Singular Perturbations and Time Scales in the Design of Digital Flight Control Systems

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Summary

This paper presents the results of applying the methodology of singular perturbations and time scales (SPATS) to the control of digital flight systems. A block-diagonalization method is described that decouples a full-order, two-time-scale (slow and fast) discrete control system into reduced-order, slow and fast subsystems. Basic properties and numerical aspects of the method are discussed. This study reveals an interesting fact that singularly perturbed discrete systems can be viewed as two-time-scale systems. The closed-loop optimal control of the full-order, two-time-scale system involves the solution of a full-order, algebraic, matrix Riccati equation. Alternatively, by using the block-diagonalization method described in this paper, the full-order system is decomposed into reduced-order, slow and fast subsystems. The closed-loop optimal control of the subsystems requires the solution of only reduced-order, algebraic, matrix Riccati equations. A composite, closed-loop, suboptimal control system is constructed as the sum of the slow and fast, optimal feedback controls. The application of this method to an aircraft model shows close agreement between the exact solution and the decoupled (composite) solution. The main advantage of the method is the considerable reduction in the overall computational requirements for the evaluation of optimal guidance and control laws. The significance of the result is that it can be used for real-time onboard simulation. This paper also contains a brief survey of digital flight systems.

1. Introduction

The dynamics of many systems is described by high-order differential equations. Frequently, the presence of small parameters such as time constants, masses, and moments of inertia is the source for the increased order of the system. A system in which the suppression of a small parameter is responsible for the degeneration of the dimension of the system is called a singularly perturbed system. Such a system possesses widely separated clusters of eigenvalues exhibiting "slow" and "fast" phenomena. The high dimensionality coupled with the two-time-scale (slow and fast) behavior makes the system computationally "stiff," with the result that extensive numerical routines are required.

The theory of singular perturbations and time scales (SPATS) in continuous control systems has reached a level of maturity (refs. 1 to 3) and has been successfully applied to aerospace problems (refs. 4 to 15). On the other hand, the subject of SPATS in digital flight control systems has not received much attention so far.

The purpose of this paper is to present a methodology of SPATS to discrete control systems with an application to an aircraft model. The basic idea underlying this method is to apply a block-diagonalization procedure to decouple a full-order, two-time-scale system into low-order, slow and fast subsystems. This procedure offers a considerable reduction in the overall computational requirements for developing optimal guidance and control laws. The significance of the result is that it has possible applications for real-time onboard implementation. Also, this paper contains a brief survey of digital flight systems. The results presented in this paper are an expanded version of those recently published by the authors in references 16 to 19.

The organization of the paper is as follows. Section 2 describes the concept of digital flight control and the previous research in this field. The ideas of SPATS in continuous and discrete control systems are given briefly in section 3. Section 4 presents a block-diagonalization procedure to decouple a high-order, two-time-scale system into low-order, slow and fast subsystems. An aircraft example is provided to illustrate the method. In section 5, the focus is on the time-scale synthesis of digital, optimal control systems. The closed-loop, linear, optimal regulation of the original two-time-scale system requires the solution of a high-order, algebraic, matrix Riccati equation. Alternatively, a composite, closed-loop, suboptimal control is constructed as the sum of slow and fast, optimal feedback controls, which require the solution of only low-order, algebraic, matrix Riccati equations. Numerical results obtained for an aircraft example show excellent agreement between the original (optimal) solution and the composite (suboptimal) solution which is computationally simpler.

Symbols

$\mathbf{A}$	system matrix
$\mathbf{A}_{ij}$	subsystem matrices $(i, j = 1, 2)$

B	input matrix
$\mathbf{B}_i$	input matrix for subsystem $(i = 1, 2)$
D	matrix of order $(m \times n)$
E	matrix of order $(n \times m)$
$e(\ )$	eigenvalue of $(\ )$
F	feedback matrix
I	identity matrix
J	performance index
k	discrete interval
m	order of state, **z**
n	order of state, **x**
P	positive, definite symmetric matrix of Riccati equation
p	eigenvalue
Q	positive, semidefinite symmetric matrix of order $(n{+}m){\times}(n{+}m)$
R	positive, definite symmetric matrix of order $(r \times r)$
r	order of control, **u**
t	time
u	control vector of order r
x	state vector of order n
y	output vector
z	state vector of order m
ε	small, positive perturbation parameter

Subscripts:

c	composite mode
f	fast mode
g	correction mode
s	slow mode
0	zero order

Superscripts:

T	transpose
(0)	zero-order solution

Abbreviations:

ADC	analog-to-digital converter
DAC	digital-to-analog converter
max	maximum
min	minimum

A dot over a symbol indicates a derivative with respect to time.

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2. Background

In this section, the idea of digital flight control systems is discussed briefly and their advantages are enumerated. The previous research in digital flight control is summarized.

A digital control system uses a digital computer to implement the control logic. The development of reliable, faster, and inexpensive microcomputers has recently aroused considerable interest in digital control systems (refs. 20 to 24). The first commercial digital computers in the 1950's were not fast enough or small enough to be placed on space vehicles. The NASA Apollo program spurred the development of smaller and faster computers for the digital control of boosters and spacecraft in the 1960's. This technology was transferred to aircraft by NASA in the 1970's. The invention of microprocessors in the 1970's made digital computers very small, fast, and inexpensive. Many military and civilian aircraft now have digital control systems (ref. 25). Digital control, which has made possible more accurate and sophisticated autopilot logic, promises to be the focal point in most improvements in navigation, guidance, and control in the future. A digital flight control system contains analog-to-digital and digital-to-analog converters (ADC and DAC, respectively) as shown in figure 1.
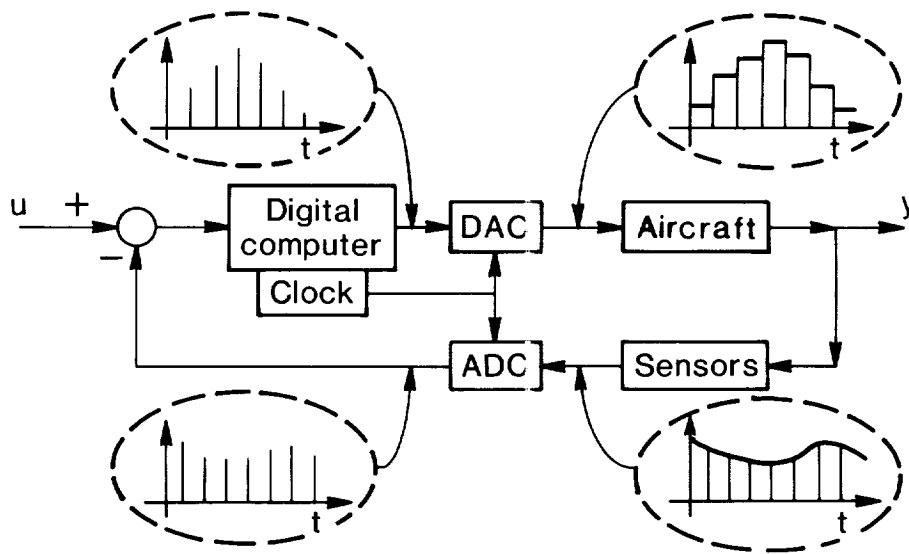

Figure 1. Concept of digital flight control.

With adequate redundancy incorporated into the design, digital flight control systems ensure adequate flight safety. The present capabilities for incorporating integrated circuits into lightweight, low-cost minicomputers and microcomputers make digital implementation of modern flight control systems especially attractive (refs. 25 to 28).

Another feature of digital implementation is the potential for the synthesis of complex control systems that involve high-order nonlinearities and that utilize time-sharing for multiple-loop control. One such complex control structure is an adaptive system that is capable of online adjustment of the control parameters in response to changing flight characteristics (ref. 29).

The advantages of digital flight control systems over their continuous counterparts are summarized (ref. 30) as follows:

(1) Use of complex control strategies
(2) Easy implementability
(3) Flexibility in adding or changing functions
(4) Repeatability of performance
(5) System integration and hardware economy
(6) Reduction in cost, size, weight, and power dissipation

In the past, there have been many applications of digital control theory for flight systems. The first application of digital technology to flight control was digital implementation of basic analog autopilot functions (refs. 31 and 32). Modal control theory has been applied to the design of digital flight control

systems such as pitch-attitude control systems and roll/yaw control systems for a short takeoff and landing (STOL) aircraft (refs. 33 to 36). An algorithm was presented for obtaining a reduced-order or simplified nonlinear model of the F-8 aircraft (ref. 37). A method for the synthesis of a nonlinear, automatic, flight control system involving an F-8 aircraft has been developed (refs. 35 and 38). A residue-measure criterion has been employed as an efficient method for model reduction in the design and analysis of the digital flight control system of NASA's Space Shuttle (ref. 39). Discrete design methods have been compared at various sample rates for several different flight modes for the terminal configured vehicle (TCV) at the Langley Research Center, which is a modified Boeing 737 aircraft (ref. 40). Research has been conducted by NASA into some advanced control laws for the F-8 digital fly-by-wire (DFBW) program (refs. 29 and 41).

The analysis and control of large systems has been a formidable task, not only because of the high order of the systems but also because the majority of these systems possess interacting phenomena of different (slow and fast) speeds. The simultaneous presence of slow and fast dynamics makes the system computationally "stiff," with the result that extensive numerical routines are required. A stiff system, having a two-time-scale character, need not necessarily be in the singularly perturbed structure.

3. Singular Perturbations and Time Scales

In this section, the definitions of singular perturbations and time scales in continuous and discrete control systems are given briefly. Broadly speaking, a continuous (discrete) system described by a differential (difference) equation containing a small parameter is called a singularly perturbed system if the order of the system is reduced by neglecting the small parameter. Such a system possesses widely separated clusters of eigenvalues and exhibits a two-time-scale (slow and fast) character.

Continuous Control Systems

Consider briefly the idea of SPATS in continuous control systems and its application to aerospace problems. Consider a linear time-invariant system such as

$$\dot{\mathbf{x}} = \mathbf{A}_{11}\mathbf{x} + \mathbf{A}_{12}\mathbf{z} + \mathbf{B}_1\mathbf{u} \qquad (\mathbf{x}(t = 0) = \mathbf{x}(0)) \tag{1a}$$

$$\varepsilon\dot{\mathbf{z}} = \mathbf{A}_{21}\mathbf{x} + \mathbf{A}_{22}\mathbf{z} + \mathbf{B}_2\mathbf{u} \qquad (\mathbf{z}(t = 0) = \mathbf{z}(0)) \tag{1b}$$

where $\mathbf{x}$ and $\mathbf{z}$ are n- and m-dimensional state vectors, respectively, and $\mathbf{u}$ is an r-dimensional control vector. The matrices $\mathbf{A}_{ij}$ and $\mathbf{B}_i$ are of appropriate dimensions, and ε is a small, positive perturbation parameter. The $(n + m)$ high-order system (eqs. (1)) is in the singularly perturbed form in the sense that by making $\varepsilon = 0$ in equations (1), the resulting system is of reduced order n only, and consequently the degenerate system does not satisfy the initial condition $\mathbf{z}(0)$. That is,

$$\dot{\mathbf{x}} = \mathbf{A}_{11}\mathbf{x}^{(0)} + \mathbf{A}_{12}\mathbf{z}^{(0)} + \mathbf{B}_1\mathbf{u} \qquad \mathbf{x}^{(0)}(t = 0) = \mathbf{x}(0) \tag{2a}$$

$$0 = \mathbf{A}_{21}\mathbf{x}^{(0)} + \mathbf{A}_{22}\mathbf{z}^{(0)} + \mathbf{B}_2\mathbf{u} \qquad \left(\mathbf{z}^{(0)}(t = 0) \neq \mathbf{z}(0)\right) \tag{2b}$$

