N89-11155

Continuum Theories for Fluid–Particle Flows: Some Aspects of Lift Forces and Turbulence [1]

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Abstract

A general framework is outlined for the modeling of fluid-particle flows. The momentum exchange between the constituents embodies both lift and drag forces, constitutive equations for which can be made explicit with reference to known single-particle analyses. Relevant results for lift are reviewed, and invariant representations are posed. The fluid and particle velocities and the particle volume fraction are then decomposed into mean and fluctuating parts to characterize turbulent motions, and the equations of motion are averaged. In addition to the Reynolds stresses, further correlations between concentration and velocity fluctuations appear. These can be identified with turbulent transport processes such as "eddy diffusion" of the particles. When the drag force is dominant, the classical convection–dispersion model for turbulent transport of particles is recovered. When other interaction forces enter, particle segregation effects can arise. This is illustrated qualitatively by consideration of turbulent channel flow with lift effects included.

Introduction

Flow of an incompressible, single-phase fluid is fully characterized by a single kinematic field, the velocity. The kinematics of a fluid-particle mixture involves the velocity of each constituent, and an additional scalar field representing the particle volume fraction, or concentration. In some flows, the latter can be quite critical. For example, since the effective viscosity of a suspension is a strong function of the concentration, particle segregation in a viscometer will violate the assumption of homogeneity required to interpret measurements. Furthermore, if the distribution of particles is in part determined by the shear *rate* (*e.g.*, Ho and Leal, 1974; McTigue, *et al.*, 1986), the apparent viscosity will be rate-dependent, and the mixture will appear to be non-Newtonian even when this may not be so locally.

A great deal of work has been done on the dynamics of a single particle in a viscous fluid; reviews are given by Happel and Brenner (1965), Goldsmith and Mason (1967), Brenner (1966, 1970), and Leal (1980). In many applications, however, it is neither practical nor even of interest to track individual particles. Rather, the

[1] This work was supported by Sandia National Laboratories under contract to the U. S. Department of Energy (DE-AC04-76DP00789).

primary concern is more often for some average characteristics of the flow. The objective of a continuum mixture theory is to provide governing equations for these average kinematic fields. Ideally, one would like to draw upon knowledge gained from single-particle analyses to guide the development of the constitutive models required by the continuum theory. As a practical matter, this process relies heavily upon empirical input as well.

This paper is intended to illustrate by example the construction of a two-phase flow model for turbulent mixtures. It is highly idealized and far from complete, but captures some interesting phenomenology. We first outline the general mechanical balance laws for a mixture. We then review in detail results from the literature on lift forces in viscous and inviscid flows. Generalizations in forms appropriate for the exchange of momentum between the constituents in a mixture are then discussed. "Exact" equations of motion are posed for the simplest forms for lift and drag interactions. Turbulent decomposition and averaging yields not only Reynolds stress terms, but other correlations of velocity and concentration fluctuations as well. Simple "eddy viscosity" and "eddy diffusivity" closure schemes are adopted to model the correlations. We then show that the classical convection-diffusion model for turbulent transport emerges naturally for the case when the drag term dominates the disperse phase momentum balance. Finally, we consider channel flow with the lift force present, and identify an equilibrium particle segregation due to a balance of lift and turbulent diffusion.

Balance Laws

The balance equations for the mass and momentum of constituent α are given by:

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \mathbf{v}_\alpha) = 0, \tag{1}$$

$$\rho_\alpha \left(\frac{\partial \mathbf{v}_\alpha}{\partial t} + \mathbf{v}_\alpha \cdot \nabla \mathbf{v}_\alpha \right) = \nabla \cdot \mathbf{T}_\alpha + \rho_\alpha \mathbf{g} + \mathbf{m}_\alpha, \tag{2}$$

where ρ_α is the density (mass of constituent α per unit volume of the mixture), $\mathbf{v}_\alpha$ is the velocity, $\mathbf{T}_\alpha$ is the stress, $\mathbf{g}$ is the acceleration due to gravity, and $\mathbf{m}_\alpha$ is a body force due to the interaction of constituent α with the other constituents present. Equations (1) and (2) take the form of the classical balance laws for a single-phase continuum, with the exception of the interaction force, or momentum exchange, $\mathbf{m}_\alpha$. Equation (1) neglects chemical interactions or phase changes, which would be embodied in mass exchange terms (*cf*, Passman, *et al.*, 1984).

We anticipate that the mixture can be represented as a single continuum, so that:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{3}$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \nabla \cdot \mathbf{T} + \rho \mathbf{g}, \tag{4}$$

where ρ, $\mathbf{v}$, and $\mathbf{T}$ are the density, velocity, and stress for the mixture. Comparison of (1)–(4) shows that the mixture quantities are related to the constituent quantities by the *summation rules*:

$$\rho = \Sigma \rho_\alpha \tag{5}$$

$$\rho \mathbf{v} = \Sigma \rho_\alpha \mathbf{v}_\alpha \tag{6}$$

$$\mathbf{T} = \Sigma[\mathbf{T}_\alpha - \rho_\alpha(\mathbf{v} - \mathbf{v}_\alpha)(\mathbf{v} - \mathbf{v}_\alpha)], \tag{7}$$

$$\Sigma \mathbf{m}_\alpha = \mathbf{0}, \tag{8}$$

where Σ indicates the summation over all constituents present. Equation (8) shows that whatever momentum is lost from one constituent is gained by the other(s).

