Revised November 1987

SOME NONLINEAR DAMPING MODELS IN FLEXIBLE STRUCTURES

A.V. Balakrishnan [†]
Electrical Engineering Department
UCLA

September 1987

Abstract

We introduce a class of nonlinear damping models with application to flexible flight structures characterized by low damping. We are able to obtain approximate solutions of engineering interest for our model using the classical "averaging" technique of Krylov and Bogoliubov. The results should be considered preliminary pending further investigation.

54

1. Introduction

The problem of characterizing the damping mechanism in flexible structures has received renewed attention in recent years in connection with the need to stabilize flexible flight structures such as antennas deployed in space. The damping models even when simplified to be linear appear to lead to rather complex mathematics if the structure is described by partial differential equations and much progress has been made (the analyticity of the generated semigroup has been shown to be essential). But experimental evidence as in SCOLE [1] seems to support the need for nonlinear models — the decrement is much smaller than predicted by linear models. Some of the difficulty inherent in handling nonlinear models is offset by the fact that damping, whatever its nature, is still small. This opens up in particular the feasibility of obtaining approximate solutions using the classical averaging method of Krylov-Bogoliubov [2].

In this paper we study a class of nonlinear models and approximate the response by the Krylov-Bogoliubov technique. We use a modal expansion and neglect off-diagonal terms. The emphasis is on useful engineering solutions rather than abstract mathematics.

We begin in Section 2 with the primary nonlinear damping model for the simplest system — the one-dimenisonal or single-mode case. We emphasize in particular one feature that emerges, viz., the potential lack of identifiability from response data. In Section 3 we generalize to the multi- (non-finite-) dimensional case. In Section 4 we show the relevance of the Krylov-Bogoliubov technique for approximating solutions to nonlinear boundary feedback. We may mention that there is much work — even classical in nature — on nonlinear oscillations such as the nonlinear pendulum where the spring constant is no longer linear; however, relatively little attention appears to have been paid to the small nonlinear damping term case.

2. Single-mode Example

To illustrate ideas, let us begin with a one-dimensional (single-mode) example:

$$\ddot{x}(t) + \varepsilon D(x, \dot{x}) + \omega^2 x(t) = 0 \tag{2.1}$$

where the dots indicate time-derivatives, as usual. We assume that:

$$D(x, \dot{x})\dot{x} \geq 0 \tag{2.2}$$

so that for $E(t)$, the energy

$$E(t) = \frac{1}{2}(\dot{x}(t)^2 + \omega^2 x(t)^2) \tag{2.3}$$

we have

$$\frac{d}{dt}E(t) = -\varepsilon D(x, \dot{x})\dot{x} \leq 0 \tag{2.4}$$

satisfying the energy nonincrease requirement. The particular choice for $D(x, \dot{x})$ we shall make is:

$$D(x, \dot{x}) = 2\omega\zeta\dot{x} + \gamma x^{2m}|x|^{\alpha} \dot{x}^{(2n+1)}|\dot{x}|^{\beta} \tag{2.5}$$

where m, n are nonnegative integers,

$$0 \leq \alpha, \beta \quad \text{and} \quad 0 \leq \alpha + \beta < 1; \quad 0 < \zeta < 1, 0 < \gamma < 1.$$

For small enough ε we may apply the averaging method of Krylov-Bogoliubov [2, 5]. Thus, we write for the approximate solution:

$$x(t) = a(t) \sin(\omega t + \phi(t)) \tag{2.6}$$

where the amplitude function $a(t)$ and the phase function $\phi(t)$ are slowly varying over the period $T = 2\pi/\omega$. According to the K-B approximation [2]:

$$\frac{da}{dt} = -\frac{\varepsilon}{\omega}K_0(a) \tag{2.7}$$

where

$$K_0(a) = \frac{1}{2\pi} \int_0^{2\pi} D(a \sin\phi, a\omega \cos\phi) \cos\phi \, d\phi \tag{2.8}$$

and

$$\frac{d\phi}{dt} = \frac{\varepsilon}{\omega a} P_0(a) \tag{2.9}$$

$$P_0(a) = \frac{1}{2\pi} \int_0^{2\pi} D(a \sin \phi, \ a\omega \cos \phi) \sin \phi \ d\phi . \tag{2.10}$$