Here, let us note that equation (2a) is a reduced-order system of order n only, with equation (2b) being an algebraic equation, and the initial condition $\mathbf{z}(0)$ is lost in the process of degeneration. The singular perturbation theory retrieves the lost initial condition. The block diagrams of the full, high-order, or perturbed system (eqs. (1)) and the degenerate, low-order, or unperturbed system (eqs. (2)) are shown in figure 2. Here we assume that the input $\mathbf{u}$ is independent of ε. Otherwise, $\mathbf{u}$ becomes $\mathbf{u}^{(0)}$. Also, $\mathbf{x}$ and $\mathbf{z}$ are predominantly slow and fast state vectors, respectively. In a general two-time-scale (slow and fast) system, the singular perturbation parameter ε does not appear explicitly as shown in equations (1). The essence of the theory of SPATS is to take advantage of the order reduction associated with the degeneration and to decouple the original system (eqs. (1)) into slow and fast subsystems of reduced order for further analysis and synthesis.
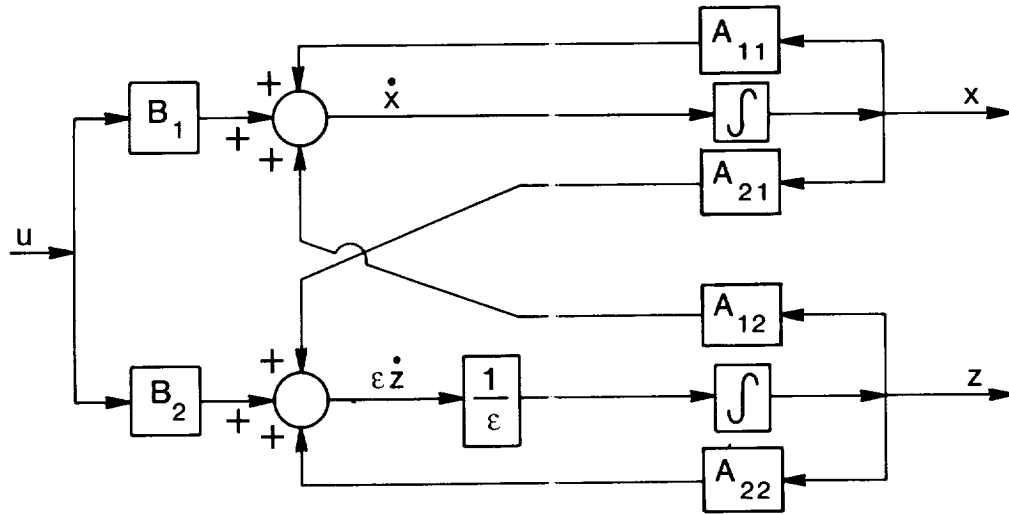
(a) Original system.
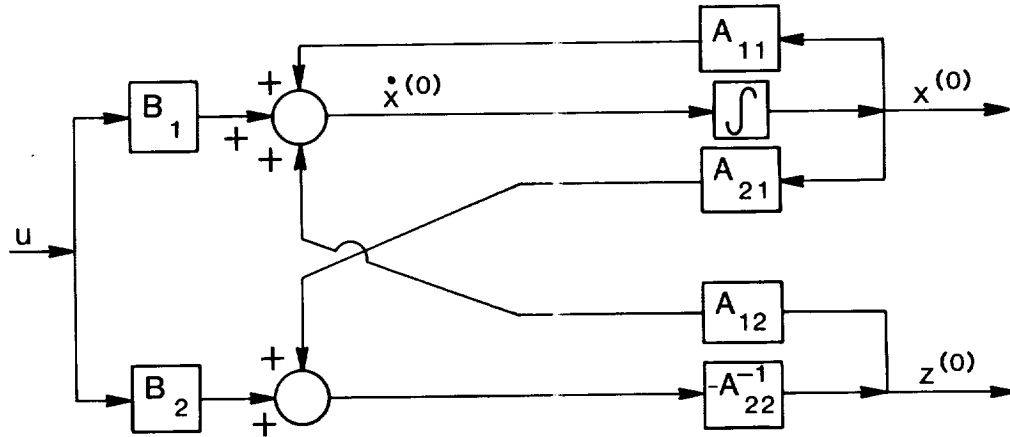
(b) Degenerate system.

Figure 2. Singularly perturbed continuous system.

The theory of SPATS has been successfully applied to aerospace problems described by continuous control systems (refs. 4 to 15). The digital flight control systems with singular perturbation (or two-time-scale) character have not received much attention so far.

Discrete Control Systems

For the purpose of synthesizing digital controllers, the continuous model (described by differential equations) of a dynamic system is transferred into a discrete model (described by difference equations) where the controls are held constant between the sampling intervals (refs. 34, 42, and 43).

The general form for a linear, shift-invariant, singularly perturbed discrete system is given by (refs. 44 to 47)

$$\mathbf{x}(k+1) = \mathbf{A}_{11}\,\mathbf{x}(k) + \varepsilon^{1-j}\mathbf{A}_{12}\,\mathbf{z}(k) + \mathbf{B}_1\,\mathbf{u}(k) \tag{3a}$$

$$\varepsilon^{2i}\,\mathbf{z}(k+1) = \varepsilon^{j}\mathbf{A}_{21}\,\mathbf{x}(k) + \varepsilon\mathbf{A}_{22}\,\mathbf{z}(k) + \varepsilon^{j}\mathbf{B}_2\,\mathbf{u}(k) \tag{3b}$$

with $0 \leq i \leq 1$ and $0 \leq j \leq 1$ where $\mathbf{x}(k)$ and $\mathbf{z}(k)$ are slow and fast state vectors of n- and m-dimensions, respectively, $\mathbf{u}(k)$ is an r-dimensional control vector, ε is the singular perturbation parameter, and $\mathbf{A}_{ij}$ and $\mathbf{B}_i$ are matrices of appropriate dimensionality. The initial-value problem with $\mathbf{x}(k{=}0) = \mathbf{x}(0)$ and $\mathbf{z}(k{=}0) = \mathbf{z}(0)$ is formulated.

The three limiting cases of equations (3) result in the following models:

(1) For the C-model $(i = 0; \; j = 0)$,

$$\mathbf{x}(k+1) = \mathbf{A}_{11}\,\mathbf{x}(k) + \varepsilon\mathbf{A}_{12}\,\mathbf{z}(k) + \mathbf{B}_1\,\mathbf{u}(k) \tag{4a}$$

$$\mathbf{z}(k+1) = \mathbf{A}_{21}\,\mathbf{x}(k) + \varepsilon\mathbf{A}_{22}\,\mathbf{z}(k) + \mathbf{B}_2\,\mathbf{u}(k) \tag{4b}$$

where the small parameter ε appears in the <u>column</u> of the system matrix.

(2) For the R-model $(i = 0; \; j = 1)$,

$$\mathbf{x}(k+1) = \mathbf{A}_{11}\,\mathbf{x}(k) + \mathbf{A}_{12}\,\mathbf{z}(k) + \mathbf{B}_1\,\mathbf{u}(k) \tag{5a}$$

$$\mathbf{z}(k+1) = \varepsilon\mathbf{A}_{21}\,\mathbf{x}(k) + \varepsilon\mathbf{A}_{22}\,\mathbf{z}(k) + \varepsilon\mathbf{B}_2\,\mathbf{u}(k) \tag{5b}$$

where the small parameter ε appears in the <u>row</u> of the system matrix.

(3) For the D-model $(i = 1; \; j = 1)$,

$$\mathbf{x}(k+1) = \mathbf{A}_{11}\,\mathbf{x}(k) + \mathbf{A}_{12}\,\mathbf{z}(k) + \mathbf{B}_1\,\mathbf{u}(k) \tag{6a}$$

$$\varepsilon\,\mathbf{z}(k+1) = \mathbf{A}_{21}\,\mathbf{x}(k) + \mathbf{A}_{22}\,\mathbf{z}(k) + \mathbf{B}_2\,\mathbf{u}(k) \tag{6b}$$

where the small parameter ε is positioned in a fashion identical to that of the continuous systems described by <u>differential</u> equations.

For the present discussion, it is enough to consider the system (eqs. (4)) as an initial-value problem with $\mathbf{x}(k = 0) = \mathbf{x}(0)$ and $\mathbf{z}(k = 0) = \mathbf{z}(0)$. The suppression of the small parameter ε in equations (4) results in the degenerate system

$$\mathbf{x}^{(0)}(k+1) = \mathbf{A}_{11}\,\mathbf{x}^{(0)}(k) + \mathbf{B}_1\,\mathbf{u}(k) \qquad \left(\mathbf{x}^{(0)}(k = 0) = \mathbf{x}(0)\right) \tag{7a}$$

$$\mathbf{z}^{(0)}(k+1) = \mathbf{A}_{21}\,\mathbf{x}^{(0)}(k) + \mathbf{B}_2\,\mathbf{u}(k) \qquad \left(\mathbf{z}^{(0)}(k = 0) \neq \mathbf{z}(0)\right) \tag{7b}$$

Here equation (7a) is a difference equation in $\mathbf{x}^{(0)}(k)$ of order n only, whereas equation (7b) is an algebraic equation. It means that once $\mathbf{x}^{(0)}(k)$ is solved from equation (7a), $\mathbf{z}^{(0)}(k)$ is automatically fixed by equation (7b) and $\mathbf{z}^{(0)}(0)$ is not, in general, equal to $\mathbf{z}(0)$. Thus, the suppression of the small parameter ε in equations (4) leads to a low-order system (eqs. (7)) with a consequent loss of the initial condition $\mathbf{z}(0)$. Hence, by definition, equations (4) are in the singularly perturbed form. The original system (eqs. (4)) and the degenerate system (eqs. (7)) are shown in figure 3. Here it is also assumed that $\mathbf{u}$ is independent of ε; otherwise $\mathbf{u}(k)$ becomes $\mathbf{u}^{(0)}(k)$ when ε is made equal to zero. The theory of SPATS in discrete systems is of recent origin (refs. 44 to 55). Attempts have been made to apply the technique to digital flight control systems, but these attempts have been limited mainly to a class of digital control of continuous systems (refs. 56 to 61).

In this section, the concepts of SPATS in continuous and discrete control systems have been discussed. In the next section, a method for decoupling a two-time-scale system is presented.

4. Time-Scale Analysis of Digital Control Systems

In this section, attention is focused on time-scale analysis (TSA) of digital control systems. This is followed by a description of a block-diagonalization procedure to develop a method to decouple a discrete system into slow and fast subsystems. The basic properties and numerical aspects of these systems are discussed. An interesting fact is provided showing that singularly perturbed discrete systems can be viewed as two-time-scale systems. Finally, an aircraft example is given to illustrate the method (ref. 16).
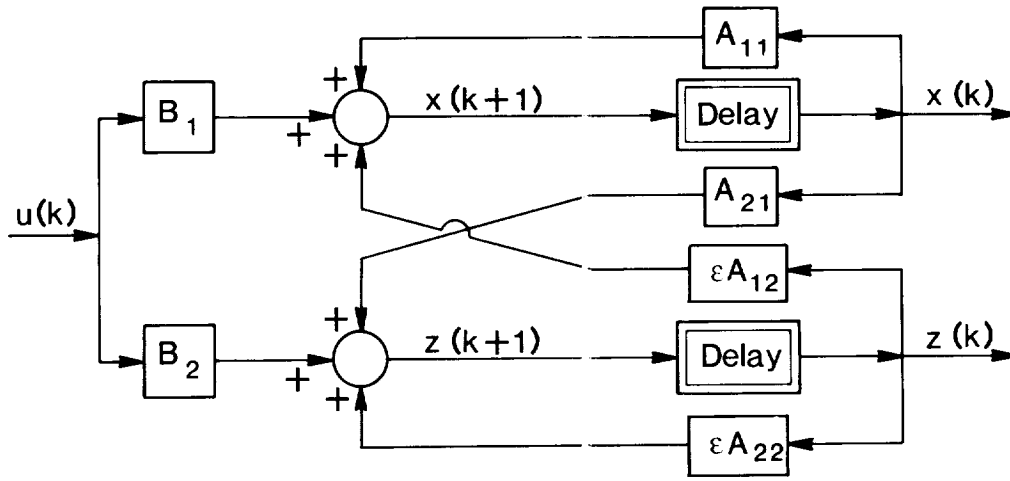
(a) Original system.
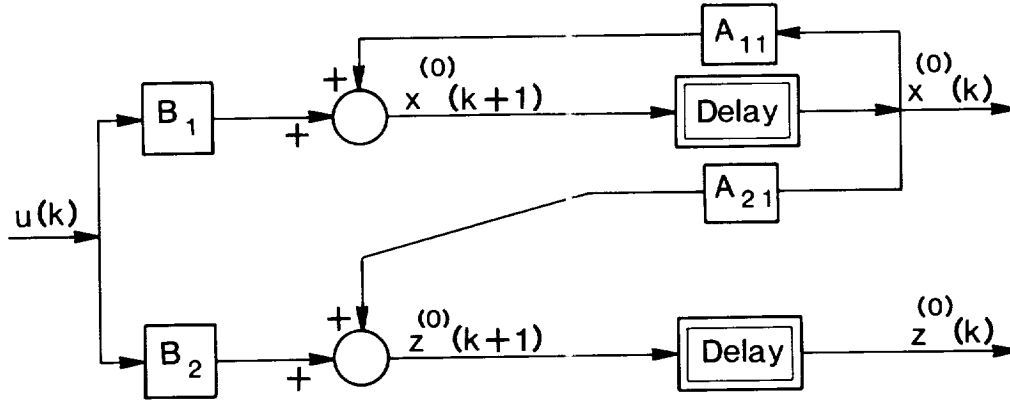
(b) Degenerate system.

