IMPROVEMENTS TO PHOTOMETRY. I. BETTER ESTIMATION OF DERIVATIVES IN EXTINCTION AND TRANSFORMATION EQUATIONS

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Abstract

Atmospheric extinction in wideband photometry is examined both analytically and through numerical simulations. If the derivatives that appear in the Strömgren-King theory are estimated carefully, it appears that wideband measurements can be transformed to outside the atmosphere with errors no greater than a millimagnitude. A numerical-analysis approach is used to estimate derivatives of both the stellar and atmospheric-extinction spectra, avoiding previous assumptions that the extinction follows a power law. However, it is essential to satisfy the requirements of the sampling theorem, to keep aliasing errors small. Typically, this means that band separations cannot exceed half of the full width at half-peak response. Further work is needed to examine higher-order effects, which may well be significant.

Introduction

The reduction of wideband photometric observations to outside the atmosphere, and thence to some standard system, are incompletely solved problems of long standing. As was shown by Bengt Strömgren [1937], and emphasized by Ivan King [1952], these problems are intimately related; for we may regard the extinction correction as a color transformation that depends on air mass.

This problem is hardly new. In the very earliest photoelectric photometry, Guthnick and Prager [1914] found that "The correction for extinction is one of the most difficult problems for such exact measurements as can be reached by photoelectric methods.... The extinction is dependent on the spectral type to a high degree. Under normal transparency conditions, the ratio of the photoelectric to the visual extinction is about 2.2 for the middle of class B, about 2.0 for class A, etc., about 1.3 for class Ma. It will apparently turn out later that these factors themselves are also functions of the zenith distance...."

Indeed, Forbes [1842] had already found "That the tendency to absorption through increasing thicknesses of air is a diminishing one.... Hence the amount of vertical transmission has always hitherto been greatly overrated,

or the value of extra atmospheric solar radiation greatly
underrated.... The physical cause of this law of absorption
appears to be the non-homogeneity of the incident rays...."

The first analytical treatment of these problems was
made by B.Strömgren [1937], who showed that if the spectral
sensitivity curve of a photometric instrument is fairly
narrow, a Taylor series expansion of the stellar spectral
irradiance curve about the instrumental centroid
wavelength allows the extinction to be expressed in terms
of the monochromatic extinction at this wavelength, with
a correction term proportional to the square of the
instrumental bandwidth. This approach was developed
further by King [1952], using a more compact notation.
King's paper is required reading for anyone who wants to
understand heterochromatic extinction.

The correction terms involve first and second derivatives
of both the stellar spectral irradiance and (in the
extinction problem) the atmospheric transmission. As is
shown by King [1952] and Young [1974], the first derivatives
may be approximated by the colors of the stars and the
reddening of the atmosphere, respectively. King [1952]
shows that the second derivative for the stars can be
neglected, and that -- under certain assumptions that will
be discussed below -- the second derivative for the

atmosphere can be related to the first derivative.

Although the corrections are traditionally made by using color indices as independent variables, Cousins and Jones [1976] found that "no equation involving B-V and U-B only will predict the extinction correctly for all luminosity types and different degrees of reddening.... The difference can exceed 0.01^m. Without more information, direct or inferred, no rigorous colour correction is possible either for extinction or for colour transformation...." Similar conclusions were reached independently by Mandwewala [1976]; see also Blanco [1957], and Fig.14 of Young [1974].

Some years ago, the accepted wisdom was that these difficulties were due to the great width of the UBV passbands, and that intermediate-passband systems such as uvby would prove far superior. However, bandwidth effects are proportional to the square of the passband width, which is about 3 times narrower for uvby than for UBV; hence, if this were the only problem, such difficulties should be nearly an order of magnitude smaller for uvby than for UBV.

But, after the most strenuous efforts at standardization, Olsen [1983] found typical systematic differences between 4-color data from northern and southern stations on the order of 0.004 mag., and unexplained systematic errors of several hundredths of a magnitude for a number

of individual stars. These figures are only about a factor of 2 better than has been done in comparably careful UBV photometry. Manfroid [1985] says that "reduction of many observing runs in the uvby system with various equipment shows that errors as high as .05 magnitude, and more, are not uncommon." Indeed, Manfroid and Sterken [1987] have recently shown systematic errors as large as a third of a magnitude in careful uvby observations, taken at a good site (La Silla), calibrated with dozens of standard stars, and reduced by reliable techniques.

Furthermore, the most precise published photometry appears to be that done in the Geneva system [see Fig.2 of Young, 1984a], which has been reproduced to better than 0.003 mag. for well-observed stars [cf. Table IV of Rufener, 1981]. This precision is all the more remarkable because the Geneva bands are comparable in width to those of the UBV system, and, like those of UBV, are defined by glass rather than interference filters and by the tail of the photocathode response function -- unlike the supposedly superior and "filter-defined" uvby bands. Clearly, there is more to precision than bandwidth alone.

