bar{s})(ch)-(ch)(\overline{sh})]}{+(\bar{c})(ch)-(ch)(\overline{ch})]+\frac{\bar{k}}{2\beta^3 L^3}[\ (s)(\bar{s})-(s)(\overline{sh})-(\bar{s})(sh)+(sh)(\overline{sh})]} \right\}$$

$$-\left\{ \frac{[\ (c)(\bar{c})-(s)(\bar{s})+(ch)(\overline{ch})+(sh)(\overline{sh})]+\frac{\overline{k_t}}{2\beta L}[(s)(\bar{c})}{[-(c)(\bar{s})-(s)(\bar{c})-(ch)(\overline{sh})-(sh)(\overline{ch})]+\frac{\overline{k_t}}{2\beta L}[(c)(\bar{c})} \right.$$

$$+(s)(\overline{ch})+(\bar{c})(sh)+(sh)(\overline{ch})]+\frac{\bar{k}}{2\beta^3L^3}[\ (c)(\bar{s})+(c)(\overline{sh})$$

$$\overline{+(c)(\overline{ch})-(\bar{c})(ch)-(ch)(\overline{ch})]+\frac{\bar{k}}{2\beta^3L^3}[-(s)(\bar{s})-(s)(\overline{sh})}$$

$$\frac{-(\bar{s})(ch)-(ch)(\overline{sh})]}{+(\bar{s})(sh)+(sh)(\overline{sh})]}\ \} = 0$$

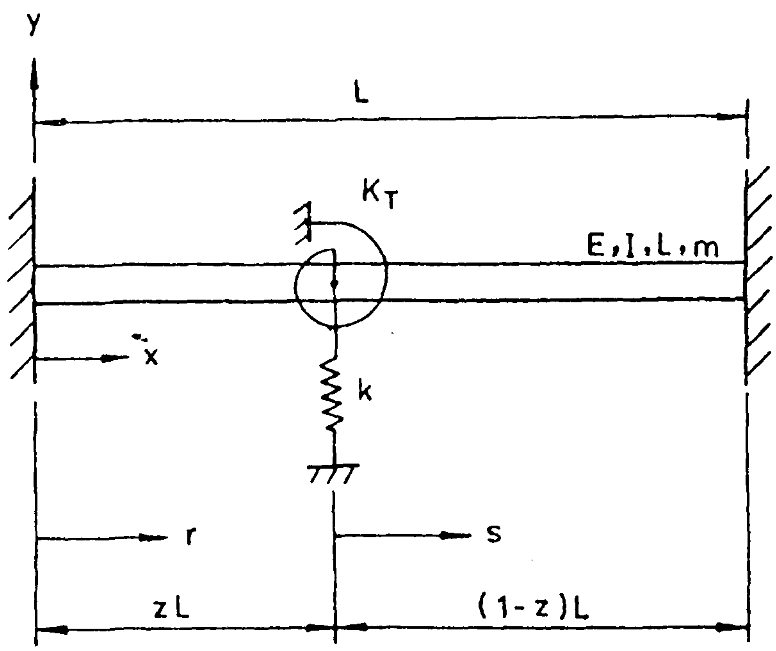
This equation was solved for the frequency parameter βL for all combinations of the following parameters:

k = 0, 1, 10, 100, 1000, 10000

k_t = 0, 1, 10, 100, 1000, 10000

z = 0.2, 0.4, 0.6, 0.8

B. Uniform Cantilever with Fixed Support at Right End



A uniform beam with both ends fixed

With the same parameters as A, the characteristic equation for this case is:

$$
\left\{ \frac{[-(c)(\bar{c})+(s)(\bar{s})+(ch)(\overline{ch})+(sh)(\overline{sh})]+\frac{\overline{K_T}}{2\beta L}[-(s)(\bar{c})+(s)(\overline{ch})-(\bar{c})(sh)+(sh)(\overline{ch})]+\frac{\bar{k}}{2\beta^3 L^3}[-(c)(\bar{s})+(c)(\overline{sh})+(\bar{s})(ch)-(ch)(\overline{sh})]}{[(c)(\bar{s})+(s)(\bar{c})-(ch)(\overline{sh})-(sh)(\overline{ch})]+\frac{\overline{K_T}}{2\beta L}[-(c)(\bar{c})+(c)(\overline{ch})+(\bar{c})(ch)-(ch)(\overline{ch})]+\frac{\bar{k}}{2\beta^3 L^3}[(s)(\bar{s})-(s)(\overline{sh})-(\bar{s})(sh)+(sh)(\overline{sh})]} \right\}
$$

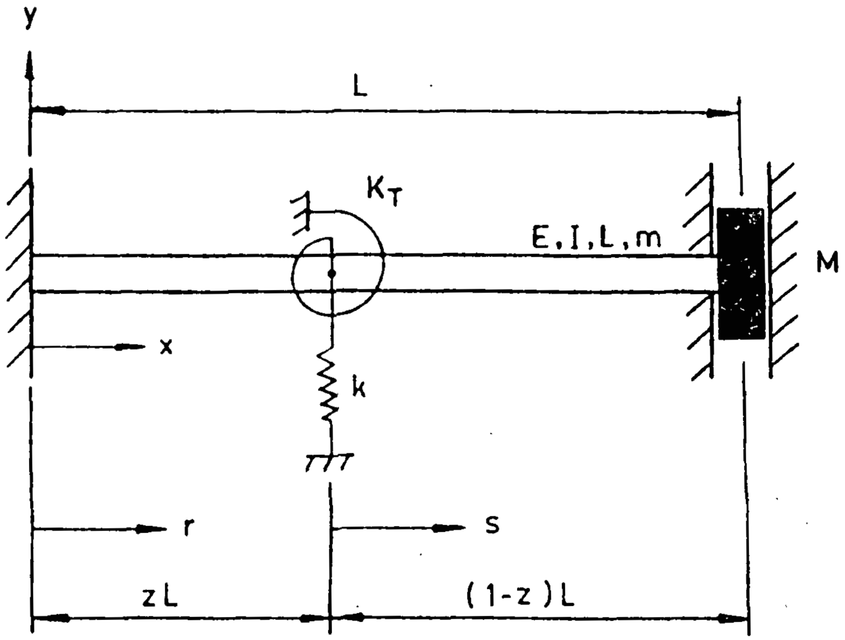
$$
- \left\{ \frac{[(c)(\bar{s})+(s)(\bar{c})+(sh)(\overline{ch})+(ch)(\overline{sh})]+\frac{\overline{K_T}}{2\beta L}[(s)(\bar{s})+(s)(\overline{sh})+(\bar{s})(sh)+(sh)(\overline{sh})]+\frac{\bar{k}}{2\beta^3 L^3}[-(c)(\bar{c})+(\bar{c})(ch)+(c)(\overline{ch})-(ch)(\overline{ch})]}{[(c)(\bar{c})-(s)(\bar{s})-(ch)(\overline{ch})-(sh)(\overline{sh})]+\frac{\overline{K_T}}{2\beta L}[(c)(\bar{s})+(c)(\overline{sh})-(\bar{s})(ch)-(ch)(\overline{sh})]+\frac{\bar{k}}{2\beta^3 L^3}[(s)(\bar{c})-(\bar{c})(sh)-(s)(\overline{ch})+(sh)(\overline{ch})]} \right\} = 0
$$

The equation was solved for the frequency parameter βL for the same combinations of parameters as for A.