The density fields, ρ_α, can vary due to changes in both the volume fraction, ϕ_α, and the local density (mass of constituent α per unit volume of that constituent), γ_α. Thus, it is convenient to introduce the decomposition:

$$\rho_\alpha = \phi_\alpha \gamma_\alpha. \tag{9}$$

Finally, we consider only *saturated* mixtures, in which all space is occupied, which imposes the requirement:

$$\Sigma \phi_\alpha = 1. \tag{10}$$

Equations (1)–(10) hold for multiphase systems with any number of constituents. For present purposes, we specialize to the case of two, a continuous fluid ($\alpha = f$) and a dispersed particulate solid ($\alpha = s$). In this case, we let

$$\phi = \phi_s = 1 - \phi_f. \tag{11}$$

We also restrict attention to mixtures comprised of incompressible constituents (γ_f and γ_s constant).

Without loss of generality, it is convenient to decompose the stresses, $\mathbf{T}_\alpha$, into an isotropic pressure, p_α, and an *extra stress*, $\mathbf{T}^\star_\alpha$:

$$\mathbf{T}_\alpha = -\phi_\alpha p_\alpha \mathbf{1} + \mathbf{T}^\star_\alpha. \tag{12}$$

There is substantial motivation to include in the momentum exchange a buoyancy force, $p_f \nabla \phi$, due to the fluid pressure acting over the interfacial surfaces (*e.g.*, Passman, *et al.*, 1984). Thus, we also define an *extra momentum exchange*, $\mathbf{m}^\star_s$, such that

$$\mathbf{m}_s = p_f \nabla \phi + \mathbf{m}^\star_s \tag{13}$$

Finally, equations (1)–(13) can be combined in the form:

$$-\frac{\partial \phi}{\partial t} + \nabla \cdot [(1 - \phi)\mathbf{v}_f] = 0, \tag{14}$$

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{v}_s) = 0, \tag{15}$$

$$\rho_f \left(\frac{\partial \mathbf{v}_f}{\partial t} + \mathbf{v}_f \cdot \nabla \mathbf{v}_f \right) = -(1-\phi)\nabla p_f + \nabla \cdot \mathbf{T}_f^* + \rho_f \mathbf{g} - \mathbf{m}_s^*, \qquad (16)$$

$$\rho_s \left(\frac{\partial \mathbf{v}_s}{\partial t} + \mathbf{v}_s \cdot \nabla \mathbf{v}_s \right) = -\phi \nabla p_f - \nabla[\phi(p_s - p_f)] + \nabla \cdot \mathbf{T}_s^* + \rho_s \mathbf{g} + \mathbf{m}_s^*, \qquad (17)$$

Note from (14) and (15) that, even though the constituents are taken to be incompressible, neither velocity field is, in general, divergence-free.

Lift Forces

Particle segregation has been observed experimentally in Poiseuille flow by a number of investigators; results have been summarized by Brenner (1966), Cox and Mason (1971), Goldsmith and Mason (1967), and Leal (1980). In general, these studies show that particles lagging the fluid motion tend to migrate toward the centerline, or region of minimum shear rate, and particles leading the fluid migrate toward the wall. Segré and Silberberg (1962) found that, for a small range of mean flow Reynolds number, neutrally buoyant particles can achieve an equilibrium position at a dimensionless radius of about 0.6.

Saffman (1956) and Bretherton (1962) have shown that a particle embedded in a steady, rectilinear Stokes flow, *i.e.* at *zero* Reynolds number, cannot experience a net force normal to the unperturbed fluid streamlines. Thus, any analysis for the cross-stream lift on a particle in a steady, rectilinear flow must take inertia into account. One approach is to introduce small inertial effects through a perturbation of the Stokes flow problem, and a number of such analyses are in the literature. Both unbounded and bounded domains have been addressed. The analyses for the former assume that the boundaries of the unperturbed flow are sufficiently far away that they do not interact with the disturbance due to the particle. Boundaries play an indirect role in this type of problem, of course, insofar as their presence may be required to establish the velocity gradient or curvature with which the particle interacts. Two well-known analyses for unbounded flows are those by Rubinow and Keller (1961) and Saffman (1968), which are summarized briefly below.

Analyses for bounded flows address configurations in which, say, a fixed wall lies within the disturbance field of the particle. Examples include the work of Ho and Leal (1974) and Vasseur and Cox (1976). It is not immediately apparent how one might adopt analyses of this type in the formulation of a continuum model. Although we have attempted previously to do so (McTigue, *et al.*, 1986) using the results of Ho and Leal, the result is not very satisfactory. The indication of difficulty is the appearance of the channel width in the expression for the lift. It would seem that this should enter through boundary conditions rather than through a constitutive equation. Obviously, this arises because the bounded-flow analyses are geometry-specific. For this reason, we consider in more detail generalizations only of lift forces in unbounded flows.

