Discretization Formulas for Unstructured Grids

Kenneth J. Baumeister
Lewis Research Center
Cleveland, Ohio

NASA

DISCRETIZATION FORMULAS FOR UNSTRUCTURED GRIDS

Kenneth J. Baumeister
National Aeronautics and Space Administration
Lewis Research Center
Cleveland, Ohio 44135

ABSTRACT

The Galerkin weighted residual technique using linear tri-
angular weight functions is employed to develop finite differ-
ence formula in cartesian coordinates for the Laplacian
operator, first derivative operators and the function for
unstructured triangular grids. The weighted residual coeffici-
ents associated with the weak formulation of the Laplacian
operator are shown to agree with the Taylor series approach on
a global average. In addition, a simple algorithm is presented
to determine the Voronoi (finite difference) area for an
unstructured grid.

INTRODUCTION

In developing unstructured finite difference equations,
Jameson [1, Eq. (4.10)] has applied the standard weighted
residual Galerkin method to obtain a time dependent discretiza-
tion of the Euler equations for an unstructured triangular
mesh. Erlebacher [2] has utilized variational methods on a
pointwise basis to establish a difference operator for the
Laplacian operator for a central difference cell similar to
that shown in Fig. 1. The present paper will utilize the
global Galerkin weighted residual technique to also generate
simple closed form finite difference approximations for the
Laplacian operator, first derivative operators, and the func-
tion itself for the general central cell in Fig. 1, as well as
the boundary element cell also shown in Fig. 1. The boundary
element cells are required in acoustics and electromagnetic
theory in establishing the far field radiation conditions.

GALERKIN WEIGHTED RESIDUAL FORMULATION

Consider the following partial derivatives:

$$\nabla \cdot \beta_1 \nabla \phi \; ; \qquad \beta_2 \frac{\delta \phi}{\delta x} \; ; \qquad \beta_3 \frac{\delta \phi}{\delta y} \; ; \qquad \beta_4 \phi \qquad (1)$$

ϕ is a scaler or potential quantity and the β's are variable property coefficients. To obtain the finite difference expressions for the terms in Eq. (1), the continuous domain D is first divided into discrete triangular areas A_e staked out by nodal (grid) points P as shown in Fig. 1. The number of connected areas (called a cell) needed to define the difference equation for node o is labeled M.
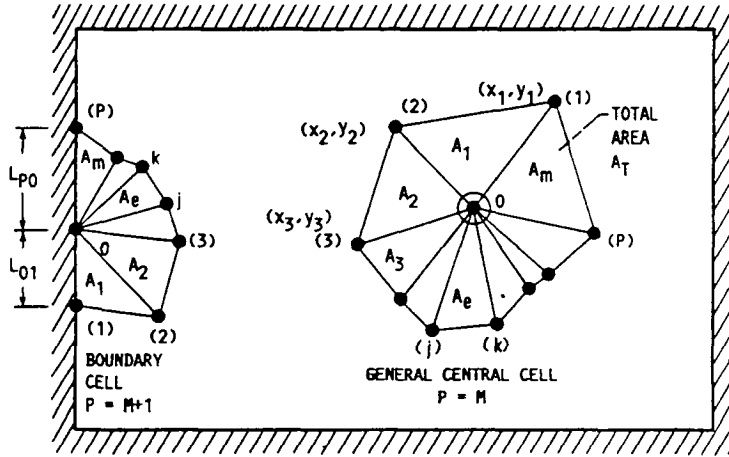

FIGURE 1. - GLOBAL MESH CONFIGURATION.