C. Uniform Cantilever with Mass in a Slot at Right End

For this case a dimensionless mass parameter was defined:

$$M^* = M/mL$$



Right end with concentrated mass in a slot

The characteristic equation is:

$$\left\{ \frac{[(c)(\bar{s})+(s)(\bar{c})+(sh)(\overline{ch})+(ch)(\overline{sh})]+\frac{\overline{k_t}}{2\beta L}[(s)(\bar{s})+(s)(\overline{sh})+(\bar{s})(sh)+(sh)(\overline{sh})]+\frac{\bar{k}}{2\beta^3 L^3}[-(c)(\bar{c})+(\bar{c})(ch)+(c)(\overline{ch})-(ch)(\overline{ch})]}{[(c)(\bar{c})-(s)(\bar{s})-(ch)(\overline{ch})-(sh)(\overline{sh})]+\frac{\overline{k_t}}{2\beta L}[(c)(\bar{s})+(c)(\overline{sh})-(\bar{s})(ch)-(ch)(\overline{sh})]+\frac{\bar{k}}{2\beta^3 L^3}[(s)(\bar{c})-(\bar{c})(sh)-(s)(\overline{ch})+(sh)(\overline{ch})]} \right\}$$

$$-\left\{ \frac{2(\beta L)^7(M^*)[-(c)(\bar{c})+(s)(\bar{s})+(ch)(\overline{ch})+(sh)(\overline{sh})]+(\beta L)^6(\overline{k_t})(M^*)[-(s)(\bar{c})+(s)(\overline{ch})-(\bar{c})(sh)+(sh)(\overline{ch})]}{2(\beta L)^7(M^*)[(c)(\bar{s})+(s)(\bar{c})-(ch)(\overline{sh})-(sh)(\overline{ch})]+(\beta L)^6(\overline{k_t})(M^*)[-(c)(\bar{c})+(c)(\overline{ch})+(\bar{c})(ch)-(ch)(\overline{ch})]} \right\}$$

$$+\frac{(\beta L)^4(\bar{k})(M^*)[-(c)(\bar{s})+(c)(\overline{sh})+(\bar{s})(ch)-(ch)(\overline{sh})]}{(\beta L)^4(\bar{k})(M^*)[\ (s)(\bar{s})-(s)(\overline{sh})-(\bar{s})(sh)+(sh)(\overline{sh})]}$$

$$+\frac{2(\beta L)^6[-(s)(\bar{c})-(c)(\bar{s})+(sh)(\overline{ch})+(ch)(\overline{sh})]}{2(\beta L)^6[-(c)(\bar{c})+(s)(\bar{s})-(ch)(\overline{ch})-(sh)(\overline{sh})]}$$

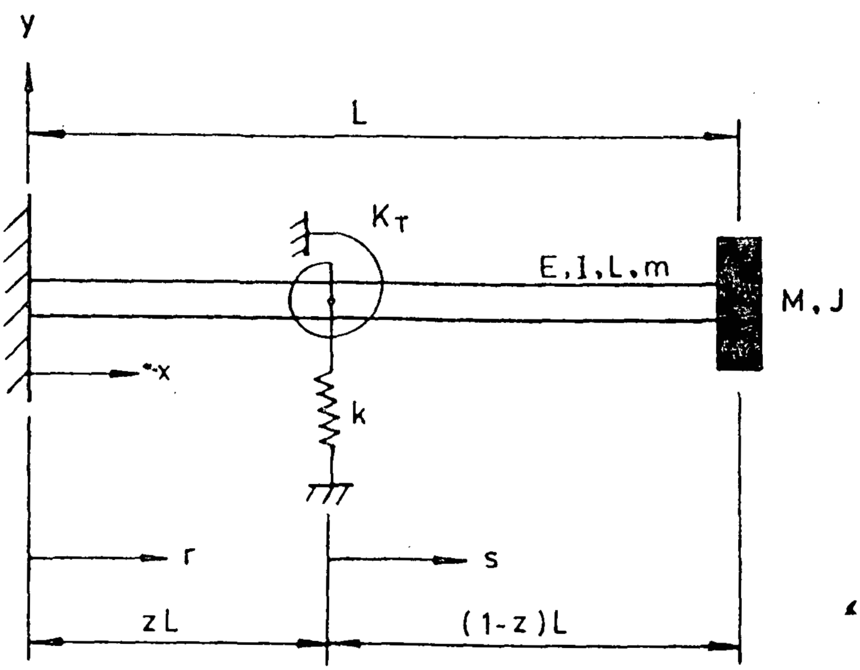
$$+\frac{(\beta L)^5(\overline{k_t})[-(s)(\bar{s})-(\bar{s})(sh)+(s)(\overline{sh})+(sh)(\overline{sh})]}{(\beta L)^5(\overline{k_t})[-(c)(\bar{s})+(\bar{s})(ch)+(c)(\overline{sh})-(ch)(\overline{sh})]}$$

$$+\frac{(\beta L)^3(\bar{k})[\ (c)(\bar{c})-(\bar{c})(ch)+(c)(\overline{ch})-(ch)(\overline{ch})}{(\beta L)^3(\bar{k})[-(s)(\bar{c})+(\bar{c})(sh)-(s)(\overline{ch})+(sh)(\overline{ch})}\ \} = 0$$