Rubinow and Keller (1961) consider a sphere spinning with angular velocity $\mathbf{\Omega}$ and translating at velocity $\mathbf{V}'$ through an incompressible viscous fluid. The fluid is assumed to be static far from the sphere. The solution takes the form of a Stokes expansion in the near field that satisfies boundary conditions at the particle, but fails far away, and an Oseen expansion in the far field that exhibits the converse behavior. An asymptotic match is performed in order to calculate the forces on the sphere. The expansions are in powers of the particle Reynolds number,

$$R_V = \frac{\gamma_f a V}{\mu}, \tag{18}$$

where a is the particle radius, V is the magnitude of the translation velocity, and μ is the fluid viscosity. The result of interest here is that for the lift force normal to the direction of translation, $\mathbf{f}_L^{(RK)}$:

$$\mathbf{f}_L^{(RK)} = \pi a^3 \gamma_f \boldsymbol{\Omega} \times \mathbf{V}'[1 + O(R_V)]. \tag{19}$$

Consider a rectilinear shearing flow, $v_{f1}(x_2)$. A force-free particle spins with the angular velocity of the fluid, so that $\Omega_3 = -\kappa/2$, where $\kappa = \partial v_{f1}/\partial x_2$ is the shear rate, and $V_1' = -V = -(v_{f1} - v_{s2})$. The "slip–spin" lift force (19) is then

$$f_{L2}^{(RK)} = \frac{1}{2}\pi a^3 \gamma_f \kappa V. \tag{20}$$

It is interesting to note that, although Rubinow and Keller's analysis is for a small inertial correction to the Stokes flow problem, the lift force is, to leading order, independent of viscosity.

Saffman (1968) considers a sphere in a simple shear flow, translating parallel to the undisturbed streamlines with a relative velocity of magnitude V'. The analysis again is based on matched asymptotic expansions. In addition to (18), two other Reynolds numbers enter the problem:

$$R_\kappa = \frac{\gamma_f a^2 \kappa}{\mu}, \quad R_\Omega = \frac{\gamma_f a \Omega}{\mu}, \tag{21}$$

and the conditions under which the analysis holds are

$$R_V \ll R_\kappa^{1/2}, \quad R_V \ll 1, \quad R_\Omega \ll 1. \tag{22}$$

For a simple shear flow given by $v_{f1} = v_0 + \kappa x_2$, Saffman obtained a "slip-shear" lift force in the form:

$$\mathbf{f}_{L2}^{(S)} = 6.46 a^2 \gamma_f^{1/2} \mu^{1/2} (\mathrm{sgn}\kappa) |\kappa|^{1/2} V. \tag{23}$$

Equation (23) indicates that a particle lagging the fluid ($V > 0$) migrates toward higher-velocity streamlines, and a particle leading the fluid ($V < 0$) migrates in the direction of decreasing fluid velocity. This qualitative behavior is in accord with experimental observations. In the rectilinear shearing flow, the ratio of the "slip–spin" lift (20) found by Rubinow and Keller to the "slip–shear" lift (23) treated by Saffman, then, scales like $R_\kappa^{1/2}$.

In a more general flow field, the applicability of the "slip-shear" analysis requires that the characteristic length scale for the disturbance field is much less than that for the variation in shear rate. Saffman (1968) suggested, from dimensional reasoning, that the lift due to interaction of the particle disturbance field with the mean-flow *curvature* takes the form

$$\mathbf{f}_{L2}^{(SC)} = ca^4\gamma_f^{2/3}\mu^{1/3}\kappa(\mathrm{sgn}\varsigma)|\varsigma|^{2/3}, \tag{24}$$

where ς is the curvature (*e.g.*, for an undisturbed mean flow $v_{f1} = v_0 + \kappa x_2 + \varsigma x_2^2/2$). Saffman noted that determination of both the sign and the magnitude of the constant c await more complete analysis. In this unbounded flow, one may define a curvature Reynolds number $R_\varsigma = \gamma_f a^3\varsigma/\mu$, in terms of which the ratio of the "shear-curvature" lift (24) to the "slip-shear" lift (23) scales like $R_\kappa^{1/2}R_\varsigma^{2/3}R_V^{-1}$. It is interesting to speculate upon the possibility that these two forces oppose one another in certain flows. Consider, for example, plane Poiseuille flow carrying a neutrally buoyant particle. The particle slips relative to the fluid due to the Faxén effect only. Thus, for symmetric flow in a channel of half-width d, the shear rate is $\kappa = -3\overline{v}x_2/d^2$, the curvature is $\varsigma = -3\overline{v}/d^2$, and the slip is $V = a^2\overline{v}/2d^2$, where $\overline{v}$ is the mean velocity. The "slip-shear" and "shear-curvature" forces are then balanced where

$$\left|\frac{x_2}{d}\right| = 0.80c^{-2}\overline{R}^{-1/3}, \tag{25}$$

where $\overline{R} = \gamma_f\overline{v}d/\mu$ is the channel Reynolds number. Segré and Silberberg (1962) observed an off-axis peak in particle concentration in flows of dilute suspensions in circular tubes. The peak occurred at a dimensionless radius of about 0.6, and was manifest in flows characterized by $\overline{R}$ of order 10. If these conditions apply to a plane geometry, the constant c would be of order 0.8. The ratio of the "slip-shear" (23) to the "shear-curvature" (24) lifts in Poiseuille flow scales like $\overline{R}^{-1/6}$. Ho and Leal (1974) also considered interaction with the mean flow curvature, but in a bounded flow. The ratio of the curvature effect discussed by Saffman in an unbounded flow (24) to that found by Ho and Leal scales like $\overline{R}^{-1/3}$.