Now we can readily calculate that for our choice, because of (2.2),

$$P_0(a) = 0$$

$$
\begin{aligned}
K_0(a) &= 2\omega\zeta a \left[\frac{1}{2\pi} \int_0^{2\pi} \omega \cos^2\phi \ d\phi \right] \\
&\quad + \omega^{1+\beta} \gamma \, a^{2m+2n+1+\alpha+\beta} \left[\frac{1}{2\pi} \int_0^{2\pi} \sin^{2m}\phi \, \cos^{2n+2}\phi \, |\sin \phi|^\alpha \, |\cos \phi|^\beta \ d\phi \right] \\
&= \omega^2 \zeta a + a^{2m+2n+1+\alpha+\beta} \gamma \, \omega^{2n+1+\beta} \mu
\end{aligned}
$$

where

$$\mu = \frac{1}{2\pi} \int_0^{2\pi} \sin^{2m}\phi \, \cos^{2n}\phi \, \cos^2\phi \, |\sin \phi|^\alpha \, |\cos \phi|^\beta \ d\phi \tag{2.11}$$

and is a constant less than $1/2$. Hence letting $p = 2m + 2n$, we have

$$\frac{da}{dt} = -\varepsilon(\omega\zeta a + a^{p+1+\alpha+\beta} \omega^{2n+\beta} \gamma\mu) \tag{2.12}$$

We may set $\varepsilon = 1$ without loss of generality since we may absorb it into ζ and γ. Then

$$t = -\int_{a(0)}^{a(t)} \frac{da}{\omega\zeta a + a^{p+1+\alpha+\beta}\omega^{2n+\beta}\gamma\mu} \tag{2.13}$$

yielding

$$a(t) = a(0)e^{-t\omega\zeta} \left[1 + a(0)^{p+\alpha+\beta} \omega^{2n-1+\beta} \frac{\gamma\mu}{\zeta} (1 - e^{-t\omega\zeta(p+\alpha+\beta)}) \right]^{-\frac{1}{p+\alpha+\beta}} \tag{2.14}$$

We can readily verify that for $\zeta = 0$, we have

$$a(t) = a(0)[1 + a(0)^{p+\alpha+\beta} \omega^{2n+\beta} \gamma\mu(p + \alpha + \beta)t]^{-\frac{1}{p+\alpha+\beta}} \tag{2.15}$$

57

The case $\gamma = 0$ is even more obvious. One salient fact that emerges immediately from (2.14) and (2.15) is that it would be difficult to resolve $p + \alpha + \beta$ into its components from response data, unless we can change ω.

Note also from (2.15) that the rate of decay is not exponential in t and further the decrement over any integral multiple of the period depends on the initial amplitude as well as the frequency of oscillation. Finally for inegral k and

$$t = \frac{2\pi k}{\omega}$$

we have, taking logarithms and setting $c = p + \alpha + \beta$

$$\log \frac{a\left(\frac{2\pi k}{\omega}\right)}{a(0)} = -2\pi k \zeta$$

$$- \frac{1}{c} \log \left[1 + \frac{a(0)^c \omega^{2n-1+\beta} \gamma \mu}{\zeta} (1 - e^{-2\pi k \zeta c}) \right] . \qquad (2.16)$$

For small ζ this is well approximated by

$$-2\pi k \zeta - \frac{1}{c} \log \left(1 + 2\pi k \lambda a(0)^c \right) ,$$

where

$$\lambda = \omega^{2n+\beta-1} \gamma \mu .$$

The slope (as a function of k)

$$= -2\pi \zeta - \frac{2\pi \lambda a(0)^c}{c(1 + 2\pi k \lambda a(0)^c)} \qquad (2.17)$$