This equation was solved for the frequency parameter βL for the same combinations of parameters as A and B with

$$M^* = 0.5,\ 1,\ 5,\ 10$$

D. Uniform Cantilever with Concentrated Mass and Mass Moment of Inertia at Right End



Right end with mass and mass moment of inertia

For this case and additional dimensionless parameter was defined:

$$J^* = J/mL^3$$

and the characteristic equation is:

$$\Bigg\{ \frac{2(\beta L)^7(M^*)[-(c)(\bar{c})+(s)(\bar{s})+(ch)(\overline{ch})+(sh)(\overline{sh})]}{2(\beta L)^7(M^*)[\;(c)(\bar{s})+(s)(\bar{c})-(ch)(\overline{sh})-(sh)(\overline{ch})]}$$

$$\frac{+(\beta L)^6(\overline{k_t})(M^*)[-(s)(\bar{c})+(s)(\overline{ch})-(\bar{c})(sh)+(sh)(\overline{ch})]}{+(\beta L)^6(\overline{k_t})(M^*)[-(c)(\bar{c})+(c)(\overline{ch})+(\bar{c})(ch)-(ch)(\overline{ch})]}$$

$$\frac{+(\beta L)^4(\bar{k})(M^*)[-(c)(\bar{s})+(c)(\overline{sh})+(\bar{s})(ch)-(ch)(\overline{sh})]}{+(\beta L)^4(\bar{k})(M^*)[\;(s)(\bar{s})-(s)(\overline{sh})-(\bar{s})(sh)+(sh)(\overline{sh})]}$$

$$\frac{+2(\beta L)^6[-(s)(\bar{c})-(c)(\bar{s})+(sh)(\overline{ch})+(ch)(\overline{sh})]}{+2(\beta L)^6[-(c)(\bar{c})+(s)(\bar{s})-(ch)(\overline{ch})-(sh)(\overline{sh})]}$$

$$\frac{+(\beta L)^5(\overline{k_t})[-(s)(\bar{s})-(\bar{s})(sh)+(s)(\overline{sh})+(sh)(\overline{sh})]}{+(\beta L)^5(\overline{k_t})[-(c)(\bar{s})+(\bar{s})(ch)+(c)(\overline{sh})-(ch)(\overline{sh})]}$$

$$\frac{+(\beta L)^3(\bar{k})[\;(c)(\bar{c})-(\bar{c})(ch)+(c)(\overline{ch})-(ch)(\overline{ch})]}{+(\beta L)^3(\bar{k})[-(s)(\bar{c})+(\bar{c})(sh)-(s)(\overline{ch})+(sh)(\overline{ch})]} \Bigg\}$$

$$- \Bigg\{ \frac{2(\beta L)^7(J^*)[(s)(\bar{c})+(c)(\bar{s})+(sh)(\overline{ch})+(ch)(\overline{sh})]}{2(\beta L)^7(J^*)[(c)(\bar{c})-(s)(\bar{s})-(ch)(\overline{ch})-(sh)(\overline{sh})]}$$

$$\frac{+(\beta L)^6(\overline{k_t})(J^*)[(s)(\bar{s})+(\bar{s})(sh)+(s)(\overline{sh})+(sh)(\overline{sh})]}{+(\beta L)^6(\overline{k_t})(J^*)[(c)(\bar{s})-(\bar{s})(ch)+(c)(\overline{sh})-(ch)(\overline{sh})]}$$

$$\frac{+(\beta L)^4(\bar{k})(J^*)[-(c)(\bar{c})+(\bar{c})(ch)+(c)(\overline{ch})-(ch)(\overline{ch})]}{+(\beta L)^4(\bar{k})(J^*)[\;(s)(\bar{c})-(\bar{c})(sh)-(s)(\overline{ch})+(sh)(\overline{ch})]}$$

$$\frac{+2(\beta L)^4[(s)(\bar{s})-(c)(\bar{c})-(sh)(\overline{sh})-(ch)(\overline{ch})]}{+2(\beta L)^4[(c)(\bar{s})+(s)(\bar{c})+(ch)(\overline{sh})+(sh)(\overline{ch})]}$$

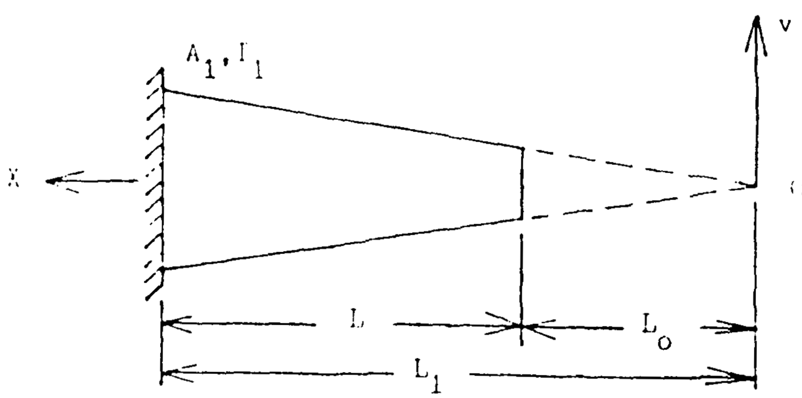
$$\frac{+(\beta L)^3(\overline{k_t})[-(s)(\bar{c})-(\bar{c})(sh)-(s)(\overline{ch})-(sh)(\overline{ch})]}{+(\beta L)^3(\overline{k_t})[-(c)(\bar{c})+(\bar{c})(ch)-(c)(\overline{ch})+(ch)(\overline{ch})]}$$

$$\frac{+(\beta L)(\bar{k})[-(c)(\bar{s})+(\bar{s})(ch)-(c)(\overline{sh})+(ch)(\overline{sh})]}{+(\beta L)(\bar{k})[\ (s)(\bar{s})-(\bar{s})(sh)+(s)(\overline{sh})-(sh)(\overline{sh})]} \quad \} = 0$$

This characteristic equation was solved for the frequency parameter βL for the same combinations of parameters as A and B with all combination of $M^* = 0, 1, 10$ and $J^* = 0, 1, 10$.

II. Truncated-Cone Tapered Beams

A simple case for the truncated-cone tapered beam or beam with the same taper ratio for width and depth is shown in the figure below. The coordinates and beam parameters are described in this figure.



Linearly Tapered Beam

E = Young's Modulus

I_1 = area moment of inertia at base of beam

A_1 = area of the beam at the base

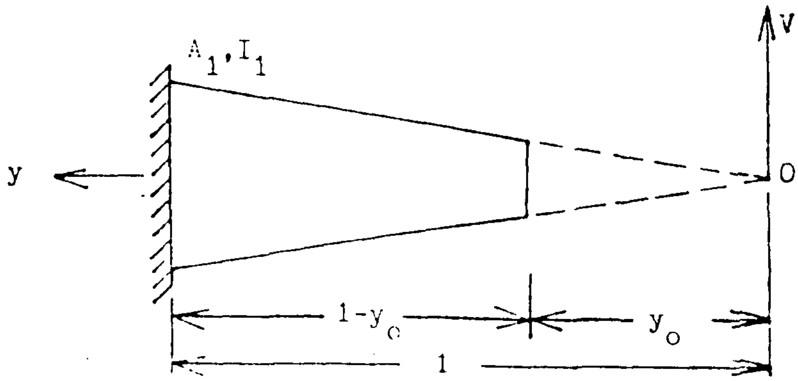
ρ = density of the beam material

L = length of the beam

A dimensionless coordinate y is introduced

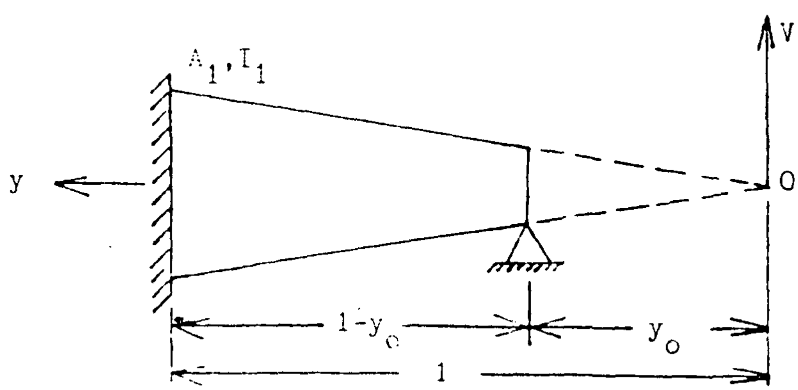
$$y = x/L_1$$

as shown in the figure below. The taper ratio is now y_o.