Both Rubinow and Keller and Saffman studied small Reynolds number effects. In the other limit, Drew and Lahey (1987) have recently considered *inviscid* rotational flow past a sphere. They obtain a lift force of exactly the same form as that found by Rubinow and Keller (19), but multiplied by a factor 4/3. The same result was obtained independently by Auton (1987). This is essentially like the classical Kutta-Joukowski lift on a two-dimensional body in a plane flow, which is just $\gamma_f U\Gamma$, where U is the velocity of the body and Γ is the circulation.

Invariant Forms for the Lift Force

The momentum exchange, $\mathbf{m}_s^\star$, includes fluid-particle interaction forces such as lift and drag. For brevity, let us decompose $\mathbf{m}_s^\star$ into drag, $\mathbf{m}_D^\star$, lift, $\mathbf{m}_L^\star$, and other components:

$$\mathbf{m}_s^\star = \mathbf{m}_D^\star + \mathbf{m}_L^\star + \ldots. \tag{26}$$

It has been suggested previously (Drew, 1976; McTigue, *et al.*, 1986; Passman, 1986) that the lift might include terms of the form

$$\mathbf{m}_L^\star = 2\alpha_2\phi\mathbf{D}_f \cdot (\mathbf{v}_f - \mathbf{v}_s) + 4\beta_2\phi\mathbf{D}_f \cdot (\nabla \cdot \mathbf{D}_f), \tag{27}$$

where $\mathbf{D}_\alpha = \mathrm{sym}\nabla\mathbf{v}_\alpha$. It is expected that α_2 and β_2 may be a functions of the particle volume fraction, ϕ, the relative speed, $|\mathbf{v}_f - \mathbf{v}_s|$, and the invariants of $\mathbf{D}_f$, $\mathbf{D}_s$, and their higher-order derivatives. In particular, if we assume that, for dilute suspensions, we should recover the single-particle results discussed in the foregoing section,[2] this function can be made explicit. For example, Saffman's result for the "slip-shear" lift (23) is recovered for the choice

$$\alpha_2 = \frac{3(6.46)}{4\pi a}\left(\frac{\gamma_f^2\mu^2}{2\mathrm{tr}\mathbf{D}_f^2}\right)^{1/4}. \tag{28}$$

The "shear-curvature" lift (24) is recovered for the choice

$$\beta_2 = \frac{3ca\gamma_f^{2/3}\mu^{1/3}}{4\pi|2\nabla \cdot \mathbf{D}_f|^{1/3}}. \tag{29}$$

Generalization of a "slip–spin" lift of the form found by Rubinow and Keller (20) poses some difficulty. It would appear that such a lift is proportional to $2\mathbf{W}_f \cdot (\mathbf{v}_f - \mathbf{v}_s)$, where $\mathbf{W}_f = \mathrm{skw}\nabla\mathbf{v}_f$ is the skew-symmetric part of the fluid velocity gradient. However, $\mathbf{W}_f$ is not invariant (*e.g.*, Truesdell, 1977, p. 115). Drew and Lahey (1987) have suggested that this dilemma can be resolved by simultaneous consideration of the virtual mass effect. The virtual mass, too, when generalized from the classical expression, is not easily put into an invariant form. However, the *combination* of the virtual mass and lift forces posed by Drew and Lahey is invariant:

$$\mathbf{m}_{VM}^\star + \mathbf{m}_L^\star = \frac{1}{2}\gamma_f\phi\left[\left(\frac{D_f\mathbf{v}_f}{Dt} - \frac{D_s\mathbf{v}_s}{Dt}\right) - 2\mathbf{W}_f \cdot (\mathbf{v}_f - \mathbf{v}_s)\right], \tag{30}$$

where $\mathbf{m}_{VM}^\star$ is the momentum exchange due to the virtual mass effect, and the substantial derivative is defined by $D_\alpha/Dt \equiv \partial/\partial t + \mathbf{v}_\alpha \cdot \nabla$. In (30), neither the virtual mass, represented by the difference in convective accelerations, nor the lift, in the form $2\mathbf{W}_f \cdot (\mathbf{v}_f - \mathbf{v}_s)$, is invariant, while their sum is. This depends upon the remarkable result that the coefficient $\gamma_f/2$ is the same for both the virtual mass and the lift. That (30) embodies the classical virtual mass effect is easily seen by specializing to an unsteady, uniform flow. That it embodies the result of Drew and Lahey for the lift can be demonstrated by specializing to steady, rectilinear shearing flow. Drew and Lahey point out that a simple regrouping of terms can yield an invariant form for the virtual mass:

$$\mathbf{m}_{VM}^\star = \frac{1}{2}\gamma_f\phi\left[\left(\frac{D_s\mathbf{v}_f}{Dt} - \frac{D_f\mathbf{v}_s}{Dt}\right) - (\mathbf{v}_f - \mathbf{v}_s) \cdot \nabla(\mathbf{v}_f - \mathbf{v}_s)\right], \tag{31}$$

and a lift in the form of the first term in (27) with $\alpha_2 = \gamma_f/2$. Equations (31) and the lift sum to recover (30).

