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NONAXISYMMETRIC VISCOUS LOWER BRANCH
MODES IN AXISYMMETRIC SUPERSONIC FLOWS

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NONAXISYMMETRIC VISCOUS LOWER BRANCH MODES IN AXISYMMETRIC SUPERSONIC FLOWS

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Abstract

In a previous paper, we have considered the weakly nonlinear interaction of a pair of axisymmetric lower branch Tollmien-Schlichting instabilities in cylindrical supersonic flows. Here the possibility that nonaxisymmetric modes might also exist is investigated. In fact it is found that such modes do exist and, on the basis of linear theory, it appears that these modes are the most important. The nonaxisymmetric modes are found to exist for flows around cylinders with nondimensional radius a less than some critical value a_c. This critical value a_c is found to increase monotonically with the azimuthal wavenumber n of the disturbance and it is found that unstable modes always occur in pairs. We show that in general, instability in the form of lower branch Tollmien-Schlichting waves will occur first for nonaxisymmetric modes and that in the unstable regime, the largest growth rates correspond to the latter modes.

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1. Introduction

Recent interest in the development of supersonic and hypersonic vehicles has stimulated research into the different instability mechanisms which might cause transition to turbulence in high speed compressible flows. Apart from Tollmien-Schlichting waves, Görtler, and crossflow vortices, the possibility of inviscid instability modes must be taken into account for compressible flows. Here an investigation of the role of nonaxisymmetric lower branch Tollmien-Schlichting waves in supersonic flows with an axis of symmetry is investigated.

Following the work of Smith (1979a,b) it is now well known that triple-deck theory provides a self-consistent asymptotic framework for the description of lower branch linear and nonlinear, two and three dimensional Tollmien-Schlichting waves. Thus it is known that finite amplitude Tollmien-Schlichting waves are stabilized by nonlinear effects as they cross the lower branch of the neutral curve. At higher disturbance amplitudes further downstream Smith and Burggraf (1985) have shown that a hierarchy of fully nonlinear states can be achieved. The planar compressible problem has been investigated using triple-deck theory by Smith (1988). At subsonic speeds lower branch disturbances are essentially unchanged from their incompressible forms. However, at supersonic speeds only three-dimensional modes can be unstable and the critical angle of propagation above which instability can occur increases with the Mach number. At hypersonic speeds Smith (1988) has shown that the modes are then fully nonparallel and a quasi-parallel theory based on triple deck theory fails. It can be shown that in this regime the modes have a structure similar to that which describes Görtler vortices in incompressible flows (see for example Hall (1983)).

In a previous paper, Duck and Hall (1988), we investigated the linear and weakly nonlinear theory of lower branch axisymmetric Tollmien-Schlichting instabilities in supersonic cylindrical flows. We found that such modes exist in pairs and that at a given Mach number they occur only for a body radius less than a critical value. In the limit of small body radius both modes have wavelength going to zero with respect to the usual triple deck streamwise length scale. In the weakly nonlinear stage it was shown that, dependent on the input frequency, either mode can lead to a stable finite amplitude state.

Here we generalize the above calculation to see if nonaxisymmetric modes can be important in the linear regime. We perturb an axisymmetric supersonic flow to three-dimensional disturbances with wavenumbers α and n in the streamwise and azimuthal directions. We show that at all supersonic speeds there is a finite band of non-dimensional body radii which can support two three-dimensional modes of the same azimuthal wavenumber and that in the limit of bod radius tending to zero one mode has wavelength tending to zero

1

whilst the other tends to infinity. Thus for these cylinders Tollmien-Schlichting waves with both short and long wavelengths compared to usual streamwise triple deck scale can occur. At higher azimuthal wavenumbers and larger body radii we show how the modes relate to the planar three dimensional modes of Smith (1988). We shall show that, on the basis of linear theory, it is the three-dimensional modes which are the most dangerous since they occur at the lowest Reynolds number and when they occur have the largest growth rates.

The procedure adopted in the rest of this paper is as follows: in §2 we formulate the linear stability problem for supersonic cylindrical flows. In §3 we determine the eigenrelations for three-dimensional TS waves around a cylinder with radius comparable with the upper deck thickness. In §4 the eigenrelation is investigated in the limit of small and large cylinders and also in the high azimuthal wavenumber limit. Finally in §5 we describe some results and draw some conclusions.

2. Formulation

We shall be concerned with the linear stability of an axisymmetric boundary layer on a cylindrical body of radius a^*, in a uniform supersonic stream of velocity U^*_∞ aligned with the axis of the cylinder.

If L^* denotes a typical streamwise length scale (for example the distance from some leading edge), ν^*_∞ the kinematic viscosity of the fluid in the far field, then the Reynolds number Re is defined to be

$$Re = U^*_\infty L^* / \nu^*_\infty. \tag{2.1}$$

It is found useful to introduce a related parameter ϵ,

$$\epsilon = Re^{-\frac{1}{8}}. \tag{2.2}$$

In this paper the Reynolds number is taken to be large, implying ϵ is small.

It is also assumed that

$$\bar{a} = \frac{a^*}{\epsilon^3 L^*} = 0(1), \tag{2.3}$$

denotes the scale of the radius of the body. This follows the scale chosen by Duck and Hall (1988), Kluwick et al. (1984), and one of the scales chosen by Duck (1984), amongst others; this turns out to be an important choice of body radius scale, with curvature terms playing a crucial role in the physics of the problem.

The problem then takes on a triple deck structure. The following non-dimensional

variables are defined

$$\bar{X} = \frac{x^* - L^*}{\epsilon^3 L^*}, \quad \bar{r} = \frac{r^*}{\epsilon^3 L^*},$$

$$\bar{p} = \frac{p^* - p_\infty^*}{\rho_\infty^* U_\infty^{*2}}, \quad \bar{u} = \frac{u^*}{U_\infty^*},$$

$$\bar{v} = \frac{v^*}{U_\infty^*}, \quad \bar{w} = \frac{w^*}{U_\infty^*}, \tag{2.4}$$

$$c = \frac{c^*}{U_\infty^*}, \quad \bar{t} = \frac{U_\infty^* t^*}{\epsilon^2 L^*}.$$

Here x^*, r^* and θ are taken to be the streamwise, radial and azimuthal co-ordinates respectively at some suitable reference point on the body, and u^*, v^*, w^* are the corresponding velocity components, c^* denotes the speed of sound, ρ_∞^* the fluid density far from the body, p^* the pressure, p_∞^* the far field pressure, and t^* the time.

We confine our study to the stability of the flow at a location on the body where the boundary layer thickness is $0(\epsilon^4 L^*)$, which is thin compared with the radius of the body. The skin friction of the undisturbed boundary layer is then taken to be $U^* \lambda \epsilon^4 / L^*$, where λ is some order one parameter. The flow is taken to be parallel, a completely rational approximation in this context because of the small, $0(\epsilon^3 L^*)$, streamwise length scale under consideration (although non-parallel effects could become important in a non-linear study).