Linearly Tapered Beam with Dimensionless Coordinates

A. Linearly Tapered Cantilever Simply Supported at the Right End



The characteristic determinant for this case is:

$$\begin{vmatrix} J_2(a) & Y_2(a) & I_2(a) & K_2(a) \\ J_4(a) & Y_4(a) & I_4(a) & K_4(a) \\ J_2(c) & Y_2(c) & I_2(c) & K_2(c) \\ J_3(c) & Y_3(c) & -I_3(c) & K_3(c) \end{vmatrix} = 0$$

$a = 2qy_o^{1/2}$ and $c = 2q$

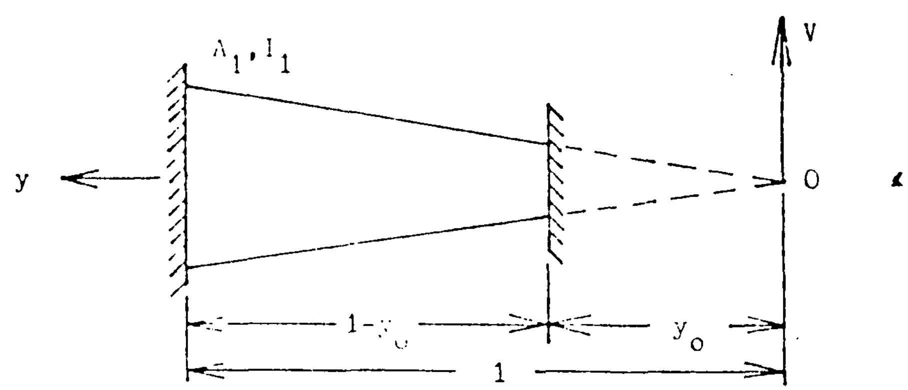
$$q^4 = \frac{\rho A_1 \omega^2 L_1{}^4}{EI_1}$$

J_2, J_3, and J_4 are Bessel functions of the first kind, second, third, and fourth order, Y_2, Y_3, and Y_4 are Bessel functions of the second kind, second, third, and fourth order. I_2, I_3, and I_4 are modified Bessel functions of the first kind, second, third, and fourth order, K_2, K_3, and K_4 are modified Bessel functions of the second kind, second, third, and fourth order.

The characteristic determinant was solved for the frequency parameter q for taper ratios of

y_o = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8 and 0.9

B. Linearly Tapered Cantilever with Right End Fixed

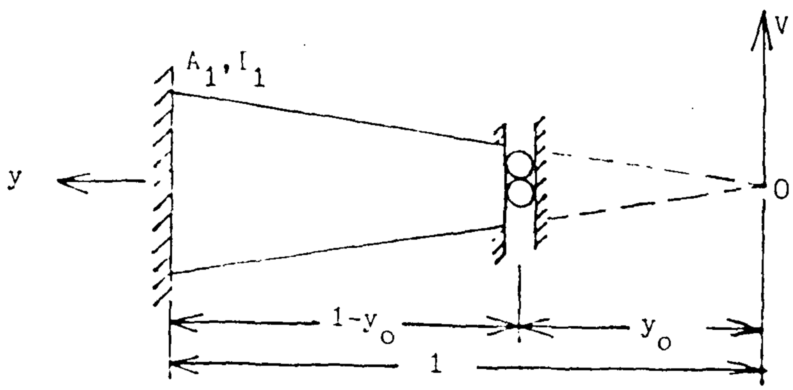


The characteristic determinant for this case is:

$$\begin{vmatrix} J_2(a) & Y_2(a) & I_2(a) & K_2(a) \\ J_3(a) & Y_3(a) & -I_3(a) & K_3(a) \\ J_2(c) & Y_2(c) & I_2(c) & K_2(c) \\ J_3(c) & Y_3(c) & -I_3(c) & K_3(c) \end{vmatrix} = 0$$

This characteristic determinant was solved for the frequency parameter q for the same values of taper ratio as A.

C. Linearly Tapered Beam with Right End in Slot

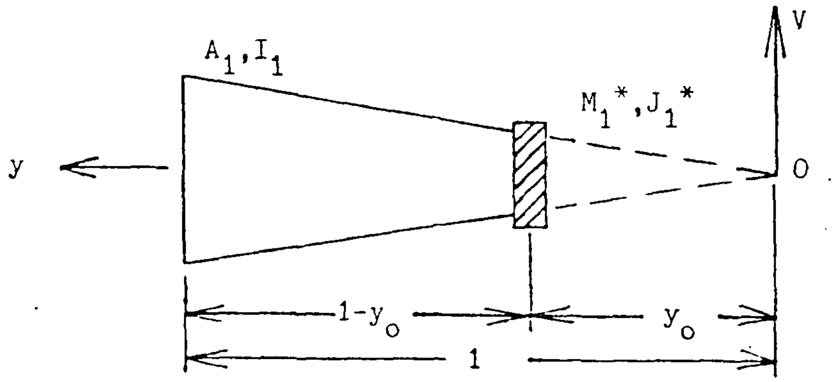


The characteristic determinant for this case is:

$$\begin{vmatrix} J_3(a) & Y_3(a) & -I_3(a) & K_3(a) \\ J_3(a) & Y_3(a) & I_3(a) & -K_3(a) \\ J_2(c) & Y_2(c) & I_2(c) & K_2(c) \\ J_3(c) & Y_3(c) & -I_3(c) & K_3(c) \end{vmatrix} = 0$$

The characteristic equation was solved for the frequency parameter q for the same values of taper ratio as for A and B.