[2]This assumption was stated by Drew (1976) as the *principle of correct low concentration limits.*

Turbulent Decomposition and Averaging

It is evident from the foregoing discussion concerning lift forces that the formulation of the necessary constitutive models necessary to complete the equations of motion (14–17) is quite formidable. It remains to specify relationships for the stresses, $\mathbf{T}_f^*$ and $\mathbf{T}_s^*$, the pressure difference, $p_s - p_f$, and momentum exchanges such as that due to drag, $\mathbf{m}_D^*$. Each of these raises subtle and complex modeling issues. For present purposes, we skirt these difficulties in order to isolate phenomena associated with the lift and drag. In particular, we assume

$$\mathbf{T}_f^* = 0, \tag{32}$$

$$\mathbf{T}_s^* = 0, \tag{33}$$

$$p_f = p_s = p, \tag{34}$$

$$\mathbf{m}_D^* = \alpha_1 \phi (\mathbf{v}_f - \mathbf{v}_s), \tag{35}$$

$$\mathbf{m}_L^* = 2\alpha_2 \phi \mathbf{D}_f \cdot (\mathbf{v}_f - \mathbf{v}_s). \tag{36}$$

We rationalize neglect of the fluid extra stress, $\mathbf{T}_f^*$ (32), by confining attention to inertially-dominated flows.[3] In a dilute suspension, it is easy to imagine that the disperse-phase extra stress, $\mathbf{T}_s^*$, vanishes (33), implying that there is no direct exchange of momentum between particles. The assumption of equal pressures (34) implies that Brownian motion (Nunziato, 1983) and certain inertial effects at the particle scale (Givler, 1987) are negligible. The drag force, $\mathbf{m}_D^*$, is written in its familiar form (35), proportional to the relative velocity. The coefficient α_1 is, in general, expected to depend upon ϕ and $|\mathbf{v}_f - \mathbf{v}_s|$, accounting for the effects of particle interference at high concentration and inertia at high relative velocity, respectively. The choice $\alpha_1 = 9\mu/2a^2$ corresponds to the classical result for Stokes drag on a single particle, and is adopted here. Finally, we take the lift in the form of (36), with α_2 assumed to be constant for simplicity. Neglect of inertial effects in the drag (35) while retaining those giving rise to the lift (36) is justified if $R_V R_\kappa^{-n} \ll 1$, where $n = 1$ for the "slip-spin" lift of Rubinow and Keller (20) and $n = 1/2$ for the "slip-shear" lift of Saffman (23).

Under these assumptions, the momentum equations (16–17) reduce to:

$$\begin{aligned} \rho_f \left(\frac{\partial \mathbf{v}_f}{\partial t} + \mathbf{v}_f \cdot \nabla \mathbf{v}_f \right) = \; & -(1-\phi)\nabla p + \rho_f \mathbf{g} \\ & -\alpha_1 \phi (\mathbf{v}_f - \mathbf{v}_s) - 2\alpha_2 \phi \mathbf{D}_f \cdot (\mathbf{v}_f - \mathbf{v}_s), \end{aligned} \tag{37}$$

$$\begin{aligned} \rho_s \left(\frac{\partial \mathbf{v}_s}{\partial t} + \mathbf{v}_s \cdot \nabla \mathbf{v}_s \right) = \; & -\phi \nabla p + \rho_s \mathbf{g} \\ & +\alpha_1 \phi (\mathbf{v}_f - \mathbf{v}_s) + 2\alpha_2 \phi \mathbf{D}_f \cdot (\mathbf{v}_f - \mathbf{v}_s), \end{aligned} \tag{38}$$

[3] The dissipative, viscous terms represented by $\mathbf{T}_f^*$ are critical, of course, to the extension of this discussion to the kinetic energy balances.

Note that these "exact" equations of motion do not contain interaction terms representing diffusive forces.

The independent field variables in (37) and (38) are each decomposed according to:

$$\mathbf{v}_a = \overline{\mathbf{v}}_a + \mathbf{v}'_a, \tag{39}$$

$$\phi = \overline{\phi} + \phi', \tag{40}$$

$$p = \overline{p} + p', \tag{41}$$

where overbars indicate mean quantities and primes indicate fluctuating quantities. By definition, $\overline{\phi'} = 0$ and $\overline{p'} = 0$. However, the averaging scheme chosen here defines the mean velocities in terms of mean *momenta*, an approach introduced originally for compressible, single-phase flows (Favre, 1965), and suggested in the multiphase context by Drew (1975):

$$\overline{\rho}_\alpha \overline{\mathbf{v}}_\alpha = \overline{\rho_\alpha \mathbf{v}_\alpha}. \tag{42}$$

Note that $\overline{\rho}_a = \gamma_a \overline{\phi}_a$ for incompressible constituents. Note also that the averages of the velocity fluctuations do not vanish, but $\overline{(1-\phi)\mathbf{v}'_f} = 0$ and $\overline{\phi \mathbf{v}'_s} = 0$.

Substitution of (39)–(41) into (14), (15), (37), and (38) and averaging yields:

$$-\frac{\partial \overline{\phi}}{\partial t} + \nabla \cdot [(1-\overline{\phi})\overline{\mathbf{v}}_f] = 0, \tag{43}$$

$$\frac{\partial \overline{\phi}}{\partial t} + \nabla \cdot (\overline{\phi}\overline{\mathbf{v}}_s) = 0, \tag{44}$$