Because the Geneva workers have been extremely careful to determine and use derivatives correctly, one suspects that a careful examination of the derivative problems

would be helpful. I have already suggested [Young, 1974; 1984b] that one problem with the traditional approach is the relatively poor approximation to the first derivatives provided by spectrally undersampled data; unfortunately, all existing photometric systems violate the requirements imposed by the sampling theorem (though the Geneva system is less undersampled than most).

This has been confirmed by Manfroid [1985], who shows that precision can be greatly improved by using a second set of filters slightly displaced from the normal set, to estimate more accurate first derivatives. I show below that both first and second derivatives can be determined accurately, and that such improved estimates lead to much more accurate extinction and transformation corrections.

In the past, precision on the order of 1 percent was regarded as "good enough" for most problems, though this involved an element of circular reasoning: the lack of better measurements prevented anyone from even considering investigations that required much better than 1 percent precision. Today, however, there are a number of problems whose photometric study clearly requires precision on the order of one millimagnitude or better: the detection of planetary systems [Borucki, 1984]; stellar seismology [Fossat, 1984; Hudson, 1984]; inventorying the Sun's comet

cloud [Meinel and Meinel, 1986]; nonlinear dynamics of pulsating white dwarfs [Auvergne and Baglin, 1986]. We may take this as a nominal goal to reach -- roughly an order of magnitude better than current practice. Heintze et al. [1984] and Schmidt-Kaler [1984] have even suggested that still smaller errors could be obtained from the ground.

It is generally accepted that major advances in ground-based photometry will require multi-channel techniques, to remove atmospheric transparency variations. However, the price that must be paid is the problem of calibrating the different channels against one another; this is essentially the transformation problem. Whether we use multichannel instrumentation or not, we cannot expect to do enormously better than 1% if the model used to represent the heterochromatic extinction is no better than 1%.

A number of distinct issues must be resolved. The question of adequate spectral sampling was raised above. But, even with properly sampled data, how should the derivatives be extracted from the data? And, even before these questions can be discussed, there is a conceptual problem with the derivatives that needs clarification; so let us begin with it.

Outline of the Theory

To see why there is a problem with these derivatives, let us review where they come from. Let $I(\lambda)$ be the stellar spectral irradiance function; let $t(\lambda, z)$ be the atmospheric transmission function

$$t(\lambda, z) = \exp[-A(\lambda) M(z)/1.086] \quad , \qquad (1)$$

which King [1952] rather inconveniently calls $Q(\lambda)$. Here $A(\lambda)$ is the wavelength-dependent extinction coefficient in magnitudes per air mass; $M(z)$ is the air-mass function of zenith distance z, discussed at length by Young [1974]; and 1.086 is short for $2.5/\ln(10) = 1.0857362...$, which is the conversion factor between natural logarithms and logs to the base 2.512 (i.e., magnitudes). As the following discussion focuses on the wavelength dependence, we shall usually omit the z dependence of $t(\lambda)$.

If the response function of the instrument (including the telescope optics) is $k(\lambda)$, the quantity measured when we observe this star is

$$L = \int I(\lambda) \, t(\lambda) \, k(\lambda) \, d\lambda \quad . \qquad (2)$$

King [1952] splits the integrand into an instrumental part, $k(\lambda)$, and the rest,

$$s(\lambda) = I(\lambda) \, t(\lambda), \qquad (3)$$

which changes from one observation to the next.

The heart of the Strömgren-King method is to expand $S(\lambda)$ in a Taylor series about some central wavelength, λ_o:

$$S(\lambda) = S(\lambda_o) + S'(\lambda_o)(\lambda - \lambda_o) + \frac{1}{2}S''(\lambda_o)(\lambda - \lambda_o)^2 + \ldots, \quad (4)$$

where primes denote wavelength derivatives. The integral (2) can then be done termwise:

$$L = S \int R(\lambda)\,d\lambda + S' \int (\lambda - \lambda_o)R(\lambda)\,d\lambda$$

$$+ \frac{1}{2}S'' \int (\lambda - \lambda_o)^2 R(\lambda)\,d\lambda + \ldots. \quad (5)$$

Thus, the part of the measured light that changes is expressed in terms of the transmitted spectral distribution S and its derivatives, evaluated at λ_o; and the invariant instrumental influence is expressed by the moments of the response function R about λ_o. Now, if λ_o is chosen to be the centroid wavelength

$$\lambda_o = \frac{\int \lambda\, R(\lambda)\,d\lambda}{\int R(\lambda)\,d\lambda}, \quad (6)$$

the S' term in (5) vanishes, and we have only

$$L = \int R(\lambda)\,d\lambda \left[S + \frac{1}{2}\mu_2^{\,2} S'' \right], \quad (7)$$

where $\mu_2^{\,2}$ is the normalized second central moment of $R(\lambda)$:

$$\mu_2^{\,2} = \frac{\int (\lambda - \lambda_o)^2 R(\lambda)\,d\lambda}{\int R(\lambda)\,d\lambda}. \quad (8)$$