D. Linearly Tapered Beam with Left End Free and a Concentrated/Rotary Inertial End Mass at Right End



Dimensionless parameters are introduced for the mass and rotary inertia of the mass:

$$J_1^* = \frac{J_1}{m_b L^2} \quad , \quad M_1^* = \frac{M_1}{m_b} \quad ,$$

where $m_b = \frac{1}{3} \rho A_1 L(1 + y_o + y_o^2)$ is the mass of the tapered beam

The characteristic determinant for this case is:

$$\begin{vmatrix} J_3(a)-XXJ_2(a) & Y_3(a)-XXY_2(a) & I_3(a)-XXI_2(a) & -K_3(a)-XXK_2(a) \\ J_4(a)-BBJ_3(a) & Y_4(a)-BBY_3(a) & I_4(a)+BBI_3(a) & K_4(a)-BBK_3(a) \\ J_3(c)-DDJ_2(c) & Y_3(c)-DDY_2(c) & I_3(c)-DDI_2(c) & -K_3(c)-DDK_2(c) \\ J_4(c)-CCJ_3(c) & Y_4(c)-CCY_3(c) & I_4(c)+CCI_3(c) & K_4(c)-CCK_3(c) \end{vmatrix} = 0$$

where $XX = (1 + y_o + y_o^2)(1 - y_o)y_o^{-2.5}qM_1^*/3$

$$BB = (1 + y_o + y_o^2)(1 - y_o)^3 y_o^{-3.5}q^3J_1^*/3$$

$$DD = -(y_o^2 + y_o + 1)(1 - y_o)qM_2^*/3$$

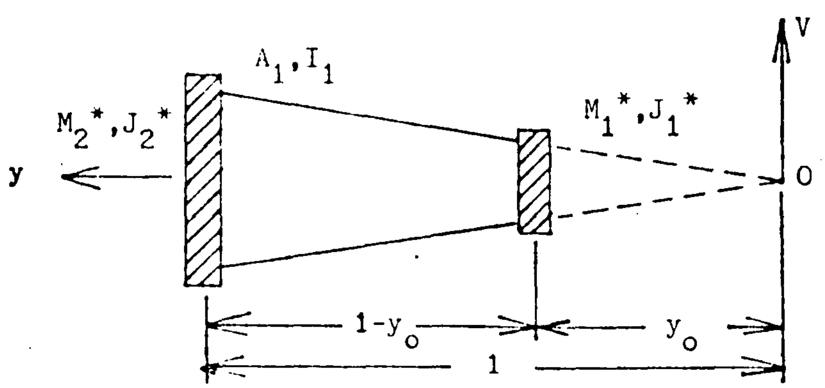
$$CC = -(y_o^2 + y_o + 1)(1 - y_o)q^3J_2^*/3$$

$$a = 2qy_o^{1/2}$$

$$c = 2q$$

This case was solved for the frequency parameter q for the same taper ratios as the previous cases and for various combinations of M^* and J^*.

E. Linearly Tapered Beam with Concentrated/Rotary Inertial End Mass on Both Ends



With the dimensionless parameters,

$$J_1^* = \frac{J_1}{m_bL^2}, \quad M_1^* = \frac{M_1}{m_b}, \quad J_2^* = \frac{J_2}{m_bL^2}, \quad M_2^* = \frac{M_2}{m_b}$$

the characteristic equation for this case is:

$$\begin{vmatrix} J_3(a)-XXJ_2(a) & Y_3(a)-XXY_2(a) & I_3(a)-XXI_2(a) & -K_3(a)-XXK_2(a) \\ J_4(a)-BBJ_3(a) & Y_4(a)-BBY_3(a) & I_4(a)+BBI_3(a) & K_4(a)-BBK_3(a) \\ J_3(c)-DDJ_2(c) & Y_3(c)-DDY_2(c) & I_3(c)-DDI_2(c) & -K_3(c)-DDK_2(c) \\ J_4(c)-CCJ_3(c) & Y_4(c)-CCY_3(c) & I_4(c)+CCI_3(c) & K_4(c)-CCK_3(c) \end{vmatrix} = 0$$

where $XX = (1 + y_o + y_o^2)(1 - y_o)y_o^{-2.5}qM_1^*/3$

$BB = (1 + y_o + y_c^2)(1 - y_o)^3 y_o^{-3.5}q^3 J_1^*/3$

$DD = -(y_o^2 + y_o + 1)(1 - y_o)qM_2^*/3$

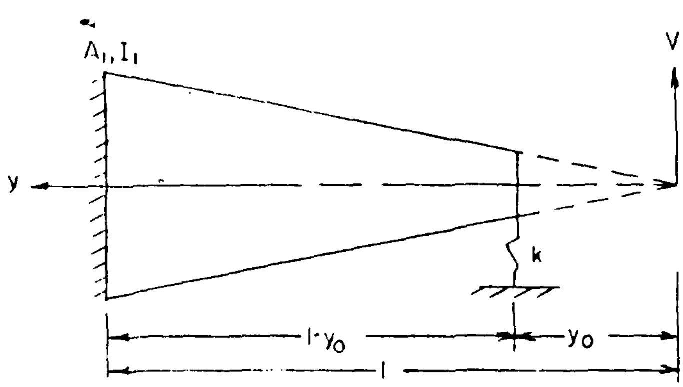
$CC = -(y_o^2 + y_o + 1)(1 - y_o)q^3 J_2^*/3$

$a = 2qy_o^{1/2}$

$c = 2q$

This characteristic equation was solved for the frequency parameter q for the same values of taper ratio as the previous cases and for various combinations of M_1^*, J_1^*, M_2^*, and J_2^*.

F. Linearly Tapered Beam with Constraining Translational Spring at Right End.



For this case a dimensionless spring constant was introduced,

$$k^* = \frac{kL^3}{EI_1}$$

The characteristic determinant is:

$$\begin{vmatrix} J_3(a) - XXJ_2(a) & Y_3(a) - XXY_2(a) & I_3(a) - XXI_2(a) & -K_3(a) - XXK_2(a) \\ J_4(a) & Y_4(a) & I_4(a) & K_4(a) \\ J_3(b) & Y_3(b) & -I_3(b) & K_3(b) \\ J_2(b) & Y_2(b) & I_2(b) & K_2(b) \end{vmatrix} = 0$$

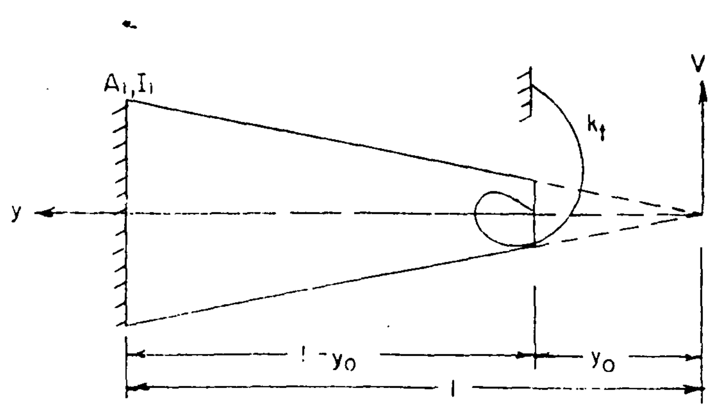
$$\text{where } XX = -\frac{k^*}{(1-y_o)^3 q^3 y_o^{5/2}}$$

$$a = 2qy_o^{1/2}$$

$$\text{and} \quad b = 2q \quad .$$

This characteristic determinant was solved for the frequency parameter q for taper ratios of 0.1, 0.5, and 0.8 and for the dimensionless spring constant equal to 1, 10, and 100.