$$\begin{aligned} \overline{\rho}_f \left(\frac{\partial \overline{\mathbf{v}}_f}{\partial t} + \overline{\mathbf{v}}_f \cdot \nabla \overline{\mathbf{v}}_f \right) = {} & -(1-\overline{\phi})\nabla \overline{p} + \overline{\phi' \nabla p'} + \nabla \cdot \mathbf{T}'_f + \overline{\rho}_f \mathbf{g} \\ & -\alpha_1 \left[\overline{\phi}(\overline{\mathbf{v}}_f - \overline{\mathbf{v}}_s) + \overline{\phi \mathbf{v}'_f} \right] \\ & -2\alpha_2 \left[\overline{\phi}\, \overline{\mathbf{D}}_f \cdot (\overline{\mathbf{v}}_f - \overline{\mathbf{v}}_s) + \overline{\mathbf{D}}_f \cdot \overline{\phi \mathbf{v}'_f} + \overline{\phi \mathbf{D}'_f} \cdot (\overline{\mathbf{v}}_f - \overline{\mathbf{v}}_s) \right] \end{aligned} \tag{45}$$

$$\begin{aligned} \overline{\rho}_s \left(\frac{\partial \overline{\mathbf{v}}_s}{\partial t} + \overline{\mathbf{v}}_s \cdot \nabla \overline{\mathbf{v}}_s \right) = {} & -\overline{\phi}\nabla \overline{p} - \overline{\phi' \nabla p'} + \nabla \cdot \mathbf{T}'_s + \overline{\rho}_s \mathbf{g} \\ & +\alpha_1 \left[\overline{\phi}(\overline{\mathbf{v}}_f - \overline{\mathbf{v}}_s) + \overline{\phi \mathbf{v}'_f} \right] \\ & +2\alpha_2 \left[\overline{\phi}\, \overline{\mathbf{D}}_f \cdot (\overline{\mathbf{v}}_f - \overline{\mathbf{v}}_s) + \overline{\mathbf{D}}_f \cdot \overline{\phi \mathbf{v}'_f} + \overline{\phi \mathbf{D}'_f} \cdot (\overline{\mathbf{v}}_f - \overline{\mathbf{v}}_s) \right] \end{aligned} \tag{46}$$

where $\mathbf{T}'_\alpha = -\overline{\rho_\alpha \mathbf{v}'_\alpha \mathbf{v}'_\alpha}$ is a Reynolds stress for constituent α, and triple correlations have been omitted.

Constitutive Models for Turbulent Correlations

The averaged mass balances (43–44) appear in forms identical to the exact equations (14–15), in part as a consequence of the definition of average velocity (42). Averaging the momentum balances, however, yields a number of correlations of fluctuating quantities. The Reynolds stress terms are familiar from single-phase turbulence, but several additional correlations arise here that are a direct consequence of the fluctuations in the particle concentration field, ϕ. Of special note is the correlation of particle concentration and fluid velocity fluctuations, $\overline{\phi \mathbf{v}'_f}$, which represents a flux of particles due to the fluid turbulence. In all subsequent developments, we neglect the pressure–concentration correlations appearing in (45) and (46).

For the present discussion, it suffices to adopt the simplest possible closure scheme, following essentially the classical "eddy viscosity" argument. That is, a correlation of some fluctuating quantity with $\mathbf{v}'_\alpha$ is taken to be proportional to the gradient of the mean of that quantity:

$$-\overline{\rho_\alpha \mathbf{v}'_\alpha \mathbf{v}'_\alpha} = u_\alpha l_{1\alpha}[\nabla \cdot (\overline{\rho}_\alpha \overline{\mathbf{v}}_\alpha)]\mathbf{1} + 2u_\alpha l_{2\alpha} \mathrm{sym} \nabla (\overline{\rho}_\alpha \overline{\mathbf{v}}_\alpha), \tag{47}$$

$$\overline{\phi \mathbf{v}'_f} = -u_f l_3 \nabla \overline{\phi}, \tag{48}$$

$$2\overline{\phi \mathbf{D}'_f} = -u_f l_4 \nabla(\nabla \overline{\phi}), \tag{49}$$

where u_α is an appropriate velocity scale for constituent α, and the ls are appropriate length scales ("mixing lengths").

Turbulent Convection and Dispersion of Particles

A commonly encountered situation for which modeling capabilities are well developed is that for particles fully entrained in the fluid. In this case, the particles are essentially "passive" tracers for the fluid, and are transported by the mean convective motion and by turbulent diffusion. It is worth considering briefly where this classical model is embedded in the mixture theory outlined here.

For $\overline{\phi} \ll 1$, the fluid mass and momentum balances (43 and 45) are approximately those for the fluid alone:

$$\nabla \cdot \overline{\mathbf{v}}_f = 0, \tag{50}$$

$$\overline{\gamma}_f \left(\frac{\partial \overline{\mathbf{v}}_f}{\partial t} + \overline{\mathbf{v}}_f \cdot \nabla \overline{\mathbf{v}}_f \right) = -\nabla \overline{p} + \nabla \cdot \mathbf{T}'_f + \gamma_f \mathbf{g}. \tag{51}$$

In this approximation, the fluid motion is unaffected by the presence of the particles, and can be solved independently. The disperse-phase mass balance (44) remains in its exact form. Suppose the drag coefficient, α_1, is large, so that the dominant terms in (46) are simply those due to drag. This can always be realized for sufficiently small particles; the ratio of the drag to lift forces discussed in the foregoing scales at least like a^{-1}. The disperse-phase momentum balance then reduces to:

$$\bar{\phi}(\bar{\mathbf{v}}_f - \bar{\mathbf{v}}_s) = -\overline{\phi \mathbf{v}'_f}, \tag{52}$$

i.e., the mean flux of particles relative to the fluid is balanced by the turbulent correlation $\overline{\phi \mathbf{v}'_f}$.