King then expresses S" in terms of the derivatives of $I(\lambda)$ and $t(\lambda)$:

$$S'' = I''t + 2 I't' + It'' \, , \tag{9}$$

which allows the measurement to be expressed in magnitudes as follows:

$$m_{obs} = m_o + A(\lambda_o)M(z) - 1.086 \ln (1 + x) \, , \tag{10}$$

where m_{obs} is the observed instrumental magnitude, m_o is the magnitude that would have been measured outside the atmosphere, and

$$x = \frac{1}{2} \left(\frac{\mu_2}{\lambda_o} \right)^2 \left[(\lambda^2 I''/I) - \frac{M(z)A(\lambda)}{1.086} \left\{ (\lambda^2 A''/A) + \right. \right.$$

$$\left. \left. (\lambda A'/A) [2(\lambda I'/I) - \frac{M(z)A(\lambda)}{1.086}(\lambda A'/A)] \right\} \right] \, . \tag{11}$$

All the parenthetical expressions are evaluated at λ_o.

The Derivative Problem

Now, the difficulty is to assign proper significance to the derivatives I' and I", evaluated at λ_o. What is intended is obviously not what is said literally, for stellar spectra are cluttered with absorption lines. If λ_o happens to fall on the sloping side of a line, I' will be enormous, and obviously untypical of the general run of the spectrum in the region a few hundred Angstroms wide that is of interest

in photometry. Both Strömgren and King neatly sidestepped this question by considering only black bodies in their examples. Yet we must deal with real stars, whose spectra plainly cannot be represented accurately by any low-order Taylor series.

Evidently, the only sensible physical interpretation of these derivatives is that they refer not to the true stellar energy distribution, which fluctuates wildly, but to some smooth function that approximates it, and that has well-behaved derivatives. As only first and second derivatives appear in (11), we may suppose that a parabolic approximation is used.

If we write the true stellar spectral irradiance as

$$I(\lambda) = I_s(\lambda) + I_r(\lambda) \; , \tag{12}$$

where I_s and I_r are the smoothed intensity and the remainder, after subtracting the smoothed irradiance from the true one, we want to choose the smoothing so that all the preceding equations are true when I_s and its derivatives are used in place of the true $I(\lambda)$, I', and I''.

In particular, suppose we use (12) in (2), so that

$$L = \int I_s(\lambda) \; t(\lambda) \; k(\lambda) \; d\lambda + \int I_r(\lambda) \; t(\lambda) \; R(\lambda) \; d\lambda \; . \tag{13}$$

Then the second term in (13) must be zero, if King's formulae are to be true for I_s. To make

$$\int I_r(\lambda) \, t(\lambda) \, R(\lambda) \, d\lambda = 0 \, , \qquad (14)$$

expand $t(\lambda)$ in a Taylor series about λ_o [chosen according to (6)]. This expansion is generally well-defined, because $t(\lambda)$ is quite smooth, provided that we avoid spectral regions containing sharp molecular absorption lines. Then

$$t(\lambda_o) \int I_r(\lambda) \, R(\lambda) \, d\lambda + t'(\lambda_o) \int (\lambda - \lambda_o) \, I_r(\lambda) \, R(\lambda) \, d\lambda +$$

$$\frac{1}{2} t''(\lambda_o) \int (\lambda - \lambda_o)^2 I_r(\lambda) \, R(\lambda) \, d\lambda = 0 \, . \qquad (15)$$

Because $t(\lambda_o)$ and its derivatives depend on air mass, and change from night to night, (15) can be generally true only if each of these integrals vanishes. For a given star, and hence a given $I(\lambda)$, equating each integral to zero provides three conditions on I_r, and hence on I_s. These are the three conditions required to choose the parabolic function $I_s(\lambda)$ uniquely.

In what follows, I assume that this choice has been made, so that all the formulae refer to I_s and its well-defined derivatives, instead of to I. The subscript s will be suppressed, but must be understood to be present throughout.

Evaluating the Derivatives

Although the discussion above clarifies the meaning of the derivatives that appear in (11), it does not provide any

way to evaluate them in practice. The whole point of broadband photometry is to avoid spreading the light of faint stars out into spectra, which would be necessary to evaluate the moment integrals in (15). We need to obtain the derivatives in (11) from the photometric data themselves.