The difference equations developed in Appendices A and B take on the form

$$
\left.
\begin{array}{l}
(\beta_2 \delta \tilde{\phi}/\delta x) \\[1ex]
(\beta_3 \delta \tilde{\phi}/\delta y) \\[1ex]
(\beta_4 \tilde{\phi})
\end{array}
\right\}
= k_o^* \phi_o + k_1^* \phi_1 + k_2^* \phi_2 + \ldots + k_n^* \phi_n + \ldots + k_p^* \phi_p
$$

$$(2)$$

$$\iint_{A_T} [-\nabla W_i \cdot \beta_1 \nabla \tilde{\phi}] \, dA = \alpha_o \phi_o + \alpha_1 \phi_1 + \alpha_2 \phi_2 + \ldots + \alpha_p \phi_p$$

$$(3)$$

where the Laplacian operator is converted to the later form by means of the weak formulation [3, p. 443] of the weighted residual approach. The values of the coefficients k and α depend only on the location of the grid points and the properties β. The superscript * represents either the derivative in x (* = x) or y (* = y) and the variable itself (* = ϕ). The coefficients k* for ϕ and its first derivatives are included in Appendix C.

The major effort of the present work is to establish the relationship between the Laplacian operator and the α coefficients from the method of weighted residuals. An expression for the Laplacian of the form

$$\nabla^2 \phi = f(\alpha_1, \alpha_2, \ldots)\qquad(4)$$

is desired. Using the result from Appendix D for a linear shape function, the Laplacian can be expressed as

$$\nabla^2 \phi = \frac{\alpha_o \phi_o + \alpha_1 \phi_1 + \alpha_2 \phi_2 + \ldots + \alpha_p \phi_p}{\dfrac{A_T}{3}} + \Gamma\qquad(5)$$

where A_T is the total area of the cell. The α values in Eq. (5) are given as

$$\alpha_n = -\left[\beta^{(nm)}/4A_{nm}\right]\left[\left(y_{n-} - y_n\right)\left(y_o - y_{n-}\right) + \left(x_n - x_{n-}\right)\left(x_{n-} - x_o\right)\right]$$

$$\times \left(1 - \delta_{n,P-M}\right) - \left[\beta^{(n)}/4A_n\right]\left[\left(y_n - y_{n+}\right)\left(y_{n+} - y_o\right) + \left(x_{n+} - x_n\right)\right.$$

$$\left.\times \left(x_o - x_{n+}\right)\right]\left(1 - \delta_{P+1-n,P-M}\right)\qquad(6)$$

$$\alpha_o = -\sum_{e=1}^{M} \left[\beta^{(e)}/4A_e\right]\left[\left(y_e - y_{e+}\right)^2 + \left(x_{e+} - x_e\right)^2\right]\qquad(7)$$

$$\Gamma = \left\{\frac{3}{A_T}\left[\beta^{(1)}\nabla\phi \cdot n\right]_{bc}^{(1)} \frac{L_{o1}}{2} + \frac{3}{A_T}\left[\beta^{(M)}\nabla\phi \cdot n\right]_{bc}^{(M)} \frac{L_{Po}}{2}\right\}\delta_{M,P-1}$$

$$(8)$$

where the subscripts are defined in Appendix C. The term Γ is the contribution from the natural boundary condition that comes automatically from the weak formulation of the method of weighted residuals and contributes only on the boundary of the domain.

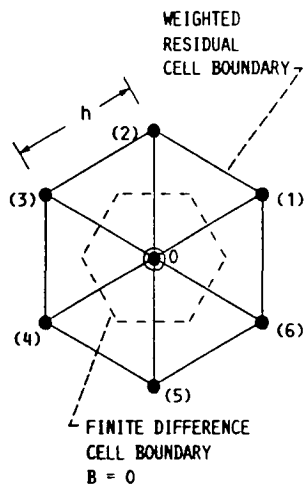DISCUSSION OF RESULTS

Uniform Grids

For the conventional six-point hexagon difference grid as shown in Fig. 2, the difference equations for the Laplacian, first derivatives, and the function as calculated from Eqs. (2) and (5) are in agreement with the standard difference operators that appear in the literature [4, p. 1114]. Similarly, the

very popular six-node system shown in Fig. 3 also shows agreement. For brevity only the Laplacian operator will be shown in Fig. 3 and on the remaining figures.