Equations (44), (50), and (52) combine to give:

$$\frac{\partial \bar{\phi}}{\partial t} + \bar{\mathbf{v}}_f \cdot \nabla \bar{\phi} = -\nabla \cdot (\overline{\phi \mathbf{v}'_f}). \tag{53}$$

Substitution of (48) into (53) yields:

$$\frac{\partial \bar{\phi}}{\partial t} + \bar{\mathbf{v}}_f \cdot \nabla \bar{\phi} = \nabla \cdot (\mathcal{D} \nabla \bar{\phi}), \tag{54}$$

where $\mathcal{D} = u_f l_3$. This recovers the classical result: the particle concentration field is governed by a convection–dispersion equation, with turbulent dispersion coefficient or "eddy diffusivity" $\mathcal{D}$. A similar discussion for the case when the gravitational body force is retained was presented by McTigue (1981, 1983).

Although this limiting case is relatively simple and quite well known, the present development is revealing. Many texts derive the turbulent diffusion equation solely from a statement of mass balance, a decomposition and averaging process, and a model for the correlation of concentration and velocity fluctuations. This tends to mask the fact that the turbulent diffusion is a *dynamic* process in response to fluid-particle interactions. Thus, the momentum equations must be considered. Indeed, it is worth reiterating here that the turbulent dispersive flux, $\overline{\phi \mathbf{v}'_f}$, appearing in in (53) arose from decomposing and averaging the drag force (35). Thus, the tendency for the particles to be convected with the mean fluid velocity and dispersed by the fluid velocity fluctuations is clearly identified with the drag. Analogous observations have been made previously with regard to molecular diffusion (*e.g.*, Müller, 1968).

We also note that the assumptions leading to (54) are quite special, and emphasize in particular the neglect of any interaction forces other than drag in writing (52). It is evident that much more complex phenomenology could be embedded in the general scheme outlined here if additional interaction forces come into play.

Channel Flow With Lift Effects

Particle segregation has been observed in turbulent jets (Laats and Frishman, 1970), and ascribed to the "Magnus" lift force (30). Here we consider plane channel flow in order to simplify the kinematics, and retain the cross-stream lift effects embodied in (46). The analysis is highly simplified and somewhat speculative, and is intended only to illustrate the type of phenomena that might be represented by a mixture model of the type sketched out here.

Consider a vertical channel, with steady upward flow. The flow is in the $+x_1$ direction, so that $\mathbf{g} = \{-g,\, 0,\, 0\}$, $x_2 = 0$ at the wall, and $x_2 = h$ at the centerline. We assume that $\gamma_s > \gamma_f$, so that gravitational settling will cause the disperse particulate phase to lag the fluid.

For a steady, rectilinear flow in the mean, the mass balances (43, 44) are identically satisfied. We expect again that, for $\overline{\phi} \ll 1$, the fluid momentum balance can be approximated by that for the fluid alone (51). Thus, in the streamwise direction, (51) becomes:

$$0 = -\frac{d\overline{p}}{dx_1} + \frac{d}{dx_2} T'_{f21} - \gamma_f g. \tag{55}$$

For a smooth-walled channel, familiar arguments for the mixing length l_{2f} (47) and the identity $u_f = u_* = (\tau_0/\gamma_f)^{1/2}$, where τ_0 is the shear stress at the wall, lead to the usual logarithmic velocity profile:

$$\frac{\overline{v}_{f1}}{u_*} = \frac{1}{\kappa} \ln \frac{u_* x_2}{\nu} + 5.5, \tag{56}$$

where $\kappa \simeq 0.4$ is the Kármán constant, and $\nu = \mu/\gamma_f$ is the kinematic viscosity. The streamwise momentum balance for the particles, from (46) and (48), and neglecting T'_{s21}, becomes:

$$0 = -\overline{\phi} \frac{d\overline{p}}{dx_1} - \gamma_s \overline{\phi} g + \alpha_1 \overline{\phi} (\overline{v}_{f1} - \overline{v}_{s1}) - \alpha_2 u_* \kappa_s x_2 \frac{d\overline{v}_{f1}}{dx_2} \frac{d\overline{\phi}}{dx_2}, \tag{57}$$

where we have assumed $u_f l_3 = u_* \kappa_s x_2$. Substituting from (55) for the mean pressure gradient in (57), and noting that the fluid shear stress gradient is simply $-\gamma_f u_*^2/h$, (57) becomes:

$$0 = \frac{\gamma_f u_*^2}{h} - (\gamma_s - \gamma_f) g + \alpha_1 (\overline{v}_{f1} - \overline{v}_{s1}) - \alpha_2 u_* \kappa_s x_2 \frac{d\overline{v}_{f1}}{dx_2} \left(\frac{1}{\overline{\phi}} \frac{d\overline{\phi}}{dx_2} \right). \tag{58}$$

Let us suppose, again for simplicity, that the second and third terms in (58) dominate. In this case, we are left with a balance between the buoyant weight and the drag, giving

$$\overline{v}_{f1} - \overline{v}_{s1} = V_\infty, \tag{59}$$

where $V_\infty = 2a^2(\gamma_s - \gamma_f) g / 9\mu$ is the Stokes settling velocity.

