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Compilation of Methods in Orbital Mechanics and Solar Geometry

James J. Buglia

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16. Abstract

This paper contains a collection of computational algorithms for determining geocentric ephemerides of Earth satellites useful for both mission planning and data reduction applications. Special emphasis is placed on the computation of sidereal time and on the determination of the geocentric coordinate of the center of the Sun, all to the accuracy found in the Astronomical Almanac. The report is completely self-contained in that no requirement is placed on any external source of information, and hence, these methods are ideal for computer application.

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Introduction

Since the launching of the first Sputnik in 1957, the use of near-Earth satellites for scientific, military, and commercial applications has progressed far beyond the expectations of even the most visionary of the early satellite pioneers. As a result, many sophisticated numerical and mathematical techniques have evolved to permit the precise prediction of the position of a satellite in time and space, and the Earth's external gravitational field has been determined to high accuracy.

However, for most mission planning purposes, and even in fact for many operational applications, ultra-high precision is neither warranted nor desired, as the computational expense generally increases quite rapidly with increased computational precision. Consequently, there is still a firm need for simple computational algorithms which produce results of reasonable accuracy at moderate expense.

Anyone who works in a specialized scientific discipline for any length of time naturally acquires over the years a computational methodology, consisting of large numbers of computational algorithms of many degrees of complexity and accuracy, other computational procedures, approximate methods, etc., with which he feels most comfortable and which have proven to be useful and accurate, even though they may not always provide the most direct way of making a particular calculation. The present text is such a compilation of procedures in orbital mechanics and spherical astronomy which the author has collected over about a 30-year period. Many of the formulas presented herein were included in an attempt to make the text as self-contained as possible, without requiring the reader to consult almanacs, ephemerides, or other reference material. Nothing but the material presented in this report is necessary to carry out any of the computations described; thus, these procedures are ideal for programming on computers.

It is not the author's intention to derive or otherwise develop all the equations and/or formulas presented herein—this task is most adequately covered in practically any good textbook on Celestial Mechanics or Spherical Astronomy. (See, for example, any of the general texts cited.) Any equations, however, which are not widely circulated in the literature, are either specifically referenced or developed to the point where a heuristic argument will suffice to convince the reader that the equation does what it is claimed to do and what its limitations are. All the comments, opinions, or other commentary are, of course, the author's own, and perhaps serve to do nothing but illustrate the author's own prejudices.

The problem of presenting a completely unambiguous and unique set of symbols is one that has plagued writers of scientific (and pseudo-scientific) documents for many years, and the present author offers no relief in this regard. In several instances, the same symbol may stand for two or more completely unrelated quantities (e.g., z describes a coordinate axis, also stands for the component of a vector along this axis, and is used as the symbol for the zenith distance angle). The philosophy adopted here is that, when discussing a particular area or discipline, the symbols most commonly used by most writers in that area are used. These are defined in the text as encountered and the use of multiply defined symbols should cause no problem as they are generally used only in the few pages in the neighborhood of where they are defined.

The layout of the text is as follows: the coordinate systems in which the remainder of the text is developed and the systems of time measurement common in the computation of satellite ephemerides are presented in chapter 1. The mathematical description of the external gravity field of the Earth is presented and the restriction in the text to the use of zonal harmonics is defended in chapter 2. Chapter 3 defines the author's preference for the set of orbital elements used to describe and advance in time the position and velocity of the spacecraft, while chapter 4 introduces the Cartesian form of the equations of motion. The Lagrange Planetary Equations are not used, but their use in theoretical developments is mentioned briefly, and some of these theoretical results are used herein. Numerical comparisons between sundry numerical procedures are presented. Chapter 5 presents numerical algorithms for accurately determining the right ascension and declination of the Sun. The word *accurately* here refers to the precision of computing a given parameter relative to that usually *required* by *most* near-Earth satellite applications and not generally to the precision available for, say, some astronomical applications.

The text concludes with a few very brief remarks on the application of the present methods to other Earth satellite problems and to the computation of the orbits of the planets and the Moon.

Chapter 1

Coordinate Systems and Time

The primary coordinate system used in the present text is a quasi-inertial system, defined by the Earth's equator and the apparent orbit of the Sun around the Earth with the origin at the center of mass of the Earth. (See fig. 1-1.) The intersection of the Sun's orbital plane, the *ecliptic*, with the Earth's equatorial plane defines a line, called the *line of nodes*. The direction defined by the center of the Earth and the node at which the Sun appears to cross the equator from south to north is called the *ascending node*, the *first point of Aries*, or more usually, the *vernal equinox* γ and defines the direction of the x-axis. The rotational axis of the Earth defines the z-axis, and the y-axis is located in the equatorial plane in such a way that the xyz coordinate system is a right-handed one. The ecliptic plane is inclined to the equator at about 23.44°, the *obliquity of the ecliptic*, and can be accurately computed from equations presented later in this chapter.

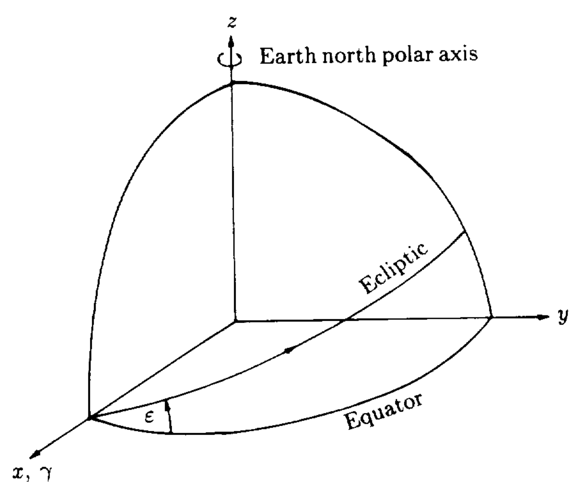


Figure 1-1. Definition of quasi-inertial x-axis by intersection of equatorial and ecliptic planes.

If the Earth and Sun were both perfectly homogeneous spheres so that their mutual gravitational attraction followed an inverse-square law and acted along a line joining their centers, if there were neither Moon nor planets, and if the Sun were either stationary in the universe or moved through it on a straight line at constant speed, then the coordinate system described in the previous paragraph would indeed be an inertial system. However, *none* of these conditions hold. The Earth's figure, both geometrically and dynamically, is closely approximated not by a sphere but by an oblate spheroid—an ellipse rotated about its minor or shorter axis—the equatorial radius being about 21 km larger than the polar radius (6378.160 km and 6356.775 km, respectively). As shown in figure 1-1, the ecliptic plane is inclined to the equator at about 23.44°, and thus, the direction of the gravity resultant of the Earth-Sun does not pass through the center of mass of the Earth but instead passes through a point on the Sun side of the line joining the centers of the Earth and Sun. This produces a gravitational couple, or torque, on the Earth, and since the Earth is spinning, this torque causes the Earth to wobble, or *precess* (fig. 1-2(a)) and *nutate* (fig. 1-2(b)). Likewise, the Earth has a rather large Moon which also orbits in a plane inclined to the equator and which also produces a gravitational torque. This combined lunisolar perturbation causes the Earth to precess and nutate, much as a spinning top does, and causes the z-axis to rotate about the ecliptic pole K in a clockwise direction as seen from the North Pole. This, in turn, results in the vernal equinox moving clockwise in the plane of the ecliptic at the rate of about 50 arc-sec/yr, the *precession of the equinoxes*. This measured value also includes a small term, called the *planetary precession*, due to the other planets, which causes the plane of the ecliptic, assumed to be fixed when computing the lunisolar precession, to wobble slightly. The combined lunisolar and planetary precessions are called the *general precession*.

(a) Precession only.   (b) Precession plus nutation.

Figure 1-2. Sketch illustrating precession only and precession plus nutation on motion of Earth's pole. Motion of γ is called *precession of equinox;* P is north pole of Earth; K is north pole of ecliptic; P moves around K.

When referring to astronomical coordinates, one generally fixes the vernal equinox (x-axis) and the equator in one of two ways—either by referring to their positions at a specific date and then requiring that these positions be frozen for the duration of the computational period, or by appending the words "of date" to the positions of the vernal equinox and equator, which means that these directions are slowly changing while the computations are being made and refer to their locations at the current time. If only the precessional effect is being taken into account (usual for much Earth satellite work) one refers to mean-of-date coordinates. If nutation is also included, then one refers to true-of-date coordinates. In the present work, mean-of-date coordinates are used, except where specifically noted, and are assumed synonymous with "inertial" coordinates.[1]

Celestial objects are located in this coordinate system by the use of two angles—right ascension and declination—analogous to the more familiar longitude and latitude, respectively (fig. 1-3). The right ascension α is measured from the vernal equinox along the equator, positive east, or counterclockwise as seen from the North Pole of the Earth. The right ascension is frequently measured in time units (practically always in astronomy), but for computational purposes, degrees are used herein. The declination δ is measured along a "meridian," north and south from the equator positive north, and in the present work, since we will consider the Earth to be geometrically a sphere but gravitationally an oblate spheroid, the declination is numerically identical with the latitude.

If R is the distance from the center of the Earth to the celestial object, then a vector to the body can be written in the right ascension-declination system as

$$\boldsymbol{r} = R \begin{bmatrix} \cos\delta\cos\alpha \\ \cos\delta\sin\alpha \\ \sin\delta \end{bmatrix} \tag{1-1a}$$

[1] In chapter 5, which discusses the determination of the position of the Sun, proper allowance for the motion of the equinox is made in the nonlinear terms of the sundry time series used in the computations. Thus, the Sun's position in the mean-of-date coordinates is accurately determined. No such allowance is made in the earlier chapters which discuss the position computation of an Earth satellite—these coordinate systems are assumed to be truly inertial. The only orbital element that is affected by this assumption is the position angle of the line of modes (defined in chapter 4). A small linear term could be subtracted from the motion of this parameter computed in chapter 4, but the error induced in reflecting this effect is negligible compared with the inherent accuracy of these simplified equations.

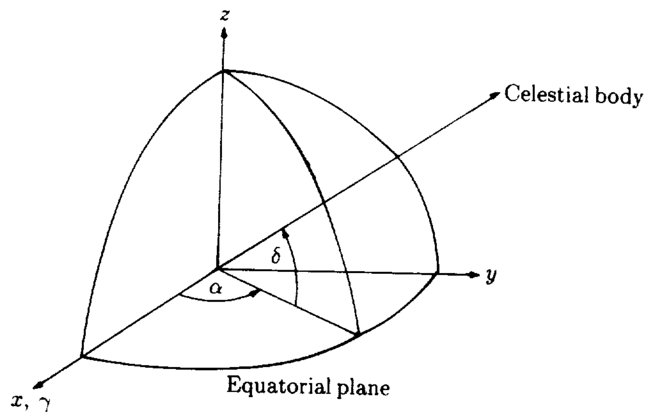


Figure 1-3. Definitions of right ascension α and declination δ of celestial body.

from which the Cartesian coordinates of the body, in our inertial system, can be computed. Also, as will frequently be done, if the Cartesian coordinates are known, equation (1-1a) can be used to compute the right ascension and declination. (Frequently, the distance of the body from the Earth center is either unknown or one has only a unit vector defining the direction of the body. Equation (1-1a) can still be used to determine the right ascension and declination simply by assuming that $|\mathbf{r}|$ and $R = 1$.)

Measurements made on the Earth's surface are most conveniently referred to a set of coordinate axes fixed with respect to the rotating Earth. The x-axis is defined by the intersection of the meridian passing through the Greenwich Observatory in England—known, cleverly enough, as the *Greenwich meridian*—with the equator. The rotating z-axis is again defined by the Earth's spin axis, and the y-axis completes the right-handed system. Objects are located on the Earth by the familiar latitude-longitude spherical coordinates. Latitude is measured positive northward from the equator, along a *meridian*. Longitude is measured in the equatorial plane, positive eastward from the Greenwich meridian. The point labeled P in figure 1-4 has latitude ψ_c and longitude λ. The Greenwich meridian is also labeled, as is the position of the nonrotating x-axis, which is labeled x, γ. Assuming a spherical Earth of radius R_E, a vector to the surface point P can be written in the rotating coordinates

$$\mathbf{r}_E = R_E \begin{bmatrix} \cos\psi_c \cos\lambda \\ \cos\psi_c \sin\lambda \\ \sin\psi_c \end{bmatrix} \qquad (1\text{-}1b)$$

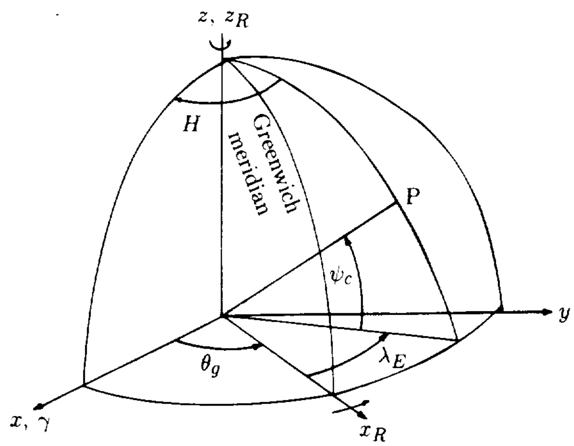


Figure 1-4. Location of Greenwich meridian and definition of geocentric latitude ψ_c and longitude λ_E. Greenwich sidereal time is θ_g.

In order to relate the coordinates of P in the Earth-fixed axes (eq. (2-1b)) to the inertial system (eq. (2-1a)), the location of the Greenwich meridian at any time relative to the vernal equinox θ_g is needed. Then, in the inertial system

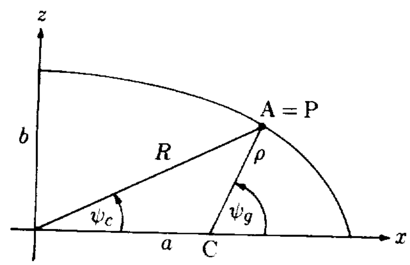
$$r = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta_g & -\sin\theta_g & 0 \\ \sin\theta_g & \cos\theta_g & 0 \\ 0 & 0 & 1 \end{bmatrix} r_E = \begin{bmatrix} R_E \cos\psi_c \cos(\theta_g + \lambda) \\ R_E \cos\psi_c \sin(\theta_g + \lambda) \\ R_E \sin\psi_c \end{bmatrix} \tag{1-2}$$

The latitude ψ_c, as determined by equation (1-2), is called the "geocentric" latitude of P (fig. 1-5(a)) and is measured from the equatorial plane along a meridian to the line joining P and the origin of coordinates.

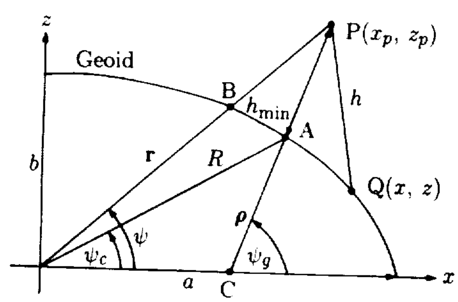
The "geodetic" latitude ψ_g of figure 1-5(b) is another measurement of latitude, usually but not always confined to measurements made on the Earth's surface and is the angle between the equatorial plane and the normal to the geoidal surface which passes through P, line PAC. The altitude of a spacecraft at P is measured along the direction of PA, the local vertical or plumb bob direction, rather than along PB. In orbit mechanics, the point A is called the subsatellite point.

If the point P were on the surface, as in figure 1-5(a), there is a simple relation between the geocentric and geodetic latitudes, namely,

$$\tan\psi_g = \frac{a^2}{b^2}\tan\psi_c \tag{1-3}$$



(a) Point P on surface.



(b) Spatial point P above surface.

Figure 1-5. Sketch of geometry and definition of geodetic latitude ψ_g and geocentric latitude ψ_c of point on and above surface of oblate spheroid.

where a and b are the equatorial and polar radii of the Earth, 6378.160 and 6356.775 km, respectively.

In the more general case shown in figure 1-5(b), no such simple relation exists, neither for the latitudes nor for the height h_{min}, and more complex methods must be resorted to. Escobal (1965) presents an iteration technique. The author's own solution is presented here.

In figure 1-5(b), let Q be any point on the surface of the ellipse cross section. The distance between P and Q is then

$$h^2 = (x - x_p)^2 + (z - z_p)^2 \tag{1-4}$$

The problem is to minimize h subject to the constraint that the point Q (x, z) lies on the ellipse

$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{1-5}$$

This is a straightforward minimization problem requiring the use of the Lagrange multiplier technique. We want to form the function

$$\Phi = (x - x_p)^2 + (z - z_p)^2 + g\left(\frac{x^2}{a^2} + \frac{z^2}{b^2} - 1\right)$$

where g is the Lagrange multiplier. The three equations

$$\frac{\partial \Phi}{\partial x} = 2\,(x - x_p) + \frac{2gx}{a^2} = 0$$

$$\frac{\partial \Phi}{\partial z} = 2\,(z - z_p) + \frac{2gz}{b^2} = 0$$

and equation (1-5) are sufficient to determine the three unknowns, x, z, and g. After a fair amount of elementary algebra we arrive at the relation

$$F(\xi) = A_4\xi^4 + A_3\xi^3 + A_2\xi^2 + A_1\xi + A_0 \tag{1-6}$$

where

$$\begin{aligned} A_4 &= \alpha(1-\Gamma)^2 \\ A_3 &= 2\alpha\Gamma(1-\Gamma) \\ A_2 &= \alpha\Gamma^2 - (1-\Gamma)^2 + \beta \\ A_1 &= -2\Gamma(1-\Gamma) \\ A_0 &= \Gamma^2 \\ \alpha &= \left(\frac{x_p}{a}\right)^2 \\ \beta &= \left(\frac{z_p}{b}\right)^2 \\ \Gamma &= \left(\frac{a}{b}\right)^2 \end{aligned}$$

and $\xi = x/x_p$ and $\eta = z/z_p$. Equation (1-6) is a quartic equation which can be solved exactly using the classical Descartes' method. (See, for example, Escobal 1965.) However, since the Earth is very nearly spherical, the author has found that the Newton-Raphson iteration technique generally converges in one or two iterations and prefers that method. In either case, once ξ is found, then

$$\eta = \frac{1}{1 - \Gamma\left(1 - \frac{1}{\xi}\right)} \tag{1-7}$$

from which

$$\left.\begin{aligned} x &= \xi x_p \\ z &= \eta z_p \end{aligned}\right\} \tag{1-8}$$

give the coordinates for point A, the subsatellite point. The height $h_{\min}$ is then found by substituting these coordinate values into equation (1-4). The geocentric latitude is found from

$$\psi_c = \tan^{-1} \frac{z}{x} \tag{1-9}$$

and the geodetic latitude from equation (1-3).