As is well known in the literature [3, p. 105], the method of weighted residual and the conventional Taylor series expansion often yield different results. For the conventional four-point square difference grid as shown in part A of Fig. 4, the difference equation for the Laplacian is different by a factor of 3/2 from the conventional Taylor series representation of the Laplacian operator [4, p. 1114]. Similarly, the eight-node system shown in part B of Fig. 4 differs by a factor of 3/4. However, since the method of weighted residual (finite element theory) is guaranteed to converge, these are acceptable formulae provided they are applied to a consistent global mesh and not a single arbitrary cell.

Relationship Between FE and FD Methods

For uniform grids, of the type displayed in Figs. 2 to 4, the finite difference and finite element expressions for the Laplacian operator will be in agreement with the conventional Taylor series difference equation provided that the bias constant B, defined by the following area rule, is identical to zero.
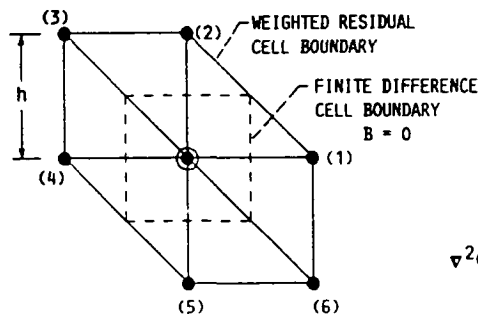
WEIGHTED
RESIDUAL
CELL BOUNDARY

$$\nabla^2 \Phi = \left[\frac{\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5 + \Phi_6 - 6\Phi_0}{\frac{3}{2}h^2} \right]$$

$$\frac{\partial \Phi}{\partial x} = \frac{\Phi_1 - \Phi_3 - \Phi_4 + \Phi_6}{2\sqrt{3}h}$$

$$\frac{\partial \Phi}{\partial y} = \frac{\Phi_1 + 2\Phi_2 + \Phi_3 - \Phi_4 - 2\Phi_5 - \Phi_6}{6h}$$

$$\Phi = \frac{\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5 + \Phi_6 + 6\Phi_0}{12}$$

FINITE DIFFERENCE
CELL BOUNDARY
B = 0

FIGURE 2. - UNIFORM 6 NODE HEXAGON GRID SYSTEM.

WEIGHTED RESIDUAL
CELL BOUNDARY

FINITE DIFFERENCE
CELL BOUNDARY
B = 0

$$\nabla^2 \Phi = \frac{\Phi_1 + \Phi_2 + \Phi_4 + \Phi_5 - 4\Phi_0}{h^2}$$

FIGURE 3. - UNIFORM 6 NODE SLANTED GRID SYSTEM.

$$\nabla^2\Phi = \frac{3}{2}\left[\frac{\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 - 4\Phi_0}{h^2}\right] \qquad \nabla^2\Phi = \frac{3}{4}\left[\frac{\Phi_1 + \Phi_3 + \Phi_5 + \Phi_7 - 4\Phi_0}{h^2}\right] \qquad B_{GLOBAL} = \frac{1}{2} \times 2h^2 + (-1/4)\ 4h^2 = 0$$
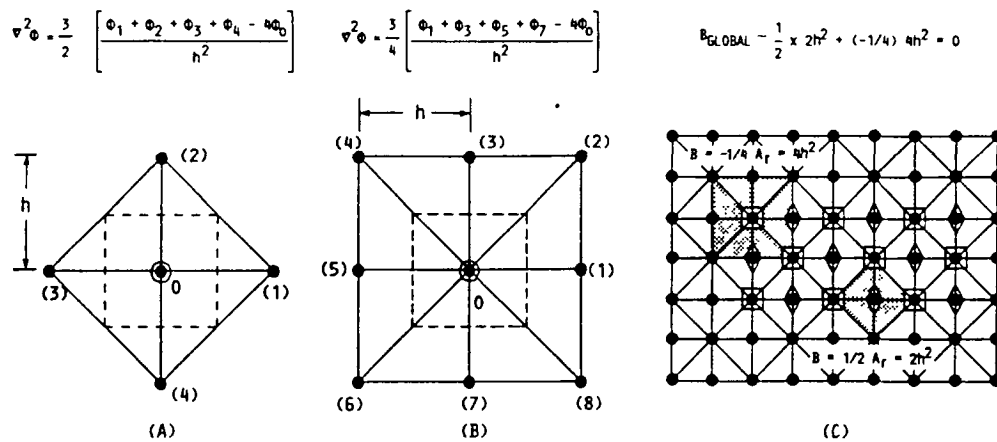
FIGURE 4. - GLOBAL CONVERGENCE EXAMPLE WITH RECTANGULAR ELEMENTS.