We consider first the upper deck of the triple deck, where $\bar{r} = 0(1)$. The solution develops as follows

$$\bar{u} = 1 + \epsilon^2 u_1(\bar{X}, \bar{r}, \theta, \bar{t}) +,$$

$$\bar{v} = \epsilon^2 v_1(\bar{X}, \bar{r}, \theta, \bar{t}) + \cdots,$$

$$\bar{w} = \epsilon^2 w_1(\bar{X}, \bar{r}, \theta, \bar{t}) + \cdots, \tag{2.5}$$

$$p = \epsilon^2 p_1(\bar{X}, \bar{r}, \theta, \bar{t}) + \cdots,$$

$$c = M_\infty^{-1} + \epsilon^2 c_1(\bar{X}, \bar{r}, \theta, \bar{t}) + \cdots.$$

M_∞ is the Mach number of the external flow. The flow in this layer turns out to be completely irrotational, and may be reduced to the Prandtl Glauert equation (expressed in cylindrical polar co-ordinates), for the pressure p_1,

$$(1 - M_\infty^2) p_{1\bar{x}\bar{x}} + \frac{1}{\bar{r}} p_{1\bar{r}} + p_{1\bar{r}\bar{r}} + \frac{1}{\bar{r}^2} p_{1\theta\theta} = 0. \tag{2.6}$$

The solution of this will be deferred, until the appropriate boundary conditions have been ascertained. We consider next the main deck, which corresponds to the transverse scale

$$y = \frac{r^* - a^*}{\epsilon^4 L^*} = 0(1). \tag{2.7}$$

3

Here curvature no longer explicitly plays a leading role, and the solution takes the form

$$\bar{u} = U_0(y) + \epsilon \bar{A}(\bar{X}, \theta, \bar{t}) U_0'(y) + \cdots,$$

$$\bar{v} = -\epsilon^2 \bar{A}_X(\bar{X}, \theta, \bar{t}) U_0(y) + \cdots,$$

$$w = \epsilon^2 \bar{D}(\bar{X}, \theta, \bar{t}) / U_0(y) + \cdots, \tag{2.8}$$

$$\rho = R_0(y) + \epsilon \bar{A}(\bar{X}, \theta, \bar{t}) R_{0y}(y) + \cdots,$$

$$p = \epsilon^2 \bar{P}(\bar{X}, \theta, \bar{t}) + \cdots,$$

where $U_0(y)$ and $R_0(y)$ are the axial velocity and density distributions (radially), corresponding to the undisturbed boundary layer. The function $\bar{D}$ can be written explicitly as

$$\bar{D}_X = -\frac{\bar{P}_\theta}{\bar{a}}. \tag{2.9}$$

The limit as $y \to \infty$ of the transverse velocity component yields an inner boundary condition for the upper deck, viz

$$p_{1\bar{r}} \mid_{\bar{r}=\bar{a}} = \bar{A}_{XX}. \tag{2.10}$$

As $y \to 0$, (2.8) violates the no-slip condition, and this is subsequently rectified by the inclusion of the third layer, the lower deck, where

$$\bar{Y} = y/\epsilon = 0(1), \tag{2.11}$$

and

$$\bar{u} = \epsilon \bar{U}(\bar{X}, \bar{Y}, \theta, \bar{t}),$$

$$\bar{v} = \epsilon^3 \bar{V}(\bar{X}, \bar{Y}, \theta, \bar{t}),$$

$$\bar{w} = \epsilon \bar{W}(\bar{X}, \bar{Y}, \theta, \bar{t}), \tag{2.12}$$

$$p = \epsilon^2 \bar{P}(\bar{X}, \theta, \bar{t}),$$

$$\rho = R_0(0) + \epsilon \rho_1(\bar{X}, \bar{Y}, \theta, \bar{t}).$$

It is now possible to scale out a number of the physical constants. Following Kluwick

4

et al. (1984) and Duck and Hall (1988), this is achieved as follows

$$\bar{X} = C^{\frac{3}{8}}\lambda^{-\frac{5}{4}}(T_w/T_\infty)^{\frac{3}{2}}X,$$

$$\bar{Y} = C^{\frac{5}{8}}\lambda^{-\frac{3}{4}}(T_w/T_\infty)^{\frac{3}{2}}Y,$$

$$\bar{P} = C^{\frac{1}{4}}\lambda^{\frac{1}{2}}P,$$

$$\bar{U} = C^{\frac{1}{8}}\lambda^{\frac{1}{4}}(T_w/T_\infty)^{\frac{1}{8}}U,$$

$$\bar{V} = C^{\frac{3}{8}}\lambda^{\frac{7}{4}}(T_w/T_\infty)^{\frac{1}{2}}V, \qquad (2.13)$$

$$\bar{W} = C^{\frac{1}{8}}\lambda^{\frac{1}{4}}(T_w/T_\infty)^{\frac{3}{2}}W,$$

$$\tilde{A} = C^{\frac{3}{8}}\lambda^{-\frac{3}{4}}(T_w/T_\infty)^{\frac{3}{2}}A,$$

$$\bar{a} = C^{\frac{7}{8}}\lambda^{-\frac{5}{4}}(T_w/T_\infty)^{\frac{3}{2}}a,$$

$$\bar{t} = C^{\frac{1}{4}}\lambda^{-\frac{3}{2}}(T_w/T_\infty)t.$$

Here C is the Chapman constant arising from the linear viscosity law

$$\mu^*/\mu_\infty^* = C(T/T_w), \qquad (2.14)$$

T_w being the non-dimensional wall temperature. The governing equations in the lower deck then turn out to be a form of the three-dimensional unsteady boundary layer equations, namely

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} + \frac{W}{a}\frac{\partial U}{\partial \theta} = \frac{\partial^2 U}{\partial Y^2} - \frac{\partial P}{\partial X},$$

$$\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial X} + V\frac{\partial W}{\partial Y} + \frac{W}{a}\frac{\partial W}{\partial \theta} = \frac{\partial^2 W}{\partial Y^2} - \frac{1}{a}\frac{\partial P}{\partial \theta}, \qquad (2.15)$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} + \frac{1}{a}\frac{\partial W}{\partial \theta} = 0,$$

with boundary conditions

$$U = V = W = 0 \quad \text{on} \quad Y = 0,$$

$$U \to Y + A(X,\theta,t) \quad \text{as} \quad Y \to \infty, \qquad (2.16)$$

$$W \to D/Y \quad \text{as} \quad Y \to \infty,$$

where

$$D_X = -\frac{1}{a}P_\theta. \qquad (2.17)$$

5

If we write the solution of (2.6) subject to (2.10) symbolically as

$$P = \mathcal{L}\{A\} \qquad (2.18)$$

then the problem is effectively closed. The system described above is inherently non-linear, and as such generally requires a numerical solution; however we now seek solutions which are small perturbations about the undisturbed state, namely

$$U = Y, \quad V = W = P = A = 0; \qquad (2.19)$$

this amounts to a linearized stability analysis of the boundary layer, and forms the basis for the following section.