Figure 3. Singularly perturbed discrete system.

Two-Time-Scale System

Consider a linear, shift-invariant, two-time-scale discrete system

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \mathbf{z}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{z}(k) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} \mathbf{u}(k) \tag{8}$$

with initial conditions $\mathbf{x}(k = 0) = \mathbf{x}(0)$ and $\mathbf{z}(k = 0) = \mathbf{z}(0)$. It is assumed that the system (eq. (8)) is asymptotically stable and that its eigenspectrum consists of a cluster of n large eigenvalues and a cluster of m small eigenvalues. Let the eigenvalues of equation (8) be arranged as

$$|1| > |p_{s1}| > \ldots > |p_{sn}| > |p_{f1}| > \ldots > |p_{fm}| \tag{9}$$

If the condition

$$\varepsilon = |p_{f1}|/|p_{sn}| << 1 \tag{10}$$

is satisfied, then the system (eq. (8)) possesses a two-time-scale (slow and fast) property (refs. 44 and 50). In other words, the stable discrete system (eq. (8)) is said to exhibit a two-time-scale behavior if the largest absolute eigenvalue of the fast eigenspectrum is much smaller than (i.e., has wide separation with) the smallest absolute eigenvalue of the slow eigenspectrum. In the literature, one usually finds stiffness being stamped on a system in which the ratio of eigenvalues is given by equation (10). This

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ratio appears explicitly as the small parameter ε in the case of a singularly perturbed system. It is also noted that the slow modes are generated by large eigenvalues and the fast modes are produced by small eigenvalues.

In order to block-diagonalize the system (eq. (8)) or separate it into slow and fast subsystems, a two-stage linear transformation is used. In the first step, the block $\mathbf{A}_{21}$ is removed to make equation (8) an upper-triangular matrix by using the transformation

$$\mathbf{z}_f(k) = \mathbf{z}(k) + \mathbf{D}\,\mathbf{x}(k) \tag{11}$$

where the $(m \times n)$ matrix $\mathbf{D}$ is a real root of the Riccati-type algebraic equation

$$\mathbf{A}_{22}\mathbf{D} - \mathbf{D}\mathbf{A}_{11} + \mathbf{D}\mathbf{A}_{12}\mathbf{D} - \mathbf{A}_{21} = 0 \tag{12}$$

Substituting equation (11) into (8) results in

$$\begin{bmatrix} \mathbf{x}(k{+}1) \\ \mathbf{z}_f(k{+}1) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_s & \mathbf{A}_{12} \\ 0 & \mathbf{A}_f \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{z}_f(k) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_f \end{bmatrix} \mathbf{u}(k) \tag{13}$$

where

$$\mathbf{A}_s = \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{D} \qquad \mathbf{A}_f = \mathbf{A}_{22} + \mathbf{D}\mathbf{A}_{12}$$
$$\mathbf{B}_f = \mathbf{D}\mathbf{B}_1 + \mathbf{B}_2$$

In the second stage, the transformation

$$\mathbf{x}_s(k) = \mathbf{x}(k) + \mathbf{E}\,\mathbf{z}_f(k) \tag{14}$$

is applied to equation (13) and the $(n \times m)$ matrix $\mathbf{E}$ is chosen as the solution of the Riccati-type algebraic equation

$$\mathbf{E}\mathbf{A}_f - \mathbf{A}_s\mathbf{E} + \mathbf{A}_{12} = 0 \tag{15a}$$
$$\mathbf{E}(\mathbf{A}_{22} + \mathbf{D}\mathbf{A}_{12}) - (\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{D})\mathbf{E} + \mathbf{A}_{12} = 0 \tag{15b}$$

Then, equation (13) reduces to

$$\begin{bmatrix} \mathbf{x}_s(k{+}1) \\ \mathbf{z}_f(k{+}1) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_s & 0 \\ 0 & \mathbf{A}_f \end{bmatrix} \begin{bmatrix} \mathbf{x}_s(k) \\ \mathbf{z}_f(k) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_s \\ \mathbf{B}_f \end{bmatrix} \mathbf{u}(k) \tag{16}$$

where

$$\mathbf{B}_s = (\mathbf{I}_s + \mathbf{E}\mathbf{D})\mathbf{B}_1 + \mathbf{E}\mathbf{B}_2$$

Now the system (eq. (16)) is in the desired decoupled form, as shown in figure 4. Note that the slow and fast variables $\mathbf{x}_s(k)$ and $\mathbf{z}_f(k)$ are related with the original state variables $\mathbf{x}(k)$ and $\mathbf{z}(k)$ by means of the transformations given in equations (11) and (14), which are combined to form

$$\begin{bmatrix} \mathbf{x}_s(k) \\ \mathbf{z}_f(k) \end{bmatrix} = \begin{bmatrix} (\mathbf{I}_s + \mathbf{E}\mathbf{D}) & \mathbf{E} \\ \mathbf{D} & \mathbf{I}_f \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{z}(k) \end{bmatrix} \tag{17}$$

and from which

$$\begin{bmatrix} \mathbf{x}(k) \\ \mathbf{z}(k) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_s & -\mathbf{E} \\ -\mathbf{D} & (\mathbf{I}_f + \mathbf{D}\mathbf{E}) \end{bmatrix} \begin{bmatrix} \mathbf{x}_s(k) \\ \mathbf{z}_f(k) \end{bmatrix} \tag{18}$$

Note that the special feature of these transformations is that equation (18) needs no inversions of the matrices, and hence the simplified computation is derived.
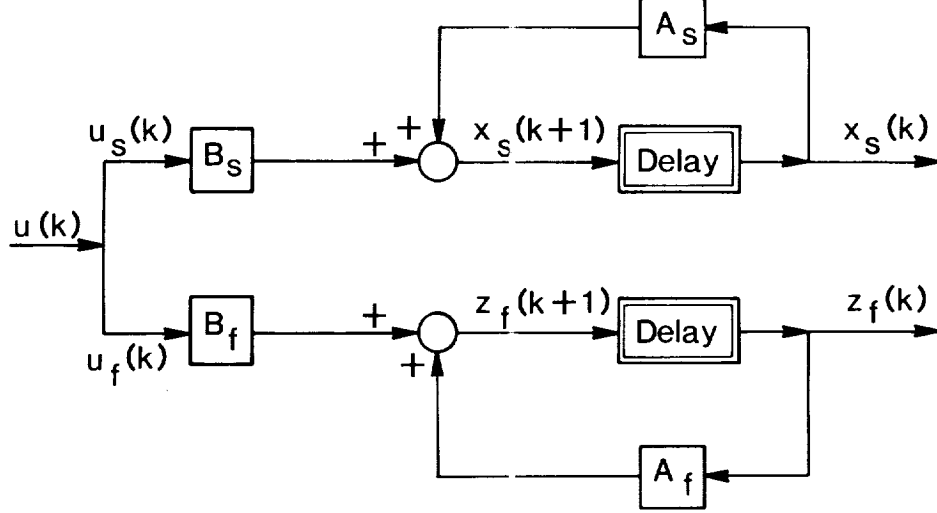
Figure 4. Decoupled slow and fast subsystems.

The condition given in equation (10) can be written as

$$\varepsilon = \max |e(\mathbf{A}_f)| / \min |e(\mathbf{A}_s)| \ll 1 \tag{19}$$

where

$$e(\mathbf{A}_f) = \{p_{f1}, \ \dots, \ p_{fm}\}$$
$$e(\mathbf{A}_s) = \{p_{s1}, \ \dots, \ p_{sn}\}$$

The iterative solution of equation (12) is given as (ref. 44)

$$\mathbf{D}_{i+1} = \left(\mathbf{A}_{22}\mathbf{D}_i + \mathbf{D}_i \mathbf{A}_{12}\mathbf{D}_i - \mathbf{A}_{21}\right)\mathbf{A}_{11}^{-1} \tag{20}$$

with an initial value of $\mathbf{D}_0 = -\mathbf{A}_{21}\mathbf{A}_{11}^{-1}$. Similarly for equations (15), we have

$$\mathbf{E}_{i+1} = \mathbf{A}_{11}^{-1}\left(\mathbf{E}_i\mathbf{A}_{22} + \mathbf{E}_i\mathbf{D}\mathbf{A}_{12} + \mathbf{A}_{12}\mathbf{D}\mathbf{E}_i + \mathbf{A}_{12}\right) \tag{21}$$

with an initial value of $\mathbf{E}_0 = \mathbf{A}_{11}^{-1}\mathbf{A}_{12}$. Substituting these initial values of $\mathbf{D}_0$ and $\mathbf{E}_0$ into equation (16) gives

$$\mathbf{A}_{s0} = \mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} \qquad \mathbf{A}_{f0} = \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \tag{22a}$$

$$\mathbf{B}_{s0} = \mathbf{B}_1 - \mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{11}^{-1}\mathbf{B}_1 + \mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}_2 \tag{22b}$$

$$\mathbf{B}_{f0} = \mathbf{B}_2 - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{B}_1 \tag{22c}$$

and

$$\begin{bmatrix} \mathbf{x}_{s0}(k+1) \\ \mathbf{z}_{f0}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{s0} & 0 \\ 0 & \mathbf{A}_{f0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{s0}(k) \\ \mathbf{z}_{f0}(k) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_{s0} \\ \mathbf{B}_{f0} \end{bmatrix} \mathbf{u}(k) \tag{23}$$

Thus, it is seen that a two-time-scale system is decoupled into slow and fast subsystems.

Singularly Perturbed Systems as Two-Time-Scale Systems

We consider the singularly perturbed system (eqs. (4)) and repeat it here for convenience. Thus,

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \mathbf{z}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \varepsilon\mathbf{A}_{12} \\ \mathbf{A}_{21} & \varepsilon\mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{z}(k) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} \mathbf{u}(k) \tag{24}$$

with initial conditions $\mathbf{x}(k=0) = \mathbf{x}(0)$ and $\mathbf{z}(k=0) = \mathbf{z}(0)$.