Sidereal Time

The angle θ_g in equation (1-2) is a very important quantity in most Earth orbit applications because it relates the rotating coordinate system, in which most *measurements* are made or related, to the inertial coordinate system, in which most *calculations* are made. This angle is variously referred to as the *Greenwich hour angle of the vernal equinox* or as the *Greenwich sidereal time*—the latter is used herein.

An approximate method for estimating the sidereal time for 0 hr GMT of any date is given as follows (more precise methods will be given following this short section). The sidereal time at 0 hr GMT is 0^h on approximately September 22 (exactly zero on passage through the autumnal equinox) and increases at the rate of $3^m56^s.5554$ per day (see the next section "Time" following the coordinate discussion). Now, $3^m56^s.5554$ is very nearly equal to $(4^m - 3^s.5)$. Therefore, if N_s is the number of days from September 22, then the sidereal time is approximately

$$\theta_{g_o} \approx (N_s \times 4) \text{ min} - (N_s \times 3.5) \text{ sec}$$

For example, find the approximate sidereal time at 0 hr GMT on March 1. September 22 is day 265, and March 1 is day 60. Then, the number of days from September 22 to March 1 is (365 − 265) + 60 or 160 days. Thus, the sidereal time for 0 hr GMT on March 1 is approximately

$$(160 \times 4) \text{ min } - (160 \times 4) \text{ sec}$$

or $10^h30^m40^s$. The Astronomical Almanac for 1985 (AA85) gives $10^h34^m50^s$. The reason for most of the difference is that in 1985, the autumnal equinox occurs at 6 hr GMT on September 21, which is 0.75 days earlier than assumed here. If we make this correction (which we usually wouldn't do, as if we knew what correction to make, we wouldn't have to use this approximate method), we would get $10^h33^m37^s$, which is just a bit more than a minute off.

Incidentally, the day of the year for any given calendar date can be computed from the equations (Almanac for Computers 1980)

$$\left.\begin{aligned} N &= \left\langle \frac{275M}{9} \right\rangle - 2\left\langle \frac{M+9}{12} \right\rangle + D - 30 \qquad \text{(Nonleap year)} \\ N &= \left\langle \frac{275M}{9} \right\rangle - \left\langle \frac{M+9}{12} \right\rangle + D - 30 \qquad \text{(Leap year)} \end{aligned}\right\} \quad (1\text{-}10)$$

where M is the month (1–12) and D is the day (1–31). The symbol $\langle\ \rangle$ means "integer value of."

In order to compute the sidereal time to an accuracy of a few hundredths of a second, needed for many space applications, we need to introduce the concept of *Julian date*.

The Julian date associated with a given calendar date is equal to the number of days which have elapsed between noon, January 1, 4713 B.C. (November 24, 4713 B.C. in the modern calendar) and the date in question. This date was selected by astronomers as preceding the earliest recorded astronomical observations so that all known observations will have a positive Julian date, making it very easy to determine the time intervals between events. With all the sundry calendars which have been in use throughout history, the Julian date is generally found by referring to tables corrected to the present Gregorian calendar. However, for the years 1801–2099 A.D., a period probably inclusive for most of us, the conversion of Gregoric calendar date to Julian date can be carried out by the following formula (Almanac for Computers 1980):

$$\mathrm{JD} = 367Y - \left\langle \frac{7\left(Y + \left\langle \frac{M+9}{12} \right\rangle\right)}{4} \right\rangle + \left\langle \frac{275M}{9} \right\rangle + D$$

$$+ 1721013.5 + \frac{UT}{24} - 0.5 \operatorname{sgn}(100Y + M - 190002.5) + 0.5 \qquad (1\text{-}11)$$

where

$$Y = \text{year } (1801 \leq Y \leq 2099)$$
$$M = \text{month } (1 \leq M \leq 12)$$
$$D = \text{day } (1 \leq D \leq 31)$$

and UT is the Universal time (Greenwich mean time, see later in this chapter for discussion). The symbol $\langle\ \rangle$ means "largest integer" function. The last two terms add up to zero for all dates after February 28, 1900, so that these terms may be omitted for all subsequent dates. (There is another algorithm quoted by Blackadar (1984), which reportedly gives the correct Julian date for any calendar date from 4713 B.C. to 3500 A.D.—some readers might find this more useful.)

Therefore, in order to compute the Greenwich sidereal time, GST, for any date and time T, use the following algorithm:

1. Set UT = 0 in equation (1-11) and find the Julian date for 0 hr UT of the date in question.

2. Find the Greenwich sidereal time at 0 hr, θ_{g_o}, from the equation (Escobal 1965, or the Explanatory Supplement to the Astronomical Ephemeris 1961, for example)

$$\theta_{g_o} = 99^\circ.6909833 + 36000^\circ.76892 T_u + 0^\circ.00038708 T_u^2 \qquad (1\text{-}12)$$

(mod(360))
where the units are in degrees, and

$$T_u = \frac{\text{JD} - 2415020.0}{36525.} \qquad (1\text{-}13)$$

is the number of Julian centuries from noon on January 1, 1900.

3. Let T be the number of minutes of mean solar time (usual clock minutes) from 0 hr UT to the actual time in question. Then, during this time interval, the Earth has rotated

$$\Delta\theta_g = 0^\circ.25068447 T \qquad (1\text{-}14a)$$

and hence the GST is

$$\theta_g = \theta_{g_o} + \Delta\theta_g \qquad (1\text{-}14b)$$

and now the Earth-fixed (rotating) coordinates can be related to the inertial coordinate system. Note the use of equation (1-12) implies a mean-of-date coordinate system, as nutational terms are not included in this equation.

Both the *mean* sidereal time (precession only) and the *apparent* sidereal time (nutation also included) are tabulated in various almanacs, ephemerides, for example. Sidereal time is generally expressed in hours-minutes-seconds, even though it is usually used in degree units. To convert from decimal degrees to decimal hours, simply divide by 15—the Earth rotates 360/24 or 15 deg/hr. To display the accuracy of equation (1-12), tables 1-1 and 1-2 show the mean sidereal time computed from equation (1-12) and the seconds only of the mean sidereal time taken from the AA85.[2] The first of each month is shown. The maximum error in table 1-1, for the year 1985, is about 0.07 second when comparing the almanac data with those computed from equation (1-12). The last column in the table shows the mean sidereal time computed from a set of formulas published in AA85 and given as follows:

$$\theta_{g_o} = 100^\circ.4606184 + 36000^\circ.77005 T_u' + 0^\circ.000387933 T_u'^2 - 0^\circ.000000258 T_u'^3 \qquad (1\text{-}15)$$

[2] The Astronomical Almanac 1985.

Table 1-1. Comparison of Computed and Published Mean Sidereal Times for First Day of Each Month for Year 1985

Date	Julian date	Mean sidereal time, θ_g, from—				
		Equation (1-12)			AA85	Equation (1-15)
		h	m	s	s	s
Jan. 1	2446066.5	6	42	21.8975	21.9674	21.9674
Feb. 1	2446097.5	8	44	35.1139	35.1838	35.1838
Mar. 1	2446125.5	10	34	58.6641	58.7341	58.7340
Apr. 1	2446156.5	12	37	11.8804	11.9505	11.9504
May 1	2446186.5	14	35	28.5413	28.6115	28.6114
June 1	2446217.5	16	37	41.7576	41.8279	41.8278
July 1	2446247.5	18	35	58.4186	58.4889	58.4888
Aug. 1	2446278.5	20	38	11.6349	11.7053	11.7052
Sept. 1	2446309.5	22	40	24.8512	24.9216	24.9216
Oct. 1	2446339.5	0	38	41.5121	41.5827	41.5826
Nov. 1	2446370.5	2	40	54.7284	54.7990	54.7990
Dec. 1	2446400.5	4	39	11.3894	11.4601	11.4600

where

$$T'_u = \frac{\text{JD} - 2451545.0}{36525.}$$

which is the number of Julian centuries from January 15, 2000 A.D. Neither equation (1-12) nor equation (1-15) is exact. Equation (1-15) gives somewhat better results when applied to post-1984 data. The reason for this is that in 1984 the International Astronomical Union adopted a new revised set of physical constants which slightly changed some of the numerical constants in the time series—compare the numerical coefficients of T_u and T'_u in equations (1-12) and (1-15), for example. These new constants include the effects of general relativity and are arguably more precise than the older, pre-1984 constants, although the absolute differences between the numerical results are of the order of hundredths of arc-seconds.

Table 1-2, taken directly from AA85, shows both the mean (precession only) and apparent (nutation included) sidereal times, and the differences between them, for the same dates.

Table 1-2. Published Values of Apparent and Mean Sidereal Times for First Day of Each Month for Year 1985

Date	Apparent sidereal time			Mean sidereal time,	Difference,
	h	m	s	s	s
Jan. 1	6	42	21.1326	21.9674	−0.8348
Feb. 1	8	44	34.4222	35.1838	−0.7616
Mar. 1	10	34	57.9653	58.7341	−0.7688
Apr. 1	12	37	11.1409	11.9505	−0.8096
May 1	14	35	27.7665	28.6115	−0.8450
June 1	16	37	41.0078	41.8279	−0.8201
July 1	18	35	57.7705	58.4889	−0.7184
Aug. 1	20	38	11.0803	11.7053	−0.6250
Sept. 1	22	40	24.2781	24.9216	−0.6435
Oct. 1	0	38	40.8663	41.5827	−0.7164
Nov. 1	2	40	54.0502	54.7990	−0.7488
Dec. 1	4	39	10.7768	11.4601	−0.6833

As seen, the differences are of the order of 1 sec, the mean sidereal time being about 1 sec later than the apparent sidereal time for this particular time interval. If the additional accuracy is needed (i.e., if the apparent sidereal time is needed), the following correction, called the *Equation of the Equinoxes*, can be added to the mean sidereal time as computed either from equation (1-12) or the sequence following it. The Equation of the Equinoxes is defined as the right ascension of the mean equinox referred to the true equator and equinox. The expression for the apparent sidereal time is

$$(\theta_g) \text{ apparent} = (\theta_g) \text{ mean} + \Delta\psi \cos \varepsilon \qquad (1\text{-}16)$$

where $\Delta\psi$, called the *nutation in longitude*, is given approximately by (Smart 1977, or Escobal 1968)

$$\Delta\psi = -(17''.233 + 0''.017T_u) \sin \Omega + 0''.209 \sin 2\Omega - 1''.273 \sin 2L - 0''.204 \sin 2C \qquad (1\text{-}17)$$

where $\Delta\psi$ is in arc-seconds in this correction formula in which (Escobal 1968)

Ω = longitude of mean ascending node of lunar orbit, measured in ecliptic plane from mean equinox of date, deg

$$= 259^\circ.132750 - 1934^\circ.1420083T_u + 0^\circ.00207778T_u^2 + 0^\circ.0000022222T_u^3 \qquad (1\text{-}18)$$

L = mean longitude of Sun, measured in ecliptic plane from mean equinox of date, deg

$$= 279^\circ.6966778 + 36000^\circ.7689250T_u + 0^\circ.000302500T_u^2 \qquad (1\text{-}19)$$

C = geocentric mean longitude of Moon, measured in ecliptic plane from mean equinox of date to mean ascending node of lunar orbit, and then along orbit, deg

$$= 270^\circ.4341639 + 481267^\circ.8831417T_u - 0^\circ.00113333T_u^2 + 0^\circ.0000018889T_u^3 \qquad (1\text{-}20)$$

ε = mean obliquity of ecliptic, deg

$$= 23^\circ.4522944 - 0^\circ.0130125T_u - 0^\circ.0000016389T_u^2 + 0^\circ.00000050278T_u^3 \qquad (1\text{-}21)$$

The correction to mean sidereal time using equations (1-16) to (1-21) is tabulated in table 1-3.

The nutation corrections computed from equations (1-16) to (1-21) agree very well with the differences presented in the last column of table 1-2, the maximum error being about 0.014 sec in November. Escobal (1968, pp. 252–260 or pp. 304–305) gives another expression for the nutation in longitude which is reportedly accurate to 0.0001 arc-sec, if such accuracy is needed.

Table 1-3. Computed Differences Between Mean and Apparent Sidereal Times for First Day of Each Month for Year 1985, With Intermediate Angular Values Included

Date	Ω, deg	L, deg	C, deg	ϵ, deg	$\Delta\psi$ cos ϵ, arc-sec (corrected)
Jan. 1	55.1508	280.5969	31.4281	23.4412	−0.8367
Feb. 1	53.5093	311.1519	79.8964	23.4412	−0.7360
Mar. 1	52.0266	338.7501	88.8355	23.4412	−0.7671
Apr. 1	50.3850	9.3052	137.3038	23.4412	−0.8125
May 1	48.7964	38.8746	172.5956	23.4412	−0.8539
June 1	47.1548	69.4296	221.0639	23.4412	−0.8243
July 1	45.5662	98.9991	256.3558	23.4412	−0.7222
Aug. 1	43.9246	129.5541	304.8241	23.4412	−0.6309
Sept. 1	42.2831	160.1092	353.2964	23.4412	−0.6443
Oct. 1	40.6944	189.6786	28.5843	23.4412	−0.7115
Nov. 1	39.0529	220.2337	77.0526	23.4412	−0.7344
Dec. 1	37.4643	249.8031	112.3445	23.4412	−0.6710

The Greenwich sidereal time was defined as the hour angle of the vernal equinox θ_g (fig. 1-4). The *local sidereal time* is defined as the hour angle of the vernal equinox measured from the observer's meridian H. Thus, from figure 1-4 it is seen that

$$H = \theta_g + \lambda_E = \text{LST} \tag{1-22}$$

and, when expressed in time measure, is the number of sidereal hours since the observer's meridian was on the vernal equinox. Note that this quantity, $\theta_g + \lambda_E$, was used in equation (1-2) which related $\boldsymbol{r}_E$ in the rotating system to the vector $\boldsymbol{r}$ in the inertial axis system.

As stated earlier, the Greenwich sidereal time θ_g allows us to relate directly quantities calculated in the "inertial coordinate" system with those in the rotating, Earth-fixed axis system (fig. 1-4).

In figure 1-6 we show the position of the vernal equinox and the Greenwich meridian, the two "x-axes" of coordinates, as well as the position of an Earth-fixed observer O and the spatial location of an Earth satellite. It must be emphasized that this picture is "frozen in time," as the Earth is rotating about the pole P, and the satellite is moving. However, at this instant, the satellite has a definite set of coordinates—right ascension α_s and declination δ_s, as measured in the inertial coordinate system, and a definite latitude $\psi_s = \delta_s$ and longitude λ_s measured in the Earth-fixed coordinates. The subsatellite point S is defined to be the point on the surface when the geocentric radius vector of the spacecraft (S/C), $\boldsymbol{r}_s$, pierces the Earth's surface. The angular coordinates of S are those of the spacecraft.

Let $\boldsymbol{R_e}$ be a vector to the observer in the rotating coordinates (ψ_o, λ_o); R_e is the magnitude of the Earth's radius,

$$\boldsymbol{R_e} = R_e \begin{bmatrix} \cos\psi_o \cos\lambda_o \\ \cos\psi_o \sin\lambda_o \\ \sin\psi_o \end{bmatrix} \tag{1-23}$$

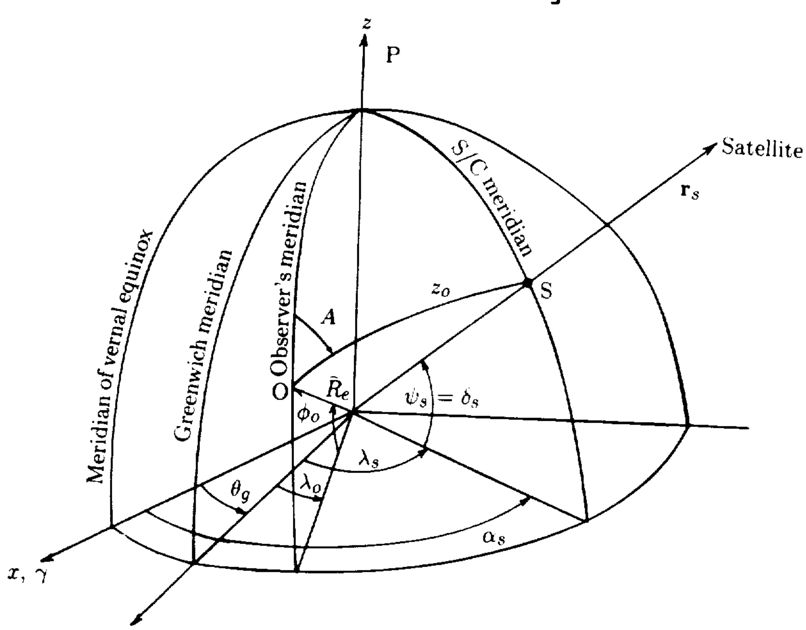


Figure 1-6. Sketch relating position of observer in rotating Earth-fixed axes to inertial position of celestial body.

The coordinates of the spacecraft are similar,

$$\mathbf{r}_s = r_s \begin{bmatrix} \cos\ \psi_s \cos\ \lambda_s \\ \cos\ \psi_s \sin\ \lambda_s \\ \sin\ \psi_s \end{bmatrix} \tag{1-24}$$

These two vectors form a plane, as shown in figure 1-7. The angle z is called the zenith distance, where z_T is the zenith distance of the spacecraft as measured at the center of the Earth, and z_o is that measured by the observer O. For near-Earth objects, such as an Earth satellite, the Moon, and for some purposes the Sun, we have $z_T \neq z_o$ (parallax effect). For observation of stars, though, as r_s approaches ∞, z_T approaches z_o, and z_o can easily be found from

$$\cos\ z_o = \frac{\mathbf{R}_e \cdot \mathbf{r}_s}{R_e r_s} = \sin\ \psi_o\ \sin\ \psi_s + \cos\ \psi_o\ \cos\ \psi_s\ \cos\ (\lambda_s - \lambda_o) \tag{1-25}$$

Define the radius from the observer to the spacecraft,

$$\boldsymbol{\rho} = \mathbf{r}_s - \mathbf{R}_e$$

and hence

$$\cos\ z_T = \frac{\mathbf{R}_e \cdot \boldsymbol{\rho}}{R_e \rho}$$

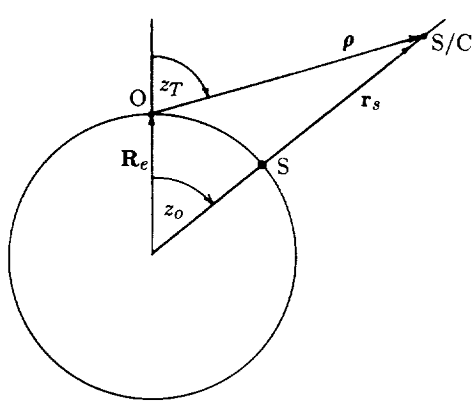


Figure 1-7. Zenith angle z_T measured by an observer at O, and one measured at center of Earth, z_o. Difference is geocentric parallax of celestial object.

which can be put into the form

$$\cos z_T = \frac{r_s \cos z_o - R_e}{(R_e^2 + r_s^2 - 2R_e r_s \cos z_o)^{1/2}} \tag{1-26}$$

From equation (1-26), it can be seen that, as r_s approaches ∞, z_o approaches z_T as expected.