King [1952] used a color index to estimate I', and this has been the traditional practice ever since. He pointed out that I" acts primarily as a zero-point shift that is independent of air mass, so that for many purposes it can be ignored. Young and Irvine [1967] and Young [1974] use the reddening power of the atmosphere to estimate A'; this is exactly analogous to using a stellar color index to estimate I'.

However, the A" term cannot be ignored. King [1952] argued that the wavelength-dependent part of the extinction is mainly Rayleigh scattering, and hence proportional to λ^{-4}. Because A' appears in (11) only in the combination (λ A'/A), which is the logarithmic derivative of A(λ), and hence a constant if A(λ) is a power law, King argued that the value of this constant is -4. His assumption of power-law extinction also allowed the second derivative A" to be expressed in terms of A'.

If we consider the scattering part of the extinction, even the Rayleigh scattering is not exactly a power law,

because the dispersions of the refractivity and the anisotropy of the polarizability make the molecular-scattering extinction steeper than λ^{-4} [Young, 1982; Bates, 1984; Nicolet, 1984]. On the other hand, the aerosol extinction is much flatter, and is usually near $1/\lambda$. Because the aerosol extinction dominates at long wavelengths, and the molecular scattering at short wavelengths, the logarithmic derivative of $A(\lambda)$ is closer to -4 in the violet and closer to -1 in the red.

On top of this, there is very strong absorption by ozone below about 350 nm, and more than 1% absorption in the Chappuis bands between about 500 and 680 nm [Vigroux, 1953; Inn and Tanaka, 1953; Griggs, 1968]. This band absorbs more than 0.1 percent in the zenith between about 450 and 850 nm.

Thus, $A(\lambda)$ cannot be regarded as a power law function in accurate work, despite the pedagogical utility of this crude approximation. However, if $A(\lambda)$ is not a power law, then not only is $(\lambda A'/A)$ wavelength-dependent, but we cannot express the second derivative in terms of the first. In fact, the A" term

$$\frac{\lambda^2 A"}{A} = \frac{d^2(\ln A)}{d(\ln \lambda)^2} + \left(\frac{d \ln A}{d \ln \lambda}\right)^2 \qquad (16)$$

involves both the first and the second logarithmic derivatives of $A(\lambda)$. If $A(\lambda)$ were a power law, its second

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logarithmic derivative would vanish, and (16) would provide

a very simple estimate for the A" term in (11). In reality,

we must estimate the second logarithmic derivative of $A(\lambda)$

in addition to the first.

In 1952, only one color index (the old International

color index) was in common use, and the plethora of modern

multicolor systems had not yet been invented. Even the UBV

system had not been formally introduced. Thus it was quite

natural for King to try to make a single color index do

everything: only one was available. But today, we need not

be so restricted.

If we measure only two points on a function, we can

fit a straight line through them. This linear fit allows

us to estimate both the function and its slope at any point.

If we have three data, we can fit a parabola, and determine

the curvature (i.e., a second derivative) as well. Even if

the data are unequally spaced, standard techniques of

numerical analysis allow us to find these derivatives at

any point.

I propose to adopt this numerical-analysis point of view,

so as to estimate the A" term. Though this may appear novel,

it is really quite similar to what is already done for

stars in a number of multicolor systems. If the bands are

nearly equally spaced in wavelength, we can use one color

index to estimate the slope of a star's spectrum, and the difference of two neighboring color indices to estimate its curvature. In fact, such curvature indices as m_1 and c_1 in the uvby system are already quite familiar to photometrists.

If three bands are exactly equally spaced in wavelength, or three samples of any function are equally spaced in its argument, it is well known that the first derivative at the central point is better estimated by the slope between the two end points than by the slope between the central one and either of its neighbors, provided that the samples are sufficiently close together that higher-order terms can be neglected. (This fact is used in the reduction of photometry in the Geneva system; this must surely be another reason for the excellent precision of the published Geneva results.) The reason is simply that three points allow parabolic approximation, which is generally better than the linear approximation through two neighboring points.

If the three points are unequally spaced, the derivative of the function can still be expressed as a simple weighted sum of the three ordinates, following standard Lagrangian interpolation methods. In the general case, the weight of the central point is not zero, as it is for equal spacing. The details of the derivation are given in the Appendix.

However, this process only makes sense if the bands overlap enough that a polynomial passing through their average intensities at their effective wavelengths is a good representation of the smoothed spectral irradiance function, as defined above. In other words, the smoothed irradiance must be sampled at intervals (i.e., band spacings) close enough to satisfy the sampling theorem [Young, 1974]. No existing photometric system does this, though the Geneva system comes close.