G. Linearly Tapered Beam with Constraining Torsional Spring at Right End



The dimensionless spring constant for this case is:

$$k_t^* = \frac{k_t L}{EI_1}$$

The characteristic determinant is:

$$\begin{vmatrix} J_4(a)-BBJ_3(a) & Y_4(a)-BBY_3(a) & I_4(a)+BBI_3(a) & K_4(a)-BBK_3(a) \\ J_3(a) & Y_3(a) & I_3(a) & -K_3(a) \\ J_3(b) & Y_3(b) & -I_3(b) & K_3(b) \\ J_2(b) & Y_2(b) & I_2(b) & K_2(b) \end{vmatrix} = 0$$

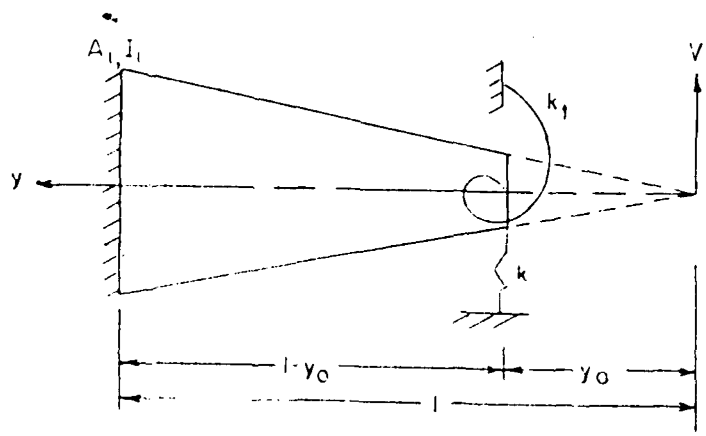
$$\text{where } BB = -\frac{k_t^*}{(1-y_o)qy_o^{7/2}} ,$$

$$a = 2qy_o^{1/2}$$

$$\text{and} \quad b = 2q \quad .$$

This characteristic determinant was solved for the frequency parameter q for taper ratios of 0.1, 0.5, and 0.8 and for the dimensionless spring constant equal to 0.001, 0.1, and 10.

H. Linearly Tapered Beam with Constraining Translational and Rotational Springs at Right End



The characteristic determinant for this case is:

$$\begin{vmatrix} J_4(a)-BBJ_3(a) & Y_4(a)-BBY_3(a) & I_4(a)+BBI_3(a) & K_4(a)-BBK_3(a) \\ J_3(a)-XXJ_2(a) & Y_3(a)-XXY_2(a) & I_3(a)-XXI_2(a) & -K_3(a)-XXK_2(a) \\ J_3(b) & Y_3(b) & -I_3(b) & K_3(b) \\ J_2(b) & Y_2(b) & I_2(b) & K_2(b) \end{vmatrix} = 0$$

$$\text{where } BB = -\frac{k_t^*}{(1-y_o)qy_o^{7/2}},$$

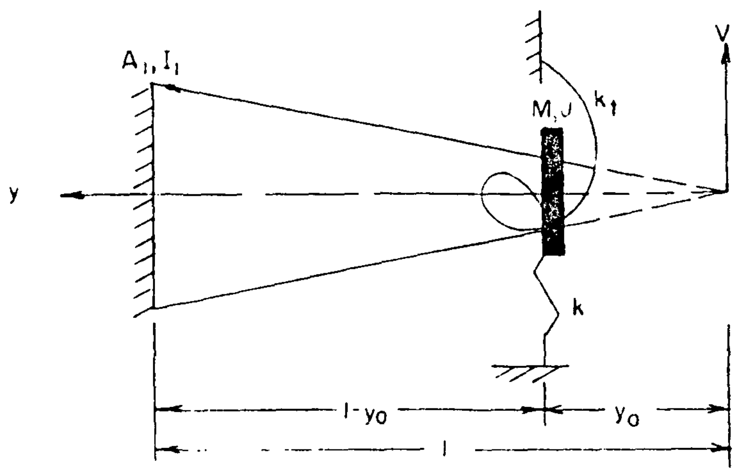
$$\text{and} \quad XX = -\frac{k^*}{(1-y_o)^3 q^3 y_o^{5/2}}$$

The characteristic determinant was solved for the frequency parameter q for taper ratios of 0.1, 0.5, and 0.8 with all combinations of the dimensionless spring constants:

$$k^* = 0.1,\ 1,\ 10$$

$$k_t = 0.001,\ 0.1,\ 10.$$

I. Linearly Tapered Beam with Constraining Springs, Concentrated/Rotary Inertial Mass at Right End.



The characteristic equation for this case is:

$$\begin{vmatrix} J_4(a)-BB_1J_3(a) & Y_4(a)-BB_1Y_3(a) & I_4(a)+BB_1I_3(a) & K_4(a)-BB_1K_3(a) \\ J_3(a)-XX_1J_2(a) & Y_3(a)-XX_1Y_2(a) & I_3(a)-XX_1I_2(a) & -K_3(a)-XX_1K_2(a) \\ J_3(b) & Y_3(b) & -I_3(b) & K_3(b) \\ J_2(b) & Y_2(b) & I_2(b) & K_2(b) \end{vmatrix} = 0$$

where $BB_1 = \frac{1}{3} q^3(1-y_o)^3 y_o^{-7/2}(1+y_o+y_o^2)J^*$

$$- y_o^{-7/2}[q(1-y_o)]^{-1}kt^* \quad ,$$

and $XX_1 = \frac{1}{3} q y_o^{-5/2}(1-y_o)(1 + y_o + y_o^2)M^*$

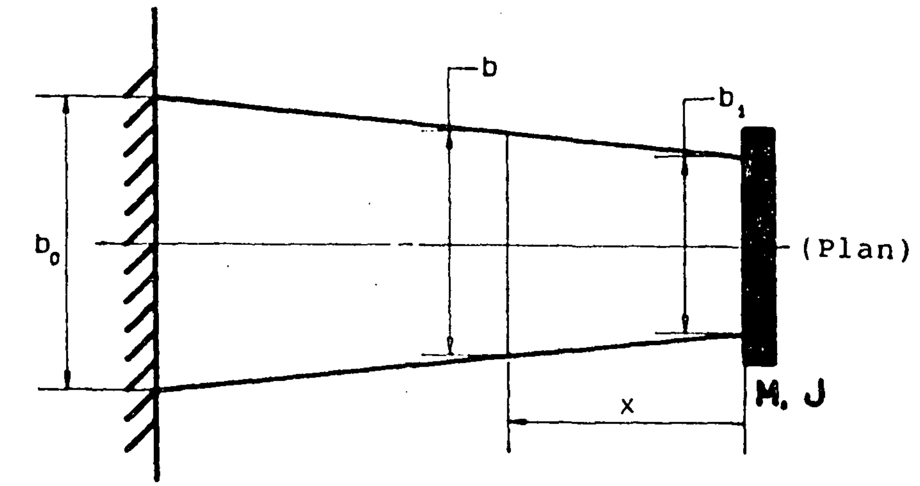
$$- y_o^{-5/2}[q(1-y_o)]^{-3}k^* \quad .$$

The characteristic determinant was solved for the frequency parameter q for taper ratios of 0.1, 0.5, and 0.8 for various combinations of the other dimensionless parameters.