3. Linear stability

Here the undisturbed flow is perturbed by a small amount $h \ll 1$, and we seek to investigate the growth or decay of these disturbances. We seek a solution of the form

$$
\begin{aligned}
U &= Y + hU_1 + \cdots, \\
V &= hV_1 + \cdots, \\
W &= hW_1 + \cdots, \\
P &= hP_1 + \cdots, \\
A &= hA_1 + \cdots,
\end{aligned}
\qquad (3.1)
$$

in particular the leading order perturbation terms are assumed to take on the following particular form

$$
\begin{aligned}
U_1 &= \hat{u}_1(Y)E_1 + c.c., \\
V_1 &= \hat{v}_1(Y)E_1 + c.c., \\
W_1 &= \hat{w}_1(Y)E_1 + c.c., \\
P_1 &= \hat{P}_1 E_1 + c.c., \\
A_1 &= \hat{A}_1 E_1 + c.c.,
\end{aligned}
\qquad (3.2)
$$

where

$$E_1 = \exp\{i[\alpha X + n\theta - \Omega t]\}, \qquad (3.3)$$

and "c.c." denotes a complex conjugate. P_1, A_1, α, Ω are in general complex constants; n is the (integral) azimuthal wavenumber.

6

The problem at $0(h)$ reduces to the following linear system

$$\hat{u}_1\{-i\Omega + i\alpha Y\} + \hat{v}_1 = i\alpha \hat{P}_1 + \hat{u}_{1YY},$$

$$\hat{w}_1\{-i\Omega + i\alpha Y\} = -\tfrac{in}{a}\hat{P}_1 + \hat{w}_{1YY}, \qquad (3.4)$$

$$i\alpha\hat{u}_1 + \hat{v}_{1Y} + \tfrac{in}{a}\hat{w}_1 = 0.$$

The solution to this system is best achieved by introducing the variable

$$\hat{\sigma} = \hat{u}_{1Y} + \frac{n}{\alpha a}\hat{w}_{1Y}. \qquad (3.5)$$

Utilizing (3.4), the following simplified system results

$$\hat{\sigma}_{YY} + [i\Omega - i\alpha Y]\hat{\sigma} = 0; \qquad (3.6)$$

the solution (which remains bounded as $Y \to \infty$) is

$$\hat{\sigma} = BAi(\xi) \qquad (3.7)$$

where

$$\xi = -\frac{(i\alpha)^{\frac{1}{3}}\Omega}{\alpha} + (i\alpha)^{\frac{1}{3}}Y. \qquad (3.8)$$

Invoking, further, the boundary conditions as $Y \to \infty$ demands

$$(i\alpha)^{-\frac{1}{3}}B\int_{\xi_0}^{\infty} Ai(\xi)d\xi = \hat{A}_1, \qquad (3.9)$$

where

$$\xi_0 = -\frac{(i\alpha)^{\frac{1}{3}}\Omega}{\alpha}. \qquad (3.10)$$

Evaluating the first two equations of (3.4) on $Y = 0$, and combining yields

$$(i\alpha)^{\frac{1}{3}}BAi'(\xi_0) = (i\alpha + \frac{in^2}{\alpha a^2})\hat{P}. \qquad (3.11)$$

To complete this solution we require the solution from the upper deck. We have

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2} - \alpha^2(1 - M_\infty^2)\right]\hat{p}_1 = 0, \qquad (3.12)$$

where

$$p_1(\bar{r}) = C^{\frac{1}{4}}\lambda^{\frac{1}{4}}[\hat{p}_1(r)E_1 + c.c.],$$

$$\bar{r} = C^{\frac{3}{8}}\lambda^{\frac{5}{4}}(T_w/T_\infty)^{\frac{3}{2}}r, \qquad (3.13)$$

subject to

$$\hat{p}_{1r}\mid_{r=a} = -\alpha^2\hat{A}_1, \qquad (3.14)$$

7

and

$$\hat{p}_1 \to 0 \quad \text{as} \quad r \to \infty. \tag{3.15}$$

The solution to (3.12), subject to (3.14), (3.15) is

$$\hat{p}_1(r) = \frac{i\alpha \hat{A}_1 K_n[i\alpha r \sqrt{M_\infty^2 - 1}]}{\sqrt{M_\infty^2 - 1} K_n'[i\alpha a \sqrt{M_\infty^2 - 1}]} \tag{3.16}$$

(assuming $M_\infty > 1$). Finally, a non-trivial solution to (3.9), (3.11), (3.16) exists only if the following (dispersion) relation is satisfied

$$\frac{A_i'(\xi_0)}{\int_{\xi_0}^\infty A_i(\xi) d\xi} = \frac{(i\alpha)^{\frac{4}{3}}[1 + \frac{n^2}{\alpha^2 a^2}] K_n[i\alpha a \sqrt{M_\infty^2 - 1}]}{\sqrt{M_\infty^2 - 1} K_n'[i\alpha a \sqrt{M_\infty^2 - 1}]}. \tag{3.17}$$

Notice that setting $n = 0$ (axisymmetric mode), and defining

$$\alpha = (M_\infty^2 - 1)^{\frac{3}{8}} \bar{\alpha},$$

$$\Omega = (M_\infty^2 - 1)^{\frac{1}{4}} \bar{\Omega},$$

$$a = (M_\infty^2 - 1)^{-\frac{7}{8}} \bar{a}, \tag{3.18}$$

$$\bar{\xi}_0 = -\frac{(i\bar{\alpha})^{\frac{1}{3}} \bar{\Omega}}{\bar{\alpha}}$$

yields the following dispersion relationship

$$\frac{Ai'(\bar{\xi}_0)}{\int_{\bar{\xi}_0}^\infty Ai(\bar{\xi}) d\bar{\xi}} = -\frac{(i\bar{\alpha})^{\frac{4}{3}} K_0[i\bar{\alpha}\bar{a}]}{K_1[i\bar{\alpha}\bar{a}]}, \tag{3.19}$$

which is the same dispersion relationship found by Duck and Hall (1988) (unlike the non-axisymmetric situation, the Mach number may be scaled from the axisymmetric problem).

In the following section, we describe various asymptotic solutions of the dispersion relationship and discuss the numerical solution of (3.17) in §5.

4. The dispersion relationship for $a \ll 1$ and $a \gg 1$

Here we shall discuss the limiting forms of the dispersion relationship in the limits of either very thin or thick cylinders. We recall that axisymmetric modes always occur in pairs and exist for a less than some critical value, with branches having α, Ω tending to infinity when $a \to 0$. Suppose then that $a \to 0$ in (3.17) in such a way that $\alpha a \to 0$. From the series expansions of K_n, K_n' we deduce that (3.17) then reduces to

$$\frac{Ai'(\xi_0)}{\chi} \simeq (i\alpha)^{\frac{1}{3}} \frac{n}{a} + \cdots,$$

8

where

$$\chi = \int_{\xi_0}^{\infty} Ai(s)ds$$

so that for neutral modes we must have that

$$\xi_0 \simeq 2.298i^{\frac{1}{3}} \quad , \quad Ai'(\xi_0)/\chi \sim 1.001i^{\frac{1}{3}}.$$

and the neutral values of α, Ω then become

$$\alpha = \{\tfrac{1.001a}{n}\}^3 + \cdots ,$$

$$\Omega = 2.298\{\tfrac{1.001a}{n}\}^2 + \cdots . \tag{4.1a,b}$$

Thus unlike the two dimensional problem there are neutral solutions for $a \ll 1$ which have $\alpha, \Omega \rightarrow 0$. This is a result of some significance because it means that for thin cylinders non-axisymmetric modes are the most important since they will occur at lower values of the Reynolds number than do the axisymmetric modes.