$$B = 3A_V/A_T - 1 \qquad\qquad (9)$$

A_V is the Voronoi neighborhood [4] or the area normally employed in the finite difference analysis. If $B \neq 0$, the coefficient in front of the finite element expression will differ from the expression for the Taylor series expression by a factor of $1 + B$. If $B = 0$, a single pattern of uniform elements can normally be placed in a regular domain.

Global Relation To Taylor Series

Although the difference equations for the Laplacian operator in Fig. 4 differ in each cell from the standard Taylor series difference equation, on a global average the Taylor series and finite element equation will average out to the same value. In this case the area average of the eight-node element combined with the four-node element will be such that the Taylor series will be valid. As shown in Fig. 4, the average global value of B is equal to zero. (B is a measure of the difference between the finite element and Taylor series as just defined.) A similar situation occurs for triangular grids, although the results are not shown.

Nonuniform Grids

Figure 5 displays the difference equations for a nonuniform five-node cell. The Laplacian operator shown in Fig. 5 is only valid when taken in conjunction with the other grid systems that surround it in a given domain. It is not to be thought of as a Taylor series approximation for the cell shown in Fig. 5.

5

FIGURE 5. - NONUNIFORM 6 NODE GRID SYSTEM.
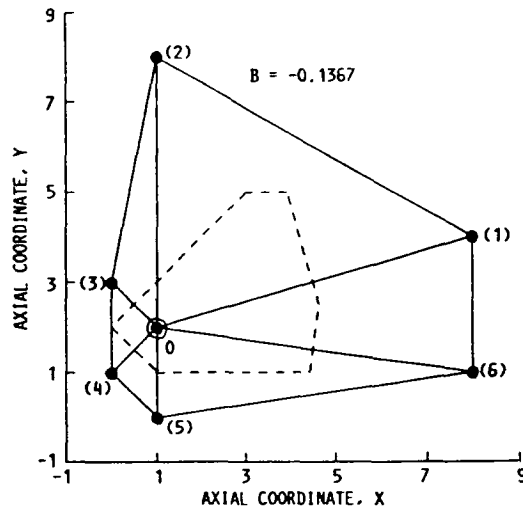
$$\nabla^2 \phi = 0.864 \left[\frac{2.307\phi_1 + \phi_2 + 19.385\phi_3 + 6.461\phi_4 + 11.077\phi_5 + 1.384\phi_6 - 41.615\phi_0}{80.884} \right]$$

Voronoi (Finite Difference) Area

The determination of the Voronoi area plays a major role in the mesh generation by triangulation of arbitrary points [2]. The coefficients used to generate the Laplacian operator can also be used to easily determine the Voronoi area without geometric construction;

$$A_V = \frac{\alpha_0\left(x_0^2 + y_0^2\right)}{4} + \frac{\alpha_1\left(x_1^2 + y_1^2\right)}{4} + \frac{\alpha_2\left(x_2^2 + y_2^2\right)}{4} + \cdots$$