The cross-stream momentum balance for the disperse phase is

$$0 = -\alpha_1 u_* \kappa_s x_2 \left(\frac{1}{\overline{\phi}} \frac{d\overline{\phi}}{dx_2} \right) + \alpha_2 \frac{d\overline{v}_{f1}}{dx_2} (\overline{v}_{f1} - \overline{v}_{s1}), \tag{60}$$

which is simply a balance between turbulent diffusion and the lift due to the mean flow. Note from (60) that the gradient of $\overline{\phi}$ vanishes at the centerline if the fluid velocity gradient vanishes there. According to the eddy viscosity model adopted for the Reynolds stress (47), the latter is in fact required by symmetry. However, of course, the logarithmic velocity profile (56) does not satisfy this condition. Therefore, we can anticipate a similar failing in the solution for $\overline{\phi}$. Substitution of (56) and (59) into (60) and integration gives

$$\overline{\phi} = \overline{\phi}(h) \exp\left[-\frac{\alpha_2 V_\infty u_\star}{\alpha_1 \kappa \mathcal{D}_h} \left(\frac{1-\eta}{\eta} \right) \right], \tag{61}$$

where $\eta = x_2/h$, and a diffusivity, $\mathcal{D}_h = u_\star \kappa_s h$ has been introduced. This profile has some of the expected characteristics: the particles are concentrated toward the center of the channel by the lift; the central peak is flattened by diffusion; and the channel margins, where the fluid velocity gradient is steepest, can be essentially clear of particles. That (61) indicates $\overline{\phi}(0) = 0$ is a result of using the logarithmic fluid velocity profile (56), which is not valid in the limit $x_2 \to 0$, in (60).

Lee and Durst (1982) conducted experiments in this configuration using glass beads in an air stream. Some of their results are in qualitative agreement with those found here: the air velocity profile (56) is little affected by the presence of the particles (at less than 0.5% mean volume fraction); the particles lag the fluid approximately by their fall velocity (59); and there is a particle-free zone near the wall. However, important phenomena are missed by this simple analysis. In particular, Lee and Durst observed that the velocity difference (59) is not uniform across the channel, but typically decreases toward zero near the wall. The particle velocity profiles, then, are more nearly uniform across the channel, suggesting that the turbulent mixing brings high-momentum particles from the core of the flow toward the boundary. For the smaller particles examined, the profiles actually cross near the wall; *i.e.*, the particles *lead* the fluid, so that the momentum exchange due to drag (35) changes sign. These phenomena are clearly not embodied in the model analysis outlined here. The limitation is most likely in the simple, Boussinesq closure scheme adopted (47–49). Kashiwa (1987) has modeled these experiments using a higher-order ($k - \epsilon$) closure, and is able to represent the cross-stream transport of streamwise particle momentum into the near-wall region.

Summary and Discussion

Treatment of turbulent suspensions in the context of the continuum theory of mixtures is currently in its most rudimentary stages. The appeal of the overall approach is that it provides an axiomatic framework on which to build. In practice, of course, one is quickly confronted with the difficulty of posing specific constitutive equations for the stresses and momentum exchange that embody the phenomena of interest. This is only compounded in the case of turbulent mixtures, in which correlations between the three kinematic fields, $\mathbf{v}_f$, $\mathbf{v}_s$, and ϕ proliferate. The intent of this paper is not to lay out a definitive set of equations of motion for such a system. Rather, we have attempted only to sketch the general spirit of the approach, and to illustrate by means of the simplest possible example. The sequence is familiar from its antecedents in classical, single-fluid flow: state balance laws, pose constitutive equations, construct "exact" equations of motion, introduce a decomposition and averaging scheme, model the resulting correlations, and, finally, solve boundary value problems. The channel flow problem considered here, exhibiting particle segregation effects, only hints at the rich and complex phenomenology that could be embedded in such a model.

Each section of the paper encounters challenges. Exact forms for lift forces, even from single-particle analyses, are not well established; those that are known are complex; and their generalizations are not immediately obvious. We emphasize in particular that *bounded* flows have been analyzed (*e.g.*, Vasseur and Cox, 1976) in which wall effects are critical, and it is not clear how one might adopt such results in a continuum model. Many of these remarks carry over to other interaction forces, as well, such as the "Basset" term (*e.g.*, Hinze, 1975, p. 463), which accounts for the history of the particle acceleration. Because no universally valid expressions for lift, drag, or other forces are available, considerable judgement is required in selecting the forms appropriate to a particular application. Constitutive equations for any concentration beyond the dilute limit are especially difficult to define; few analytical results are available (*e.g.*, Batchelor's (1972) work on "hindered settling") and resort is usually made to empiricism.

Perhaps the greatest challenge encountered in constructing a model for turbulent mixtures is the "closure" problem, familiar from single-phase turbulence, but magnified here by the presence of additional fluctuating fields. As in single-phase problems, some simple configurations can be addressed through classical Boussinesq models (*e.g.*, 47–49) and simple scaling arguments. However, it is also clear, even from the highly idealized channel flow problem addressed here, that such an approach is severely limited. For example, the Boussinesq model for the Reynolds stresses (47) does not embody normal stress effects in rectilinear flows, while one might easily imagine that such effects could be important. Higher-order closure schemes are obviously called for, and steps in this direction have been taken with some success. Scheiwiller (1986) has developed a $k - \epsilon$ model to represent snow avalanches, and has achieved excellent agreement with laboratory experiments. Kashiwa (1987) has used a similar approach, and successfully captures some of the unusual phenomena observed by Lee and Durst (1982) in the vertical channel flow discussed in the foregoing section.

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