The sine and cosine of the azimuth of the spacecraft, A, can be found by applying the law of sine and the law of cosine for sides to the spherical triangle POB of figure 1-6

$$\left.\begin{aligned} \sin A &= \frac{\cos \psi_s \sin (\lambda_s - \lambda_o)}{\sin z_o} \\ \cos A &= \frac{\sin \psi_s - \sin \psi_o \cos z_o}{\cos \psi_o \sin z_o} \end{aligned}\right\} \tag{1-27}$$

Don't forget α_s, δ_s, z_o, z_T, and A are time-dependent quantities.

Another coordinate system that is very often encountered is a local axis set fixed in some definable way to the orbiting spacecraft. Measurements with on-board instruments and/or measurement direction vectors are made in this local system, and one must frequently transform vectoral quantities back and forth between this local system and either the inertial system or the Earth-fixed system defined earlier.

The most fundamental local system is one which is defined completely in terms of the dynamic variables $\boldsymbol{r}$ and $\boldsymbol{v}$, the position and velocity vectors of the spacecraft given at some time t in the inertial coordinate system. These can perhaps best be visualized by thinking of the spacecraft as an airplane flying in the "normal" flight position. The unit vector

$$\hat{e}_r = \frac{\boldsymbol{r}}{r} = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix} \tag{1-28}$$

defines a unit vector in the "up" or local vertical direction, positive when pointing away from the center of the Earth.

A second vector

$$\hat{e}_n = \frac{\boldsymbol{v} \times \boldsymbol{r}}{|\boldsymbol{v} \times \boldsymbol{r}|} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} \tag{1-29}$$

is a unit vector pointing out the "right wing" of the spacecraft and is opposite in direction to the orbital angular momentum vector. The third vector

$$\hat{e}_v = \hat{e}_r \times \hat{e}_n = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \tag{1-30}$$

is a unit vector pointing "forward." If the orbit were exactly circular, then $\hat{e}_v$ would be parallel to the velocity vector $\boldsymbol{v}$.

These three mutually orthogonal unit vectors are sketched in figure 1-8.

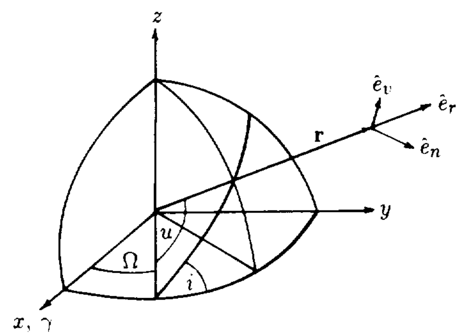


Figure 1-8. Sketch showing relation of spacecraft-fixed unit vectors $\hat{e}_r$, $\hat{e}_n$, and $\hat{e}_v$ to inertial axes.

If one has a vector $\boldsymbol{v}_I$ in the inertial system, then its components in the local S/C system are simply

$$\left.\begin{aligned} v_n &= \hat{e}_n \cdot \boldsymbol{v}_I \\ v_v &= \hat{e}_v \cdot \boldsymbol{v}_I \\ v_r &= \hat{e}_r \cdot \boldsymbol{v}_I \end{aligned}\right\} \tag{1-31}$$

or in matrix form

$$\boldsymbol{v}_L = \begin{bmatrix} v_n \\ v_v \\ v_r \end{bmatrix} = \begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} v_{Ix} \\ v_{Iy} \\ v_{Iz} \end{bmatrix} \tag{1-32}$$

The matrix of equation (1-32) is a pure rotation matrix, and hence transformation from the local S/C axis system to the inertial axes is readily accomplished by using the matrix transpose. In particular, let $\boldsymbol{v}_L$ be a vector defined in the local S/C system by its magnitude and by the azimuth angle A_L and elevation angle γ_L as shown in figure 1-9.

The local vector components are

$$\left.\begin{aligned} v_n &= v_L \cos \gamma_L \sin A_L \\ v_v &= v_L \cos \gamma_L \cos A_L \\ v_r &= v_L \sin \gamma_L \end{aligned}\right\} \qquad (1\text{-}33)$$

and the components of this vector in the inertial system are then

$$\begin{bmatrix} v_{Lx} \\ v_{Ly} \\ v_{Lz} \end{bmatrix} = \begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} v_n \\ v_v \\ v_r \end{bmatrix} \qquad (1\text{-}34)$$

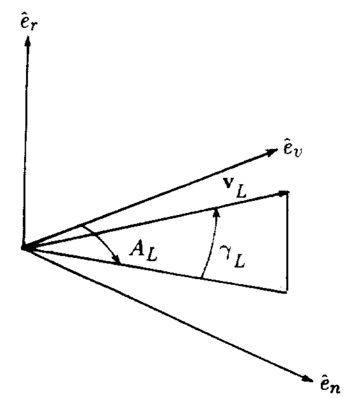


Figure 1-9. Description of orbiting vector v_L in local spacecraft axes defining azimuth A_L and elevation angle γ_L.

Equation (1-2) then provides the link between the inertial coordinate system and the Earth-fixed coordinate system, and hence, vector quantities can easily be transformed between the Earth-centered coordinate and the local S/C system.

Other instrument or spacecraft-specific coordinate transformations can, of course, be readily defined relative to the local dynamic coordinate system defined by equations (1-33). One has merely to chain the sundry transformation matrices together to provide transformation links between any of them and the inertial or Earth-fixed systems.

Time

The concept of time is one that most of us take pretty much for granted. But *time* is one of the most difficult concepts to resolve in orbital mechanics and astronomy because of the many very small effects that creep into its measurement and definition. As precision methods were developed, many time concepts have had to be redefined because parameters which were once thought to be constants or absolutes were found to be indeed variables. Entire books have been written on this subject alone. Consequently, the following discussion will treat only the main features of time and will serve only to permit the gross features of and differences between *sidereal time* (or time with respect to the fixed stars) and *solar time* (time with respect to the Sun) to be understood and appreciated. Chapter 10 of Green (1985) contains a fairly thorough discussion of the various times considered in modern astronomy. However, these concepts are generally much more stringent than those required in most Earth orbital applications of orbit mechanics theory. (See also Smart 1977, chap. VI.)

In figure 1-10 is shown a portion of the Earth's orbit around the Sun. Suppose that, at point ①, there are two observers on the Earth located exactly 180° apart in longitude. Observer A is watching the stars and observer B is watching the Sun. Suppose further that they are in constant communication with each other (relativity effects are ignored) and that at the exact instant that observer A reports a star on his meridian, observer B reports that the center of the Sun is exactly on *his* meridian. Now, observer A has been watching the stars for years. He has constructed a clock, based on stellar time, which is extremely (infinitely) accurate. He has defined 24 hr of sidereal time as the interval between two successive passages of the same star over his meridian (see the note at the end of this chapter). Each hour is divided into 60 sidereal min and each minute into 60 sidereal sec. Observer A has set and calibrated an identical clock which he has given to B. So, at the instant ①, both A and B start their clocks.

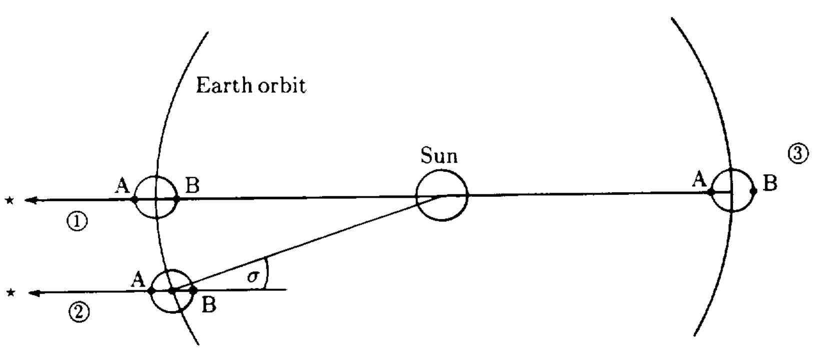


Figure 1-10. Position of Earth relative to Sun on two consecutive days, ① and ②, and 6 months later, ③.

At point ② (fig. 1-10), which occurs one "day" later, observer A calls out that his star is again on his meridian and that his clock reads 24 hr. Observer B notices that, sure enough, 24 hr has passed on his clock too, but the Sun has not yet appeared on his meridian. In fact, he finds that he must wait an additional 3 min and 56 sec for the Sun to line up in his instrument. The following day, he finds that he must wait 7 min 52 sec, etc., with each day being about 3 min 56 sec longer than the day before. At the end of one full revolution of the Earth around the Sun, 1 yr, observer B finds that his observation of the Sun's center on the meridian is exactly 24 sidereal hr late.

When the Earth arrives again at point ①, 1 yr later, observer B would find that he has made 365.2422 revolutions with respect to the Sun. Observer A, of course, would find that he has made 366.2422 revolutions with respect to the fixed stars. (When B makes one daily rotation with respect to the Sun, A has made one full revolution with respect to the stars plus a little more—in fact, 3^m56^s more, or σ in figure 1-10. These "little mores" add up to exactly 1 full day in 1 yr. If the Earth were not rotating, the Sun would still make one apparent revolution around the Earth in 1 yr.

Now, observer B finds that the amount he must wait each day for the Sun to appear on his meridian is not exactly 3^m56^s every day. The interval is somewhat longer in December and shortest in September (see fig. 1-11), but the mean, averaged over 1 year, is 3 min 56 sec

$$\frac{1440 \text{ min/day}}{365.2422} = 3^m56^s.5554$$

Observer B correctly attributes these daily variations to two factors—the apparent orbit of the Sun around the Earth is not a circle but an ellipse and the ecliptic, the plane in which the Sun appears to move, is inclined to the equator. Both these factors cause a nonuniform motion of the Sun about the Earth (fig. 1-11).

Figure 1-11 shows a plot of the difference between the solar day and 86400 mean solar sec for each day of the year. The solar day is longest in late December for two reasons: (1) the Earth is near perihelion and has the largest angular velocity in its orbit and (2) the Sun is also near the winter solstice, and consequently, is moving essentially parallel to the equator. The solar day is shortest in mid-September (and almost as short in mid-March) because the Sun is crossing the equator at these times and hence has the smallest component of its angular velocity projected onto the equator.

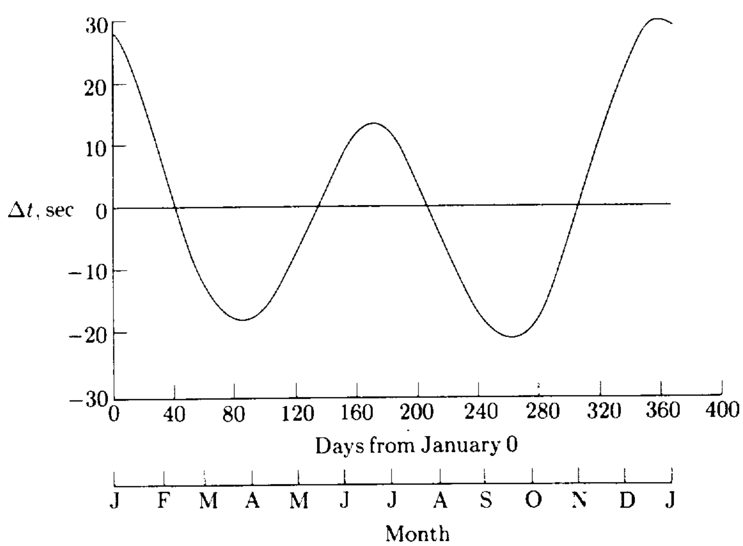


Figure 1-11. Plot of quantity Δt = Length of solar day − 86400, in mean solar seconds, versus day of year.

Observer A now asks the very legitimate question: Why can't B use A's clock to keep his (B's) time? The answer is, of course, that he can, but B would not find this aesthetically pleasing and it would, in fact, cause him considerable "headaches" later on. If B used A's clock, then at the starting position shown at ① in figure 1-10, B would define his time as local noon and A's time as local midnight. A new day would start for A and would start for B 12 hr later. However, 6 months later, A and B would be at position ③ of figure 1-10. Now, keeping A's time, midnight for B would occur when the Sun is directly overhead on his meridian, that is, at midday. Also, during the 6 months which have elapsed, if B retains the notion to start a new day 12 hr after the Sun is directly overhead for him, he would find that the change from one day to another would occur at various times of day throughout the year. This would certainly lead to some confusion if not some real practical difficulties.

Long before the beginnings of recorded history, man has regulated his everyday affairs by the Sun—working, hunting, etc., by day and sleeping at night. Since even now, most people work during the day, it is convenient to have days change from one to the other during the hours of darkness, that is, at midnight. Therefore, a time geared to the Sun would be extremely useful. However, the direct use of the observed

Sun would produce days which vary in length throughout the year for the reasons mentioned earlier.

In order to construct a clock which keeps uniform *solar* time of some sort, B proposes the use of a fictitious Sun, the *fictitious mean Sun*, which orbits the Earth in the *equatorial* plane, rather than the ecliptic, and moves at a uniform rate such that the fictitious mean Sun completes one revolution about the Earth in exactly the same time period as does the true Sun—i.e., in 365.2422 *mean solar days*. Since the fictitious Sun moves in the Earth's equatorial plane at a uniform rate, its right ascension is increasing at a uniform rate. The difference between the right ascensions of the mean Sun and the true Sun is called the *equation of time* (see fig. 1-12):

$$E_T = \text{RAMS} - \text{RATS} \tag{1-35}$$

and can readily be computed from simple orbit mechanics (see, for example, Smart 1977) and, to terms of the second order in the Earth's eccentricity, is given in units of radians by

$$E_T = y \sin 2L - 2e_e \sin M_s + 4ye_e \sin M_s \cos 2L - \frac{1}{2}y^2 \sin 4L - \frac{5}{4}e_e^2 \sin 2M_s + \ldots \tag{1-36}$$

where

$$y = \tan^2 \frac{\varepsilon}{2} \tag{1-37}$$

ε = obliquity of ecliptic, equation (1-21)

e_e = eccentricity of Earth orbit

$$= 0.0167514 - 0.0000418T_u - 0.000000126T_u^2 \tag{1-38}$$

and where the right ascension of the mean Sun, RAMS, and the Sun's apparent mean anomaly (see chap. 4 for the definition of some of the orbital element concepts given here) are given by (Escobal 1968)

$$L = \text{RAMS (eq. 1-19) or } L + \Delta\psi \cos \varepsilon \text{ if correction needed} \tag{1-39a}$$

$$M_s = 358^\circ.475844 + 35999^\circ.04975T_u - 0^\circ.00015T_u^2 - 0^\circ.000003333T_u^3 \tag{1-39b}$$

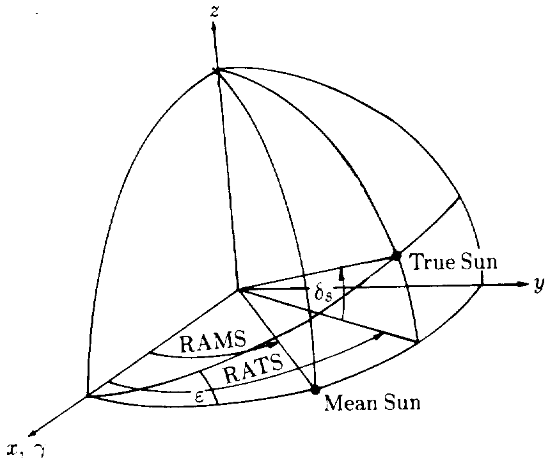


Figure 1-12. Sketch showing geometrical relationship between true Sun and mean Sun.

The nonlinear terms in this equation are due to the easterly motion of the Earth's perihelion.

See chapter 10 of Green (1985) for a modern definition of E_T and an equation for its calculation. The concept of the equation of time is no longer used in modern astronomy, although as a computational tool, it is still quite useful. The fall from grace of this concept came about when it was discovered that the rotational speed of the Earth about its axis was not truly a constant. The time definitions we have given are, therefore, only approximately correct, as they hinge on the constancy of this quantity. Equation (1-35) is still correct, however, if ephemeris time is used. Ephemeris time is the time used in the differential equations of motion. It is thus based on a dynamical concept, rather than the geometrical concepts used up to now.

When B sets his clock to the fictitious mean Sun, he is now measuring mean solar time, or more commonly, the ordinary clock time, with which we're all familiar. The international time, universal time, formerly called Greenwich mean time, is computed as mean solar time, with Greenwich mean noon being the instant that the mean Sun is on the Greenwich meridian.

As pointed out, many of the simplified time concepts discussed here are only approximately true because they depend on the constancy of the rotational rate of the Earth with respect to the fixed stars. Within the last half-century or so, it was found that this premise is not strictly true, and other definitions of time, which do not depend on the Earth's rotation, have been introduced. These very small differences, however, are generally of little importance in applications of these equations to Earth orbit mission design and studies and are generally ignored. These concepts are quite accurate enough for determining the position of the Sun, for example, as will be shown in chapter 5. Most of the new texts on Spherical Astronomy address this question of time most adequately. (See, for example, Green 1985, Taff 1981, Taff 1985, and the edition of Smart's classical text revised by Green listed herein as Smart 1977.)

In practice, of course, it would be impractical for every location on the surface to keep its own mean solar time. To standardize this somewhat, the Earth was, by international agreement, divided into 24 time zones of approximately 15° of longitude. The time zone centered at 0° of longitude, the Greenwich meridian, and extending for 7.5° on either side of the 0° meridian keeps Greenwich mean time. The time zone at 15°± 7.5° *west* is the first time zone, and so on. For the United States, eastern standard time is referenced to the 75th meridian, central time to 90°, mountain time to 105°, and Pacific standard time to 120° west longitude. Thus, since 75° W corresponds to 5 hr of time we can convert standard times in the four time zones to GMT as follows:

$$\left\{\begin{array}{ll}\text{Eastern standard time} & +\ 5\ \text{hours}\\ \text{Central standard time} & +\ 6\ \text{hours}\\ \text{Mountain standard time} & +\ 7\ \text{hours}\\ \text{Pacific standard time} & +\ 8\ \text{hours}\end{array}\right\} = \text{GMT}$$

(subtract 1 hour from these numbers for daylight savings time).