and hence the linear damping term is yielded by the asymptotic slope as $k \to \infty$, while for small k there is a marked curvature which depends also on the initial amplitude $a(0)$. The inital (at $k = 0$) slope
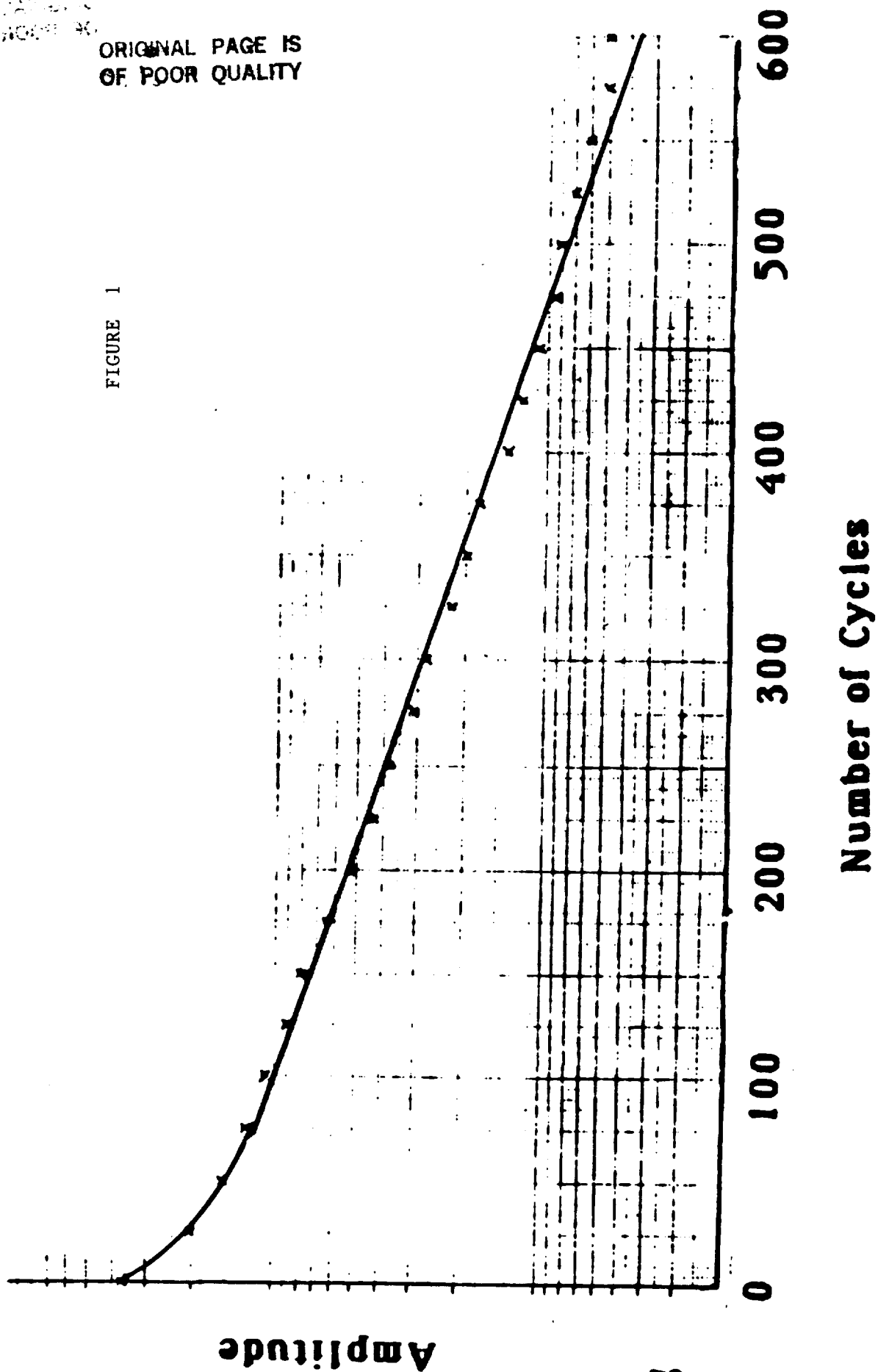
$$= -2\pi \zeta - \frac{2\pi \lambda a(0)^c}{c} \qquad (2.18)$$

is larger (in absolute value). The second derivative being positive, the curve is convex — CUP. This is in excellent qualitative agreement with SCOLE damping data: see Figure 1 where amplitude is plotted on logarithmic scale (period = 5 seconds).

SCOLE DAMPING

FIGURE 1

Amplitude

Number of Cycles

59

To get another version of (2.16) we may replace (2.7) by the more exact formula

$$\frac{a(t+T) - a(t)}{T} = -\frac{\varepsilon}{\omega} K_0(a(t)) \tag{2.19}$$

and hence using

$$a_k = a(kT) \tag{2.20}$$

we would have

$$a_{k+1} = a_k - \frac{\varepsilon T}{\omega} K_0(a_k) \tag{2.21}$$

so that

$$\log\left(\frac{a_k+1}{a_k}\right) = \left(1 - \frac{\varepsilon T}{\omega} \frac{K_0(a_k)}{a_k}\right)$$