$$+ \frac{\alpha_n\left(x_n^2 + y_n^2\right)}{4} \qquad (10)$$

where α's are given by Eqs. (6) and (7). The rationale for development of this simple algorithm to predict the Voronoi area comes from the fact that replacing $A_T/3$ by A_V in Eq. (5) fortuitously satisfies the Taylor series approximation for uniform grids. (However, it does not in general for non-uniform grids, and $A_T/3 = A_V$ when $B = 0$.) Therefore, if ϕ is assumed to vary as $[x^2 + y^2]/4$ then the Laplacian for uniform grids will have a value of unity and Eq. (10) results. To validate this hypothesis, Eq. (10) was checked against the nonuniform cell shown in Fig. 6, as well as a large variety of other nonuniform cells, and found to always be in exact agreement with geometrical calculations. For certain complex cells, the Voronoi boundaries can cross as shown in Fig. 6. In these cases, the algorithm will consider reversed areas as negative, in that it senses the direction of the path of the Voronoi region. Again the algorithm agrees with the geometric calculation.
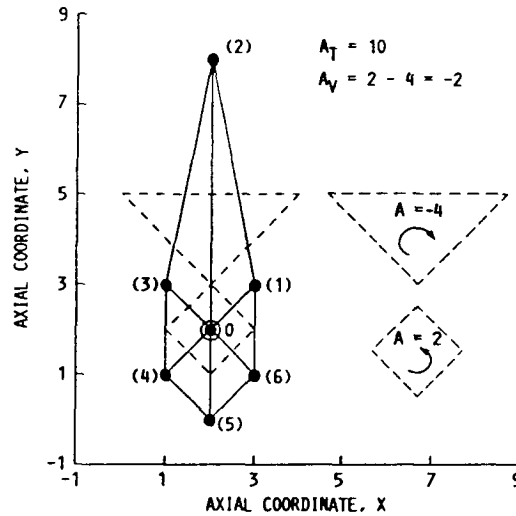
FIGURE 6. - CALCULATION OF VORONOI AREA FOR
6 NODE GRID WITH REVERSAL.

Local Convergence

If $B = 0$, Eq. (5) combined with Eq. (9) for central cells yields

$$\nabla^2 \phi = (\alpha_0 \phi_0 + \alpha_1 \phi_1 + \alpha_2 \phi_2 + \ldots + \alpha_p \phi_p)/A_V \qquad (11)$$

where A_V is the Voronoi area of the cell and the α values are given in Eqs. (6) and (7). Equation (11) was checked against a number of nonuniform grids (six or less nodes) where $B \neq 0$, and found to satisfy the test functions 1, x, y, x^2, and y^2. However, the test function xy was not satisfied when $B \neq 0$. For the special case, as shown in Fig. 5, where $B \sim 0$, Eq. (11) nearly satisfied the xy function test. Therefore, Eq. (11) may be valid on a pointwise bases for the special condition of $B = 0$. More testing will be required for validation.

CONCLUSIONS

The Galerkin weighted residual technique using linear triangular weight functions is employed to develop finite difference formulae in cartesian coordinates for the Laplacian operator, first derivative operators and the function for unstructured triangular grids.

REFERENCES

1. Jameson, A.: Successes and Challenges in Computational Aerodynamics, AIAA Paper 87-1184, AIAA 8th Computational Fluid Dynamics Conference, Honolulu, Hawaii, June 9-11, 1987.

2. Erlbacher, G.: Finite Difference Operators on Unstructured Triangular Meshes, Paper in Lecture Notes in Physics, Vol. 238, Springer-Verlag, New York, 1985, pp. 22-53, editors Fritts, M. J., Crowley, W. P., and Trease, H.

3. Lapidus, L. and Pinder, G. F.: Numerical Solution of Partial Differential Equations in Science and Engineering, John Wiley & Sons, New York, 1982.