In fact there is another asymptotic solution of (3.17) available in the limit $a \gg 1$. The second mode again has $\alpha a \ll 1$ but $| \xi_0 |$ now tends to infinity. For large values of $| \xi_0 |$ we can replace $\frac{Ai'(\xi_0)}{\chi}$ by $-\xi_0\{1 - \xi_0^{-\frac{3}{2}} + \cdots\}$ and we again approximate $\frac{K_n(z)}{K'_n(z)}$ using the series expansion of the modified Bessel function. After equating the dominant real and imaginary parts of (3.17) we then deduce that

$$\alpha = \left\{\frac{2^{n+\frac{1}{2}}n!n-1!}{\pi(M_\infty^2-1)^n}\right\}^{\frac{1}{2n+\frac{1}{2}}} a^{-\left(\frac{4n-3}{4n+1}\right)} + \cdots ,$$

$$\Omega = \tfrac{n\alpha}{a} + \cdots . \tag{4.2a,b}$$

Thus $\alpha \sim a^{-\left(\frac{4n-3}{4n+1}\right)}$ in this limit and since $\frac{4n-3}{4n+1}$ increases monotonically from $\frac{1}{5}$ to 1 when $n \rightarrow \infty$ it follows that the largest values of α correspond to n tending to infinity. It follows that the distances between the upper and lower branches of (3.17) for $a \rightarrow 0$ increases with n. Thus the band of unstable wavenumbers increases with the azimuthal wavenumber n; this does not of course necessarily mean that the maximum growth rate will occur for $n \gg 1$. We will return to the latter point in the next section.

Now we consider the structure of (3.17) in the limit $a \rightarrow \infty$. Firstly we note that if we let $a \rightarrow \infty$ with n fixed then, if $| \alpha a | \gg 1$, (3.17) reduces to

$$\frac{Ai'(\xi_0)}{\chi} = -\frac{(i\alpha^{\frac{4}{3}})}{(M_\infty^2 - 1)^{\frac{1}{2}}} + \cdots$$

which has no neutral solutions. Thus we must instead consider the double limit $n \rightarrow \infty, a \rightarrow \infty$ but with $\frac{n}{a}$ held fixed. This means that the wavelength in the azimuthal

direction is comparable with that in the streamwise direction. The asymptotic form for a modified Bessel function of large argument and order yields

$$\frac{K_n(i\alpha a\sqrt{(M_\infty^2-1)})}{K_n'(i\alpha a\sqrt{(M_\infty^2-1)})} = \frac{-i\alpha a\sqrt{M_\infty^2-1}}{n\{1-\frac{\alpha^2 a^2}{n^2}[M_\infty^2-1]\}^{\frac{1}{2}}} + \cdots,$$

where we have assumed that $\frac{\alpha^2 a^2}{n^2}[M_\infty^2-1] < 1$. It follows that neutral solutions of (3.1) again occur when $\xi_0 \simeq -2.298 i^{\frac{1}{3}}$ so that

$$1.001\{1-\frac{\alpha^2 a^2}{n^2}[M_\infty^2-1]\}^{\frac{1}{2}} = \frac{\alpha^{\frac{7}{3}}a}{n}\{1+\frac{n^2}{\alpha^2 a^2}\} + \cdots,$$

or if we write $\beta = n/a$ we obtain

$$1.001\{1-\frac{\alpha^2}{\beta^2}[M_\infty^2-1]\}^{\frac{1}{2}} = \alpha^{\frac{7}{3}}\beta^{-1}\{1+\frac{\beta^2}{\alpha^2}\} + \cdots, \tag{5.3}$$

which is the supersonic planar neutral dispersion relationship of Smith (1988).

It is an easy matter to show that (3.17) has no neutral solutions for $n, a \gg 1$ with $\frac{\alpha^2 a^2}{n^2}[M_\infty^2-1] > 1$. Thus neutral disturbances exist only for azimuthal wavenumbers satisfying

$$n^2 > \alpha^2 a^2\{M_\infty^2-1\}. \tag{5.4}$$

We shall discuss the above asymptotic results after describing our numerical results for the dispersion relationship (3.17).

5. Results and Discussion

The dispersion relationship (3.17) was solved for the neutral state ($Imag\{\alpha\} = Imag\{\Omega\} = 0$), for prescribed M_∞, n, using a straightforward Newton iterative scheme. Results for $M_\infty = \sqrt{2}$ are shown in Figs 1a ($\Omega - a$ curve), 1b ($\alpha - a$ curve), for $M_\infty = 5$ in Figs 2a ($\Omega - a$ curve), 2b ($\alpha - a$ curve), for $M_\infty = 10$ in Figs 3a ($\Omega - a$ curve), 3b ($\alpha - a$ curve).

As with the axisymmetric case considered by Duck and Hall (1988), for fixed a less than a_c, the critical body radius above which the flow is stable) two possible modes exist. However, there is an important difference between this and the axisymmetric case, namely the behavior of the "lower branch," which here has $\Omega \to 0, \alpha \to 0$ as $a \to 0$, whilst in the axisymmetric case it is found that $\Omega \to \infty, \alpha \to \infty$ as $a \to 0$.

Further observations regarding these $n \neq 0$ results are :-

1. as n increases, a_c increases (indeed, all the a_c determined here were greater than the a_c obtained in the $n = 0$ case).

2. as M_∞ increases, for n fixed, a_c decreases ;

10

3. kinks were consistently observed on the upper branch of the $\alpha - a$ curve for the smaller value of n.

In Figures 1, we have also shown (as broken curves) the lower and upper branches predicted by the asymptotic analysis of §4. We see that the asymptotic theory accurately predicts the neutral wavenumbers and frequencies over a wide range of values of a. A similar agreement is found for the higher Mach number cases but is not shown in the figures.

In Fig. 4, we have shown the result of some non-neutral calculations corresponding to $M_\infty = \sqrt{2}$ and $a = 2$. The frequency Ω is taken to be real and when the corresponding complex value of the wavenumber α is calculated we see that the curves are unstable for frequencies between the two neutral values where the latter exist. Furthermore, the growth rates initially increase with n so that, at least at the values of a and M_∞ chosen, the non-axisymmetric modes become progressively more unstable with increasing n. In fact the most dangerous mode, i.e., the one with the largest growth rate occurs for $n = 6$ for $a = 2, M_\infty = \sqrt{2}$. A similar trend was found at higher Mach numbers and in general we found that the value of n for the most dangerous mode increased with M_∞.

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LIST OF CAPTIONS

Fig. 1a. Neutral $\Omega - a$ curve, $M_\infty = \sqrt{2}$.

Fig. 1b. Neutral $\alpha - a$ curve, $M_\infty = \sqrt{2}$.