which under our "small damping" assumption, may be replaced by

$$\log\frac{a_k+1}{a_k} = -\frac{\varepsilon T}{\omega} \frac{K_0(a_k)}{a_k} , \tag{2.22}$$

$$= -\varepsilon 2\pi(\zeta + a_k^c \gamma\mu\omega^{2n-1+\beta}) . \tag{2.23}$$

3. Multidimensional Generalization

Analogous entirely to the one-dimensional case, we may write the general nonlinear dynamic equation for flexible structures [2] as

$$M\ddot{x}(t) + \mathcal{D}(x(t), \dot{x}(t)) + Ax(t) = 0 \tag{3.1}$$

where the state $x(t)$ ranges in a separable (real) Hilbert space $\mathcal{H}$; M is a self-adjoint positive definite (with bounded inverse) operator on $\mathcal{H}$ onto $\mathcal{H}$; A is a self-adjoint nonnegative definite closed linear operator with domain dense in $\mathcal{H}$ and with compact resolvent; we shall (for simplicity) assume that zero is in the resolvent set of A. In the *linear* case

$$\mathcal{D}(x(t), \dot{x}(t)) = D\dot{x}(t) \tag{3.2}$$

where D is also a self-adjoint nonnegative definite closed linear operator whose domain includes that of $\sqrt{A}$. In the most important case we further specify that

$$D\phi_k = 2\zeta_k\omega_k M\phi_k \tag{3.3}$$

where $\{\phi_k\}$ are the M-orthonormalized eigenfunctions of A with eigenvalues ω_k^2 such that

$$A\phi_k = \omega_k^2 M\phi_k . \tag{3.4}$$

Here ζ_k is the damping ratio. If $\zeta_k = \zeta$ and we have strict proportional damping — see [3] for more — D is then essentially the positive square root of A (except for M). More generally we require that $\lim\limits_{k} \zeta_k \geq \zeta > 0$. In the nonlinear analogue of (2.5) we set

$$\mathcal{D}(\phi_j, \phi_k) = 0 \qquad j \neq k \tag{3.5}$$

and more generally for x, y such that

$$\sum \omega_k^2 b_k^2 + \sum a_k^{4m+2\alpha} b_k^{2n+2+2\beta} \gamma_k^2 < \infty \tag{3.6}$$

where

$$a_k = [x, \phi_k] ; \qquad b_k = [y, \phi_k] \tag{3.7}$$

we define:

$$[\mathcal{D}(x, y), \phi_k] = \gamma_k a_k^{2m} |a_k|^\alpha b_k^{2n+1} |b_k|^\beta + 2\zeta_k\omega_k b_k \tag{3.8}$$

where, as before, m and n are nonnegative integers and that

$$0 \leq a , \quad \beta < 1 ; \qquad \alpha + \beta < 1 ; \qquad 0 \leq \gamma . \tag{3.9}$$

Note that

$$[\mathcal{D}(x, y), y] \geq 0$$

for every x and y. Hence if

$$E(t) = \tfrac{1}{2} \{[A x(t), x(t)] + [M\dot{x}(t), \dot{x}(t)]\}$$

we have that

$$\frac{d}{dt}E(t) = -[\mathcal{D}(x(t), \dot{x}(t)), \dot{x}(t))] \leq 0 . \tag{3.10}$$

Or, the energy is nonincreasing. Using the modal expansion

$$x(t) = \sum a_k(t)\, \phi_k \qquad (3.11)$$

we see that for each k

$$\ddot{a}_k(t) + \omega_k^2\, a_k(t) + a_k(t)^2\, |a_k(t)|^\alpha\, \dot{a}_k(t)\, |\dot{a}_k(t)|^\beta + 2\zeta_k \omega_k\, \dot{a}_k(t) = 0 . \qquad (3.12)$$

We can therefore invoke the K-B averaging procedure obtaining the approximate solution

$$a_k(t) = A_k(t)\, \sin(\omega_k t + \phi_k) \qquad (3.13)$$

$$A_k(t) = A_k(0)\, e^{-t\zeta_k \omega_k}\left[1 + A_k(0)^c\, \omega_k^{2n+\beta-1}\, \frac{\gamma_k \mu}{\zeta}\, (1 - e^{-t\zeta_k \omega_k c})\right]^{-\frac{1}{c}} . \qquad (3.14)$$