With all the perturbations and undulations which time can take, the really important thing to remember for our purpose is that a sidereal year has 366.2422 mean sidereal days, and the mean solar year has 365.2422 mean solar days. Thus,

$$\begin{aligned}\text{24 hours of mean solar time} &= 24 \times \frac{366.2422}{365.2422}\\ &= 24^h 3^m 56^s.555 \text{ of mean sidereal time}\end{aligned}$$

and

$$\begin{aligned}\text{24 hours of mean sidereal time} &= 24 \times \frac{365.2422}{366.2422}\\ &= 23^h 56^m 4^s.091 \text{ of mean solar time}\end{aligned}$$

The ratio 365.2422/366.2422 is, of course, the ratio of any (mean solar time interval)/(mean sidereal time interval). This is the source for the weird-looking constant 0.25068447 in equation (1-14). The Earth rotates 0.25° in 1 sidereal min. Therefore, in 1 mean solar min it rotates

$$0.25 \times \frac{366.2422}{365.2422} = 0.25068447 \text{ deg/mean solar min}$$

Note: the scenario between the two astronomers was, of course, highly simplified in many ways, and in fact, one concept used was intentionally erroneous at that point for clarity. A sidereal day is actually defined as the time interval between two successive passages of the vernal equinox *over the same meridian, rather than the interval between two successive passages of a fixed star (one whose proper motion is essentially zero). Since, as mentioned earlier, the vernal equinox is moving westward at nominally 50 arc-sec/yr, the sidereal day is actually a bit shorter than it would be if the fixed star were used for time definition. The precessional constant is 50.2564 arc-sec/yr and is measured along the ecliptic. Its component along the equator is thus 50.2564 cos 23.44 = 46.1091 arc-sec/yr, or 0.12624 arc-sec/mean sidereal day, and hence, the day as defined is 0.12624/15 or 0.008416 sidereal seconds shorter than it would be if it were defined relative to the "fixed" stars. This amounts to about 1/120 sec. (See, for example, Motz and Duveen 1966.)*

Chapter 2

Gravitational Field of the Earth

The external field of the Earth can be described mathematically in many ways. Because of its rotation, and its plasticity, especially during its early formative years, the Earth is very nearly an oblate spheroid. Hence, for many analytical purposes, it is convenient to expand the Earth's external gravity field in a series of spherical harmonics (Heiskanen and Moritz 1967, p. 59):

$$V = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[A_{nm} \frac{R_{nm}(\psi, \lambda)}{r^{n+1}} + B_{nm} \frac{S_{nm}(\psi, \lambda)}{r^{n+1}} \right] \tag{2-1}$$

in which ψ and λ are the latitude and longitude, respectively, and r is the distance from the origin to the point at which the potential is to be computed. The constants A_{nm} and B_{nm} are integrals determined by the internal mass distribution of the Earth, and

$$R_{nm}(\psi, \lambda) = P_{nm}(\sin\ \psi)\ \cos\ m\lambda \tag{2-2}$$

$$S_{nm}(\psi, \lambda) = P_{nm}(\sin\ \psi)\ \sin\ m\lambda \tag{2-3}$$

in which P_{nm} are associated Legendre polynomials, are the spherical harmonics.

The integrals A_{nm} and B_{nm} are in practice determined from the precision tracking of large numbers of Earth satellites and then performing statistical fittings to inverted data to determine the values for these constants, which best fit the observed tracking data. The Goddard Space Flight Center (GSFC) has done much of this work to high precision.

Equation (2-1) essentially assumes that the Earth is a distorted sphere. The $n = 0$ term is the spherical part of the gravity field (the inverse square part), and the subsequent terms are needed to describe the departure of the body from sphericity—the greater this departure, the more terms are needed to describe the external potential to a specific accuracy, and the larger are the constants A_{nm} and B_{nm}.

The central force part of the gravity field is usually described by the term

$$V_{cf} = \frac{\mu}{r} \tag{2-4}$$

where μ is the gravitational constant. If one takes the $n = 0$ terms out of equation (2-1) and writes successively

$$\begin{aligned} V &= \frac{A_{00}}{r} + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left[A_{nm}\frac{R_{nm}(\psi,\lambda)}{r^{n+1}} + B_{nm}\frac{S_{nm}(\psi,\lambda)}{r^{n+1}}\right] \\ &= \frac{\mu}{r}\left\{1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(\frac{R_e}{r}\right)^n [J_{nm}\ R_{nm}(\psi,\lambda) + K_{nm}\ S_{nm}(\psi,\lambda)]\right\} \end{aligned} \tag{2-5}$$

where

$$\begin{aligned} A_{00} &= \mu \\ R_e^n J_{nm} &= \frac{A_{nm}}{A_{00}} \\ R_e^n K_{nm} &= \frac{B_{nm}}{A_{00}} \end{aligned}$$

and R_e is the "equatorial radius" of the body, then equation (2-5), as written, describes the gravity field as a central force term (the term μ/r) plus a "perturbation potential" (the summation terms of eq. (2-5)).

If the origin of the coordinate system coincides with the center of the mass of the central body, a usual assumption, then (Heiskanen and Moritz 1967, p. 63)

$$J_{10} = J_{11} = K_{11} = 0$$

and we write equation (2-5) as

$$V = \frac{\mu}{r}\left\{1 + \sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{R_e}{r}\right)^n [J_{nm}\ R_{nm}(\psi,\lambda) + K_{nm}\ S_{nm}(\psi,\lambda)]\right\} \tag{2-6}$$

The various types of harmonics in equation (2-6) are sketched in figure 2-1 and are identified by

$m = 0$, zonal harmonics

$m = n \neq 0$, sectorial harmonics

$m \neq n \neq 0$, tesseral harmonics

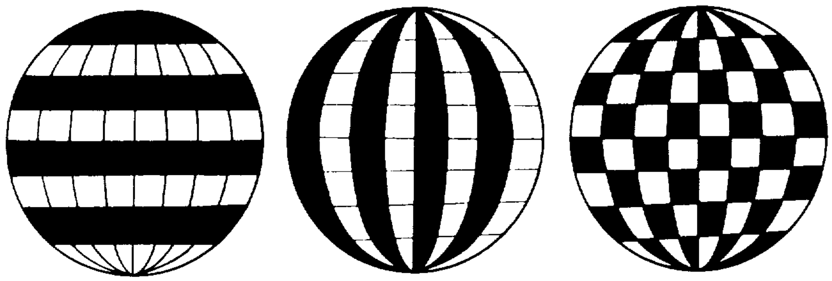


Figure 2-1. Sketch showing various types of harmonic coefficients. White area implies elevation above; black, elevation below mean spherical surface.

The zonal harmonics are functions of latitude only and hence can only reflect north-south variations in the gravity field.

Many of the precision orbit prediction programs used by GSFC, NOAA, and others use large numbers of terms from equation (2-6). GSFC, for example, has (at least) two such programs, one of which goes up to $n = m = 8$ and the other to $m = n = 21$. The GEM-8 model is the one used for SAGE I and II and SAM orbit work (these programs include other than gravitational effects, e.g., drag, atmosphere models which are affected by sunspot activity, relativistic effects, other planetary effects).

For many applications it is generally found that the sectorial and tesseral harmonic terms in equation (2-6) can be neglected and that only the first few zonals are needed. Thus, if we retain only the $m = 0$ terms, then all the K_{nm} terms vanish, and letting $J_{n0} = J_n$, equation (2-6) can be written as

$$V = \frac{\mu}{r}\left[1 + \sum_{n=2}^{\infty}\left(\frac{R_e}{r}\right)^n J_n\ P_n(\sin\ \psi)\right] \tag{2-7}$$

where P_n is the nth-order Legendre polynomial. The first few of these are (Heiskanen and Moritz 1967, p. 23)

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

and the rest can be generated from the recursion formula

$$P_n(x) = -\frac{(n-1)}{n}P_{n-2}(x) + \frac{(2n+1)}{n}xP_n(x) \qquad (2\text{-}8)$$

If we go as high as J_6, we can write equation (2-7) as (Escobal 1965)

$$\begin{aligned} V = \frac{\mu}{r}\Bigg\{ & 1 + \frac{J_2}{2}\left(\frac{R_e}{r}\right)^2 (1 - 3\ \sin^2\ \psi) + \frac{J_3}{2}\left(\frac{R_e}{r}\right)^3 (3 - 5\ \sin^2\ \psi) \\ & \times \sin\ \psi - \frac{J_4}{8}\left(\frac{R_e}{r}\right)^4 (3 - 30\ \sin^2\ \psi + 35\ \sin^4\ \psi) - \frac{J_5}{8}\left(\frac{R_e}{r}\right)^5 \\ & \times (15 - 70\ \sin^2\ \psi + 63\ \sin^4\ \psi)\ \sin\ \psi \\ & + \frac{J_6}{16}\left(\frac{R_e}{r}\right)^6 (5 - 105\ \sin^2\ \psi + 315\ \sin^4\ \psi - 231\ \sin^6\ \psi) + \ldots \Bigg\} \end{aligned} \qquad (2\text{-}9)$$

The constants in equation (2-9), as obtained from GSFC in March 1986, are

$$\begin{aligned} \mu &= 398600.64\ \text{km}^3/\text{sec}^2 \\ R &= 6378.14\ \text{km} \\ J_2 &= +1082.6271E\text{-}6 \\ J_3 &= -2.5358868E\text{-}6 \\ J_4 &= -1.6246180E\text{-}6 \\ J_5 &= -0.22698599E\text{-}6 \\ J_6 &= +0.54518572E\text{-}6 \end{aligned}$$

The potential (eq. (2-9)) and these constants are used in some of the author's orbit programs to generate short-term ephemeris data for use in the SAGE II and SAM II data reduction. (See chap. 4.)

Chapter 3
Orbital Elements

As will be seen in chapter 4, the equations of motion in Cartesian coordinates of a satellite orbiting about an oblate central body (zonal harmonics only) are rather simple to write, and with modern computers, it is possible to numerically integrate these rapidly and accurately.

However practical for number generation, Cartesian coordinates are not the most useful coordinates for visualizing or otherwise describing a spacecraft orbit. A time sequence of spacecraft position and velocity vectors by itself has little pictorial value and, consequently, conveys little information about the evolution of the orbit.

There are several sets of "orbital elements" used in astronomy, astrophysics, space sciences, etc., each of which is most useful in the specialized application for which it was conceived. It takes six independent coordinates to completely specify the state of an orbiting spacecraft and permit the determination of its future (or past) state (for example, three position and three velocity Cartesian components), and hence, it also takes six independent orbital elements. For Earth orbit analysis or "Keplermanship" (Escobal's term for playing games with the two-body equations), the set described below is the one favored by this writer, in both its utilitarian and interpretive senses.

From the first of Kepler's laws, we know that (for central body motion) the orbit of an Earth satellite is an ellipse with the center of the Earth at one of the foci. The ellipse has two axes—the major axis AB and the minor axis CD in figure 3-1. The origin is at point O, the center of the Earth. The spacecraft S is located by the radial position r and the angle f measured from the major axis and where $f = 0$ is defined by the radius OA where the spacecraft is closest to the Earth, the *perigee*. The equation of this ellipse is

$$r = \frac{a(1 - e^2)}{1 + e \cos f} \tag{3-1}$$

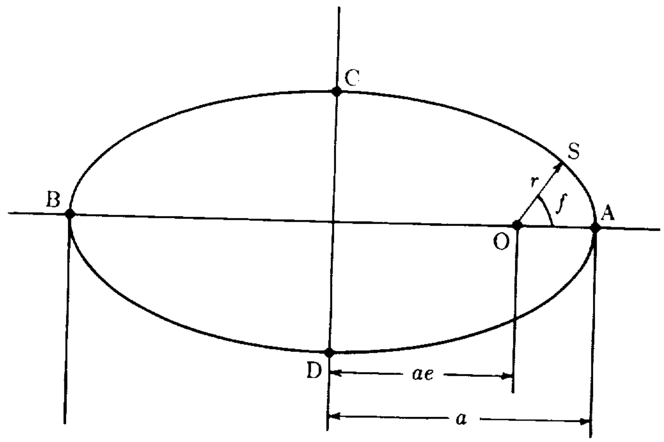


Figure 3-1. Geometry of ellipse in plane.

where the two *orbital elements,*

$$a = \text{semimajor axis, km}$$
$$e = \text{eccentricity}$$

determine the size and shape of the ellipse.

For central force motion, the orbit lies in a fixed plane, passing through the center of the Earth. Two additional orbital elements

$$\Omega = \text{right ascension of ascending node, deg}$$
$$i = \text{inclination to equator, deg}$$

locate the position and orientation of this plane in the "inertial" coordinate system defined earlier. (See fig. 3-2.) Note that by general agreement, if $i < 90°$, the orbit is called a "posigrade" orbit, whereas for $90° \leq i \leq 180°$, the orbit is referred to as "retrograde."

The fifth orbital element,

$$\omega = \text{argument of perigee, deg}$$

(fig. 3-2), locates the position of the major axis of the orbit in this plane (more specifically, the position of perigee) and is measured in the direction of motion from the ascending node of the orbit.

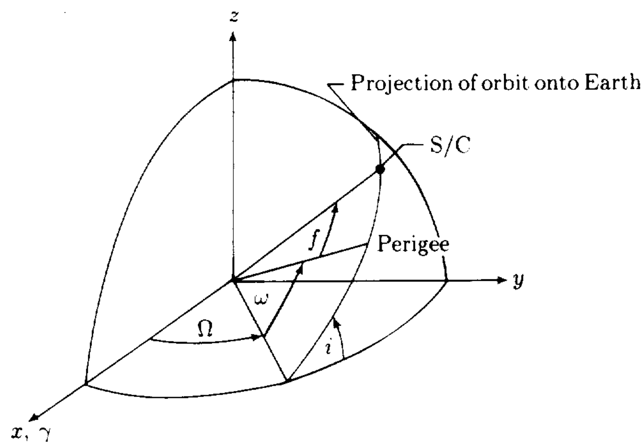


Figure 3-2. Sketch defining location of orbital plane and perigee point in inertial coordinate system.

The sixth element must somehow relate the position of the spacecraft in its orbit to some specific *time*. The angle f, called the *true anomaly*, is awkward to use for this purpose. According to the second of Kepler's laws, the radius vector of the spacecraft sweeps out equal areas in equal time increments. Since the spacecraft is closer to the Earth at perigee ($f = 0$) than it is at apogee ($f = 180$), the spacecraft must move faster at perigee than it does at apogee. This means that the angular rate df/dt is not constant around the orbit. A *mean* angular rate of the spacecraft can be shown to be

$$n = \frac{2\pi}{\text{Period}} = \sqrt{\frac{\mu}{a^3}} \tag{3-2}$$

and is a constant of the orbit for a central force. Let t_o be the time the spacecraft last passed through perigee. Then the angle

$$M = n(t - t_o) \tag{3-3}$$

is an angle which has the desirable property of increasing at a uniform rate. This angle is called the *mean anomaly* and is selected here as the sixth orbital element. In order to relate the mean anomaly to the true anomaly, we must first introduce a third anomaly, the *eccentric anomaly* E, which relates f to M as follows:

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\,\tan\frac{f}{2} \tag{3-4}$$

and Kepler's equation

$$M = E - e\sin E \tag{3-5}$$

These transcendental equations are still not directly solvable (i.e., to find f given t). However, for orbits with small eccentricity (near-circular orbits) as most near-Earth orbits are, the following series (Smart 1977) is useful—angles are in radians:

$$E = M + \left(e - \frac{e^3}{8}\right)\sin M + \frac{e^2}{2}\sin 2M + \frac{3e^3}{8}\sin 3M + \ldots \tag{3-6}$$

$$f = M + \left(2e - \frac{e^3}{4}\right)\sin M + \frac{5e^2}{4}\sin 2M + \frac{13e^3}{12}\sin 3M + \ldots \tag{3-7}$$

If the eccentricity is large, or if extreme accuracy is needed, the Taylor expansion of equation (3-5) gives the iteration formula

$$E_{n+1} + E_n = \frac{M_T - M_n}{1 - e\cos E_n}$$

in which, given the true value of M_T, find E_n from equation (3-6) and M_n from equation (3-5). Then equation (3-8) gives a corrected value E_{n+1}. For a second iteration, set $E_n = E_{n+1}$ from the last iteration, and repeat to convergence. This procedure generally converges to six or seven decimals in three or four iterations, even for values of e near unity.

There are two basic advantages for using orbital elements. First, the orbit is much easier to visualize, as the orbital elements describe the total geometry of the orbit—its size, shape, and orientation in space. Second, and perhaps more important, if the Earth were a perfect sphere (i.e., the gravity field describable by the inverse square law only) and there were no other external forces acting on the spacecraft, then five of the six orbital elements—a, e, i, Ω, and ω—would be *constants*, only M would change with time, and this change would be a simple linear one. Even for a spherical Earth, all six of the Cartesian components change continually and dramatically with time at each integration step.

It seems reasonable to expect, then, that if the gravitational potential field of the central body only deviates very slightly from that of a sphere as does the Earth, the orbital elements a, e, i, Ω, and ω would change very slowly with time, if at all, and these changes would be predictable to varying orders analytically. This is, of course, precisely what happens for an Earth satellite. Very detailed and complex analytical studies (e.g., Kozai 1959, Brouwer 1959, King-Hele 1958, Garfinkel 1959, Merson 1961) show that it is possible to account for practically all the oblateness perturbative effects of the Earth with a single term (J_2), and to first order it is found that a, e, and i are constants, and Ω and ω change linearly and slowly with time (order of a few degrees per day, or less, depending on the inclination of the orbit). These changes are easy to compute (Escobal 1965 or McCuskey 1963). After the equations of motion are introduced in chapter 4, a brief series of numerical examples will show the effects of various harmonics on both the Cartesian components and on the orbital elements.

It is, of course, possible, if not imperative, that we be able to transfer from Cartesian coordinates to orbital elements and vice versa. The following algorithms, applicable to circular and/or elliptical orbits only, are used for this purpose (see, e.g., Escobal 1965 or McCuskey 1963).