Fig. 2a. Neutral $\Omega - a$ curve, $M_\infty = 5$.

Fig. 2b. Neutral $\alpha - a$ curve, $M_\infty = 5$.

Fig. 3a. Neutral $\Omega - a$ curve, $M_\infty = 10$.

Fig. 3b. Neutral $\alpha - a$ curve, $M_\infty = 10$.

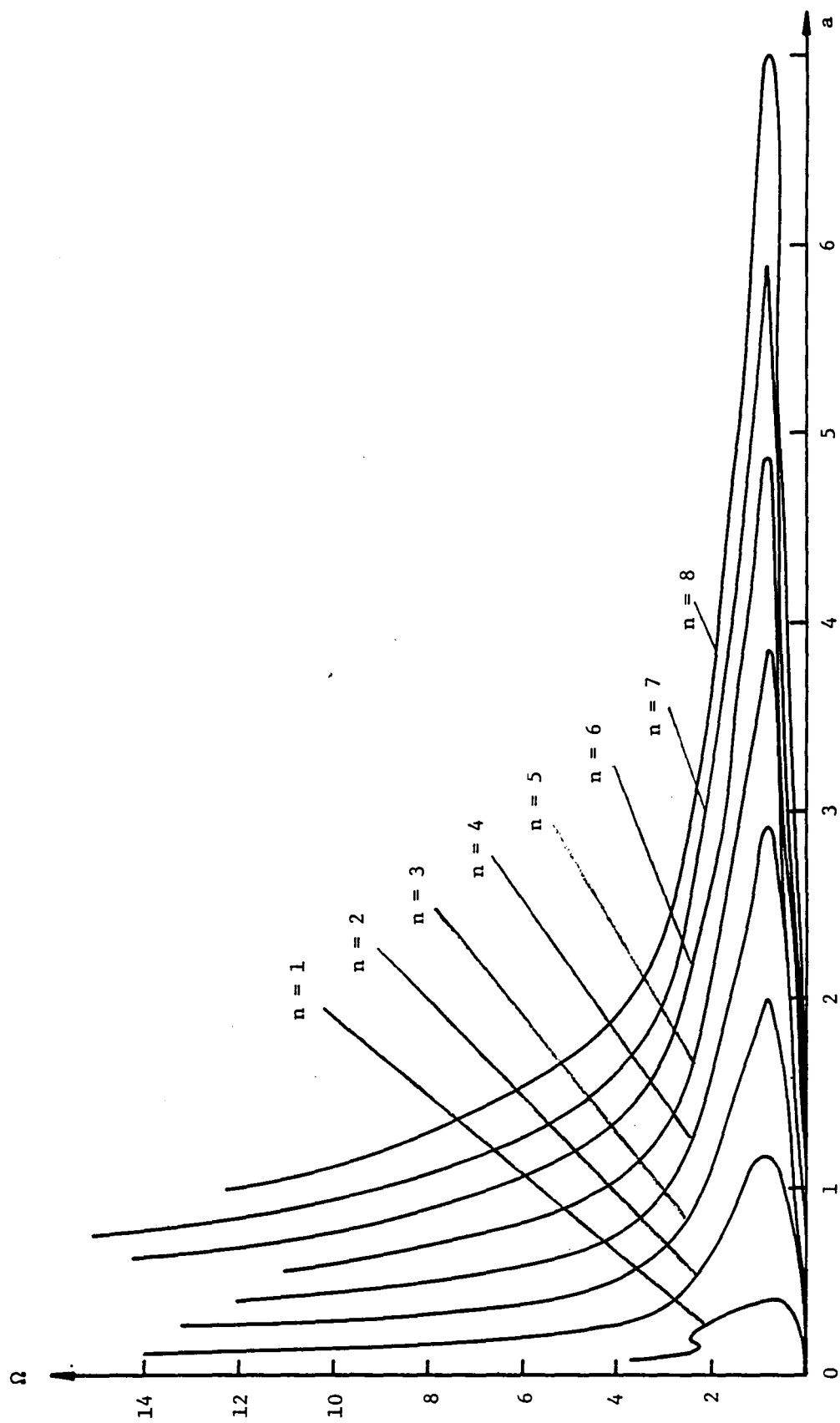Fig. 4.   Spatial growth rates (α_i), Ω real, $M_\infty = \sqrt{2}$, $a = 2$, n as indicated.

Figure 1a.

Fig. 1b.

15

Fig. 2a.

16

Fig. 2b.

17

Fig. 3a.

18

Fig. 3b.

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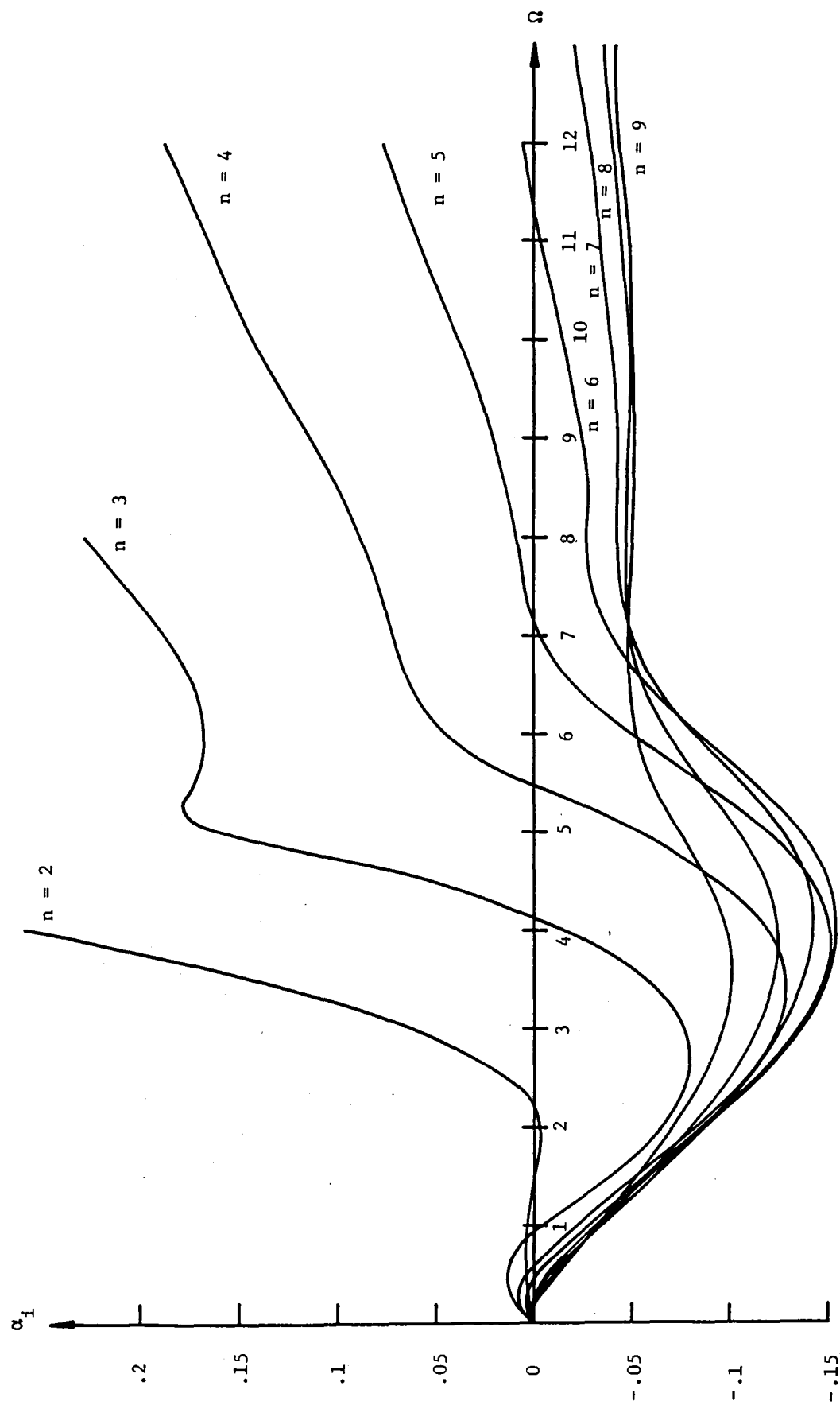
Fig. 4.