And for $\zeta_k = 0$,

$$A_k(t) = A_k(0)\left[1 + A_k(0)^c\, \omega_k^{2n+\beta}\, \gamma_k \mu c t\right]^{-\frac{1}{c}}$$

where, as before,

$$\mu = \frac{1}{2\pi} \int_0^{2\pi} \sin^{2m}\phi\, \cos^{2n}\phi\, |\sin\phi|^\alpha\, |\cos\phi|^\beta\, \cos^2\phi\; d\phi . \qquad (3.15)$$

For $\alpha + \beta = 0$, we can give a kernel representation. Thus

$$z = \mathcal{D}(x, y) = \sum \gamma_i\, \phi_i[\phi_i, x]^{2m}\, [\phi_i, y]^{2n+1}$$

where

$$\sum \gamma_i^2 < \infty ; \qquad \gamma_i \geq 0 ;$$

and for the concrete realization $\mathcal{H} = L_2(0, L)$, the corresponding "kernel" would be

$$W(s, \sigma_1, ..., \sigma_{2m}, s_1, ..., s_{2n+1}) = \sum_i \gamma_i\, \phi_i(s)\, \phi_i(\sigma_1) \cdots \phi_i(\sigma_{2m})\, \phi_i(s_1) \cdots \phi_i(s_{2n+1}) \qquad (3.16)$$

and

$$z(s) = \int_0^L \cdots \int_0^L W(s, \sigma_1, ..., \sigma_{2m}, s_1, ..., s_{2n+1})\, x(\sigma_1)\, x(\sigma_2) \cdots x(\sigma_{2m})$$
$$\times\; y(s_1) \cdots y(s_{2n+1})\; d\sigma_1\, d\sigma_2 \cdots d\sigma_{2m}\, ds_1 \cdots ds_{2n+1} . \qquad (3.17)$$

A plausible model in this case would be to rewrite (3.1) as

$$M\ddot{x}(t) + \mathcal{D}(x(t), D\dot{x}(t)) + 2\zeta D\dot{x}(t) + Ax(t) = 0 . \qquad (3.18)$$

which will satisfy (3.10), since the γ_i in (3.16) are nonnegative. In the notation of (3), the "roll" equations for example will have the form:

$$\rho A \ddot{u}_\phi(t, s) \;+\; EI_\phi u_\phi''''(t, s) \;-\; 2\zeta\sqrt{\rho AEI_\phi}\;\frac{\partial^3 u_\phi(t, s)}{\partial t\,\partial^2 s}$$

$$-\;\int_0^L \cdots \int_0^L W(s, \sigma_1, ..., \sigma_{2m}, s_1, ..., s_{2n+1}) \;\times\; u_\phi(t, \sigma_1) \cdots u_\phi(t, \sigma_{2m})$$

$$\times\;\frac{\partial^3 u_\phi(t, s_1)}{\partial t\,\partial^2 s} \cdots \frac{\partial^3 u_\phi(t, s_{2n+1})}{\partial t\,\partial^2 s}\; d\sigma_1 \cdots d\sigma_{2m}\; ds_1 \cdots ds_{2n+1}$$

$$= 0\;.$$

It is clear that we may generalize (3.17) without recourse to modes. The "nonlocal" nature of the operator should hardly be surprising, since this is already so in the linear case if we want strict proportionality ($\zeta_k = \zeta$) for example.

4. Application to Nonlinear Boundary Feedback

In this section we shall apply the K-B averaging technique to obtain approximate solution to the response of a flexible structure to nonlinear boundary feedback control. The control effort is small so that the K-B approximation is reasonable. We follow [4] for the model where the "boundary" is finite-dimensional. Thus we have in the same setting as Section 3, but omitting the natural damping term:

$$M\ddot{x}(t) \;+\; Bf(B^*\dot{x}(t)) \;+\; Ax(t) \;=\; 0 \tag{4.1}$$

where B means R^m onto $\mathcal{H}$ and $f(\cdot)$ maps R^m into R^m and is such that

$$[f(u), u] \;>\; 0 \qquad \text{for } u \neq 0\;. \tag{4.2}$$

Using the modal expansion as in Section 3:

$$x(t) \;=\; \sum a_k(t)\,\phi_k$$

we obtain

$$\ddot{a}_k(t) \;+\; \omega_k^2 a_k(t) \;+\; [f(\sum \dot{a}_j(t)b_j),\; b_k] \;=\; 0 \tag{4.3}$$

where

$$B^*\phi_k = b_k .$$

Taking the approximation

$$[f(\sum \dot{a}_j(t)b_j), \; b_k] = [f(\dot{a}_k(t)b_k), \; b_k]$$

we see that setting

$$a_k(t) = A_k(t) \sin(\omega_k t + \phi_k(t))$$

that

$$\frac{d}{dt}\phi_k(t) = 0$$

$$\frac{d}{dt}A_k(t) = -\frac{K_0(A_k(t))}{\omega_k}$$

where

$$K_0(a) = \frac{1}{2\pi} \int_0^{2\pi} [f(a\omega_k \cos\phi \, b_k), \; b_k] \cos\phi \; d\phi .$$