Now, substituting the transformation of equation (17) into (24) and replacing $\mathbf{E}$ by $\varepsilon\mathbf{E}$ allows the decoupled subsystem to become

$$\begin{bmatrix} \mathbf{x}_s(k+1) \\ \mathbf{z}_f(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_s & 0 \\ 0 & \varepsilon\mathbf{A}_f \end{bmatrix} \begin{bmatrix} \mathbf{x}_s(k) \\ \mathbf{z}_f(k) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_s \\ \mathbf{B}_f \end{bmatrix} \mathbf{u}(k) \tag{25}$$

where

$$\mathbf{A}_s = \mathbf{A}_{11} - \varepsilon\mathbf{A}_{12}\mathbf{D} \qquad \mathbf{A}_f = \mathbf{A}_{22} + \mathbf{D}\mathbf{A}_{12}$$
$$\mathbf{B}_s = (\mathbf{I}_s + \varepsilon\mathbf{E}\mathbf{D})\mathbf{B}_1 + \varepsilon\mathbf{E}\mathbf{B}_2 \qquad \mathbf{B}_f = \mathbf{D}\mathbf{B}_1 + \mathbf{B}_2$$

and $\mathbf{D}$ and $\mathbf{E}$ are the solutions of the Riccati-type algebraic equations

$$\varepsilon\mathbf{A}_{22}\mathbf{D} - \mathbf{D}\mathbf{A}_{11} + \varepsilon\mathbf{D}\mathbf{A}_{12}\mathbf{D} - \mathbf{A}_{21} = 0 \tag{26}$$

$$\varepsilon\mathbf{E}(\mathbf{A}_{22} + \mathbf{D}\mathbf{A}_{12}) - (\mathbf{A}_{11} - \varepsilon\mathbf{A}_{12}\mathbf{D})\mathbf{E} + \mathbf{A}_{12} = 0 \tag{27}$$

whose iterative solutions start with the initial values of $\mathbf{D}_0 = -\mathbf{A}_{21}\mathbf{A}_{11}^{-1}$ and $\mathbf{E}_0 = \mathbf{A}_{11}^{-1}\mathbf{A}_{12}$. For a sufficiently small ε, the zero-order approximate solutions are

$$\mathbf{A}_{s0} = \mathbf{A}_{11} \qquad \mathbf{A}_{f0} = \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \tag{28a}$$

$$\mathbf{B}_{s0} = \mathbf{B}_1 \qquad \mathbf{B}_{f0} = \mathbf{B}_2 - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{B}_1 \tag{28b}$$

so that the decoupled subsystem becomes

$$\begin{bmatrix} \mathbf{x}_{s0}(k+1) \\ \mathbf{z}_{f0}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & 0 \\ 0 & \varepsilon\mathbf{A}_{f0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{s0}(k) \\ \mathbf{z}_{f0}(k) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_{f0} \end{bmatrix} \mathbf{u}(k) \tag{29}$$

with

$$\mathbf{x}_{s0}(k=0) = \mathbf{x}(0)$$
$$\mathbf{z}_{f0}(k=0) = \mathbf{z}(0) - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{x}(0)$$

Now the system (eq. (29)) is obtained in an alternative way from equation (24) if the singular perturbation method (SPM) is used. The degenerate (slow) subsystem is obtained by making $\varepsilon = 0$ in equation (24). That is,

$$\mathbf{x}^{(0)}(k+1) = \mathbf{A}_{11}\mathbf{x}^{(0)}(k) + \mathbf{B}_1\mathbf{u}^{(0)}(k) \qquad \left(\mathbf{x}^{(0)}(k=0) = \mathbf{x}(0)\right) \tag{30a}$$

$$\mathbf{z}^{(0)}(k+1) = \mathbf{A}_{21}\mathbf{x}^{(0)}(k) + \mathbf{B}_2\mathbf{u}^{(0)}(k) \qquad \left(\mathbf{z}^{(0)}(k=0) \neq \mathbf{z}(0)\right) \tag{30b}$$

Here, it is noted that $\mathbf{z}(k)$ has lost its initial condition $\mathbf{z}(0)$ in the process of degeneration. In order to recover this lost initial condition, a correction subsystem is used. The transformations between the original and correction variables are

$$\mathbf{x}_g(k) = \mathbf{x}(k)/\varepsilon^{k+1} \qquad \mathbf{z}_g(k) = \mathbf{z}(k)/\varepsilon^k \tag{31a}$$

$$\mathbf{u}_g(k) = \mathbf{u}(k)/\varepsilon^{k+1} \tag{31b}$$

Substituting equations (31) into (24) allows the transformed system to become

$$\varepsilon\mathbf{x}_g(k+1) = \mathbf{A}_{11}\mathbf{x}_g(k) + \mathbf{A}_{12}\mathbf{z}_g(k) + \mathbf{B}_1\mathbf{u}_g(k) \tag{32a}$$

$$\mathbf{z}_g(k+1) = \mathbf{A}_{21}\mathbf{x}_g(k) + \mathbf{A}_{22}\mathbf{z}_g(k) + \mathbf{B}_2\mathbf{u}_g(k) \tag{32b}$$

The zero-order approximation ($\varepsilon = 0$) of equations (32) becomes

$$0 = \mathbf{A}_{11}\,\mathbf{x}_g^{(0)}(k) + \mathbf{A}_{12}\,\mathbf{z}_g^{(0)}(k) + \mathbf{B}_1\,\mathbf{u}_g^{(0)}(k) \tag{33a}$$

$$\mathbf{z}_g^{(0)}(k+1) = \mathbf{A}_{21}\,\mathbf{x}_g^{(0)}(k) + \mathbf{A}_{22}\,\mathbf{z}_g^{(0)}(k) + \mathbf{B}_2\,\mathbf{u}_g^{(0)}(k) \tag{33b}$$

Rewriting equations (33) produces

$$\mathbf{x}_g^{(0)}(k) = -\mathbf{A}_{11}^{-1}\left[\mathbf{A}_{12}\,\mathbf{z}_t^{(0)}(k) + \mathbf{B}_1\,\mathbf{u}_g^{(0)}(k)\right] \tag{34a}$$

$$\mathbf{z}_g^{(0)}(k+1) = \mathbf{A}_{g0}\,\mathbf{z}_g^{(0)}(k) + \mathbf{B}_{g0}\,\mathbf{u}_g^{(0)}(k) \tag{34b}$$

where

$$\mathbf{A}_{g0} = \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}$$

$$\mathbf{B}_{g0} = \mathbf{B}_2 - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{B}_1$$

The total solution consists of the sum of the outer solution given by equations (30) and the correction solution of equations (34). For the present, to simplify the presentation, u(k) and its associated functions are omitted. Then (ref. 47),

$$\mathbf{x}(k) = \mathbf{x}^{(0)}(k) + \varepsilon^{k+1}\mathbf{x}_g^{(0)}(k) \tag{35a}$$

$$\mathbf{z}(k) = \mathbf{z}^{(0)}(k) + \varepsilon^{k}\mathbf{z}_g^{(0)}(k) \tag{35b}$$

The current interest is only zero-order approximations. i.e., the slow part $\mathbf{x}^{(0)}(k)$ of $\mathbf{x}(k)$ and the fast part $\varepsilon^k\mathbf{z}_g^{(0)}(k)$ of $\mathbf{z}(k)$. Thus, from equations (30) and (34) (omitting input functions),

$$\mathbf{x}^{(0)}(k+1) = \mathbf{A}_{11}\,\mathbf{x}^{(0)}(k) \tag{36a}$$

$$\mathbf{z}_g^{(0)}(k+1) = \mathbf{A}_{g0}\,\mathbf{z}_g^{(0)}(k) \tag{36b}$$

or

$$\mathbf{z}_r^{(0)}(k+1) = \varepsilon\mathbf{A}_{g0}\,\mathbf{z}_r^{(0)}(k) \tag{36c}$$

where

$$\mathbf{z}_r^{(0)}(k) = \varepsilon^k\,\mathbf{z}_g^{(0)}(k) \tag{37a}$$

$$\mathbf{x}^{(0)}(k = 0) = \mathbf{x}(0) \tag{37b}$$

$$\mathbf{z}_r^{(0)}(k = 0) = \mathbf{z}_g^{(0)}(0)$$

$$= \mathbf{z}(0) - \mathbf{z}^{(0)}(0)$$

$$= \mathbf{z}(0) - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\,\mathbf{x}(0) \tag{37c}$$

By comparing equations (29) and (36), it is seen that to a zero-order approximation, the degenerate and correction subsystems of the singular perturbation method are the same as the slow and fast subsystems of the time-scale analysis. This is a result similar to that in continuous systems where the singular perturbation and time-scale approaches give identical results to a zero-order approximation. The method is now demonstrated by an aircraft example.

Application to an Aircraft

The methodology described in the previous sections is illustrated by the application to an aircraft. The longitudinal dynamic equations of an aircraft model are obtained based on a rigid-body assumption (ref. 62). Also, the angle of attack is assumed to be small. The linearized model possesses a two-time-scale property in the sense that pitch angle, velocity, and altitude are the slow variables and angle

of attack and pitch rate are the fast variables, corresponding to phugoid and short-period modes, respectively. For digital implementation, a zero-order hold is used to obtain the discrete model. The aircraft under consideration is a twin-engine general aviation aircraft. The discrete model is given by (ref. 63)

$$\mathbf{y}(k+1) = \mathbf{A}\,\mathbf{y}(k) + \mathbf{B}\,\mathbf{u}(k) \tag{38}$$

where

$$\mathbf{y}(k) = \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{z}(k) \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$$

and

$x_1(k)$	velocity, ft/sec
$x_2(k)$	pitch angle, deg
$x_3(k)$	altitude, ft
$z_1(k)$	angle of attack, deg
$z_2(k)$	pitch angular velocity, deg/sec
$u_1(k)$	elevator deflection, deg
$u_2(k)$	flap deflection, deg
$u_3(k)$	throttle position, deg

$$\mathbf{A} = \begin{bmatrix} 0.923701 & -0.308096 & 0 & 0.053043 & -0.090367 \\ 0.039705 & 0.995525 & 0 & -0.107454 & 0.588883 \\ 0.087127 & 1.899490 & 1.0 & -0.635270 & 0.394015 \\ -0.035537 & 0.010123 & 0 & 0.007748 & 0.137407 \\ 0.069562 & -0.012706 & 0 & -0.097108 & 0.287411 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0.042825 & -0.000395 & -0.154048 \\ -0.484628 & -0.515359 & -0.002237 \\ -0.161525 & -0.067522 & -0.005257 \\ -0.202010 & -0.289303 & 0.005061 \\ -0.806770 & -0.852161 & -0.006353 \end{bmatrix}$$

The states $x_1(k)$ and $x_3(k)$ are scaled down by a factor of 100 to facilitate easy representation. The eigenvalues of the discrete (original) system are given by 1.0, $0.962103 \pm 0.175342j$, 0.217297, and 0.072882, suggesting that there are three slow modes ($x_1(k)$, $x_2(k)$, and $x_3(k)$) and two fast modes ($z_1(k)$ and $z_2(k)$) with the eigenvalue separation ratio given as $|p_{f1}|/|p_{s3}| = 0.222196$. The results are now summarized below for two cases in which the two-time-scale approach is used.

Case I—Zero iterations. Here the initial values $\mathbf{D}_0$ and $\mathbf{E}_0$ are themselves used in obtaining the decoupled subsystems without resorting to any iterative solutions of the Riccati-type algebraic equations (26) and (27). Thus,

$$\mathbf{D}_0 = \begin{bmatrix} 0.038398 & 0.001715 & 0 \\ -0.074861 & -0.010405 & 0 \end{bmatrix}$$

$$\mathbf{E}_0 = \begin{bmatrix} 0.021141 & 0.098166 \\ -0.108780 & 0.587615 \\ -0.430486 & -0.730705 \end{bmatrix}$$

$$\mathbf{A}_{s0} = \begin{bmatrix} 0.914899 & -0.309128 & 0 \\ 0.087915 & 1.001837 & 0 \\ 0.141016 & 1.904675 & 1.0 \end{bmatrix}$$

$$\mathbf{A}_{f0} = \begin{bmatrix} 0.009600 & 0\ 134947 \\ -0.099961 & 0\ 288048 \end{bmatrix}$$

$$\mathbf{B}_{s0} = \begin{bmatrix} -0.040446 & -\ 0.089654 & -0.153556 \\ -0.935733 & -\ 0.981356 & 0.000913 \\ 0.513256 & 0.676144 & -0.008690 \end{bmatrix}$$

$$\mathbf{B}_{f0} = \begin{bmatrix} -0.201197 & -0.290202 & -0.000858 \\ -0.804933 & -0.846769 & 0.005203 \end{bmatrix}$$

The eigenvalues of the slow subsystem are 1.0 and $0.9\flat8368 \pm 0.159020j$, and the eigenvalues of the fast subsystem are 0.225595 and 0.072053 with an eigenvalue separation ratio of 0.232222.