4. Rektorys, K., Ed.: Survey of Applicable Mathematics, MIT Press, Cambridge, MA, 1969.

APPENDIX A - DERIVATION GALERKIN WEIGHTED RESIDUAL EQUATIONS

In accordance with the method of weighted residuals, weight functions W_O in the form of a linear pyramid are introduced (one for each grid point), such that the integration of the product of each term in Eq. (1) times the weight W_O over the cell area shown in Fig. 1 yields

$$\iint_{A_T} (-\nabla W_i \cdot \beta_1 \nabla \phi) dA = \sum_{e=1}^{M} \beta_1^{(e)} \iint_{A_e} \left[-\nabla N_O^{(e)} \cdot \left[\nabla_N^{(e)} \right] \{\phi^{(e)}\} \right] dA$$

(A1)

$$\iint_{A_T} W_i \beta_2 \frac{\partial \phi}{\partial x} dA = \sum_{e=1}^{M} \beta_2^{(e)} \iint_{A_e} \left[N_O^{(e)} \left[N_x^{(e)} \right] \{\phi^{(e)}\} \right] dA \qquad \text{(A2)}$$

where the subscript x stands for the x derivatives of the known shape functions and N stands for the linear interpolation functions whose x and y dependence is presented in most finite element texts [3, p. 111, Eq. (3.1.4)]. The equations for the derivative of ϕ with respect to y and the function itself are similar to Eq. (A2). The Laplacian operator in Eq. (A1) has utilized the weak formulation of the weighted residual formulation [3, p. 443].

APPENDIX B - AVERAGED OPERATORS

Assuming averaged finite difference values over the weighting function area allows the function and first derivatives to be written explicitly. For example,

$$\beta_2 \frac{\delta\phi}{\delta x} \approx \frac{3}{A_T} \sum_{e=1}^{M} \beta_2^{(e)} \iint_{A_e} \left[N_o^{(e)} \left[N_x^{(e)} \right] \{\phi^{(e)}\} \right] dA \qquad (B1)$$

In this case, the integral of the weight W_o over the cell area is equal to $A_T/3$, which was determined with the aid of the standard area integration formula [3, p. 112]. Equation (B1) as well as the equation for the Laplacian, the function and the derivative of the function with respect to y can now be easily evaluated using conventional finite element theory to obtain the difference equations given in Appendix C and Eqs. (6) and (7) in the body of the report.

APPENDIX C – COEFFICIENTS FOR FUNCTION AND FIRST DERIVATIVES

To develop the difference equations, the Kronecker delta is defined as

$$\delta_{\alpha,\mu} = 0, \qquad \alpha \neq \mu; \quad \text{or} \quad \delta_{\alpha,\mu} = 1, \qquad \alpha = \mu \qquad (C1)$$

and the following circular indices are also defined

$$n- = n - 1 + P\delta_{n,1} \qquad (C2)$$

$$n+ = n + 1 - P\delta_{n,P} \qquad (C3)$$

$$e+ = e + 1 - M\delta_{e,P} \qquad (C4)$$

$$nm = n - 1 + M\delta_{n,1} \qquad (C5)$$

with

$$A_e = [1/2] \, x_o(y_j - y_k) + [1/2] \, x_j(y_k - y_o) + [1/2] \, x_k(y_o - y_j)$$

$$(C6)$$

[3, p. 110, Eq. (3.1.2)]. Using this notation, the coefficients for the following operators can be written as follows:

x derivative operators

$$k_n^x = \left[\beta_2^{(nm)}/2A_T \right]\left(y_o - y_{n-} \right)\left(1 - \delta_{n,P-M} \right)$$

$$+ \left[\beta_2^{(n)}/2A_T \right]\left(y_{n+} - y_o \right)\left(1 - \delta_{P+1-n,P-M} \right) \qquad (C7)$$

$$k_o^x = \sum_{e=1}^{M} \left[\beta_2^{(e)}/2A_T \right]\left(y_e - y_{e+} \right) \qquad (C8)$$

y derivative operators

$$k_n^y = \left[\beta_3^{(nm)}/2A_T\right]\left(x_{n-} - x_o\right)\left(1 - \delta_{n,P-M}\right)$$

$$+ \left(\beta_3^{(n)}/2A_T\right)\left(x_o - x_{n+}\right)\left[1 - \delta_{P+1-n,P-M}\right] \qquad (C9)$$

$$k_o^y = \sum_{e=1}^{M} \left(\beta_3^{(3)}/2A_T\right)\left[x_{e+} - x_e\right] \qquad (C10)$$