To simplify matters further let us take

$$m = 1 .$$

Then

$$[f(a\omega_k b_k \cos\phi), \; b_k] = b_k f(ab_k\omega_k \cos\phi) .$$

We shall take:

$$f(u) = \lambda \tan^{-1} u$$

which is consistent with (4.2). Then (4.3) becomes

$$\ddot{a}_k(t) + \omega_k^2 a_k(t) + \lambda b_k^2 \tan^{-1} \dot{a}_k(t) = 0 \tag{4.4}$$

$$K_0(a) = \frac{1}{2\pi} \int_0^{2\pi} \lambda b_k \tan^{-1}(ab_k\omega_k \cos\phi) \cos\phi \; d\phi$$

$$= \frac{\lambda}{2\omega_k} [\sqrt{1 + a^2 b_k^2 \omega_k^2} - 1]$$

Hence

$$\frac{A_k(t) \; dA_k(t)}{[\sqrt{1 + a^2 b_k^2 \omega_k^2} - 1} = -\frac{\lambda}{\omega_k^2} dt .$$

To solve this, let

$$z(t) = 1 + A_k(t)^2 b_k^2 \omega_k^2 \tag{4.5}$$

so that

$$\frac{dz(t)}{2(\sqrt{z(t)} - 1)} = -b_k^2 \lambda \, dt \, . \tag{4.6}$$

Let

$$F(z) = e^{\sqrt{z}} (\sqrt{z} - 1) \, , \qquad z \geq 1 \, . \tag{4.7}$$

Then

$$F'(z) > 0 \qquad \text{for } z > 1$$

and hence we may define the inverse function

$$F(z) = y \, ; \qquad z = F^{-1}(y) \, .$$

Thus (4.6) has the solution:

$$z(t) = F^{-1}[F(z(0))e^{-b_k^2 \lambda t}] \, , \tag{4.8}$$

$$b_k \omega_k A_k(t) = \sqrt{F^{-1}(F(1 + A_k(0)^2 b_k^2 \omega_k^2)e^{-b_k^2 \lambda t}) - 1} \tag{4.9}$$

where

$$z(0) = 1 + A_k(0)^2 b_k^2 \omega_k^2 > 1$$

unless $A_k(0) = 0$. Note that

$$F^{-1}(y) \to 1 \qquad \text{as } y \to 0$$

and hence $z(t)$ decreases monotonically to 1 and hence the amplitude $A_k(t)$ decays to zero asymptotically.

Note that the decay rate depends on the control effort λb_k^2 as well as the initial amplitude. Of course we have in (4.4) yet another nonlinear damping model. Following (2.22) we have:

$$\log \frac{a_{j+1}}{a_j} = \frac{2\pi\lambda}{2\omega_k^2} \left[\frac{\sqrt{1 + a_j^2 b_k^2 \omega_k^2} - 1}{a_j} \right]$$

where

$$a_j = A_k(jT) \, ; \qquad T = \frac{2\pi}{\omega_k}$$

65

References

[1] *Proceedings of the 3rd Annual NASA SCOLE Workshop, November 1986.* NASA Technical Memorandum 89075. Edited by L.W. Taylor.

[2] N. Krylov and N. Bogoliubov. *Introduction to Non-Linear Mechanics*, Princeton: Princeton University Press, 1949.

[3] A.V. Balakrishnan. Control of Flexible Flight Structures. In: Volume dedicated to J.L. Lions. To be published by Gauthier-Villars, Paris, 1988.

[4] A.V. Balakrishnan. Stability Enhancement of Flexible Structures by Nonlinear Boundary Feedback. In: *Proceedings of IFIP Working Conference on Boundary Control and Boundary Variations, Nice, June 1986.*

[5] S.C. Hayashi. *Nonlinear Oscillations in Physical Systems.* New York: McGraw-Hill, 1964.