Case II—Five iterations. Here the values of $\mathbf{D}_5$ and $\mathbf{E}_5$, obtained after five iterations of the Riccati equations (26) and (27), are used for the decoupled subsystems. The following matrices are obtained:

$$\mathbf{D}_5 = \begin{bmatrix} 0.023371 & -0.006771 & 0 \\ -0.110005 & -0.028\flat61 & 0 \end{bmatrix}$$

$$\mathbf{E}_5 = \begin{bmatrix} -0.026374 & 0.215\flat47 \\ -0.178397 & 0.741\flat58 \\ -0.144412 & -1.487\flat36 \end{bmatrix}$$

$$\mathbf{A}_{s5} = \begin{bmatrix} 0.912520 & -0.3103\flat7 & 0 \\ 0.106996 & 1.011676 & 1.0 \\ 0.145317 & 1.906477 & 1.0 \end{bmatrix}$$

$$\mathbf{A}_{f5} = \begin{bmatrix} 0.009715 & 0.131307 \\ -0.099864 & 0.28047\flat \end{bmatrix}$$

$$\mathbf{B}_{s5} = \begin{bmatrix} -0.123959 & -0.173428 & -0.151789 \\ -1.041054 & -1.09\flat552 & 0.005406 \\ 1.053315 & 1.219171 & -0.021322 \end{bmatrix}$$

$$\mathbf{B}_{f5} = \begin{bmatrix} -0.197728 & -0.285\flat22 & 0.001476 \\ -0.797591 & -0.837\flat47 & 0.010657 \end{bmatrix}$$

The eigenvalues of the slow subsystem are 1.0 and $0.96\flat098 \pm 0.175345j$, and the eigenvalues of the fast subsystem are 0.217307 and 0.072882 with an eigenvalue separation ratio of 0.222207.

The summary of eigenvalues for the original system and the decoupled subsystems is shown in table I. The eigenvalue separation ratio (consisting of the maximum of absolute values of fast eigenvalues and the minimum of absolute values of slow eigenvalues) is shown in table II.

The responses of the original system and the decoupled subsystems are shown in figures 5 to 9 for case I with zero iterations and in figures 10 to 14 for case II with five iterations. From both the figures and the tables, it is clear that the decoupled subsystems are close to the original system with considerable reduction in total computational requirements.

In this section it has been shown that a full-order, two-time-scale system can be decoupled into reduced-order, slow and fast subsystems. The technique has been demonstrated by applying it to an aircraft model. The next section considers the decoupling of optimal discrete control systems.

Table I. Summary of Eigenvalues

Original system	1.0	$0.962103 \pm 0.175342j$	0.217297	0.072882
Slow subsystem	Case I	1.0	$0.958368 \pm 0.159020j$		
	Case II	1.0	$0.962098 \pm 0.175345j$		
Fast subsystem	Case I			0.225595	0.072053
	Case II			0.217307	0.072882

Table II. Summary of Eigenvalue Separation Ratio

Original system		0.222196
Decoupled subsystems	Case I	0.232222
	Case II	0.222207
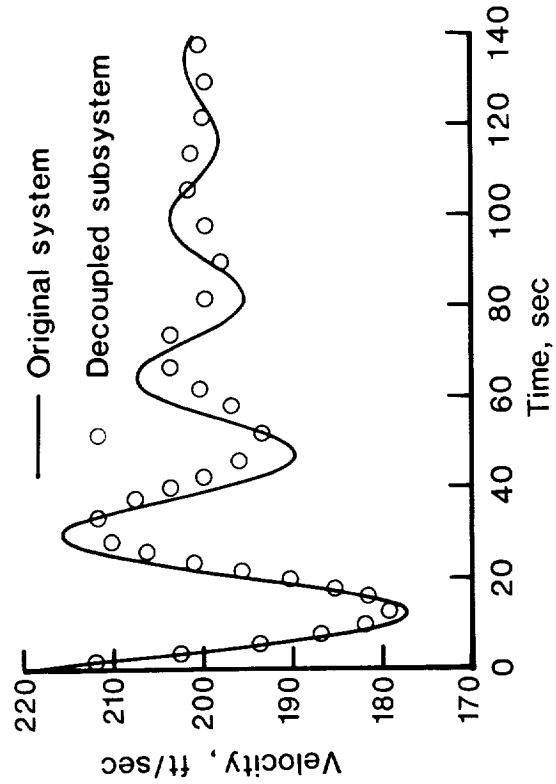
Figure 5. Comparison of solutions for velocity for case I.
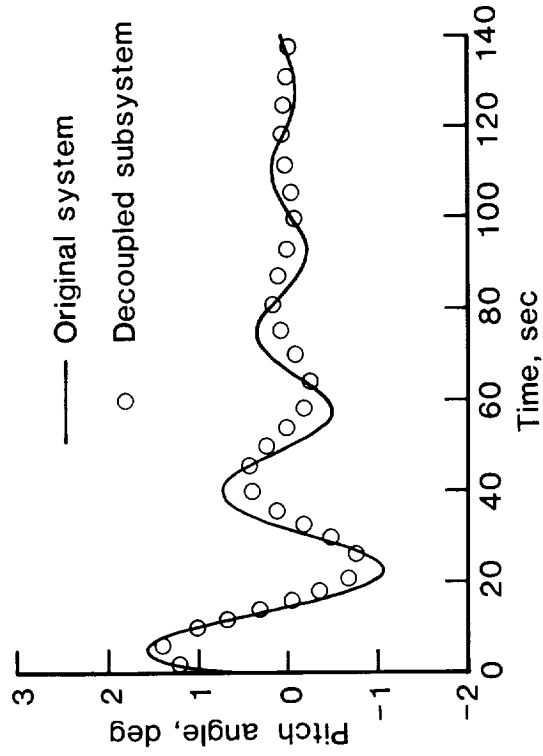
Figure 6. Comparison of solutions for pitch angle for case I.
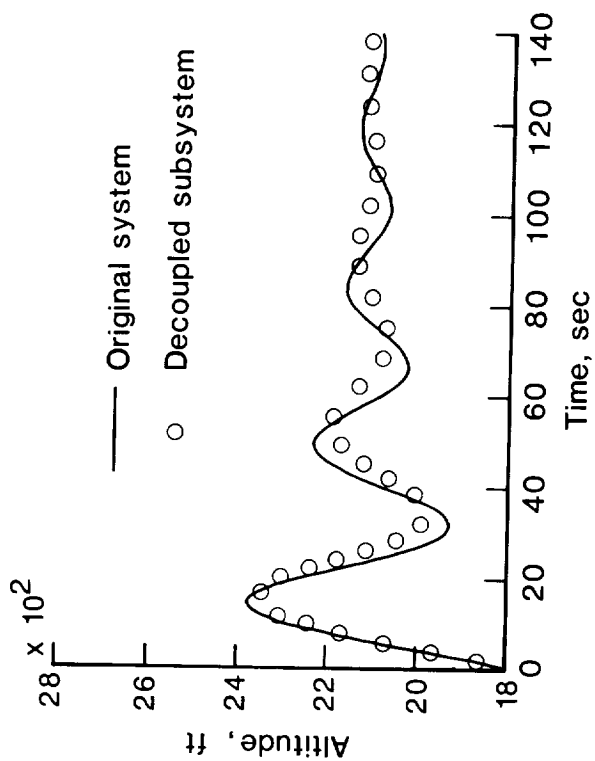
Figure 7. Comparison of solutions for altitude for case I.
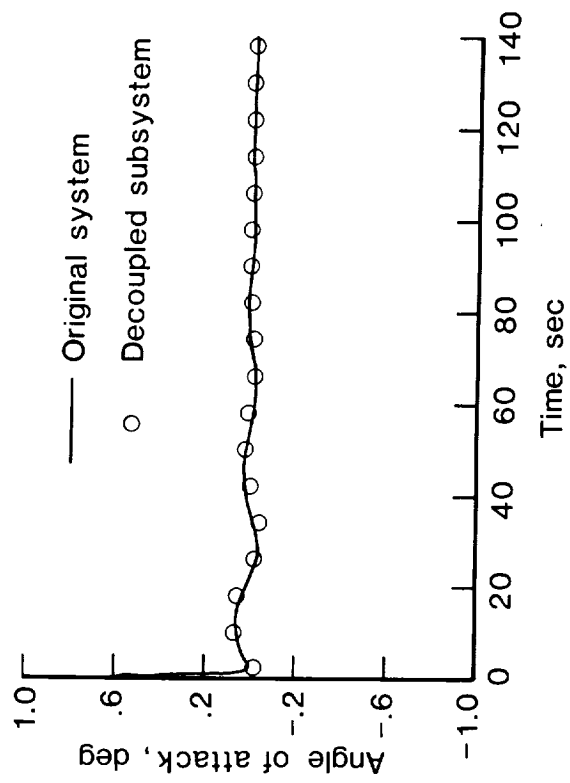
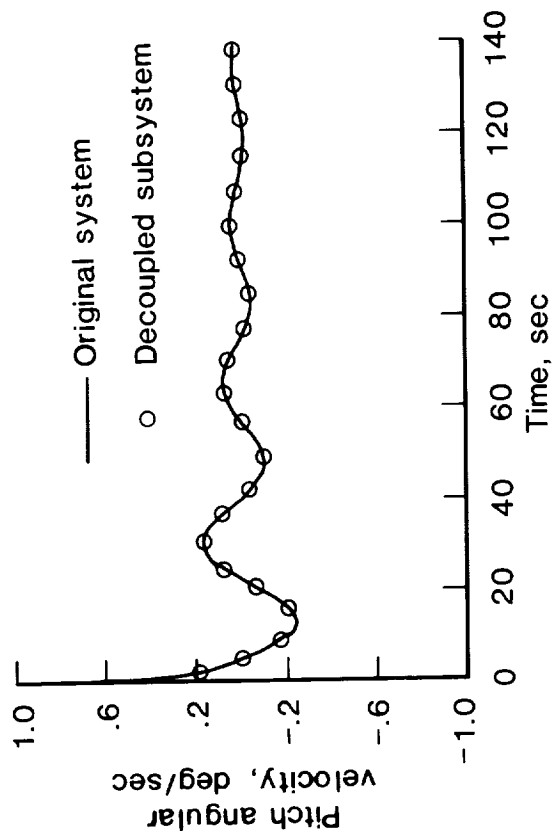Figure 8. Comparison of solutions for angle of attack for case I.
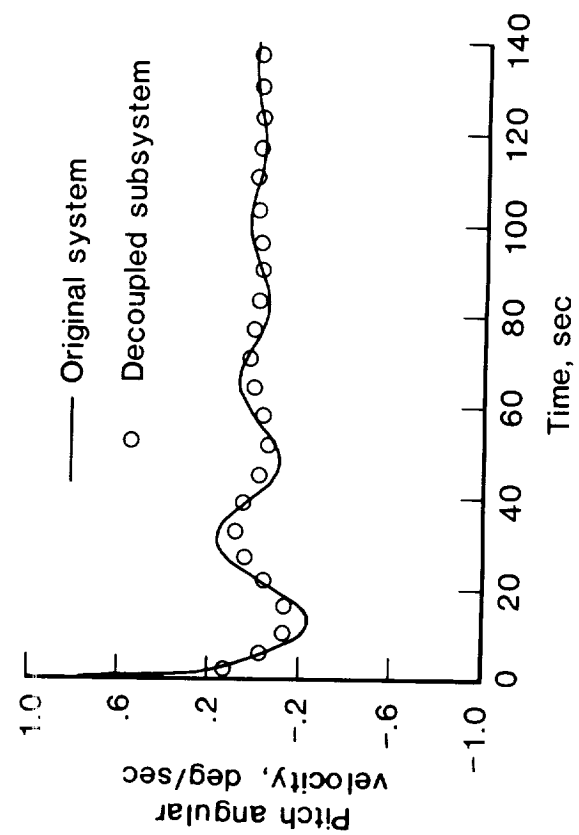
Figure 9. Comparison of solutions for pitch angular velocity for case I.
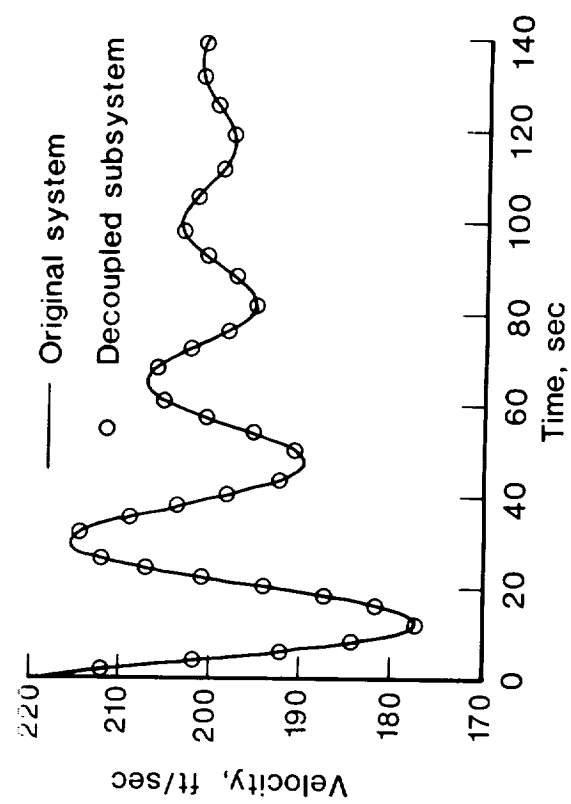
Figure 10. Comparison of solutions for velocity for case II.