Numerical Simulations

To illustrate the improvements possible with adequate sampling and accurate data reduction, I have done several simple numerical simulations of wideband photometry and reductions. For simplicity, I used symmetrical passbands, to keep third-order effects negligible. Both inherently smooth spectra (a set of black bodies, and a set of artificial continua parabolic in the logarithm of spectral irradiance) and realistic spectra taken from the tables of Gunn and Stryker [1983] were multiplied by standard atmospheric transmission functions for 1.0, 1.5, 2.0, and 2.5 air masses; multiplied by cosine-squared response functions 500 Angstroms wide at half maximum; and integrated, to give synthetic observational data. Similar calculations without the atmospheric transmission gave true extra-atmospheric

values for each instrumental system.

In each case, the central band of a 3-band system was kept fixed at 4500 Angstroms; band spacings of 100, 200, 300, and 500 Angstroms were used for the stars. The columns of Table I show the standard deviation (root-mean-square residual per degree of freedom); the maximum residual in the middle (4500 A) band -- a rough measure of internal error; and the maximum error in the extra-atmospheric magnitude in the middle band calculated from the fitted parameters. This last column is a rough estimate of external error.

The table concentrates on the results for the middle band, because it is the same for all cases; the outer bands move as the band spacing changes, and so are not strictly comparable from case to case. Nevertheless, it is worth remarking that the errors in the shortest-wavelength band are about double those for the middle band so long as the spectra are smooth, or the bands are closer than 300 Angstroms. For 300 A spacing, the errors at the shortest wavelength are about 4 times those for the middle band; for 500 A spacing, they are about 5 times larger.

Thus, the Table suggests that sampling for real stellar spectra is adequate at 200 A spacing; marginal at 300 A; and wholly inadequate at 500 A, which is the full width at half-maximum of the bands. The aliasing errors are thus small

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for band spacings below half the FWHM, and increase rapidly at larger spacings. Furthermore, as only a dozen stars have been used, and we may expect the worst error from a larger set of stars to be somewhat larger, it appears that millimagnitude accuracy can be achieved with bands as broad as 500 A if their spacing is about 250 A.

The reader should bear in mind that these bands were perfectly symmetrical, so that third-order terms (which involve the third central moment of the passband) were eliminated. Real filters always produce markedly asymmetrical passbands, so we may well need to include the next-order terms in the expansions. Unfortunately, the resources available for this work did not allow a thorough investigation of the spacing required to reach a given level of precision with realizable passbands.

As the atmosphere is part of the instrumental system, the success in transforming these pseudo-observations from inside to outside the atmosphere to millimagnitude accuracy suggests that the transformation problem between different instrumental systems can also be satisfied with properly sampled data, using this same numerical-interpolation approach.

Conclusions

Bandwidth effects do not seem to be a serious limitation

to the precision and accuracy of broadband photometry, if
they are modelled correctly. This requires both a more
detailed understanding of the derivatives that appear in
the classical Strömgren-King type of analysis, and the use
of well-established numerical-analysis methods to determine
all the required derivatives directly from the
observational data.

Numerical simulation experiments show that
millimagnitude accuracy can be achieved -- roughly an order
of magnitude improvement over conventional methods -- even
with bands as broad as 500 Angstroms (full width at half
maximum). This is similar to the width of the UBV bands.
However, much closer spacing (about 200 Angstroms) is
required than the roughly 1000 Angstrom spacing of the UBV
bands. Thus, the low accuracy of UBV photometry seems
primarily to be due to its violation of the sampling
theorem, as pointed out earlier [Young, 1974].

As the uvby 4-color system is even more undersampled,
one would expect even larger aliasing errors to occur in
it; and, indeed, errors exceeding a tenth of a magnitude
are reported by Manfroid and Sterken [1987], even in careful
work where many standards are used and the extinction is
well determined.

Appendix

The extension of the Strömgren-King method to higher-order terms requires derivatives of both the smoothed stellar spectral irradiance I_s and the atmospheric extinction A, as functions of wavelength. Suppose we sample these functions at three unequally spaced wavelengths determined by the instrumental filters. We need to know the relative spacings of the samples (i.e., the photometric passbands), which are required in the Lagrangian interpolation and differentiation formulae.

Let us suppose that the middle sample (band) is displaced a fraction f of the separation of the outer two from their midpoint. Thus, f may run from $-1/2$ at the shortest of the three wavelengths to $+1/2$ at the longest; it would be 0 if the middle band were exactly midway between the others. (Obviously, we will try to choose filters that make f small.)