1. Orbital elements to Cartesian coordinates (CONCAR)
   (a) Find E from (M, e) as described earlier (eq. (3-6))
   (b) Then

$$x_\omega = a(\cos E - e)$$
$$z_\omega = a\sqrt{1-e^2}\sin E$$
$$\dot{x}_\omega = -a\dot{E}\sin E$$
$$\dot{z}_\omega = a\dot{E}\sqrt{1-e^2}\cos E$$

where

$$\dot{E} = \frac{n}{1 - e\cos E}$$

   (c) Compute the unit vectors

$$\hat{P} = \begin{bmatrix} \cos\omega\cos\Omega - \sin\omega\sin\Omega\cos i \\ \cos\omega\sin\Omega + \sin\omega\cos\Omega\cos i \\ \sin\omega\sin i \end{bmatrix}$$

$$\hat{Q} = \begin{bmatrix} -\sin\omega\cos\Omega - \cos\omega\sin\Omega\cos i \\ -\sin\omega\sin\Omega + \cos\omega\cos\Omega\cos i \\ \cos\omega\sin i \end{bmatrix}$$

(d) Then compute the spacecraft position and velocity vectors from

$$\boldsymbol{r} = x_\omega \hat{P} + y_\omega \hat{Q}$$
$$\dot{\boldsymbol{r}} = \dot{x}_\omega \hat{P} + \dot{y}_\omega \hat{Q}$$

2. Cartesian coordinates to orbital elements (CARCON)

(a) From angular momentum vector

$$\boldsymbol{h} = \boldsymbol{r} \times \dot{\boldsymbol{r}} = \begin{bmatrix} y\dot{z} - z\dot{y} \\ z\dot{x} - x\dot{z} \\ x\dot{y} - y\dot{x} \end{bmatrix} = \begin{bmatrix} h \sin i \sin \Omega \\ -h \sin i \cos \Omega \\ h \cos i \end{bmatrix}$$

Find h, i, and Ω

(b) With $r = |\boldsymbol{r}|$, $v = |\boldsymbol{v}|$, find the semimajor axis a from

$$\frac{1}{a} = \frac{2}{r} - \frac{v^2}{\mu}$$

where for the Earth, $\mu = 398600.64 \text{ km}^3/\text{sec}^2$.

(c) Find the eccentricity

$$e = \sqrt{1 - \frac{h^2}{\mu a}}$$

(d) Define $u = \omega + f$, then

$$r \cos u = x \cos \Omega + y \sin \Omega$$
$$r \sin u = \frac{z}{\sin i}$$

which gives the proper quadrant for u

(e) Find the true anomaly f

$$e \cos f = \frac{p}{r} - 1$$
$$e \sin f = \frac{p}{h} \left[\frac{\boldsymbol{r} \cdot \dot{\boldsymbol{r}}}{r} \right]$$

where $p = a(1 - e^2)$

(f) Finally, then,

$$\omega = u - f$$

(g) Find E from equation (3-4) and M from equation (3-5).

This completes the orbital element set.

Note that these equations even hold for circular orbits ($e = 0$), if we agree to set $\omega = 0$ and measure f (or M, E) from the ascending node of the orbit.

The algorithm, CONCAR, changing from orbital elements to Cartesian coordinates, is quite useful in determining time histories of the Cartesian coordinates, which parameters are frequently used in determining other geometric parameters and properties in which the mission design engineers might be interested. Given the set of orbital elements at time t_o, the mean anomaly M is found at $t + \Delta t$ from equation (3-3) as

$$M = M_o + n \,\Delta t$$

The eccentric anomaly is found from equation (3-6), as described earlier, and then the Cartesian state vector is found from the algorithmic equations. This process can be repeated at any time interval Δt, and the results hold as long as only a central force field (all $J_n = 0$) can be assumed. In actual practice, this time might be of the order of 10 min or so, and then the small perturbations begin to make themselves felt, and another algorithm, to be described in chapter 4, must be used. This requires using the analytical expressions for $\dot{\omega}$ and $\dot{\Omega}$ explained in chapter 4 to continuously update ω and Ω, thus enabling the algorithm CONCAR to be used over extended time periods.

A third algorithm, especially useful if one wants to preserve the identity of the initial Cartesian coordinates, is the so-called f- and g-series method. (See, for example, Escobal 1965.) Here, the radius vector $\boldsymbol{r}(t)$ and velocity vector $\dot{\boldsymbol{r}}(t)$ at any time t are expressed in terms of the initial values of the position and velocity vectors, and the $f(t, t_o)$ and $g(t, t_o)$ parameters

$$\dot{\boldsymbol{r}}(t) = f(t, t_o)\ \boldsymbol{r}_o + g(t, t_o)\dot{\boldsymbol{r}}_o \tag{3-8}$$

$$\dot{\boldsymbol{r}}(t) = \dot{f}(t, t_o)\ \boldsymbol{r}_o + \dot{g}(t, t_o)\dot{\boldsymbol{r}}_o \tag{3-9}$$

where $\boldsymbol{r}_o$ and $\dot{\boldsymbol{r}}_o$ are the initial position and velocity vectors, respectively. The f and g terms were originally derived as an infinite series involving the zeroth, first, and second derivatives of the position vector as constant coefficients and the time as the independent variable. However, for central force motion, these series can be expressed in analytical form as

$$f(t, t_o) = 1 - \frac{a}{r_o}\,[1 - \cos\,(E - E_o)] \tag{3-10}$$

$$g(t, t_o) = t - \sqrt{\frac{\mu}{a^3}}\,[(E - E_o) - \sin\,(E - E_o)] \qquad (3\text{-}11)$$

and

$$\dot{f}(t, t_o) = -\sqrt{\frac{\mu a}{r r_o}}\,\sin\,(E - E_o) \qquad (3\text{-}12)$$

$$\dot{g}(t, t_o) = 1 - \frac{a}{r}\,[1 - \cos\,(E - E_o)] \qquad (3\text{-}13)$$

in which

$$r = a(1 - e\,\cos\,E) \qquad (3\text{-}14)$$

The parameters a, e, and E_o are computed from algorithm 2 (CARCON) at $t = 0$, and E at any subsequent (or previous) time can be found from equation (3-6) with the help of the iteration process described previously. This procedure offers no real computational advantage over that of algorithm 2 alone. It has, in fact, the disadvantage of not readily incorporating the oblateness effects through the $\dot{\Omega}$ and $\dot{\omega}$ terms as described earlier. It does, however, retain the identity of the initial Cartesian state, as mentioned, and is quite useful in studying the effects of errors in the initial conditions, as the variance-covariance matrix of errors in the values of future coordinates can be constructed directly from equations (3-8) and (3-9).

As pointed out earlier, five of the six orbital elements would be constant if the Earth were a perfect sphere and there were no other outside perturbations acting on the orbit. The effects of the nonspherical components (in the spherical harmonic sense) of the gravity field, the perturbation forces, are very small for a typical Earth satellite, and hence, the orbital elements, instead of being constant, vary very slowly in time. The most elaborate analyses referenced earlier (Kozai 1959 and the others) indicate that these changes are combinations of the following:

Secular changes: these are linear, or at best quadratic, changes, which always proceed in the same direction. The elements Ω, ω, and M are the only ones which experience secular changes.

Long-period terms: periodic changes of small amplitude whose periods are of the order of 80–100 days and longer.

Short-period terms: periodic changes of small amplitude whose periods are small (integer or half-integer) multiples of the orbital period.

These terms are sketched in figure 3-3.

Escobal (1965, chap. 10) discusses perturbations of the orbital elements thoroughly, as does the classical text by Moulton (1914). A typical orbital element can be most generally represented by an equation of the form

$$q = q_o + \dot{q}(t - t_o) + K_1 \cos 2\omega + K_2 \sin(2f + 2\omega) + \text{H.O.T.}$$

where

$\dot{q} =$ secular term

$q_o =$ mean value of the element

$K_1 =$ long-period term

$K_2 =$ short-period term

Higher order terms (H.O.T.) in other multiples of ω and f are, of course, also present, and any of the constants can be zero.

In any event, as a result of these perturbations, the orbit of an Earth satellite is never a true ellipse. However, as seen in the preceding algorithms, the specification of the spacecraft state vector permits the computation of a unique set of orbital elements, and thus at each instant of time, a unique set of orbital elements can be associated with the orbit. If at a given instant of time all the gravity perturbations were turned off and only the central force part of the gravity field were allowed to remain, the spacecraft would then truly be in the elliptical orbit defined by the orbital elements at that moment. This constantly changing set of orbital elements is referred to as the set of *osculating elements* and is specified at a unique time. Point a at time t in figure 3-3 identifies the osculating element at this time. It has been found through many sets of calculations that, if one determines the set of osculating elements at one time, say T, and then uses that set as described above to advance the position of the spacecraft along the orbit, then considerable error in the predicted Cartesian state may develop quite rapidly. The reason for this is that there may be considerable deviation in the value of a given element in different points of the orbit. (Compare points a and b in fig. 3-3.) For example, in a spacecraft orbit of small eccentricity at an altitude of 600 km and inclination of 57°, the mean value of the semimajor axis is 6981 km. However, the actual value can range from a minimum value of about 6976 km to a maximum of 6986 km. Table 3-1 shows the differences in mean anomaly that one would compute at various time intervals using these values of a.

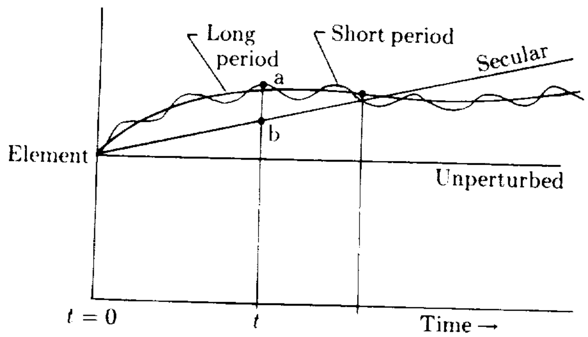


Figure 3-3. Sketch illustrating various orders of perturbations.

Table 3-1. Change in Mean Anomaly for ±5-km Change in Semimajor Axis

a, km	n, deg/min	M at t = 100 min, deg
6976	3.725059	372.5059
6981	3.721058	372.1058
6986	3.717064	371.7064

Since 1° represents a down track distance of about 122 km for this orbit, table 3-1 represents a potential difference of about 97 km in only 100 min, which is about 1 revolution for this orbit. Even if the first-order perturbations are included as described in chapter 4, rather large errors still ensue after only a short time.

Experience has shown that much greater accuracy is maintained over long time periods if the *mean* elements are used. Perhaps a better way of saying this is to say that a more acceptable error results which allows the simple algorithm to be used over a longer period of time. Relations for the osculating elements in terms of the mean elements can be found in the references by Kozai (1959), Brouwer (1959), et al. cited earlier. These same relations can be used iteratively to compute the mean elements from the osculating set at any time.

Chapter 4

Equations of Motion

The development of the general equations of motion for a satellite of negligible mass orbiting about a massive primary, or central body, can be found in practically any textbook on Celestial Mechanics. (See, for example, Escobal 1965, Smart 1977, Dubyago 1961, or the classical text by Moulton (1914).) In vector form

$$\frac{d^2\mathbf{r}}{dt^2} = \nabla V \tag{4-1}$$

where $\mathbf{r}$ is the radius vector from the center of the primary (the origin of coordinates for this development) to the spacecraft, t is time, and V is the gravitational potential of the central body. If we restrict ourselves to the 6th-degree zonal expansion of equation (2-9) and recall that

$$\sin\,\psi\ \frac{z}{r}$$

then we can write the three components of equation (4-1) as (see, for example, Escobal 1965)

$$\begin{aligned}\frac{d^2x}{dt^2} = \frac{\partial V}{\partial x} = -\frac{\mu x}{r^3}\Bigg\{1 - \frac{3}{2}J_2\left(\frac{R_e}{r}\right)^2\left[1-5\left(\frac{z}{r}\right)^2\right] \\ -\frac{5}{2}J_3\left(\frac{R_e}{r}\right)^3\left[3-7\left(\frac{z}{r}\right)^2\right]\frac{z}{r} \\ +\frac{5}{8}J_4\left(\frac{R_e}{r}\right)^4\left[3-42\left(\frac{z}{r}\right)^2+63\left(\frac{z}{r}\right)^4\right] \\ +\frac{3}{8}J_5\left(\frac{R_e}{r}\right)^5\left[35-210\left(\frac{z}{r}\right)^2+231\left(\frac{z}{r}\right)^7\right]\frac{z}{r} \\ -\frac{1}{16}J_6\left(\frac{R_e}{r}\right)^6\Bigg[35-945\left(\frac{z}{r}\right)^2+3465\left(\frac{z}{r}\right)^4 \\ -3003\left(\frac{z}{r}\right)^6\Bigg]\Bigg\}\end{aligned} \tag{4-2}$$

$$\frac{d^2y}{dt^2} = \frac{\partial V}{\partial y} = \frac{y}{x}\left(\frac{d^2x}{dt^2}\right) \tag{4-3}$$

$$\begin{aligned}\frac{d^2z}{dt^2} = \frac{\partial V}{\partial z} = -\frac{\mu z}{r^3}\Bigg\{1 &- \frac{3}{2}J_2\left(\frac{R_e}{r}\right)^2\left[3-5\left(\frac{z}{r}\right)^2\right] \\ &- \frac{5}{2}J_3\left(\frac{R_e}{r}\right)^3\left[6-7\left(\frac{z}{r}\right)^2\right]\frac{z}{r} \\ &+ \frac{5}{8}J_4\left(\frac{R_e}{r}\right)^4\left[15-70\left(\frac{z}{r}\right)^2+63\left(\frac{z}{r}\right)^4\right] \\ &- \frac{3}{8}J_5\left(\frac{R_e}{r}\right)^5\left[105-315\left(\frac{z}{r}\right)^2+231\left(\frac{z}{r}\right)^4\right]\frac{z}{r} \\ &- \frac{1}{16}J_6\left(\frac{R_e}{r}\right)^6\left[245-220\left(\frac{z}{r}\right)^2+4851\left(\frac{z}{r}\right)^4 \right. \\ &\left. - 3003\left(\frac{z}{r}\right)^6\right]\Bigg\} - \frac{\mu}{r^2}\left[\frac{3}{2}J_3\left(\frac{R_e}{r}\right)^3 \right. \\ &\left. - \frac{15}{8}J_5\left(\frac{R_e}{r}\right)^6\right]\end{aligned} \tag{4-4}$$

and according to GSFC, the latest (as of March 1986) values of the constants are

$$\begin{aligned}\mu &= 398600.64\ \text{km}^3/\text{sec}^2 \\ R_e &= 6378.140\ \text{km} \\ J_2 &= +1082.28E\text{-}6 \\ J_3 &= -2.5358868E\text{-}6 \\ J_4 &= -1.6246180E\text{-}6 \\ J_5 &= -0.22698599E\text{-}6 \\ J_6 &= +0.54518572E\text{-}6\end{aligned}$$

Equations (4-2) to (4-4) can readily be programmed on a modern computer and integrated using any number of accurate, rapid numerical methods. A modified 7th-order Runge-Kutta technique, one of several available canned routines at the Langley Research Center (LaRC), was used for the numerical examples of the present text and was also used in some of the operational versions of the integration routines.

The equations of motion can also be written directly in terms of the orbital elements (the Lagrange Planetary Equations.) (See, for example, Escobal 1965, Smart 1977, Dubyago 1961, Moulton 1914.) The perturbative gravity field, however, becomes very messy when expressed in terms of the orbital elements. Consequently, this form of the equations of motion, while occasionally used in numerical work, is mainly used in theoretical developments, in which usually only the J_2 or the J_3 term of the zonal expansion is used in an attempt to derive purely analytical expressions for the response of the spacecraft orbit to these terms. (See, for example, Kozai 1959, Brouwer 1959, Garfinkel 1959, King-Hele 1958.) Escobal (1965) expounds somewhat on Kozai's method and gives a lucid physical explanation of the effects of the perturbations on the orbital elements. (To this end, see also the excellent presentation by Moulton (1914).)

As mentioned earlier, the perturbing potential is difficult to express in terms of the orbital elements and, with modern computers and numerical techniques, it is the author's contention that, if one is merely seeking accurate numbers with which to work, then the integration of equations (4-2) to (4-4), or in their more complete form equation (2-6) for the potential, is as good a way as any. Of course, if one is faced with the problem of predicting the position or orbital characteristics far in the future—for example, for several weeks or months—obviously integration of the equations of motion would become quite expensive. In this case, one would probably be well advised to sacrifice some accuracy for economy and speed and use one of the theoretical models alluded to earlier or the simple mean element model shown later in this chapter which uses only the J_2 term.

The first numerical examples are shown to illustrate the accuracy of equations (4-2) to (4-4). An initial state vector

$$
\begin{aligned}
x &= 3211.365 \text{ km} \\
y &= -4680.423 \text{ km} \\
z &= -4081.154 \text{ km} \\
\dot{x} &= 2.326315 \text{ km/sec} \\
\dot{y} &= 5.555629 \text{ km/sec} \\
\dot{z} &= -4.545389 \text{ km/sec}
\end{aligned}
$$

was picked from an ephemeris tape prepared by GSFC for the SAGE II experiment. The GSFC prediction program, as mentioned earlier, uses an $n = m = 8$ gravity field (eq. (2-6)) and serves as a standard here with which other computations are compared.

The same initial conditions were used in the integration of equations (4-2) to (4-4) with three runs:

1. All J_2–J_6 used
2. $J_2 \neq 0$, J_3–$J_6 = 0$
3. All J_2–$J_6 = 0$

Run 3, of course, corresponds to the purely central force field. The results of these runs are given in table 4-1 as follows: the first column lists the GSFC results after 48 hr (2880 min), 96 hr (5760 min), and 144 hr (8640 min). Columns 2–4 list the differences

(LaRC results of runs) − (GSFC results)

and the LaRC results were computed from the modified Runge-Kutta numerical method with a 60-sec time increment for the computation interval.

Table 4-1. Comparison Between GSFC and LaRC Ephemerides

[Run 1 includes all J_2–J_6; Run 2 uses J_2 only; Run 3 is spherical Earth result]

	GSFC	Differences in LaRC and GSFC results for—		
		Run 1	Run 2	Run 3
$t = 48$ hr				
x, km . . .	−2414.451	−3.087	−3.686	591.649
y, km . . .	−5520.263	4.505	5.144	−57.877
z, km . . .	3521.274	4.912	4.763	257.880
$\dot{x}$, km/sec .	3.1777850	−0.002647	−0.002981	0.504577
$\dot{y}$, km/sec .	−4.6332890	−0.006146	−0.006592	0.140997
$\dot{z}$, km/sec .	−5.0583560	0.003965	−0.003546	0.223293
$t = 96$ hr				
x, km . . .	−2767.378	4.915	5.892	−788.005
y, km . . .	3806.603	10.928	12.370	−212.267
z, km . . .	5141.751	−4.985	−4.834	−341.438
$\dot{x}$, km/sec .	−2.997119	−0.005224	−0.006453	1.376789
$\dot{y}$, km/sec .	−6.258721	0.007867	0.008922	−0.183703
$\dot{z}$, km/sec .	3.014856	0.010240	0.009824	0.608086
$t = 144$ hr				
x, km . . .	3164.478	6.164	7.876	−2011.458
y, km . . .	5901.433	−7.421	−6.162	354.769
z, km . . .	−1980.466	−13.117	−11.255	−894.553
$\dot{x}$, km/sec .	−2.952138	0.008289	0.009102	−1.038673
$\dot{y}$, km/sec .	3.573608	0.015006	0.013410	−0.310911
$\dot{z}$, km/sec .	5.968511	−0.005320	−0.005758	−0.443209

It is seen from table 4-1 that the use of only the J_2–J_6 terms of the gravity field expansion is not too shabby when compared with the full-blown 8 by 8 model of GSFC and is certainly accurate enough for many satellite applications. Equally obvious from the table is the fact that this model is probably *not* accurate for other applications—for example, long-range prediction (i.e., several months) of the ground or geodetic position of the spacecraft to be used for ground truth location studies. However, even for 6 days of integration, the differences between the standard GSFC model and the truncated model used herein are only of the order of 10 km or so in position and of the order of 15 m/sec in velocity.