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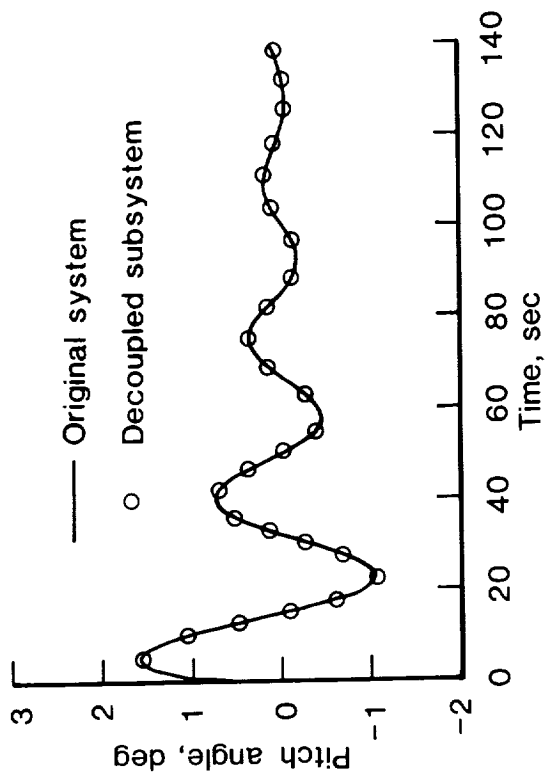
Figure 11. Comparison of solutions for pitch angle for case II.
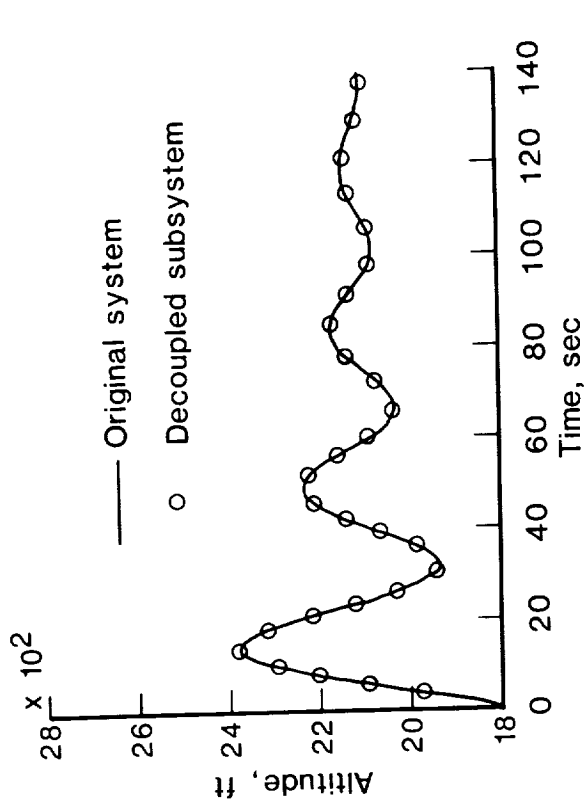
Figure 12. Comparison of solutions for altitude for case II.
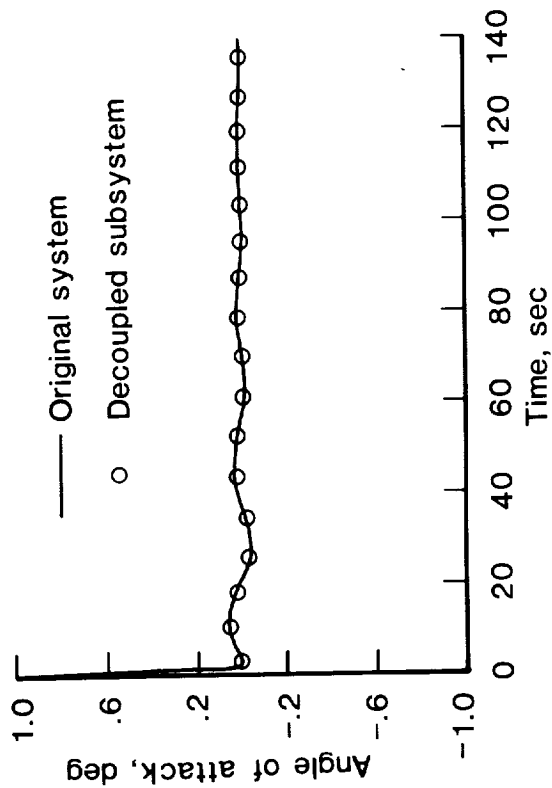
Figure 13. Comparison of solutions for angle of attack for case II.

Figure 14. Comparison of solutions for pitch angular velocity for case II.

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5. Time-Scale Synthesis of Digital Optimal Control Systems

In this section a two-time-scale discrete control system is considered. The closed-loop, optimal, linear, quadratic regulator for the system requires the solution of a full-order, algebraic Riccati equation. Alternatively, the original system is decomposed into reduced-order, slow and fast subsystems. The closed-loop optimal control of the subsystems requires the solution of two algebraic Riccati equations of lower order than that required for the full-order system. A composite, closed-loop suboptimal control is formed from the sum of the slow and fast feedback optimal controls. The main advantage of the method is a considerable reduction in the overall computational requirements for the closed-loop optimal control of digital flight systems (refs. 18 and 64 to 67).

For the sake of convenience, let us repeat here the stable, linear, shift-invariant discrete control system given as equation (8):

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \mathbf{z}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{z}(k) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} \mathbf{u}(k)$$

where the various state variables and the matrices have been previously described. The performance index to be minimized is

$$J = \frac{1}{2} \sum_{k=0}^{\infty} \left[\mathbf{y}^T(k)\, \mathbf{Q}\, \mathbf{y}(k) + \mathbf{u}^T(k)\, R\, \mathbf{u}(k) \right] \tag{39}$$

where

$$\mathbf{y}^T(k) = \begin{bmatrix} \mathbf{x}^T(k), & \mathbf{z}^T(k) \end{bmatrix}$$

and

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{12}^T & \mathbf{Q}_{22} \end{bmatrix}$$

is a positive, semidefinite symmetric matrix of order $(n + m) \times (n + m)$. The matrix $\mathbf{R}$ denotes a positive, definite symmetric matrix of order $(r \times r)$.

The closed-loop optimal control is given by (ref. 68)

$$\mathbf{u}(k) = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} \left(\mathbf{I} + \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} \right)^{-1} \mathbf{A}\, \mathbf{y}(k) \tag{40}$$

where $\mathbf{P}$, of order $(n+m) \times (n+m)$, is the positive, definite symmetric solution of the algebraic, matrix Riccati equation

$$\mathbf{P} = \mathbf{A}^T\mathbf{P} \left(\mathbf{I} + \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} \right)^{-1} \mathbf{A} + \mathbf{Q} \tag{41}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$$

The closed-loop optimal system is given by

$$\mathbf{y}(k+1) = (\mathbf{A} - \mathbf{BF})\, \mathbf{y}(k) \tag{42}$$

where

$$\mathbf{F} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} \left(\mathbf{I} + \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} \right)^{-1} \mathbf{A}$$

Instead of directly tackling the original regulator problem described by equations (8) and (39), one should appropriately decompose it into two regulator problems (approaches) for slow and fast subsystems. For this, it is necessary to separate the original performance index into the sum of two performance indices for slow and fast subsystems.

Separation of Original Performance Index

Approach 1. In approach 1 the original performance index (eq. (39)) has to be represented as the sum of the performance indices of the slow and fast subsystems. Using the transformation in equation (18) between the original state variables ($x(k)$ and $z(k)$) and the subsystem variables ($x_s(k)$ and $z_f(k)$) in equation (39) and using $u_s(k) = u_f(k) = u(k)$ as shown in figure 4 results in

$$J = \frac{1}{2} \sum_0^\infty \left[x_s^T(k) \, Q_{ss} \, x_s(k) + z_f^T(k) \, Q_{ff} \, z_f(k) + u_s^T(k) \, R_s \, u_s(k) \right.$$
$$\left. + x_s^T(k) \, Q_{sf} \, z_f(k) + z_f^T(k) \, Q_{fs} \, x_s(k) + u_f^T(k) \, R_f \, u_f(k) \right] \tag{43}$$

where

$$Q_{ss} = [I_s, \; -D^T] \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} I_s \\ -D \end{bmatrix}$$

$$Q_{ff} = [E^T, \; -(I_f + DE)^T] \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} E \\ -(I_f + DE) \end{bmatrix}$$

$$Q_{sf} = [I_s, \; D^T] \begin{bmatrix} -Q_{11} & Q_{12} \\ Q_{12}^T & -Q_{22} \end{bmatrix} \begin{bmatrix} E \\ (I_f + DE) \end{bmatrix}$$

$$Q_{fs} = [E^T, \; (I_f + DE)^T] \begin{bmatrix} -Q_{11} & Q_{12} \\ Q_{12}^T & -Q_{22} \end{bmatrix} \begin{bmatrix} I_s \\ D \end{bmatrix}$$

Note that $Q_{fs} = Q_{sf}^T$ and $R_s = R_f = R/2$.

Since J has to be represented as the sum of J_s and J_f, it is necessary to neglect Q_{sf} and Q_{fs}. Then,

$$J_s = \frac{1}{2} \sum_0^\infty \left[x_s^T(k) \, Q_{ss} \, x_s(k) + u_s^T(k) \, R_s \, u_s(k) \right] \tag{44}$$

$$J_f = \frac{1}{2} \sum_0^\infty \left[z_f^T(k) \, Q_{ff} \, z_f(k) + u_f^T(k) \, R_f \, u_f(k) \right] \tag{45}$$

As Q_{sf} and Q_{fs} have been neglected, this certainly introduces an error in that J will not be equal to the sum of J_s and J_f. To take this into account, it is necessary to readjust Q_{ss} and Q_{ff} (ref. 65).

Approach 2. Since the present method is simply a design or synthesis problem, first select the performance indices of the subsystems and then formulate the total performance index of the original system. Thus, define

$$J_s' = \frac{1}{2} \sum_0^\infty \left[x_s^T(k) \, Q_{ss}' \, x_s(k) + u_s^T(k) \, R_s' \, u_s(k) \right] \tag{46}$$

$$J_f' = \frac{1}{2} \sum_0^\infty \left[z_f^T(k) \, Q_{ff}' \, z_f(k) + u_f^T(k) \, R_f' \, u_f(k) \right] \tag{47}$$

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Using the transformation in equation (17) between the subsystem variables $x_s(k)$ and $z_f(k)$ and the original system variables $x(k)$ and $z(k)$ produces

$$J = J'_s + J'_f$$
$$= \frac{1}{2} \sum_0^\infty \left[y^T(k) \, Q' \, y(k) + u^T(k) \, R' \, u(k) \right] \tag{48}$$

where

$$Q' = \begin{bmatrix} Q'_{11} & Q'_{12} \\ Q'^T_{12} & Q'_{22} \end{bmatrix} \qquad \left(R' = 2R'_s = 2R'_f \right)$$

$$Q'_{11} = (I_s + ED)^T Q'_{ss}(I_s + ED) + D^T Q'_{ff} D$$
$$Q'_{12} = (I_s + ED)^T Q'_{ss} E + D^T Q'_{ff}$$
$$Q'_{22} = D^T Q'_{ss} E + Q'_{ff}$$

Thus, in equations (46) to (48), first select Q'_{ss}, Q'_{ff}, R'_s, and R'_f; and then by using D and E, one obtains Q' and R'. Here it is possible to decouple J J_s, and J_f exactly without any approximation. The original J is dependent on D and E, the decoupling matrices, which may not be of practical advantage.