For the sake of generality, let us use x for the independent variable and y for the dependent variable, rather than wavelength, spectral irradiance, extinction, or any other specific quantity. Our three samples are at x_0, x_1, and x_2, and the function values are y_0, y_1, and y_2. In terms of f, the middle sample is at

$$x_1 = \frac{1}{2}(x_0 + x_2) + f(x_2 - x_0) . \qquad (A1)$$

Now, if we want the function value somewhere in the interval

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from x_0 to x_2, let us similarly specify the interpolation position by a parameter g defined like f, so that g runs from $-1/2$ at x_0 to $+1/2$ at x_2; $g = f$ at the middle sample. In terms of f and g, the interpolating polynomial is

$$y(g) = \frac{(g - f)(g - 1/2)}{(f + 1/2)} y_0 + \frac{(g + 1/2)(g - 1/2)}{(f + 1/2)(f - 1/2)} y_1$$

$$+ \frac{(f - g)(g + 1/2)}{(f - 1/2)} y_2 , \tag{A2}$$

and its derivative is

$$y'(-1/2) = -\frac{(f + 3/2)}{(f + 1/2)} y_0 - \frac{1}{(f + 1/2)(f - 1/2)} y_1$$

$$+ \frac{(f + 1/2)}{(f - 1/2)} y_2 \tag{A3a}$$

at x_0;

$$y'(f) = +\frac{(f - 1/2)}{(f + 1/2)} y_0 + \frac{2 f}{(f + 1/2)(f - 1/2)} y_1$$

$$- \frac{(f + 1/2)}{(f - 1/2)} y_2 \tag{A3b}$$

at x_1; and

$$y'(+1/2) = -\frac{(f - 1/2)}{(f + 1/2)} y_0 + \frac{1}{(f + 1/2)(f - 1/2)} y_1$$

$$+ \frac{(f - 3/2)}{(f - 1/2)} y_2 \tag{A3c}$$

at x_2. At all three points, the second derivative is

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$$y'' = + \frac{2}{(f + 1/2)} y_0 - \frac{2}{(f + 1/2)(f - 1/2)} y_1$$

$$+ \frac{2}{(f - 1/2)} y_2 \; , \tag{A4}$$

Independent of g, as the second derivative of a parabola is a constant.

Now we can write Eq.(11) in the text as

$$x = \frac{1}{2} \left(\frac{\mu_2}{\lambda_0} \right)^2 \left[m'' + (m')^2 - \frac{M(z)A(\lambda)}{1.086} \left\{ a'' + (a')^2 + a'[2 m' - \frac{M(z)A(\lambda)}{1.086} a'] \right\} \right] , \tag{A5}$$

where

$$m' = \frac{d \ln I}{d \ln \lambda} \; ; \qquad a' = \frac{d \ln A}{d \ln \lambda} \; ;$$

$$m'' = \frac{d^2 (\ln I)}{(d \ln \lambda)^2} \; ; \qquad a'' = \frac{d^2 (\ln A)}{(d \ln \lambda)^2} \; ;$$

and all expressions are evaluated at λ_0.

We now set $a_i = \ln A_i$, for $i = 1, 2,$ and 3 . Then, for $g = f$ (the middle band), Eq. (A3b) gives

$$a' = \left[\ln \left(\frac{\mu_2}{\lambda_0} \right) \right]^{-1} \left[\frac{(f - 1/2)}{(f + 1/2)} a_0 + \frac{2 f}{(f + 1/2)(f - 1/2)} a_1 \right.$$

$$\left. - \frac{(f + 1/2)}{(f - 1/2)} a_2 \right] \tag{A6}$$

and there will be similar expressions, based on Eqs. (A3a) and (A3c), for the other two bands. For all three, we find

$$a'' = \left[\ln\left(\frac{\mu_2}{\lambda_0}\right)\right]^{-2}\left[\frac{2}{(f + 1/2)}a_0 + \frac{2}{(f + 1/2)(f - 1/2)}a_1 + \frac{2}{(f - 1/2)}a_2\right].\tag{A7}$$

Because magnitudes are negative logs to the base 2.512... instead of natural logarithms, the equations for m' and m'' are similar to these, but contain additional factors of $-1.0857...$. The magnitudes, unlike the extinction coefficients (which are measured on an absolute scale), contain additive zero-point terms due to the instrumental sensitivity differences among bands. Thus, in terms of the extra-atmospheric monochromatic magnitudes m_0, m_1, and m_2, we have

$$m' = \left[\ln\left(\frac{\mu_2}{\lambda_0}\right)\right]^{-1}\left[\frac{(f - 1/2)}{(f + 1/2)}m_0 + \frac{2f}{(f + 1/2)(f - 1/2)}m_1 - \frac{(f + 1/2)}{(f - 1/2)}m_2\right]\left(\frac{-1}{1.0857}\right)\tag{A8}$$

for the middle band at $g = f$, and corresponding equations, mutatis mutandis, for the other two bands. Finally,

$$m'' = \left[\ln\left(\frac{\mu_2}{\lambda_0}\right)\right]^{-2}\left[\frac{2}{(f + 1/2)}m_0 + \frac{2}{(f + 1/2)(f - 1/2)}m_1\right.$$

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$$+ \frac{2}{(f - 1/2)} m_2 \Bigg] \left(\frac{1}{1.0857}\right)^2 \qquad (A9)$$

at all three bands. Note that the factor of −1.0857... is

squared here, and hence positive. Also, the magnitudes m_0

to m_2 here implicitly contain zero-point terms that must

be evaluated; a constraint such as $z_0 + z_1 + z_2 = 0$

must be imposed to prevent the matrix of the normal

equations from being singular.