The J_2 term clearly dominates the perturbation effect, a fact which has been known since the early days of satellite flight. In fact, examination of the table shows that, at least over the 6-day period used for the example, the use of the J_3–J_6 coefficients is not warranted at all. However, only further study could determine the time span in which their exclusion should be adopted; for example, their effect would doubtless show up for an integration span of several weeks.

If one now converts the Cartesian components of table 4-1 to orbital elements using the CARCON algorithm, one finds the results of table 4-2. Here, the complete values are presented, not merely the differences between the LaRC and the GSFC results. The actual orbital elements at $t = 0$ hr were

$$\begin{aligned} a &= 6981.471516 \text{ km} \\ e &= 0.00141817 \\ i &= 57^\circ.002219 \\ \Omega &= 96^\circ.623064 \\ \omega &= 58^\circ.316978 \\ M &= 165^\circ.753617 \end{aligned}$$

First, note the last column of table 4-2, which shows the unperturbed elements. The agreement of these elements (except, of course, M) with the initial conditions illustrates the accuracy and stability of the numerical integration routine and serves to rule out integration errors as a cause of some of the differences in the Cartesian coordinates of table 4-1.

Second, the orbital elements appear to be very close in all cases. The percent differences in all elements except ω and the anomalies are very small (order of 0.001 percent) compared with the percent error in the Cartesian coordinates (order of 0.2 percent). The maximum errors are in ω and f individually. However, and much more important, the errors in the sum $\omega + f$ are rather small, and it is the sum which determines the angular distance of the spacecraft from the ascending node. The error in ω is mainly due to the small eccentricity of this orbit. As the CARCON algorithm of chapter 3 shows, the computation of $u = \omega + f$ does not explicitly involve the eccentricity, but the computation of f does. Therefore, u ought to be determined with reasonable accuracy, even for small eccentricities. However, when the eccentricity is small in absolute value, even very small absolute errors are very large relatively, or percentagewise, and hence the accuracy to which f, and consequently ω, are individually computed is questionable. If these are used to determine the Cartesian coordinates, though, the results are still reasonable, as u is the quantity which is used, and this is determined with some confidence.

If one considers only the J_2 part of the perturbed gravity field and further restricts oneself to first-order theory, then one finds that a, e, and i are constants in time for a given orbit. The longitude of the ascending node, Ω, and the argument of perigee, ω, are found to vary linearly and are given by the following equations:

$$\Omega = \Omega_o + \dot{\Omega}(t - t_o) \tag{4-5}$$

$$\omega = \omega_o + \dot{\omega}(t - t_o) \tag{4-6}$$

$$M = M_o + \bar{n}(t - t_o) \tag{4-7}$$

where

$$\dot{\Omega} = -\frac{3}{2} J_2 \left(\frac{R_e}{p}\right)^2 \bar{n} \cos i \tag{4-8}$$

$$\dot{\omega} = \frac{3}{2} J_2 \left(\frac{R_e}{p}\right)^2 \left[2 - \frac{5}{2} \sin^2 i\right] \bar{n} \tag{4-9}$$

$$\bar{n} = \sqrt{\frac{\mu}{a^3}} \left[1 + \frac{3}{2} J_2 \left(\frac{R_e}{p}\right)^2 \sqrt{1 - e^2} \left(1 - \frac{3}{2} \sin^2 i\right)\right] \tag{4-10}$$

and in which

$$p = a(1 - e^2) \tag{4-11}$$

If equations (4-5) and (4-6) are used to compute Ω and ω and equations (4-7) and (4-10) are used to compute M, it is found that the element-to-Cartesian coordinate algorithm given earlier will predict the Cartesian components reasonably accurately for longer periods of time than can be computed from the central force alone (order of a few days), but of course the acceptable time period depends heavily on the intended use and accuracy requirements.

Table 4-2. Comparison Between GSFC and LaRC Osculating Orbital Elements Computed at 48, 96, and 144 hr

[Run 1 includes all J_2–J_6; Run 2 uses J_2 only; Run 3 is spherical Earth result]

	GSFC	LaRC results for—		
		Run 1	Run 2	Run 3
		$t = 48$ hr		
a, km	6983.088680	6983.070167	6983.090200	6981.471471
e	0.00186271	0.00184027	0.00181924	0.00141816
i, deg	57.006807	57.006451	57.006509	57.002221
Ω, deg	88.674902	88.675025	88.667845	96.623060
ω, deg	72.635991	72.802224	74.944251	58.318410
M, deg	70.179204	69.955204	67.817778	81.313131
f, deg	70.380168	70.153465	68.010984	81.473819
$\omega + f$, deg	143.016159	142.955684	142.955235	139.992229
		$t = 96$ hr		
a, km	6977.549839	6977.616167	6977.629441	6981.472607
e	0.00116235	0.00113651	0.001140987	0.00141831
i, deg	56.992038	56.991955	56.991988	57.002216
Ω, deg	80.804759	80.806862	80.792607	96.623066
ω, deg	89.510732	91.289731	98.225995	58.319535
M, deg	332.149893	330.265486	323.337990	356.872952
f, deg	332.087589	330.200812	323.259833	356.864070
$\omega + f$, deg	61.598321	61.490543	61.485828	55.183605
		$t = 144$ hr		
a, km	6986.459458	6986.385047	6986.376400	6981.471158
e	0.00146659	0.00140939	0.00127645	0.00141834
i, deg	57.014001	57.015303	57.015294	57.002222
Ω, deg	72.866127	72.869378	72.847804	96.623062
ω, deg	50.919996	52.026907	58.879086	58.318779
M, deg	289.476609	288.229329	281.390348	272.434666
f, deg	289.318070	288.075816	281.246913	272.272271
$\omega + f$, deg	340.238066	340.102753	340.125999	330.591050

Equations (4-5) to (4-10) were used to generate the next set of numerical data, again for $t = 48$, 96, and 144 hr. The set of orbital elements given above was assumed at $t = 0$. The orbital elements were updated as prescribed, and the CONCAR algorithm was used to generate the Cartesian components from the orbital elements. The results are shown in table 4-3. The first two thirds of the table give the results of the appropriate calculations, the top third giving the orbital elements and the middle third giving the Cartesian coordinates. The bottom third gives the difference in the Cartesian coordinates between the above results and the reference GSFC results from column 1 of table 4-1. The bottom third also displays the difference between the sum $w + f$ computed from the exact Cartesian coordinates and that computed from the approximate orbital elements.

The differences shown in table 4-3 are about an order of magnitude greater than those found using only J_2 with the integration routine. (The integration of the equation of motion includes *all* orders of effect and not just the first-order effects predicted by equations (4-5) to (4-7), with a, e, and i constant.) Comparison between tables 4-2 and 4-3 shows that the nodal point Ω is predicted rather well by equation (4-5), and the quantity $\omega + f$ is the prime error source. However, we must remember that the approximate orbital element treatment using equations (4-5) to (4-10) is expressly to be used with *mean* orbital elements, while we are here applying these equations to *osculating* elements. There may be some instances where we have no choice as the only initial conditions available may be in Cartesian form, and these almost always refer to osculating elements. Proceeding with this caveat, note that the two orbital planes line up rather well but the position of the spacecraft in the true orbit is slightly ahead of the position in the approximate orbit. This is, in turn, traced to the fact that the first-order theory assumes that a is a constant. However, for this orbit, a varies between 6975.2 and 6987.6 km, and hence, the timing between the true and approximate (through the mean anomaly) orbit is off somewhat and, as can be seen in the last column of table 4-3, the true spacecraft position is gaining on the approximate position by about 0.25 deg/day.

The approximate first-order analysis can thus also be reasonably accurate for a few days or so. If one starts off with a set of *mean* elements, then the spatial position can be predicted reasonably well for periods of several weeks to several months, depending on the accuracy constraints imposed on the desired results. This algorithm is rather simple to program and compute when compared with the integration routine described earlier.

Table 4-4 demonstrates the application of this modified algorithm. The mean elements at $t = 0$ were computed using the method of Kozai (1959). (These equations are rather lengthy and hence are not reproduced here. The original reference should be consulted.) The mean elements a, e, and i were then held constant, and the remaining three mean elements updated using the following rates computed from equations (4-5) to (4-11):

$$\bar{n} = 223.234095 \text{ deg/hr}$$
$$\dot{\Omega} = -0.16475043 \text{ deg/hr}$$
$$\dot{\omega} = 0.073098627 \text{ deg/hr}$$

Table 4-3. Orbital Elements, Cartesian Coordinates, and Errors in Cartesian Coordinates Computed With CONCAR Algorithm Using J_2 Term and Starting Conditions as Osculating Elements at $t = 0$

	$t = 48$ hr	$t = 96$ hr	$t = 144$ hr
a, km	6981.471516	6981.471516	6981.471516
e	0.00141817	0.00141817	0.00141817
i, deg	57.002219	57.002219	57.002219
Ω, deg	88.712177	80.801189	72.890102
ω, deg	61.826844	65.336711	68.846577
M, deg	80.515297	355.276977	170.038657
$\omega + f$, deg	142.502536	60.600278	338.722663
x, km	-2437.812650	-2718.177361	3232.649280
y, km	-5484.269655	3907.133828	5811.739703
z, km	3563.455210	5094.024327	-2124.784121
$\dot{x}$, km/sec	3.157658	-3.050433	-2.863125
$\dot{y}$, km/sec	-4.681182	-6.184601	3.739829
$\dot{z}$, km/sec	-5.023572	3.114084	5.908481
Δx, km	23.362	-49.201	-68.171
Δy, km	-35.993	-100.201	89.693
Δz, km	-42.181	-47.717	144.318
$\Delta\dot{x}$, km/sec	0.020127	0.053314	-0.89013
$\Delta\dot{y}$, km/sec	0.047892	-0.074120	-0.166221
$\Delta\dot{z}$, km/sec	-0.034784	-0.099828	0.600030
$\Delta(w + f)$, deg	-0.513623	-0.998043	-1.515403

The top third of table 4-4 gives the mean elements at $t = 0$, 48, 96, and 144 hr. The middle third of the table gives the Cartesian coordinates computed from the CONCAR algorithm and using the mean elements, and the bottom third gives the differences between the LaRC and the GSFC results as was done in tables 4-1 and 4-3. It can be seen that these differences in table 4-4, using the mean elements at $t = 0$, are not substantially different from those of runs 1 and 2 of table 4-1, being within a factor of 2 or 3 of the integrated results, and certainly much better than the 1 to 2 orders of magnitude differences displayed in table 4-3 in which the oscillating elements were used at $t = 0$. It can perhaps be implied that the perturbation-modified algorithm using mean elements can be used, with only a slight degradation in accuracy, over approximately the same span as the numerical integration using only the zonal harmonics. It may just be a personal prejudice, but the author believes that the numerical integration algorithm is more defensible than the modified mean-element approach, especially if the integration time exceeds a few weeks. However, the mean-deviant algorithm has been successfully applied to the prediction of both SAGE II and SAM II ground truth sites with lead times of as much as 4–6 weeks.

Table 4-4. Orbital Elements, Cartesian Coordinates, and Errors in Cartesian Coordinates Computed With CONCAR Algorithm Using J_2 Term and Starting Conditions as Mean Elements at $t = 0$

	$t = 0$ hr	$t = 48$ hr	$t = 96$ hr	$t = 144$ hr
a, km	6981.26555	6981.26555	6981.26555	6981.26555
e	0.00254626	0.00254256	0.00254626	0.00254626
i, deg	56.997801	56.997801	56.997801	56.997801
Ω, deg	96.601960	88.693939	80.785919	72.877898
ω, deg	71.220024	74.728758	78.237492	81.746226
M, deg	152.821231	68.057791	343.294351	258.530911
x, km	2215.11242	−2409.67576	−2755.911096	3176.32012
y, km	−4679.87474	−5520.91264	3819.199893	5889.11718
z, km	−4089.14043	3515.54137	5130.239109	−2004.32760
$\dot{x}$, km/sec	2.32563390	3.18182136	−3.00963556	−2.94169646
$\dot{y}$, km/sec	5.55145098	−4.63367603	−6.25485145	2.59164655
$\dot{z}$, km/sec	−4.53986239	−5.06051117	3.03212667	5.95682454
Δx, km	3.747	4.775	11.467	11.842
Δy, km	0.548	0.133	12.597	−12.316
Δz, km	−7.986	−5.733	−11.512	−23.868
$\Delta\dot{x}$, km/sec	−0.0006811	0.0040364	−0.0125166	0.0104415
$\Delta\dot{y}$, km/sec	−0.004178	−0.003870	0.0038695	0.0180386
$\Delta\dot{z}$, km/sec	0.0055266	−0.0021552	0.0172707	−0.0116862

Chapter 5

Where Is the Sun?

Most of the time series relations needed to determine the position of the Sun have already been given in chapter 1. The remaining geometrical concepts will be developed in this chapter, as needed.

Perhaps the best way to introduce this material is through numerical examples. The following problems, typical of those encountered in spherical astronomical applications, will be addressed in this chapter, and a complete numerical example will be worked out for each to illustrate the method. The text procedures are not the only approach, of course, but are the ones that, for now unknown reasons, appealed to the writer at the time he first encountered these problems. The number of significant figures shown in the numerical results is generally that required for the accuracy given; 0.1 arc-sec is $0^\circ.000028$, and calculations to a few hundredths of an arc-sec are frequently required.

The most fundamental problem which arises concerning any computation of the Sun's position is the determination of the right ascension and declination of the center of the Sun, given any date of the year and time of day. These coordinates, along with the Greenwich sidereal time, are central to the solution of all the problems listed below. Therefore, the problems to be considered in the present chapter are the following (all the problems presented require the day of the year and/or the time of day to be given; some also require the specification of a particular location of the Earth's surface—the coordinates of an "observer;" these are assumed given):

1. Given any calendar date in the year and time of day, what are the right ascension and declination of the Sun? Note: *Only mean values will be determined as the extra computation will serve no useful purpose here.*
2. What is the Greenwich sidereal time for a given universal time, and what is the local sidereal time at a specified geographic location?
3. What are the geographic coordinates of the subsolar point at a given time of day?

4. For a given observer, what are the azimuth and elevation angles of the Sun at a given local (zone) time of day (with and without refraction)?
5. What is the local zone time when the Sun is directly on the local meridian (zonal time of local noon)?
6. What is the elevation angle of the Sun at local noon?
7. What are the local zone times of sunrise and sunset?
8. What are the azimuth angles of the Sun at sunrise and sunset?

Note that if the Cartesian coordinates of a spacecraft are known in the inertial coordinate system, then its right ascension and declination can be found from equation (1-1a). Then problems 2–8 can be applied to satellite motion by substituting the satellite coordinates for those of the Sun, taking into account, when necessary, the much more rapid time changes of these coordinates.

As a specific application of some of the concepts discussed here—given the semimajor axis and eccentricity of a satellite orbit, what must be the location of the line of nodes of a Sun-synchronous orbit which crosses the equator at a specific local time? (See, e.g., Brooks 1977.)

The numerical example will be worked out for the time of April 6, 1985, at 2:37 PM EST in Hampton, Virginia, whose latitude and longitude, respectively, are approximately 37°N and 76°W.

The chapter will terminate with the solution of a problem arising in connection with Sun-viewing satellites (e.g., SAM, SAGE) and specifically the problem of reducing the data from these satellites—namely, what is the minimum altitude of a ray from the center of the Sun (or indeed any other specific celestial location) to the spacecraft above the oblate Earth, and what are the geographic coordinates of the "subtangent" point on the surface of the Earth? (See Brooks 1980.)

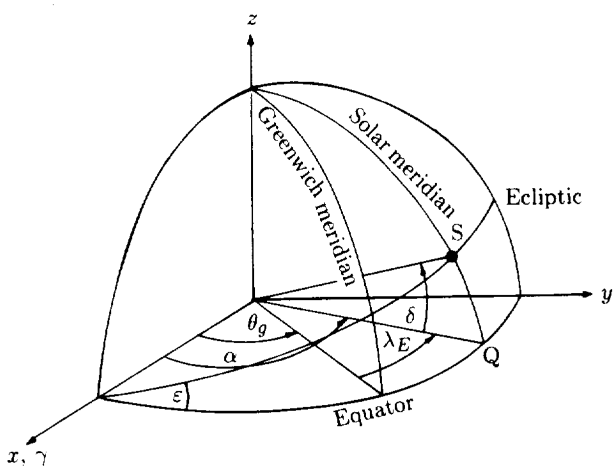
Problems and Examples

<u>Problem 1:</u>

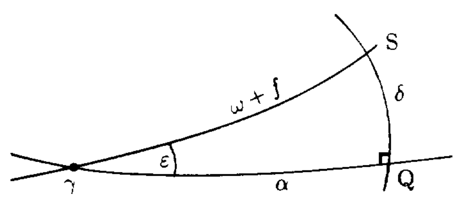
Figure 5-1(a) shows the inertial coordinates defined earlier in chapter 1. The coordinates of the Sun are the right ascension α and declination δ. Figure 5-1(b) shows the spherical triangle γSQ.

For the given date, find the Julian date as outlined in chapter 1. Then proceed by the following steps:

(a) Compute the mean longitude of the Sun, $L = M + \tilde{\omega}$, from equation (1-19)



(a) Sun in inertial coordinate system and its relation to Greenwich meridian.



(b) Blowup of spherical triangle δSQ of figure 5-1(a).

Figure 5-1. Location of Sun in both inertial and Earth-fixed coordinate systems.

(b) Find the mean anomaly of the Sun, M, from equation (1-39) and the eccentricity of the solar orbit from equation (1-38)

(c) Compute the quantity $f - M$ from equation (3-7) and convert to degrees

(d) Then, $(\tilde{\omega} + f) = (\tilde{\omega} + M) + (f - M)$

(e) Compute the obliquity of the ecliptic, ε, from equation (1-21)

Then from figure 5-1(b), the mean right ascension is

$$\tan\ \alpha = \cos\ \varepsilon\ \tan\ (\tilde{\omega} + f) \tag{5-1}$$

and the mean declination

$$\sin\ \delta = \sin\ \varepsilon\ \sin\ (\tilde{\omega} + f) \tag{5-2}$$

The quadrant of α is always the quadrant of $(\tilde{\omega} + f)$.