Optimal Control of Subsystems

Using the transformation in equation (17) allows the original system in equation (8) to decompose into slow and fast subsystems as

$$x_s(k+1) = A_s \, x_s(k) + B_s \, u_s(k) \tag{49}$$
$$z_f(k+1) = A_f \, z_f(k) + B_f \, u_f(k) \tag{50}$$

Now these slow and fast subsystems are optimized with respect to their corresponding performance indices given in equations (44) and (45), respectively. The slow regulator problem consists of the slow subsystem of equation (49) and the performance index of equation (44). The fast regulator problem consists of the fast subsystem of equation (50) and the performance index of equation (45). For convenience, let $Q_{ss} = Q_s$ and $Q_{ff} = Q_f$.

The optimal feedback control of the slow subsystem of equation (49) is given by

$$u_s(k) = -R_s^{-1} B_s^T P_s \left(I_s + B_s R_s^{-1} B_s^T P_s \right)^{-1} A_s \, x_s(k) \tag{51}$$

where P_s, of order $(n \times n)$, is a positive, definite symmetric solution of the reduced-order, algebraic Riccati equation

$$P_s = A_s P_s \left(I_s + B_s R_s^{-1} B_s^T P_s \right)^{-1} A_s + Q_s \tag{52}$$

Similarly, the optimal feedback control of the fast subsystem of equation (50) becomes

$$u_f(k) = -R_f^{-1} B_f^T P_f \left(I_f + B_f R_f^{-1} B_f^T P_f \right)^{-1} A_f \, z_f(k) \tag{53}$$

where P_f, of order $(m \times m)$, is a positive, definite symmetric solution of the reduced-order, algebraic Riccati equation

$$P_f = A_f P_f \left(I_f + B_f R_f^{-1} B_f^T P_f \right)^{-1} A_f + Q_f \tag{54}$$

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The control laws of equations (51) and (53) are rewritten, respectively, as

$$u_s(k) = -\mathbf{F}_s\, \mathbf{x}_s(k) \tag{55}$$

$$u_f(k) = -\mathbf{F}_f\, \mathbf{z}_f(k) \tag{56}$$

where

$$\mathbf{F}_s = \mathbf{R}_s^{-1}\mathbf{B}_s^T\mathbf{P}_s\left(\mathbf{I}_s + \mathbf{B}_s\mathbf{R}_s^{-1}\mathbf{B}_s^T\mathbf{P}_s\right)^{-1}\mathbf{A}_s$$

$$\mathbf{F}_f = \mathbf{R}_f^{-1}\mathbf{B}_f^T\mathbf{P}_f\left(\mathbf{I}_f + \mathbf{B}_f\mathbf{R}_f^{-1}\mathbf{B}_f^T\mathbf{P}_f\right)^{-1}\mathbf{A}_f$$

It is noted that the control laws in equations (51) and (53) are optimal only with respect to the slow and fast subsystems of equations (49) and (50). It is, however, computationally simpler to determine these control laws than the optimal control law of equation (40) of the original system (refs. 51 and 52).

Composite Control

The composite control is formulated as the sum of the slow and fast feedback controls given by equations (51) and (53). That is,

$$\begin{aligned} u_c(k) &= u_s(k) + u_f(k) \\ &= -\left[\mathbf{F}_s\, \mathbf{x}_s(k) + \mathbf{F}_f\, \mathbf{z}_f(k)\right] \end{aligned} \tag{57}$$

Applying the transformation of equation (17) between the slow and fast variables and the original variables to equation (57) produces

$$u_c(k) = -\left[\mathbf{F}_{cs}\, \mathbf{x}(k) + \mathbf{F}_{cf}\, \mathbf{z}(k)\right] = -\mathbf{F}_c\, \mathbf{y}(k) \tag{58}$$

where

$$\mathbf{F}_{cs} = \mathbf{F}_s(\mathbf{I}_s + \mathbf{ED}) + \mathbf{F}_f\mathbf{D}$$

$$\mathbf{F}_{cf} = \mathbf{F}_s\mathbf{E} + \mathbf{F}_f$$

$$\mathbf{F}_c = [\mathbf{F}_{cs},\ \mathbf{F}_{cf}]$$

Using the composite control of equation (58) in the original system in equation (8) results in
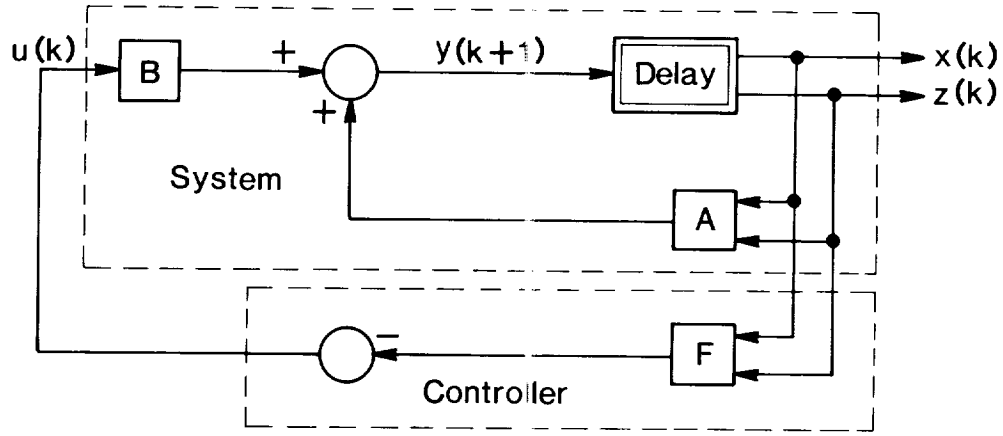
$$\mathbf{y}_c(k+1) = (\mathbf{A} - \mathbf{BF}_c)\, \mathbf{y}_c(k) \tag{59}$$

as shown in figure 15. It is known that minimizing the original performance index of equation (39) with respect to the composite system of equation (59) results in the suboptimal performance index (refs. 65 and 69),
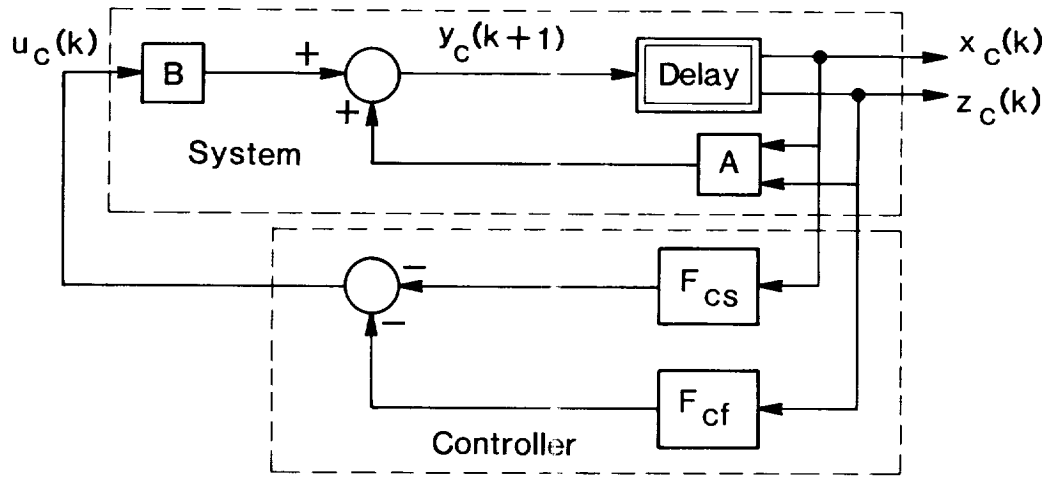
$$J_c = \frac{1}{2}\, \mathbf{y}^T(0)\, \mathbf{P}_c\, \mathbf{y}(0) \tag{60}$$

where $\mathbf{P}_c$ is the positive, definite symmetric solution of the discrete Liapunov equation

$$\mathbf{P}_c = (\mathbf{A} - \mathbf{BF}_c)^T\mathbf{P}_c(\mathbf{A} - \mathbf{BF}_c) + \mathbf{Q} + \mathbf{F}_c^T\mathbf{RF}_c \tag{61}$$

(a) Optimal feedback control.

(b) Composite feedback control.

Figure 15. Original and composite systems.

Aircraft Example

Consider the same aircraft model as previously given. The eigenvalues of the discrete (original) model are 1.0, $0.962103 \pm 0.175342j$, 0.217297, and 0.072882, an indication that there are three slow modes $(x_1(k), x_2(k),$ and $x_3(k))$ and two fast modes $(z_1(k)$ and $z_2(k))$ with an eigenvalue separation ratio of 0.222196. The performance measures in equation (39) are taken as $Q_{11} = Q_{22} = I$ and $R_s = R_f = 1$.

Using the method described in the previous section gives the results summarized below. The positive, definite matrix P of equation (41) is

$$
P = \begin{bmatrix}
4.822583 & 0.427518 & 0.709917 & -0.272287 & 0.059216 \\
0.427518 & 11.654952 & 3.732418 & -2.665579 & 3.042291 \\
0.709917 & 3.732418 & 2.699923 & -1.096067 & 0.882337 \\
-0.272287 & -2.665579 & -1.096067 & 1.742674 & -0.689301 \\
0.059216 & 3.042291 & 0.882337 & -0.689301 & 2.015052
\end{bmatrix}
$$

Using the above P, the optimal control of equation (40) is obtained with F as

$$
F = \begin{bmatrix}
-0.021105 & -1.136159 & -0.268959 & 0.260873 & -0.507318 \\
-0.110915 & -0.700095 & -0.101750 & 0.142104 & -0.410844 \\
-0.629305 & -0.025190 & -0.100651 & 0.033007 & 0.003334
\end{bmatrix}
$$

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The closed-loop optimal system of equation (42) has the eigenvalues 0.843545, 0.258790 $\pm$ 0.326649j, and 0.06457 $\pm$ 0.047911j.

For the slow and fast subsystems, the corresponding values are

$$\mathbf{P}_s = \begin{bmatrix} 4.621454 & 1.051044 & 0.945863 \\ 1.051044 & 13.073895 & 4.463674 \\ 0.945863 & 4.463674 & 2.710497 \end{bmatrix}$$

and

$$\mathbf{P}_f = \begin{bmatrix} 1.204794 & 0.244345 \\ 0.24435 & 1.914426 \end{bmatrix}$$

For the composite control, the feedback matrix $\mathbf{F}_c$, obtained from equation (58), is

$$\mathbf{F}_c = \begin{bmatrix} -0.010457 & -1.138702 & -0.270112 & 0.262312 & -0.507373 \\ -0.110238 & -0.699239 & -0.101678 & 0.142112 & -0.410528 \\ -0.654236 & -0.018681 & -0.100062 & 0.031065 & 0.004602 \end{bmatrix}$$

and the positive, definite matrix $\mathbf{P}_c$, obtained from the discrete Liapunov equation (61), is

$$\mathbf{P}_c = \begin{bmatrix} 5.045733 & 0.338736 & 0.671223 & -0.205383 & -0.013507 \\ 0.338736 & 11.734399 & 3.769142 & -2.716177 & 3.103791 \\ 0.671223 & 3.769142 & 2.714962 & -1.128617 & 0.918364 \\ -0.205383 & -2.716177 & -1.128617 & 1.790493 & -0.744196 \\ -0.013507 & 3.103791 & 0.918364 & -0.744196 & 2.083318 \end{bmatrix}$$

The corresponding closed-loop, suboptimal composite system has eigenvalues 0.840132, 0.259351 $\pm$ 0.334107j, and 0.063358 $\pm$ 0.04656j. The performance indices of the original optimal system and the composite suboptimal system are obtained as 612.1122 and 616.1734, respectively, with a percentage error of 0.66347. The central processing time on the CDC CYBER computer required for the original system is 5.42 sec, whereas that required for the composite system is only 0.44 sec. Throughout the paper, the computations are performed by using the linear-quadratic-regulator (LQR) design package ORACLS and commercial mathematical library (ref. 70).