Thus, the equation of condition for photometric

reductions is the result of combining Eqs. (A5 − A9) with

Eq. (10) in the text. In this combination, note that the

terms containing a', a", m', and m" always involve either

second derivatives alone, or squares or products of first

derivatives, so that the factor

$$W = \left[\frac{\mu_2 / 1.086}{\lambda_1 \ln (\lambda_2 / \lambda_0)} \right]^2 \qquad (A10)$$

may be removed from all terms. The quantity W in Eq. (A10)

plays a role similar to that of the old parameter of the

same name in Eqs. (3.1.56) and (3.1.57) of Young [1974], but

the new equations are more exact and involve fewer

approximations. In particular, I now evaluate the second

derivatives of A and I explicitly from the data, as well as

keeping the logarithm function intact.

The greater complexity of these equations is not a serious obstacle to photometric reductions. If we are to approach the precision that has long been the prerogative of the astrometrists, it is only reasonable that our equations must begin to approach theirs in complexity. In any case, computers are now so large and fast that there is no difficulty in solving for a slightly larger number of parameters: here we have 3 magnitudes for each star; 3 extinction coefficients for each night; two independent instrumental zero points (which should remain fixed if the instrument is well designed and constructed); the three bandwidth parameters W, one for each band; and the parameter f that specifies the relative band spacing.

Thus, only the last 6 parameters describe the instrumental system. As Manfroid and Heck [1983, 1984] have shown, even more instrumental parameters can be well determined if data from several nights are combined. And, in fact, numerical experiments show that these parameters can be determined adequately with a modest number of observations. A particular advantage of this more precise model is that the data are represented more closely than with the older approximations, so that (if the observations are carefully done) the residuals from the least-squares fit are smaller. This means that fewer observations per parameter are required to reach a given level of precision.

However, one must bear in mind that the model is useful only if the bands overlap enough to satisfy the sampling theorem. I have pointed out before [Young, 1974] that no existing system does this. Further work is needed to determine the necessary spacing; but preliminary numerical experiments suggest that bands should be spaced about 1/2 of their full width at half maximum. Thus, for bands as wide as those of the UBV system, a spacing on the order of 200 A is suggested.

Acknowledgments

References

Auvergne, M., and A. Baglin, The dynamics of a
variable star. Phase-space reconstruction
and dimensional computation, Astron.
Astrophys., 168, 118-124, 1986.

Bates, D. R., Rayleigh scattering by air, Planet.
Space Sci., 32, 785-790, 1984.

Blanco, V. M., Band-width effects in photoelectric

photometry, Ap. J., 125, 209-212, 1957.

Borucki, W. J. (1984), Photometric precision needed
for planetary detection, in Proceedings of
the Workshop on Improvements to Photometry,
edited by W. J. Borucki and A. T. Young (NASA CP-2350), pp.
15-27, NASA Ames Research Center, Moffett Field, 1984.

Cousins, A. W. J., and D. H. P. Jones, Numerical
simulation of natural photometric systems,
Mem. R. Astron. Soc., 81, 1-23, 1976.

Heintze, J. R. W., C. de Jager, and W. van der Veen, The
accuracy limit of ground-based stellar
photometry, in Space Research Prospects in
Stellar Activity and Variability, edited by
F. Praderie, pp. 157-167, Observatoire de Paris,
Meudon, 1984.

Forbes, J. D., On the transparency of the
atmosphere and the law of extinction of the
solar rays in passing through it, Phil. Trans.,
132, 225-273, 1842.

Fossat, E., Should one measure radial velocity or
brightness fluctuations?, in Space Research
Prospects in Stellar Activity and Variability,
edited by F. Praderie, pp. 77-88, Observatoire
de Paris, Meudon, 1984a.

Fossat, E., Atmospheric limitations in stellar
seismology -- should one measure radial
velocity or brightness fluctuations?, in
Proceedings of the Workshop on Improvements to
Photometry, edited by W. J. Borucki and A. T. Young
(NASA CP-2350), pp. 68-78, NASA Ames Research Center,
Moffett Field, 1984b.

Griggs, M., Absorption coefficients of ozone in
the ultraviolet and visible regions, J. Chem.

Phys., <u>49</u>, 857-859, 1968.

Gunn, J. E., and L. L. Stryker, <u>Astrophys. J. Suppl.</u>, <u>52</u>, 121, 1983.

Guthnick, P., and R. Prager, Photoelektrische Untersuchungen an spektroskopischen Doppelsternen und an Planeten, <u>Veroff. Kgl. Sternwarte Berlin-Babelsberg</u>, <u>1</u>, 1-68, 1914.