Example:

(1-11)[3] $\quad \text{JDO} = 728495 - 3475 + 122 + 6 + 1721013.5$

$= 2446161.5$ at 0 hr GMT on April 6, 1985

Hampton, Virginia, is 5 time zones from Greenwich—<76/15>—and therefore 2:37 PM EST is 2:37 + 12 + 5 = 19:37 GMT. Then,

$\text{JD} = 2446161.5 + 19.6167/24 = 2446162.31761$

(1-13) $\quad T_u = 0.852630181$

Note on accuracy: T_u can be accurately computed from equation (1-13) if one groups the terms as

$$T_u = \frac{[(\text{JDO} - 2415020.) + UT/24]}{36525.}$$

Continuing

(1-19) $\quad L = 15^\circ.0390181$

(1-39) $\quad M = 92^\circ.35203707$

(1-38) $\quad e = 0.0167156694$

(1-21) $\quad \varepsilon = 23^\circ.44119896$

(3-7) $\quad f - M = 1^\circ.911865208$

(Step (d)) $\quad f + \tilde{\omega} = 16^\circ.95088331$

(5-1) $\quad \alpha = 15^\circ.62304219 = 1^h02^m29^s.5$

(5-2) $\quad \delta = 6^\circ.660242901 = 6^\circ39'36''.87$

The procedure described here gives the right ascension and declination relative to the mean equator and equinox of date. The apparent right ascension and declination (referred to the *true* equator and equinox of date) may be obtained from the mean values by applying the proper corrections for nutation and planetary aberration. (See, for example, Smart 1977, chaps. 8 and 10.) The numerical values tabulated in the Astronomical Almanac are apparent values. Linear interpolation in AA85 gives $\alpha = 1^h02^m27^s.5$ and $\delta = 6^\circ39'25''.56$; thus, for this example, $\Delta\alpha = (29.5 - 27.5) \times 15 = 30$ arc-sec, and $\Delta\delta = 36.87 - 25.36 = 11.51$ arc-sec.

Planetary aberration (correction for the finite speed of light) gives a correction of about $-20'' \pm 2''$ on the right ascension, where the

[3] The number in parentheses at the left of the equation is the number of the equation used to calculate the quantity.

correction of 2″ is sinusoidal with a period of 1 year. The aberration correction to declination is also a yearly sinusoid with an amplitude of about 8″. The nutation in right ascension is a sinusoid with a period of just over 18 yr with an amplitude of about 20″. The nutation in declination is a multiple-frequency sinusoidal sum, but its maximum magnitude is about 9″.5. Using the formal equations (e.g., Smart 1977, chap. 8), these data for the present example are

	Nutation	Aberration	Sum
$\Delta\alpha$ . . .	$-12''.74$	$-18''.92$	$-31''.66$
$\Delta\delta$ . . .	$-4''.06$	$-8''.02$	$-12''.08$

If these are added to the mean values computed above, we get

$$\alpha = 1^h 02^m 27^s 39$$
$$\delta = 6^\circ 39' 24''.29$$

which are very close to the tabulated AA85 values.

For *most* Earth orbital applications, it is probably not necessary to add the nutation and aberration corrections; the mean values should suffice.

Problem 2:

Recall that the sidereal time is the hour angle of the vernal equinox. If it is measured from the Greenwich meridian, it is known as Greenwich sidereal time, and if it is measured from any other location, it is known as local sidereal time. Figure 5-2 shows the Greenwich meridian, with the Greenwich sidereal time θ_g. The observer at O has an east longitude λ_o and latitude ψ_o.

(a) Find the sidereal time at 0 hr GMT from equation (1-12) or (1-15) using the appropriate T_u

(b) Correct for time of day using equation (1-14a,b)

(c) From figure 5-2, the local sidereal time is

$$\text{LST} = \theta_g + \lambda_o \qquad (\lambda_o + \text{East}) \qquad (5\text{-}3)$$

Example:

JDO = 2446161.5

(1-13) $T_u = 0.8526078029$

(1-12) $\theta_{g_o} = 194^\circ.2277554 = 12^h 56^m 54^s.6613$ (AA85 gets $12^h 56^m 54^s.7273$)

$= 19^h 37^m \text{GMT} = 1177$ min

(1-14a) $\Delta\theta_g = 295^\circ.0556212$

(1-14b) $\theta_g = 129^\circ.2833766$

(5-3) $\text{LST} = 53^\circ.28337659 = 3^h33^m08^s.0100$

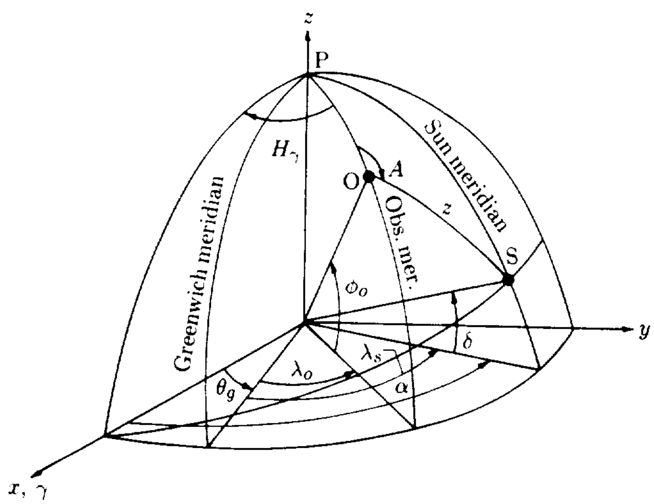


Figure 5-2. Diagram similar to figure 5-1 but including observing meridian. Defines azimuth A and zenith distances z of celestial body S relations to observer O.

Problem 3:

From figure 5-2, the east longitude of the subsolar point, λ_s, is found from

$$\alpha = \theta_g + \lambda_s \tag{5-4}$$

The geocentric latitude of the subsolar point is identical to the declination, and the geographic latitude is found from equation (1-3) with $a = 6378.160$ km and $b = 6356.775$ km.

Example:

(5-4) $\lambda_s = 246^\circ.3396656$

$\psi_c = 6^\circ.660242901$

(1-3) $\psi_g = 6^\circ.704796079$

As a rough check, note that the Sun is $(360^\circ - 76^\circ) - 246^\circ.3$ or about 38° west of Hampton at this time. If we assume that the Sun is directly overhead at 1200 hr local time, which is not exactly true, then in 2^h37^m, the Sun will move to the west, at 15 deg/hr, a distance of about 39°.25, which is close to the 38° above.

Problem 4:

In figure 5-2 at the observer's meridian O, the azimuth angle A is measured clockwise from geographic north. The solar zenith angle z, arc OS in the figure, is the complement of the altitude angle, and hence,

$$\gamma = 90^\circ - z \tag{5-5}$$

The zenith angle z is found by applying the law of cosines for sides to the spherical triangle POS (the quadrant of z is no problem, since if $\cos z < 0.$, the object (Sun) is below the horizon):

$$\cos z = \sin \psi_o \sin \delta + \cos \psi_o \cos \delta \cos (\lambda_s - \lambda_o) \tag{5-6}$$

Both sine and cosine of the azimuth angle must be used to determine the quadrant of A from the triangle,

$$\sin A = \frac{\cos \delta \sin (\lambda_s - \lambda_o)}{\sin z} \tag{5-7}$$

and also

$$\cos A = \frac{\sin \delta - \cos z \sin \psi_o}{\sin z \cos \psi_o} \tag{5-8}$$

Note: *Some users prefer to measure azimuth east or west of due south—most spherical astronomy texts define it this way.* This azimuth A' is found from the above as

$$A' = A - 180^\circ \tag{5-9}$$

and is west if A' is positive, east if A' is negative.

Example:

(5-6) $z = 45^\circ.7516467$
(5-5) $\lambda = 44^\circ.2483533$
(5-7) $\sin A = -0.8471830565$
(5-8) $\cos A = -0.5313011094$
$A = 237^\circ.9065922$
(5-9) $A' = 57^\circ.90659223$ W

To correct for the effects of refraction, the following law, derived for a spherical atmosphere (Smart 1977, Green 1985, for example),

$$z - z_R = R = A \tan z_R + B \tan^3 z_R \tag{5-10}$$

is used, where the constants A and B, while defined by analytic expressions, are usually found empirically. Smart (1977) gives

$$A = 58''.16$$
$$B = -0''.067$$

whereas Green (1985) gives

$$A = 60''.29$$
$$B = -0''.06688$$

The above constants assume standard atmospheric conditions at the surface. Green (1985) gives the following expressions for A and B, in which the temperature and pressure profiles of the atmosphere are allowed to vary. Define

$$\rho_o H_o = \int_0^\infty \rho \, dh$$

where ρ is the density at altitude h. The parameter H_o is thus the density scale height of the atmosphere. Then Green gives the correction equations

$$A = (n_o - 1)\left(1 - \frac{H_o}{R_e}\right) \qquad (5\text{-}11)$$

$$B = -(n_o - 1)\left[\frac{H_o}{R_e} - \frac{1}{2}(n_o - 1)\right] \qquad (5\text{-}12)$$

where the units of A and B are radians, and n_o is the index of refraction of air at the surface. This can be computed with acceptable accuracy from the well-known Edlen formula (Edlen 1953)

$$(n_o - 1) \times 10^6 = \left(77.46 + \frac{0.459}{\lambda^2}\right)\frac{P}{T} - \frac{P_{H_20}}{1013}\left(43.49 - \frac{0.347}{\lambda^2}\right) \qquad (5\text{-}13)$$

where

P = atmospheric pressure, mb
T = atmospheric temperature, K
P_{H_20} = partial pressure of water vapor, mb
λ = wavelength, μm

Given R, then the observed zenith as seen at the surface of the Earth is

$$z_R = z - R \qquad (5\text{-}14)$$

It is seen that if extreme accuracy is required, an iteration is required. Assume $z_R = z$, then compute z from equation (5-6). Use this in equation (5-10) to get an estimate of R; if further accuracy is needed (seldom), use z_R computed from equation (5-14) to go back to equation (5-10) and recompute R.

The refraction correction described above (equally applicable, of course, to the moon, stars, satellites, or any other celestial body) is sufficiently accurate whenever $z_R \leq 75°$. For zenith angles greater than this, tables of refraction correction are generally used. The following equation (see any Astronomical Almanac, Section B) has been quoted as being reliable for $z_R \geq 75°$. Let γ equal 90 − z, the altitude. Then,

$$R = P\frac{(0.1594 + 0.0196\gamma + 0.00002\gamma^2)}{(273 + T)(1 + 0.505\gamma + 0.0845\gamma^2)} \tag{5-15}$$

where

$$P = \text{surface pressure, mb}$$
$$T = \text{surface temperature, K}$$
$$\gamma = \text{altitude, deg}$$

and in this formula, R is in degrees. For $z = 75°$, and at STP ($P = 1013$ mb, $T = 0°$), the above formula gives $R = 0°.06159$ or $221''.724$. Equation (5-10) gives $213''.573$ using Smart's constants and $221''.529$ using Green's.

Example: Use Smart's constants.
Assume $z_R = 45.7516467$
then
(5-10) $R = 59''.633854$
(5-14) $z_R = 45°.735081 = 45°44'06''.29$

The new R would be $59''.599454$ and the iterated $z_R = 45°.735091$ or $45°44'06''.33$, or a change in zenith distance of $0''.04$.

Problem 5:

In figure 5-2, the observer's meridian is rotating from west to east at the rate of 15 deg/mean sidereal hr. From the figure it is obvious that the condition

$$\theta_g + \lambda_o = \alpha$$

or

$$\theta_{g_o} + \dot{\theta} t_{\min} + \lambda_o = \alpha \tag{5-16}$$

is the condition for the Sun to be on the local meridian. We know that, within a maximum error of $7°.5$ or 30 min in time, the mean Sun is directly overhead at 12 hr LMT. Thus for EST, for example, local noon will occur near 17 hr GMT. Let us first find the right ascension of the Sun at 17 hr GMT and assume that it is fixed at this position for this calculation. Proceeding exactly as in the first example, for 17 hr GMT,

$$\alpha = 15^\circ.52331572$$
$$\delta = 6^\circ.619146553$$

From equation (5-16),

$$194^\circ.2277554 + 0.25068447t_{\text{min}} + 284^\circ.0 = 15^\circ.52331572$$

Solve for t_{min} (using arithmetic mod 360), we find that t_{min} = 1026.3725 min after 0 hr GMT, or $17^h06^m22^s$ GMT. Since our observer is in an Eastern Standard Time zone, the local time of the Sun's meridian passage is $12^h06^m22^s$. Interpolation in the tables given in AA85 gives $12^h06^m29^s$, and our approximate method here is only 6 sec in error (or else AA85 is 6 sec in error).

Problem 6:

From equation (5-6), at local noon, $\lambda_s = \lambda_o$, and

$$\cos\ z_{\text{noon}} = \cos\ \psi_o\ \cos\ \delta + \sin\ \psi_o\ \sin\ \delta = \cos\ (\psi_o - \delta)$$
$$z_{\text{noon}} = \psi_o - \delta$$

Example:

(5-11) $z_{\text{noon}} = 37^\circ.0 - 6^\circ.660242901 = 30^\circ.3397571$

(5-5) $\gamma_{\text{noon}} = 59^\circ.6602429$

Problem 7:

The angle measured clockwise (looking down at the north pole) between the observer's meridian and the meridian of any celestial body is called the local hour angle of that body—for example, in figure 5-2, the angle OPγ is the local hour angle of the vernal equinox, H_γ, and the angle OPS is $(360 - H_s)$, where H_s is the hour angle of the Sun; that is,

$$360 - H_s = \lambda_s - \lambda_o \qquad (5\text{-}17)$$

in figure 5-2. In terms of the hour angle of the Sun, we can write equation (5-6) as

$$\cos\ z = \sin\ \psi_o\ \sin\ \delta + \cos\ \psi_o\ \cos\ \delta\ \cos\ H_s \qquad (5\text{-}18)$$

If the Earth had no atmosphere, the $z = 0^\circ$ would put the center of the Sun right on the horizon, and thus from figure 5-2

$$\cos H_{r/s} = -\tan \psi_o \tan \delta \qquad (5\text{-}19)$$

would give the rise/set value of the hour angle, from which the rise/set times from the time of local noon could be determined.

The Earth does have an atmosphere, however, and there is refraction to be considered. Consequently, there are three different ranges of twilight generally computed in the almanacs:

(a) Civil twilight begins when the top of the Sun just touches the horizon; this requires $z = 90°50'$ (i.e., $16'$ for the half-angle of the Sun plus $34'$ refraction correction)

(b) Nautical twilight begins when $z = 96°$

(c) Astronomical twilight begins when $z = 102°$ and ends when $z = 108°$

To calculate the various times, put the appropriate values of z into equation (5-12) to compute the hour angle, divide the hour angle by 15 to get the number of hours from local noon, and add or subtract this time interval from the local time of meridian passage (problem 5) to get the respective time of sunrise/sunset.

Example:

We know that sunrise/sunset will occur at about 6 hr before and 6 hr after local noon, respectively. For the EST time zone, these will occur at about 11 hr GMT and 23 hr GMT. These times are used as in problem 1 to get the declinations of the sun at sunrise/sunset,

$$\delta_{\text{rise}} = 6°.524818313$$
$$\delta_{\text{set}} = 6°.713346807$$

These are used in equation (5-18) to give the values in table 5-1. (The numbers in parentheses are the minutes determined by a double interpolation in AA85 for the example latitude and times.)

Problem 8:

From equations (5-7) and (5-8), we can now set $z = 90°$, since here we're concerned with the Sun center and refraction can be neglected,

$$\sin A = \cos \delta \sin H \qquad (5\text{-}20)$$

$$\cos A = \frac{\sin \delta - \sin \psi_o}{\cos \psi_o} \qquad (5\text{-}21)$$

in which, of course, the appropriate rise/set values are used for H and δ.

Table 5-1. Hour Angles and Rise/Set Times Measured From the Time of Local Meridian Passage for Example 7

z	AM		PM	
	H_{rise}, deg	t_{rise}, hr	H_{set}, deg	t_{set}, hr
90°50′	95.99935	6.38248	96.144453	6.39213
96°	102.58716	6.82047	102.73768	6.83048
102°	110.37838	7.33847	110.53816	7.34909
108°	118.40111	7.87186	118.57461	7.88339

z	Sunrise	Sunset
90°50′	5^h43^m *(38^m)	18^h30^m *(28^m)
96°	5^h17^m (11^m)	18^h56^m (53^m)
102°	4^h46^m (41^m)	19^h21^m (24^m)
108°	4^h14^m (10^m)	19^h59^m (57^m)

*Numbers in parentheses are the minutes determined by double interpolation in AA85 for the example latitude and times. The present technique appears to be accurate to within 5 or 6 min when compared with AA85.

Example:

Sunrise

$$\delta = 6^\circ.524818313$$
$$H = 94^\circ.9443029$$
$$\sin A = 0.9898252808$$
$$\cos A = -0.6112693926$$
$$A = 127^\circ.681345$$

Sunset

$$\delta = 6^\circ.713346807$$
$$H = -95^\circ.08883614$$
$$A = 232^\circ.6143557$$

As one example for the application of this material to an orbit problem, consider the following frequently posed problem: a satellite is to be placed in a circular ($e = 0$) orbit at some given altitude (a is thus known), such that it is Sun synchronous and the ascending node crosses the equator at some specified local time, t hr, either before or after local noon. What must be the inclination of the orbit and where should the ascending node be placed at orbit insertion?

Recall that the mean daily angular travel of the Sun about the Earth is 360/365.2422, or 0.98564733 deg/mean solar day, and that this motion is west to east in the inertial coordinate system. Equation (4-8) gives the angular rate of the ascending node of the orbit: $\dot{\Omega}$ must thus be +0.98564733 deg/mean solar day in order for the orbit to be Sun synchronous. Equation (4-8) then defines the inclination of the orbit, which must be greater than 90° in this case—i.e., retrograde—in order to be Sun synchronous. (See Brooks 1977, for example.)

Finally, for a PM crossing, the node must be located to the east of the solar meridian ($15 \times t$, where t is the time from local noon in hours), whereas for an AM crossing, the node must be located the same angular distance to the west of the solar meridian.

For instance, suppose we want to put a satellite into a circular Sun-synchronous orbit with a semimajor axis of 6978 km. We want to have the ascending node cross the equator at 2 PM local time. What are the inclination and the right ascension of the ascending node at orbit insertion for this case? Assume the insertion date is the same date as we've used so far for numerical examples.