The responses for the various states and controls for the exact (optimal) system and the composite (suboptimal) system are shown in figures 16 to 23. These responses are obtained for the case when the aircraft trajectories are to be regulated to the equilibrium conditions of $x_{1e} = 190.66$ ft/sec, $x_{2e} = 0$, $x_{3e} = 2000$ ft, and $z_{1e} = z_{2e} = 0$ with equilibrium controls.

The above results clearly show an excellent agreement between the exact system and the composite system. Once again it is noted that the composite control is obtained from the lower-order, slow and fast subsystems with a considerable reduction in the overall computation (refs. 16 to 18).

Concluding Remarks

This paper describes a time-scale method for digital control systems with an application to an aircraft. The idea of digital flight control systems was briefly discussed and their advantages were enumerated. The previous research in digital flight control was summarized. The concepts of singular perturbations and time scales (SPATS) in continuous and discrete control systems were introduced.

Attention was focused on the time-scale analysis of digital control systems. A block-diagonalization procedure was described for developing a method to decouple a discrete system into slow and fast subsystems. Basic properties and numerical aspects of the method were discussed. It was shown that singularly perturbed discrete systems can be viewed as two-time-scale systems. Finally, an aircraft example was given to illustrate the method.

The important problem of optimal control of two-time-scale discrete systems was discussed. The closed-loop, optimal, linear, quadratic regulator for the system requires the solution of a full-order, algebraic, matrix Riccati equation. Alternatively, the original system was decomposed into reduced-order slow and fast subsystems. The closed-loop optimal control of the subsystems requires the solution
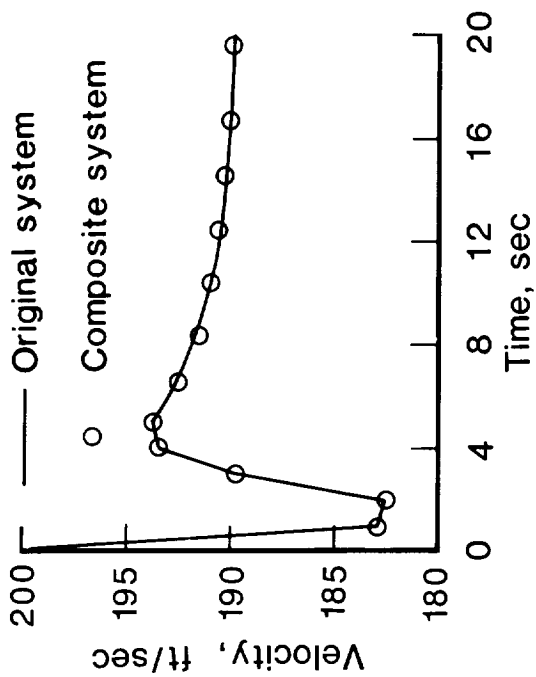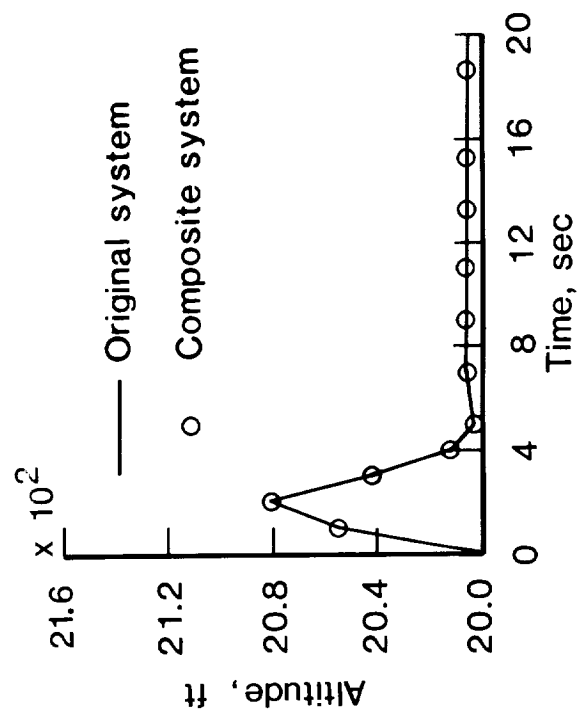
22

Figure 17. Comparison of solutions for pitch angle.

Figure 19. Comparison of solutions for angle of attack.
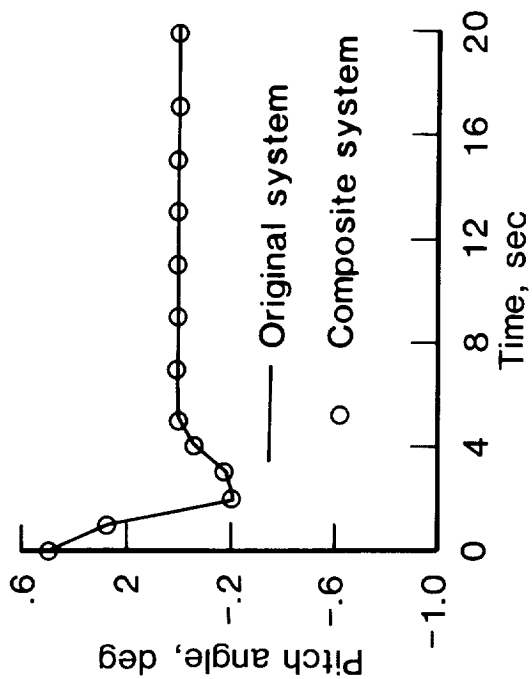

Figure 16. Comparison of solutions for velocity.

Figure 18. Comparison of solutions for altitude.
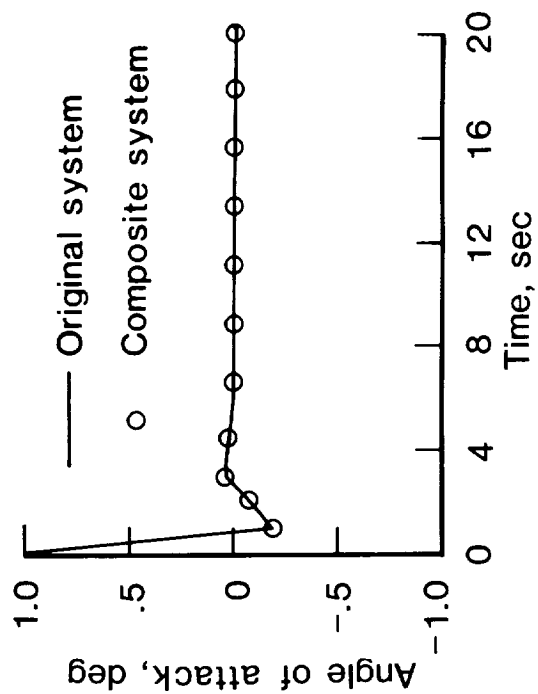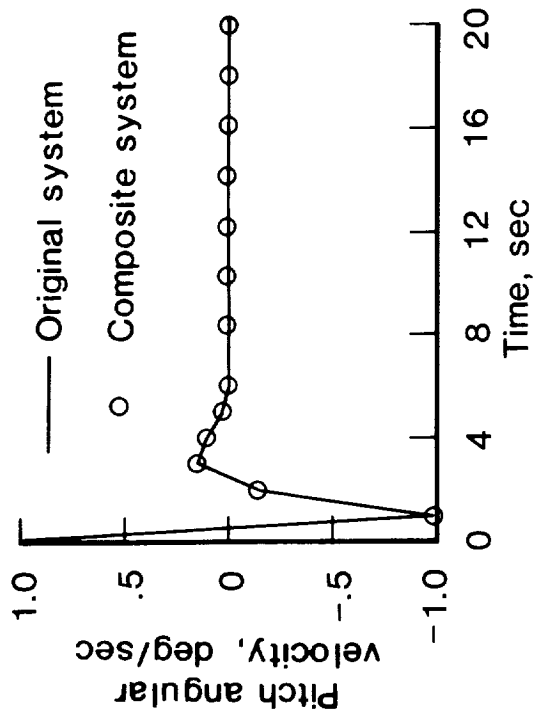
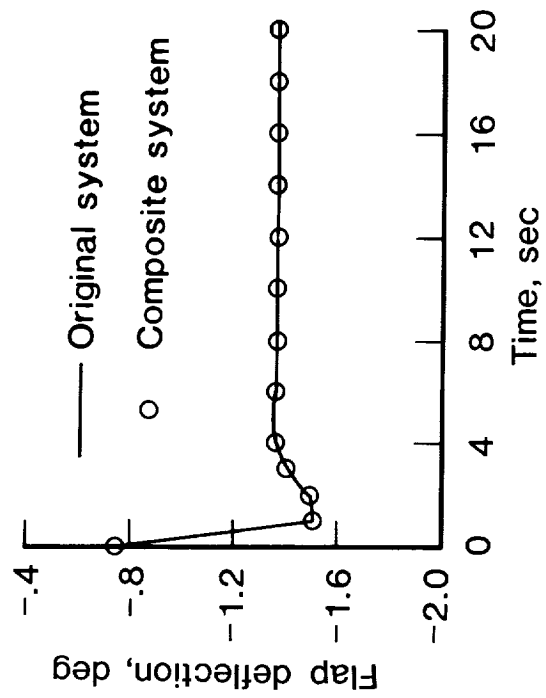Figure 20. Comparison of solutions for pitch angular velocity.

Figure 21. Comparison of solutions for elevator deflection.

Figure 22. Comparison of solutions for flap deflection.
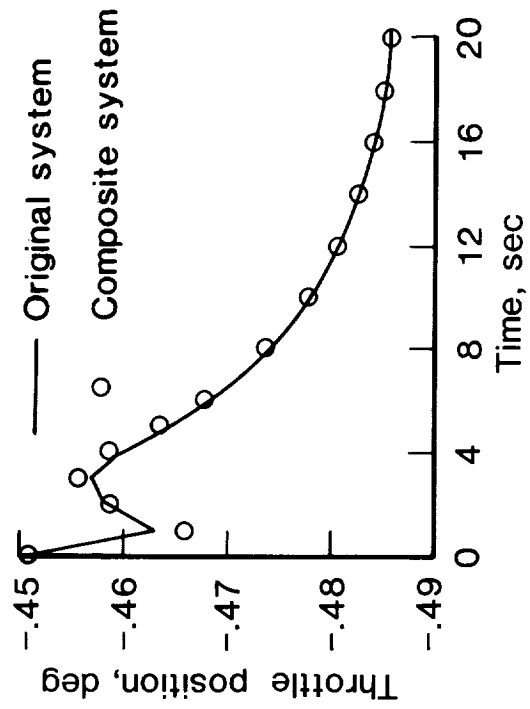
Figure 23. Comparison of solutions for throttle position.

of two algebraic Riccati equations of an order lower than that required for the full-order system. A composite, closed-loop, suboptimal control has been formed from the sum of the slow and fast optimal feedback controls. Again, numerical results obtained for an aircraft model showed very close agreement between the exact (optimal) solution and the composite (suboptimal) solution which is computationally simpler. The main advantage of the method is a considerable reduction in the overall computational requirements for obtaining the closed-loop, optimal control laws of digital flight systems. The significance of the methodology is that it can be used for generating optimal guidance and control strategies for real-time onboard simulation.

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16. Abstract

This paper presents the results of applying the methodology of singular perturbations and time scales (SPATS) to the control of digital flight systems. A block-diagonalization method is described that decouples a full-order, two-time-scale (slow and fast) discrete control system into reduced-order, slow and fast subsystems. Basic properties and numerical aspects of the method are discussed. A composite, closed-loop, suboptimal control system is constructed as the sum of the slow and fast, optimal feedback controls. The application of this technique to an aircraft model shows close agreement between the exact solution and the decoupled (composite) solution. The main advantage of the method is the considerable reduction in the overall computational requirements for the evaluation of optimal guidance and control laws. The significance of the result is that it can be used for real-time onboard simulation. This paper also contains a brief survey of digital flight systems.

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