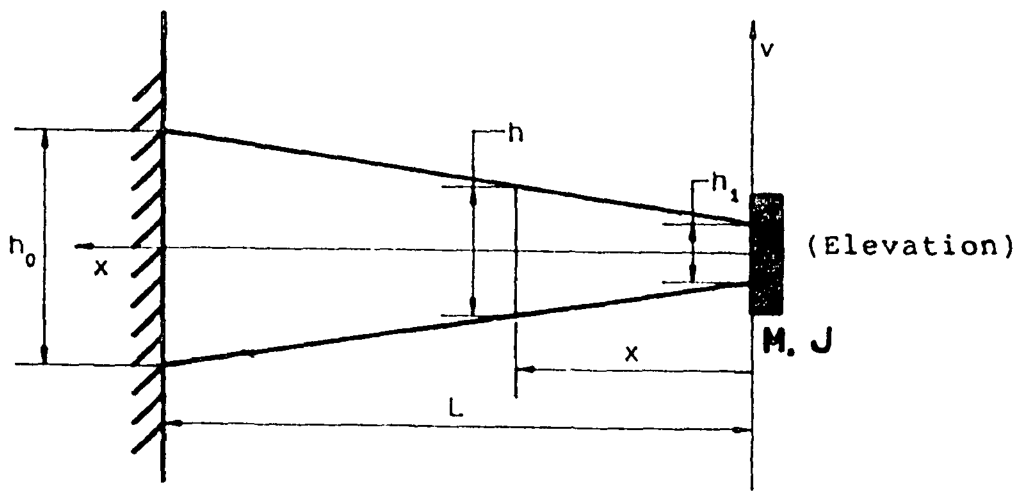
All the cases of tapered beams were solved for at least the first three frequency parameters, in some cases for the first five. In addition the mode shapes were determined for many cases.

III. Beams with Unequal Tapers

A cantilever beam with unequal taper and a concentrated mass M and rotary inertia J at the end is shown in the figure below. This problem is more complex that a truncated-cone beam, but a few of the simpler cases have been solved in this project.



$$\text{Taper Ratio in Horizontal Plane} = \beta = \frac{b_0}{b_1}$$



$$\text{Taper Ratio in Vertical Plane} = \alpha = \frac{h_0}{h_1}$$

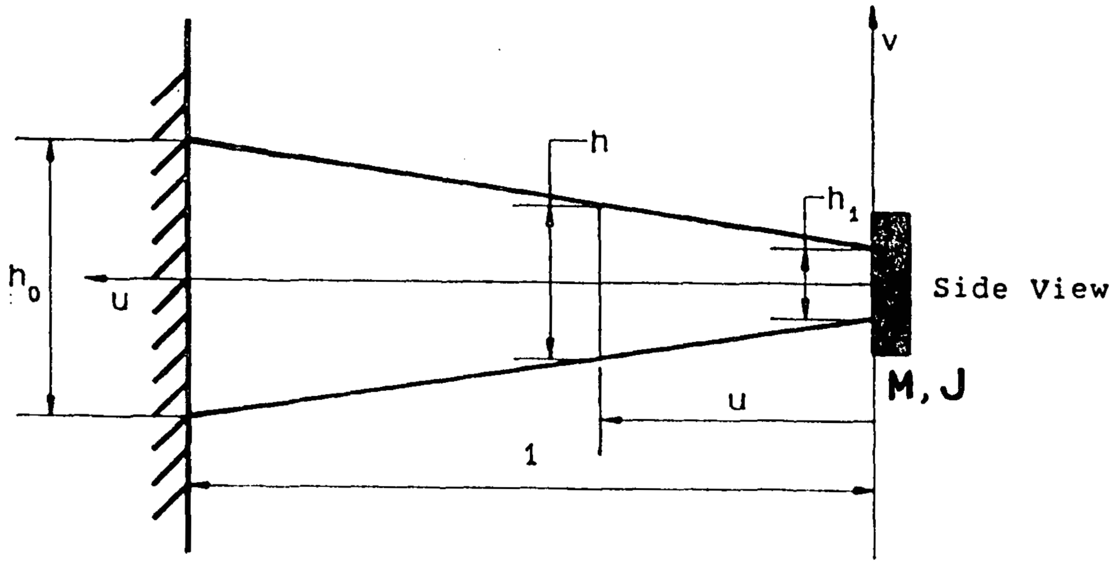
Tapered Cantilever Bar Carrying a Concentrated End Mass M Having a Rotary Inertia J

A. Cantilever Bar of Constant Width and Thickness Taper Ratio α with a Concentrated End Mass

With a dimensionless coordinate introduced,

$$u = x/L,$$

this bar is as shown below.



The dimensionless parameters are as before

$$J^* = \frac{J}{m_b L^2}, \qquad M^* = \frac{M}{m_b},$$

where:

$m_b = \rho A_1 L[1+\frac{1}{2}(\alpha+\beta-2)+\frac{1}{3}[\alpha-1)(\beta-1)]$ is the mass of the tapered bar with unequal tapers.

The characteristic determinant is:

$$\begin{vmatrix} J_3(b)-BBJ_2(b) & Y_3(b)-BBY_2(b) & I_3(b)+BBI_2(b) & K_3(b)-BBK_2(b) \\ J_2(b)-AAJ_1(b) & Y_2(b)-AAY_1(b) & I_2(b)-AAI_1(b) & -K_2(b)-AAK_1(b) \\ J_2(a) & Y_2(a) & -I_2(a) & K_2(a) \\ J_1(a) & Y_1(a) & I_1(a) & K_1(a) \end{vmatrix} = 0$$

where

$$BB = \frac{1}{2}q^3 J^*(\alpha+1),$$

$$AA = \frac{1}{2}qM^*(\alpha+1), \qquad q^4 = \frac{\rho A_1 \omega^2 L^4}{EI_1}$$