ϕ function

$$k_n^\phi = \left[\beta_4^{(nm)}\left(A_{nm}/4A_T\right)\right]\left(1 - \delta_{n,P-M}\right)$$

$$+ \left[\beta_4^{(n)} \left(A_n/4A_T\right)\right]\left(1 - \delta_{P+1-n,P-M}\right) \qquad (C11)$$

$$k_o^\phi = (1/2\ A_T) \sum_{e=1}^{M} \beta_4^{(e)} A_e \qquad (C12)$$

APPENDIX D – SOLUTION OF THE WEIGHTED RESIDUAL EQUATIONS

Consider the partial differential equation of the form

$$\nabla^2\phi + \frac{\delta\phi}{\delta x} = 0 \qquad (D1)$$

Employing the weak solution [3, p. 443] for the four-node cell in part A of Fig. 4 yields

$$(\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi_o) + h[(\phi_1 - \phi_3)/3] = 0 \qquad (D2)$$

To put Eq. (D2) in a more familiar form which coincides with the standard Taylor series finite difference approach, Eq. (D2) is multiplied by $3/A_T$, which corresponds to $3/2h^2$ for the geometry shown in Fig. 4. Thus, Eq. (D2) becomes

$$3/2\ [(\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi_o)/h^2] + (\phi_1 - \phi_3)/2h = 0 \qquad (D3)$$

The $3/A_T$ term was chosen as the multiplying constant because it will always convert the function and its first derivatives so that they match the Taylor series expression. Consequently, the $A_T/3$ term appears in Eq. (5) in the body of this report.

Report Documentation Page

1. Report No	2. Government Accession No.	3. Recipient's Catalog No.
NASA TM-101298		

4. Title and Subtitle	5. Report Date
Discretization Formulas for Unstructured Grids	6. Performing Organization Code

7. Author(s)	8. Performing Organization Report No.
Kenneth J. Baumeister	E-4094
	10. Work Unit No.
	505-62-21

9. Performing Organization Name and Address	11. Contract or Grant No.
National Aeronautics and Space Administration Lewis Research Center Cleveland, Ohio 44135-3191	13. Type of Report and Period Covered

12. Sponsoring Agency Name and Address	
National Aeronautics and Space Administration Washington, D.C. 20546-0001	Technical Memorandum
	14. Sponsoring Agency Code

15. Supplementary Notes

Prepared for the Second International Conference on Numerical Grid Generation in Computational Fluid Dynamics cosponsored by the National Aeronautics and Space Administration, the U.S. Air Force Office of Scientific Research, and the University of Miami, Miami Beach, Florida, December 5-8, 1988.

16. Abstract

The Galerkin weighted residual technique using linear triangular weight functions is employed to develop finite difference formula in cartesian coordinates for the Laplacian operator, first derivative operators and the function for unstructured triangular grids. The weighted residual coefficients associated with the weak formulation of the Laplacian operator are shown to agree with the Taylor series approach on a global average. In addition, a simple algorithm is presented to determine the Voronoi (finite difference) area for an unstructured grid.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
Finite difference, Finite element, Laplacian, Varonoi, Area rule	Unclassified - Unlimited Subject Category 34

19. Security Classif. (of this report)	20. Security Classif (of this page)	21. No of pages	22. Price*
Unclassified	Unclassified	12	A02

NASA FORM 1626 OCT 86   *For sale by the National Technical Information Service, Springfield, Virginia 22161