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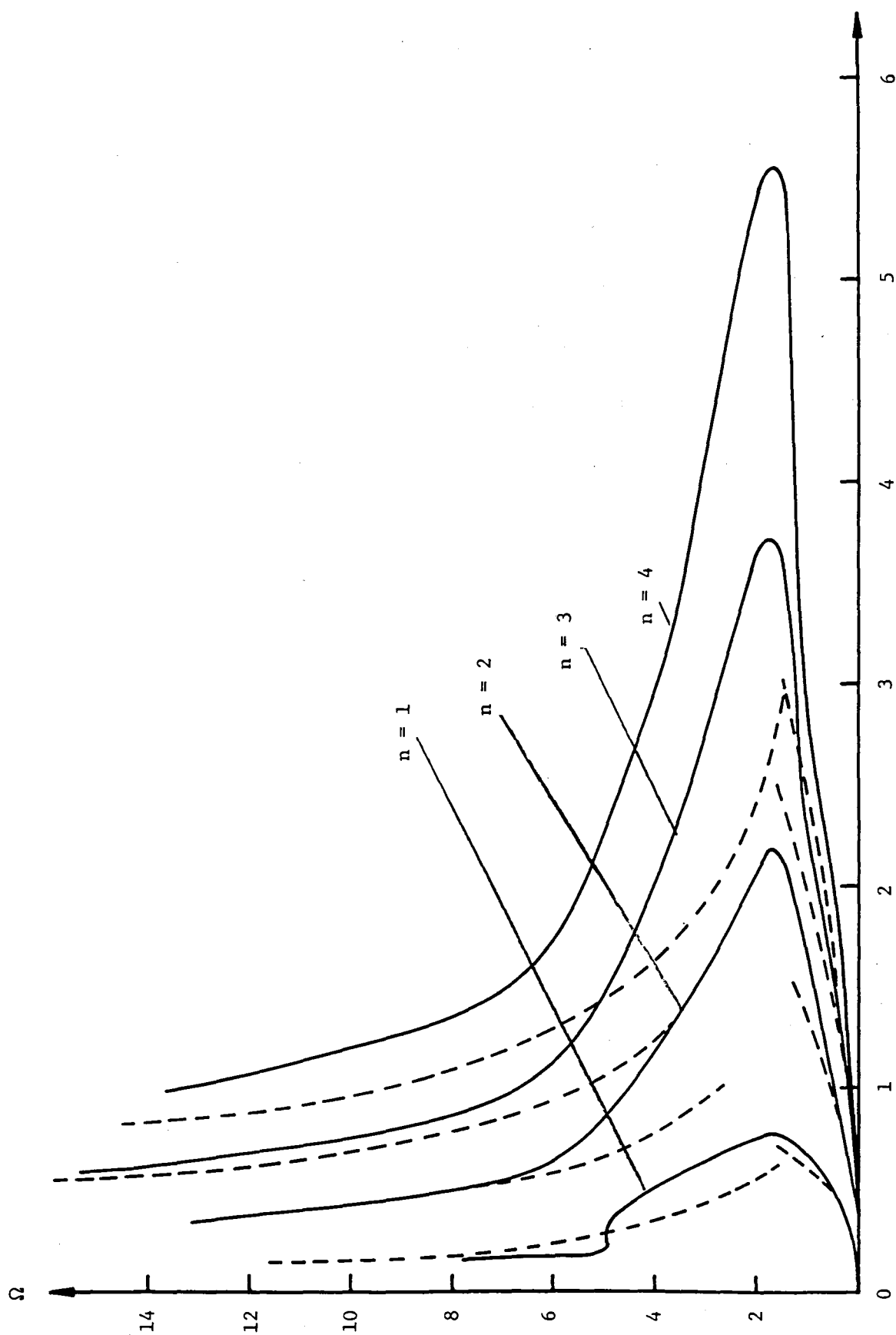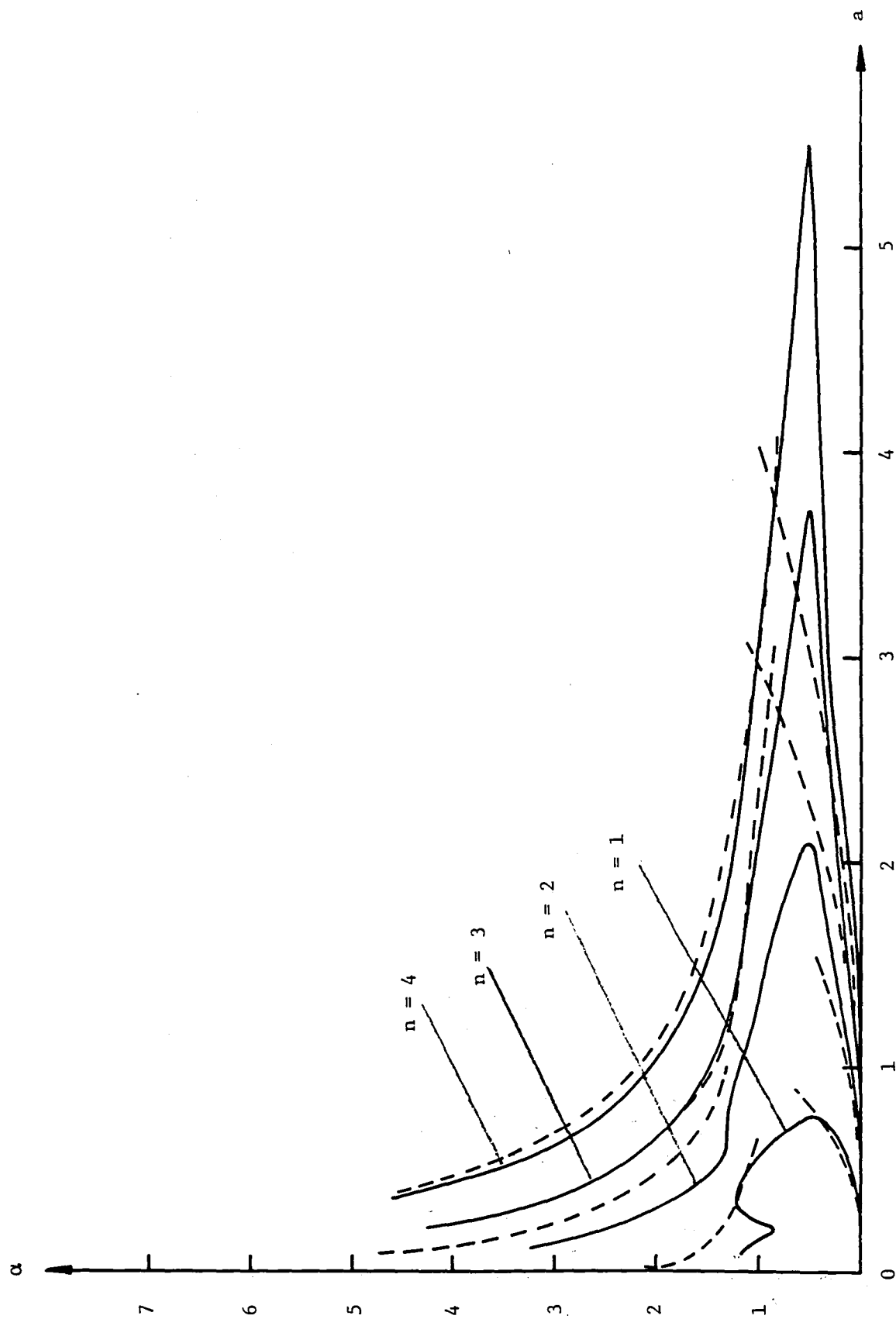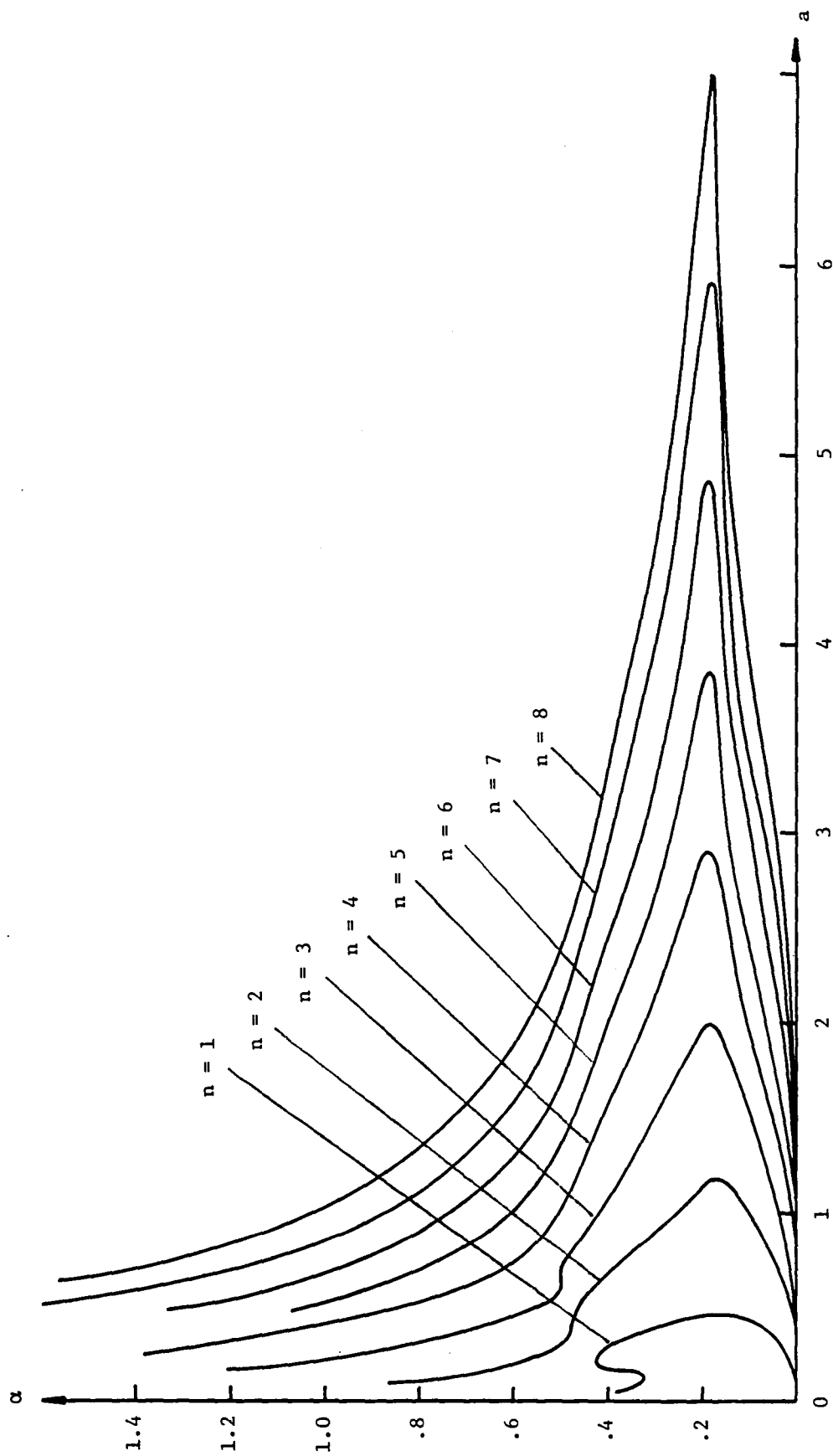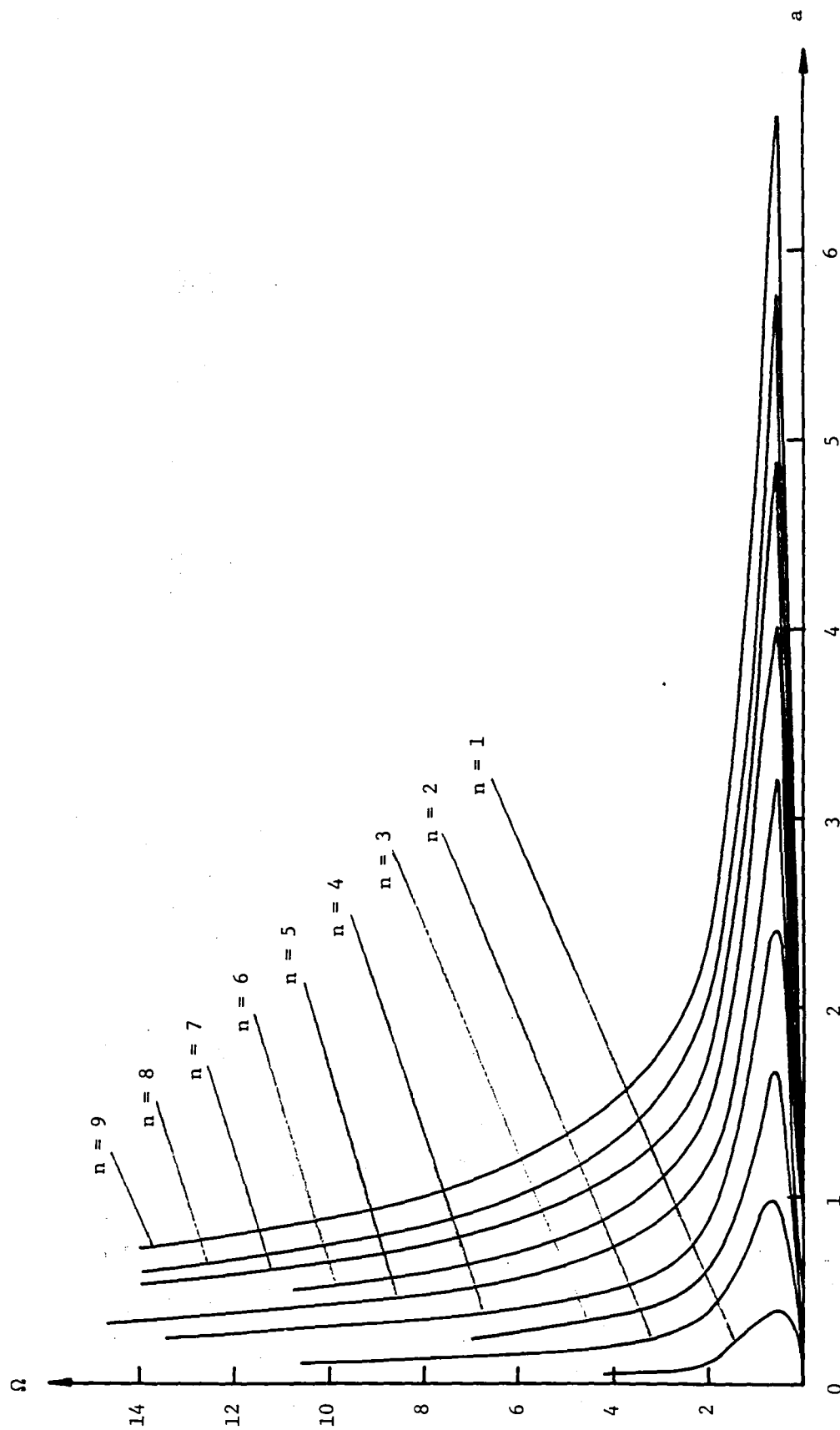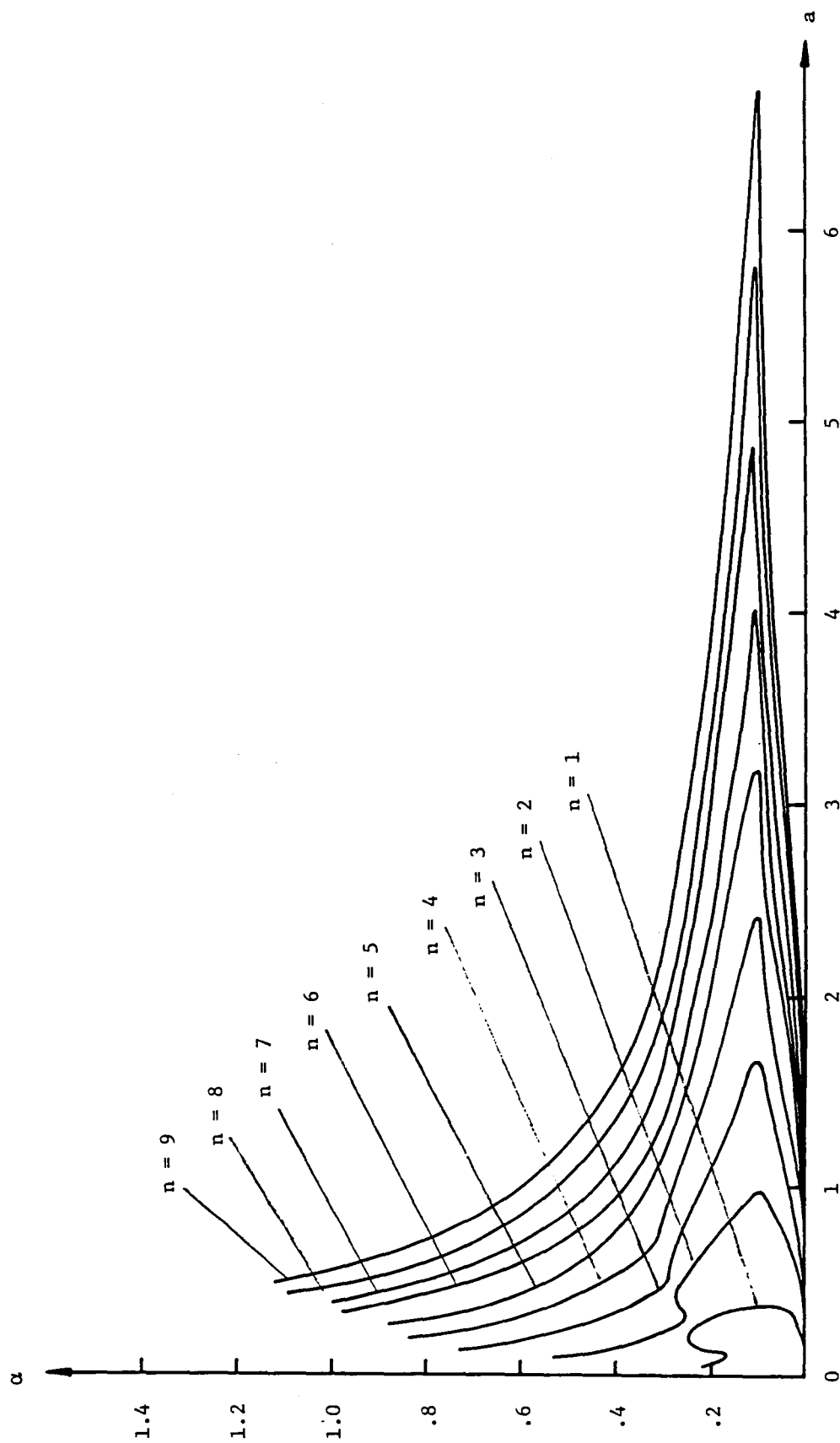
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16. Abstract

   In a previous paper, we have considered the weakly nonlinear interaction of a pair of axisymmetric lower branch Tollmien-Schlichting instabilities in cylindrical supersonic flows. Here the possibility that nonaxisymmetric modes might also exist is investigated. In fact it is found that such modes do exist and, on the basis of linear theory, it appears that these modes are the most important. The nonaxisymmetric modes are found to exist for flows around cylinders with nondimensional radius a less than some critical value a_c. This critcal value a_c is found to increase monotonically with the azimuthal wave-number n of the disturbance and it is found that unstable modes always occur in pairs. We show that in general, instability in the form of lower branch Tollmien-Schlichting waves will occur first for nonaxisymmetric modes and that in the unstable regime, the largest growth rates correspond to the latter modes.

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