$$b = 2q/(\alpha-1),$$

$$a = 2q\alpha^{0.5}/(\alpha-1).$$

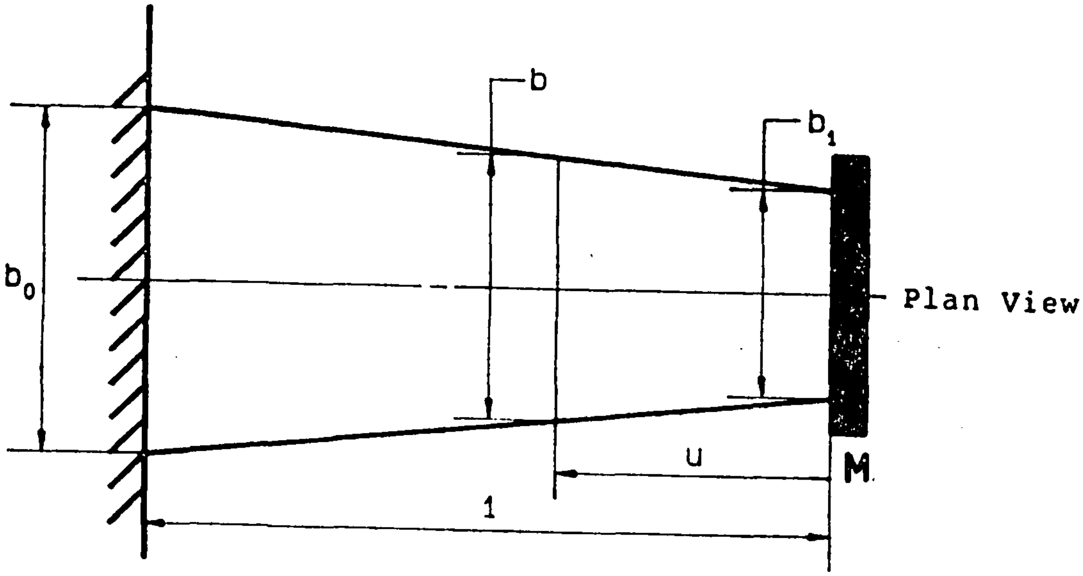
The characteristic determinant was solved for the frequency parameter q for the first three modes for all combinations of:

taper ratio, α = 1.2, 1.4, 1.6, 1.8, 2.0, 2.5, 3.0, 3.5, 4.0, 5.0, 10.0

M^* = 0.1, 0.3, 0.5, 1.0, 5.0, 10.0

and J^* = 0.01, 0.1, 1.0.

B. Tapered Bar of Constant Thickness and Width Taper Ratio β with a Concentrated End Mass



For this case the differential equation is such that a closed form solution is not available. This case was solved by a finite difference technique. Solutions were obtained for the frequency parameter q for the first five modes for all combinations of:

taper ratio β = 1.2, 1.4, 1.6, 1.8, 2.0, 2.5, 3.0, 3.5, 4.0, 5.0, 10.0

M^* = 0.1, 0.3, 0.5, 1.0, 5.0, 10.0

The mode shapes were also determined for many of the cases of beams of unequal taper.

C. Tapered Bar of Variable Thickness and Width Taper Ratios with a Concentrated End Mass

This is essentially the same type of solution as B. except that both taper ratios were varied. Solutions were obtained for the frequency parameter q for the first five modes for all combinations of:

taper ratio α = 1.0, 1.2, 1.4, 1.6, 2.0, 2.5, 3.0, 4.0, 5.0

taper ratio β = 1.0, 1.2, 1.4, 1.6, 2.0, 2.5, 3.0, 4.0, 5.0

M^* = 0, 0.01, 0.1, 0.3, 0.5, 1.0, 2.0, 5.0, 10.0

The mode shapes were also determined for many of the cases of beams with unequal taper.

IV. Truncated-Cone Beams Using the Rayleigh-Ritz Method

The Rayleigh-Ritz method was used as an independent method of solving some of the truncated-cone beam problems, as well as some of the uniform beam problems, to check the solutions. The Rayleigh-Ritz method produces an upper bound on the frequency parameters, and solutions were checked for the following cases:

Case I.D Uniform Cantilever with Constraining Springs and Concentrated Mass and Mass Moment of Inertia at Right End

Cases II. F, G, H, and I Linearly Tapered Cantilevers with a Constraining Translational Spring, a Constraining Rotational Spring, a Concentrated Rotary Inertial Mass at Right End

PROBLEMS IN PROGRESS

Since this research will continue, the problems in progress at the present time are listed here.

I. Tapered Beams with Unequal Tapers and End Masses Using the Rayleigh-Ritz Method

The Rayleigh-Ritz method is being used to check the answers of these problems already solved. This method is a completely different method that uses more computer time, but is a excellent way to check some of the solutions.

II. Tapered Beams with Unequal Tapers, Concentrated and Inertial End Mass

We are continuing to solve the problem with unequal tapers in both depth and width, with both concentrated and inertial end mass. At present we are not able to use the numerical method used for cases III. B and C because of some complications introduced by the inclusion of the mass moment of inertia of the end mass. Even if this is not resolved, these problems can be worked using the Rayleigh-Ritz method.

III. Tapered Beams with Unequal Tapers and Constraining Springs at One Position Along the Beam

Constraining springs, one linear and one rotational, situated at a given position along the length of a tapered beam results in a characteristic determinant of order eight rather that four. This determinant has been obtained for a beam with constant width and variable thickness, and other cases are being pursued.

PERSONNEL, EQUIPMENT, AND BUDGET

Personnel: The three students who have received scholarships through this grant have received their Master's degrees. The students, their thesis titles, and dates of graduation are:

> John Michael Lucero, "Vibrational Analysis of Constrained Tapered Cantilever Beams with Nonclassical Boundary Conditions," August 1987.
>
> Manuel Lazos Jr., "Transverse Vibrational Analysis of Linearly Tapered Beams with Classical and Nonclassical Boundary Conditions," August 1987
>
> David Raymundo Serna, "Vibration Frequencies of a Double-Tapered Beam with Concentrated End Mass," December 1987.

Another student, Jose Antonio Nava, began to receive a NASA scholarship January 15, 1988.

Budget: The following is a preliminary accounting of the expenditures that have been made from June 1, 1986 through June 30, 1988. A nocost extension for this grant was awarded for six months beginning January 1, 1988. During the time of this extension, several transfers were made between the accounts of the grant, and the only money expended was for scholarships. The following is an approximation of expenditures based on the information that I have at hand. An official statement will be forthcoming when the Business Office closes this account.

	expenditures (through 6/88)
Salaries & Wages -------- (Dr. Lionel Craver)	$24,885.60
Fringe Benefits --------- (Dr. Lionel Craver)	4,605.66
Stipends (Manuel Lazos, -- (John Lucero, David Serna, Jose Nava)	38,500.00
Equipment ----------------	6,009.50
Materials ----------------	97.34
Computer Time ------------	5,000.00
Travel -------------------	2,797.91
Overhead -----------------	16,777.83
Total Expenditures -------	98,673.84
Original Budget ----------	98,734.00

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