Hudson, H. S., A survey for photometric variability from space, in <u>Proceedings of the Workshop on Improvements to Photometry</u>, edited by W. J. Borucki and A. T. Young (NASA CP-2350), pp. 43-54, NASA Ames Research Center, Moffett Field, 1984.

Inn, E. C. Y., and Y. Tanaka, Absorption coefficient of ozone in the ultraviolet and visible regions, <u>J. Opt. Soc. Amer.</u>, <u>43</u>, 870-873, 1953.

King, I., Effective extinction values in wide-band photometry, <u>Astron. J.</u>, <u>57</u>, 253-258, 1952.

Mandwewala, N. J., Analysis of Rufener's method for the atmospheric extinction reduction, <u>Publ. Obs. Genève Ser. A, Fasc. 82 (Arch. Sci, Genève 29)</u> 119-148, 1976.

Manfroid, J., On photometric standards and color transformation, in <u>I. A. U. Symposium 111, Calibration of Fundamental Stellar Quantities</u>, edited by A. G. Davis Philip, pp. 495-497, Reidel, Dordrecht, 1985.

Manfroid, J., and A. Heck, A generalized algorithm for efficient photometric reductions, <u>Astron. Astrophys.</u>, <u>120</u>, 302-306, 1983.

Manfroid, J., and A. Heck, On the accuracy of the reduction of uvby photometry, <u>Astron.</u>

Astrophys., <u>132</u>, 110-114, 1984.

Manfroid, J., and C. Sterken, Instrumental effects
and the Strömgren photometric system, Astron.
Astrophys. Suppl., <u>70</u>, ???-???, 1987.

Meinel, A. B., and M. P. Meinel, Close star encounters,
Bull. Amer. Astron. Soc., <u>18</u>, 711, 1986.

Nicolet, M., On the molecular scattering in the
terrestrial atmosphere: an empirical
formula for its calculation in the
homosphere, Planet. Space Sci., <u>32</u>, 1467-1468, 1984.

Olsen, E. H., Four-colour uvby and H-beta photometry
of A5 to G0 stars brighter than 8ᵐ.3, Astron.
Astrophys. Suppl. Ser., <u>54</u>, 55-134, 1983.

Rufener, F., Third catalogue of stars measured in
the Geneva Observatory Photometric System,
Astron. Astrophys. Suppl. Ser., <u>45</u>, 207-366, 1981.

Schmidt-Kaler, T., Ground-based solar and stellar
intensity measurements: practical limits of
accuracy, in Space Research Prospects in
Stellar Activity and Variability, edited by
F. Praderie, pp. 169-174, Observatoire de Paris,
Meudon, 1984.

Strömgren, B., in Handbuch der Experimentalphysik,
Band 26, Astrophysik, edited by B. Strömgren,
pp. 321-564, Akademische Verlagsgesellschaft,
Leipzig, 1937.

Vigroux, E., Contribution à l'étude expérimentale
de l'absorption de l'ozone, Ann. Phys. Paris (12), <u>8</u>,
709-762, 1953.

Young, A. T., and W. M. Irvine, Multicolor
photoelectric photometry of the brighter
planets. I. Program and procedure, Astron. J., <u>72</u>,

945-950, 1967.

Young, A. T., in Methods of Experimental Physics, Vol.
    12 (Astrophysics, Part A: Optical and Infrared),
    edited by N. Carleton, pp. 123-192, Academic
    Press, New York, 1974.

Young, A. T., Rayleigh scattering, Phys. Today, 35, 42-48,
    1982.

Young, A. T., History of photometric precision, in
    Proceedings of the Workshop on Improvements to
    Photometry, edited by W. J. Borucki and A. T. Young
    (NASA CP-2350), pp. 8-14, NASA Ames Research Center, Moffett
    Field, 1984a.

Young, A. T., Filter and passband problems, in
    Proceedings of the Workshop on Improvements to
    Photometry, edited by W. J. Borucki and A. T. Young
    (NASA CP-2350), pp. 217-221, NASA Ames Research Center,
    Moffett Field, 1984b.

Table 1. Standard Deviations, Maximum Residuals, and Maximum
Extrapolation Errors of Extinction Fits.

Data set	band separation	overall std.dev.	middle band max.resid.	middle ban max. error
Black bodies	300 A	0.000023	0.000033	0.0002
Parabolic	300 A	0.000078	0.00011	0.0006
Gunn-Stryker	100 A	0.000071	0.00023	0.0005
Gunn-Stryker	200 A	0.000092	0.00022	0.0006
Gunn-Stryker	300 A	0.00020	0.00025	0.0008
Gunn-Stryker	500 A	0.00067	0.00050	0.0018