At a first guess, assume that $\bar{n} = n = \sqrt{\mu/a^3}$ in equation (4-8). This gives n = 5361.7792 deg/day and, with $\dot{\Omega}$ = 0.98564 deg/day, equation (4-8) yields $i = 97^\circ.789976$. To improve on this slightly, use this value of i in equation (4-10) and compute $\bar{n}$ = 5358.3436 deg/day (this is only a 0.6-percent decrease in mean angular rate). Using this value for $\bar{n}$ in equation (4-8) gives i = 97.795001 deg/day, and obviously further iteration is unwarranted, as the inclination at orbit insertion cannot be achieved with this accuracy.

On April 6, 1985, at 0 hr GMT, the right ascension of the Sun is 15°2161669. For a 2 PM local time of crossing, the ascending node must be 2×15 or 30° to the east of the Sun's meridian, or the right ascension of the ascending node must be at 45.216°. Due to the nonuniform motion of the Sun throughout the year, the 2 PM crossing cannot be maintained exactly, but neglecting the effects of the higher order perturbations, this orbit is the best that can be done. Just as the differences in right ascension between the mean and true Sun are given by the equation of time, the same equation will predict the differences between the actual nodal crossing time and 2 PM. These differences will vary from −14 to +16 min throughout the year. Figure 5-3 shows a rough sketch of the resulting orbital geometry.

Omnipresent β Angle

The β angle is a parameter that comes up again and again in mission design literature, as many mission parameters are directly related to this angle or functionally dependent on it (see, e.g., Buglia 1986). The

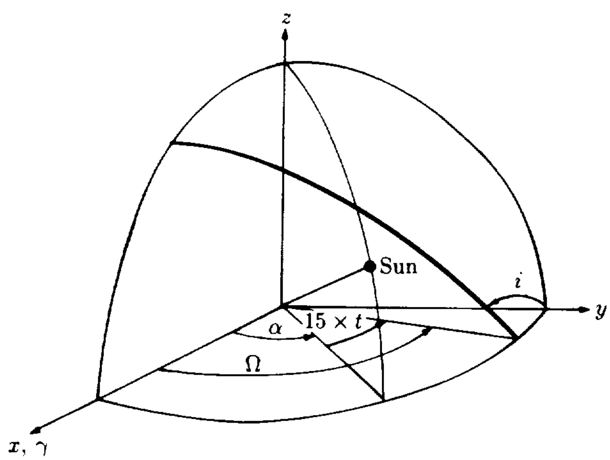


Figure 5-3. Sketch of geometry of retrograde Sun-synchronous orbit, with equatorial crossing time t hr PM.

β angle is defined (see fig. 5-4) as the angle between the Sun vector and the orbital plane, and as can easily be seen, many problems related to temperature or cooling, Sun viewing or the prevention of solar energy entering the optics of an instrument, and many other such mission parameters are closely related to this quantity.

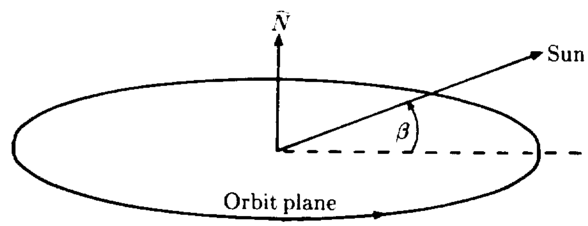


Figure 5-4. Definition of β angle.

If the spacecraft Cartesian coordinates are given in the inertial coordinate system, the unit vector normal to the plane of the orbit (along the angular momentum vector) is given by

$$\hat{N} = \frac{\boldsymbol{r} \times \dot{\boldsymbol{r}}}{|\boldsymbol{r} \times \dot{\boldsymbol{r}}|} \tag{5-22}$$

If $\hat{e}_s$ is a unit vector to the center of the Sun, then

$$\hat{e}_s = \begin{bmatrix} \cos\delta \cos\alpha \\ \cos\delta \sin\alpha \\ \sin\delta \end{bmatrix} \tag{5-23}$$

and thus

$$\sin\beta = \hat{N} \cdot \hat{e}_s \tag{5-24}$$

In terms of the elements Ω and i of the orbit, the normal to the orbit can be written in the alternate form from equation (5-14) (Escobal 1965) as

$$\hat{N} = \begin{bmatrix} \sin\Omega \sin i \\ -\cos\Omega \sin i \\ \cos i \end{bmatrix} \tag{5-25}$$

Equations (5-23) and (5-25) then permit equation (5-24) to be written in the form

$$\sin\beta = \cos i \sin\delta + \sin i \cos\delta \sin(\Omega - \alpha) \tag{5-26}$$

As stated above, many mission parameters are directly related to the β angle. For example, solar panels require that they view the Sun, whereas some measurement instruments may require that they never view the Sun because direct sunlight may either permanently damage the instrument or saturate it so that it may become inoperable for a short time. If the orbit insertion date is specified, the position angles of the Sun are known. Thus, if the β angle is specified by mission constraints, the right ascension of the orbit can be determined from equation (5-26).

How Long Is the Solar Day?

The solar day and its variable length were discussed briefly in chapter 1, and figure 1-11 shows a plot of the difference between the length of the solar day and 86400 sec, the length of the mean solar day. We are now able to do the calculations which led to that figure.

At local noon of some day, the Sun is at position ① of figure 5-5 on the observer's meridian. The next day at noon, the Sun is again in the observer's meridian at position ②. Thus, during the solar day, the Earth has turned through the angle $(\alpha_2 - \alpha_1) + 360°$, where α_2 and α_1 are the respective values of the right ascension of the Sun. The angular rotation rate of the Earth is 0.25068447 deg/mean solar min (see chap. 1), or $4.178074622 \times 10^{-3}$ deg/mean solar sec. Thus, the length of the solar day is

$$\frac{(\alpha_2 - \alpha_1) + 360°}{4.178074622 \times 10^{-3}} \text{ mean solar sec}$$

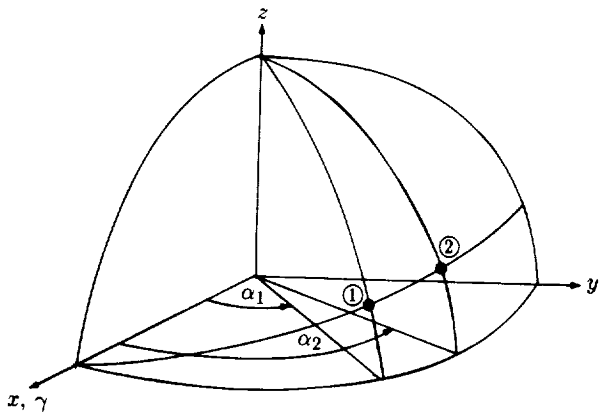


Figure 5-5. Location of Sun on observer's meridian on two consecutive days at local noon.

Example:

From figure 1-11 the longest solar day occurs on about December 23 and the shortest on about September 17. From the methods discussed earlier in this chapter we find that from noon on December 23 to noon on December 24, the right ascension of the Sun changes by 1°.1098751, and hence the length of the solar day at this time is 86429.733 mean solar sec. Similarly, from September 17 to 18, the right ascension changes by only 0°.8967917, and the length of this solar day is 86378.733 mean solar sec. The solar day on December 23 is thus about 51 sec longer than the solar day on September 17.

Minimum Height of Ray Above Oblate Spheroid

The Earth can, for most space applications, be adequately represented by an oblate spheroid, with equatorial radius a and polar radius b. Suppose we are given a spacecraft position vector $\mathbf{r}_p$, and an arbitrary direction from the spacecraft represented by the unit vector $\hat{e}$. This vector could be, for example, a unit vector to the center of the Sun, to another satellite, or to any other celestial body. The question we address here is, then, what is the minimum distance between the ray defined by $\hat{e}$ and the surface of the spheroid, h_{min}, and what are the geographic coordinates of the subtangent point A. (See fig. 5-6.)

For a spherical Earth, this problem is almost trivial. (See fig. 5-7.) The angle ρ is given by

$$\rho = \cos^{-1} \frac{\mathbf{r}_p \cdot \hat{e}}{r_p} \tag{5-27}$$

and the distance c from the spacecraft to the normal to $\hat{e}$ is given by

$$c = r_p \cos (180 - \rho) \tag{5-28}$$

Hence, the tangent point vector $\mathbf{r}_T$ is

$$\mathbf{r}_T = \mathbf{r}_p + c\hat{e} \tag{5-29}$$

and thus h_{min} is simply

$$h_{\text{min}} = r_T - R_e \tag{5-30}$$

where R_e is the mean radius of the Earth.

In terms of the right ascension and declination of point A, we can write

$$\left.\begin{aligned} x_T &= r_T \cos \delta_T \cos \alpha_T \\ y_T &= r_T \cos \delta_T \sin \alpha_T \\ z_T &= r_T \sin \delta_T \end{aligned}\right\} \tag{5-31}$$

from which the right ascension α_T and declination δ_T of the subtangent point can readily be found. The longitude of A is then found from (see problem 2)

$$\lambda_{\text{A}} = \alpha_T - \theta_g \tag{5-32}$$

where θ_g is the Greenwich sidereal time.

For an oblate spheroid, the problem is still directly solvable, but somewhat more cumbersome in form. The author's solution is presented below.

The vectors $\mathbf{r}_p$ and $\hat{e}$ define a plane. In this plane, define new axes $\bar{x}$ and $\bar{y}$, where $\bar{x}$ is parallel to $\hat{e}$ and $\bar{y}$ is normal to $\hat{e}$. This plane intersects the ellipsoidal surface in a curve given by

$$f(\bar{x}, \bar{y}) = 0$$

and the position of the subtangent point (and hence the minimum altitude) is defined by the condition

$$\frac{d\bar{y}}{d\bar{x}} = 0$$

as can be seen from figure 5-6. The coordinates of A in the "barred" coordinate axis system, $\bar{x}_A$ and $\bar{y}_A$, are then rotated back into the "unbarred" coordinate axes (the original axis system), and hence A is located in this system.

Compute r_T in exactly the same way as required by equation (5-29), using ρ from equation (5-27) and c from equation (5-28). Then, construct the unit vectors

$$\hat{e}_{\bar{x}} = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix} \tag{5-33}$$

is given and

$$\hat{e}_{\bar{y}} = \frac{\boldsymbol{r}_T}{r_T} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} \tag{5-34}$$

The unit vector $\hat{e}_{\bar{z}}$ normal to the $\bar{x}$-$\bar{y}$ plane is

$$\hat{e}_{\bar{z}} = \hat{e}_{\bar{x}} \times \hat{e}_{\bar{y}} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \tag{5-35}$$

Now, any vector $\boldsymbol{r}(x, y, z)$ in the unbarred axis system has components in the barred axis system given by

$$\left.\begin{aligned} \bar{x} &= \boldsymbol{r} \cdot \hat{e}_{\bar{x}} \\ \bar{y} &= \boldsymbol{r} \cdot \hat{e}_{\bar{y}} \\ \bar{z} &= \boldsymbol{r} \cdot \hat{e}_{\bar{z}} \end{aligned}\right\} \tag{5-36}$$

or, in matrix form,

$$\begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} = \begin{bmatrix} \ell_1 \; \ell_2 \; \ell_3 \\ m_1 \; m_2 \; m_3 \\ n_1 \; n_2 \; n_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{5-37}$$

In the present application, we're interested in the inverse transformation to equation (5-37)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \ell_1 \; m_1 \; n_1 \\ \ell_2 \; m_2 \; n_2 \\ \ell_3 \; m_3 \; n_3 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} \tag{5-38}$$

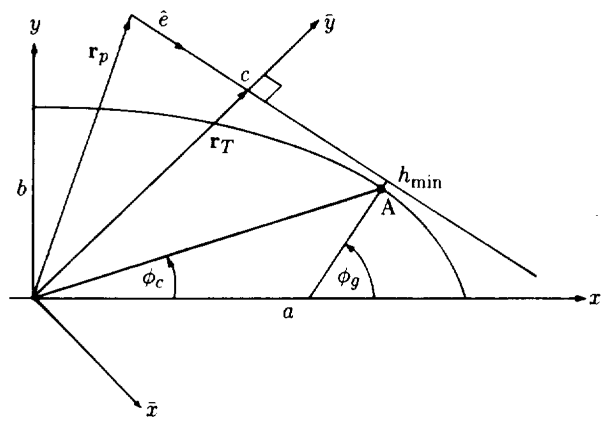


Figure 5-6. Geometry of unrefracted ray from celestial body to spacecraft. Reciprocal direction is defined by unit vector ê; minimum altitude of ray (subtangent point) above oblate spheroid is defined along with its geodetic and geocentric latitude, ϕ_g and ϕ_c, respectively.

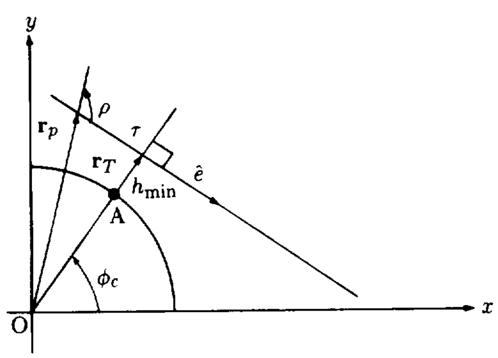


Figure 5-7. Geometry of unrefracted ray from celestial body to spacecraft. Reciprocal direction is defined by unit vector ê; minimum altitude of ray (subtangent point) along the extension of geocentric radius OA is defined.

and, since we're interested only in the $\bar{x}$-$\bar{y}$ plane, $\bar{z} = 0$, and

$$\left.\begin{aligned} x &= \ell_1\bar{x} + m_1\bar{y} \\ y &= \ell_2\bar{x} + m_2\bar{y} \\ z &= \ell_3\bar{x} + m_3\bar{y} \end{aligned}\right\} \tag{5-39}$$

In the unbarred axis system, the equation of the ellipsoid representing the Earth is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$$

or

$$x^2 + y^2 + \frac{a^2}{b^2}z^2 = a^2 \tag{5-40}$$

If we substitute equations (5-39) into equation (5-40) we get the equation of the curve defined by the intersection of the r_p-$\hat{e}$ plane with the ellipsoid; that is, $f(\bar{x}, \bar{y}) = 0$, or

$$(\ell_1\bar{x} + m_1\bar{y})^2 + (\ell_2\bar{x} + m_2\bar{y})^2 + \frac{a^2}{b^2}(\ell_3\bar{x} + m_3\bar{y})^2 = a^2 \tag{5-41}$$

The coordinates $\bar{x}_A$, $\bar{y}_A$ of the subtangent point are defined by the condition $\frac{d\bar{y}}{d\bar{x}} = 0$. Differentiate equation (5-41) with respect to $\bar{x}$, set $\frac{d\bar{y}}{d\bar{x}} = 0$, and collect terms to get

$$\bar{x}_A\left(\ell_1^2 + \ell_2^2 + \frac{a^2}{b^2}\ell_3^2\right) + \bar{y}_A\left(\ell_1 m_1 + \ell_2 m_2 + \frac{a^2}{b^2}\ell_3 m_3\right) = 0 \tag{5-42}$$

or

$$\bar{x}_A = \Phi\bar{y}_A \tag{5-43}$$

where

$$\Phi = \frac{-\left(\ell_1 m_1 + \ell_2 m_2 + \frac{a^2}{b^2}\ell_3 m_3\right)}{\left(\ell_1^2 + \ell_2^2 + \frac{a^2}{b^2}\ell_3^2\right)} \tag{5-44}$$

Use equation (5-43) to eliminate $\bar{x}_A$ in equation (5-42) and solve for $\bar{y}_A$,

$$\bar{y}_A = \frac{a}{\left[(\ell_1\Phi + m_1)^2 + (\ell_2\Phi + m_2)^2 + \frac{a^2}{b^2}(\ell_3\Phi + m_3)^2\right]^{1/2}} \tag{5-45}$$

and then $\bar{x}_A$ follows from equation (5-43), and the coordinates of the subtangent point A in the barred coordinates are known.

Reference to figure 5-6 shows that the minimum altitude h_{min} is given simply by

$$h_{min} = |r_T| - \bar{y}_A \tag{5-46}$$

Finally, the coordinates of A in the unbarred coordinate axis system are found by putting $\bar{x}_A$ and $\bar{y}_A$ (with $\bar{z} = 0$) into equation (5-38), giving

$$\begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} = \begin{bmatrix} \ell_1 \; m_1 \\ \ell_2 \; m_2 \\ \ell_3 \; m_3 \end{bmatrix} \begin{bmatrix} \bar{x}_A \\ \bar{y}_A \end{bmatrix} \tag{5-47}$$

The right ascension and declination are then found from equation (5-31) and the longitude from equation (5-32).

Of course, the declination is identical with geocentric latitude, and finally the geodetic latitude is found from equation (1-3).

Final Remarks

It goes without saying that there are myriad other problems, not mentioned herein, that are solvable by the methods and equations of the present text. With regard to Earth satellites, for example, these include such diverse problems as

1. Rise/set times of a satellite with respect to a given ground station
2. Range, range-rate, and angular data of a satellite with respect to a given ground station
3. Rise/set times of one satellite with respect to another satellite
4. Range, range-rate, and angular data of one satellite with respect to another
5. Computation of ground tracks of satellites
6. Area coverage of satellite borne sensors
7. Entry/exit with regard to sunlight conditions (Escobal (1965 and 1968) and Green (1985) discuss these and other problems in "Keplermanship;" see also Buglia 1986 and Brooks 1977)

The planets move in essentially elliptic orbits about the Sun. A consistent set of the time-varying orbital elements for all the planets is given by Escobal (1968). These orbital elements can be used with the present equations to determine the heliocentric position of any planet. If the orbit of the Earth is also computed, then a few elementary coordinate transformations are all that is necessary to yield the right ascension and declination of the planet with respect to the standard Earth-centered coordinates described in chapters 1 and 4, and hence the positions of the planets with respect to either a fixed ground station or a satellite orbiting the Earth can be readily computed by the methods of chapter 5, with the planet's position being substituted for that of the Sun.

The Moon is the Earth's nearest natural celestial body, and except for the musings of poets, lovers, and other eccentrics, the motion of our nearest neighbor is quite complex due to the rather large perturbations imposed on its motion by the oblate Earth and the Sun. However, Escobal (1968) presents an algorithm, based on earlier work by G.W. Hill and E.W. Brown, which is reported to yield an accuracy of about 30 arc-sec. The Moon subtends an angle of about $1/2°$ as seen from the Earth, or a radial displacement of 900 arc-sec. The Escobal algorithm is thus accurate to about $1/30$ of the angular radius of the Moon. This is not too shabby and is certainly close enough to find the Moon.

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