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ON THE COUPLING OF HYPERBOLIC AND PARABOLIC SYSTEMS: ANALYTICAL AND NUMERICAL APPROACH

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ABSTRACT

We deal with the coupling of hyperbolic and parabolic systems in a domain Ω divided into two disjoint subdomains Ω^+ and Ω^-. Our main concern is to find out the proper interface conditions to be fulfilled at the surface separating the two domains. Next, we will use them in the numerical approximation of the problem. The justification of the interface conditions is based on a singular perturbation analysis, that is, the hyperbolic system is rendered parabolic by adding a small artificial "viscosity". As this goes to zero, the coupled parabolic-parabolic problem degenerates into the original one, yielding some conditions at the interface. These we take as interface conditions for the hyperbolic-parabolic problem. Actually, we discuss two alternative sets of interface conditions according to whether the regularization procedure is variational or non-variational. We show how these conditions can be used in the frame of a numerical approximation to the given problem. Furthermore, we discuss a method of resolution which alternates the resolution of the hyperbolic problem within Ω^- and of the parabolic one within Ω^+. The spectral collocation method is proposed, as an example of space discretization (different methods could be used as well); both explicit and implicit time-advancing schemes are considered. The present study is a preliminary step toward the analysis of the coupling between Euler and Navier-Stokes equations for compressible flows. [1]

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1. INTRODUCTION

In this work we deal with (initial) boundary value problems for partial differential equations (or systems) which change their character within the domain under consideration. Precisely, we consider problems which are of hyperbolic type in a subdomain Ω^- of the whole domain Ω and of parabolic type in the complement Ω^+.

This interest is motivated by various applications. Among others, we emphasize the case of fluid dynamical problems for viscous, compressible flows in presence of a body, governed by the system of Navier-Stokes equations (see, e.g., [CM], [S]). A convenient numerical approach to the solution to these problems relies upon the splitting of the physical region where flow occurs in two computational domains, one (say, Ω^+) close to the body, where viscous terms are to be taken into account and another far away (say, Ω^-), where viscous terms may be neglected.

This leads precisely to a coupled problem involving Euler equations (hyperbolic) in the far region and the complete Navier-Stokes equations in the near region.

From a computational point of view, this splitting procedure carries obvious advantages. In particular, we mention the possibility of using different solvers for the two subproblems. Of course, a crucial point in this framework is how to relate the two problems to each other at the interface separating the two subregions. This feature must be investigated for the differential problem, first: suitable conditions at the interface will be derived. Whatever numerical scheme is used, it must take these conditions into account.

Identifying such conditions is generally understood whenever the two differential problems in the subregions are of the same kind. For instance, for the interaction of two second order elliptic problems, the interface conditions consist in requiring the continuity of the unknowns and of the flux (these conditions are transferred to any numerical scheme easily). Analogously, when coupling two Navier-Stokes problems, we must impose the continuity of the velocity and of the normal stress at the interface. Eventually, when coupling two hyperbolic systems of first order, we request the continuity of the unknown at the interface (unless the interface is a discontinuity line, in which case Rankine-Ugoniot equations ought to be fulfilled).

When coupling Euler and Navier-Stokes equations, the proper interface conditions are not obvious, in advance. A possible way of deducing them is to see the coupled problem as a limit of two coupled Navier-Stokes problems with vanishing viscous terms in Ω^-.

This approach can be adopted also in a simplified version of the problem, namely considering a coupling between hyperbolic and parabolic linear systems in one space variable (as well as their stationary counterpart). In this framework, the hyperbolic-parabolic problem is seen as limit of a coupling between two parabolic problems endowed with the usual transmission conditions at the interface. As we shall see, this procedure yields certain interface conditions for the limit problem. Although these conditions are somewhat physically meaningful, there is a sensible loss of continuity in passing from the approaching problems to the limit one. In what follows we shall detail also a different type of limit procedure, which maintains a higher order of regularity for the limit problem.

To be more precise, let us state the problem we will discuss in this paper:

In the interval (a,c), find $\mathbf{w}$ such that (b is any point internal to (a,c))

$$\mathbf{w}_t^- + A\,\mathbf{w}_x^- + B\,\mathbf{w}^- = \mathbf{F} \quad \text{for } x \in \Omega^- = (a,b),\ t > 0, \tag{1.1}$$

$$\mathbf{w}_t^+ - (\mu\,\mathbf{w}_x^+)_x + A\,\mathbf{w}^+{}_x + B\,\mathbf{w}^+ = \mathbf{F} \quad \text{for } x \in \Omega^+ = (b,c),\ t > 0, \tag{1.2}$$

with an initial condition and proper boundary conditions at $x = a$ and $x = c$. Here A and B are two constant 3×3 matrices, while $\mathbf{F}$ is a given vector function with three components ($\mathbf{w}$ is an unknown three dimensional vector); $\mu = \mu(x,t) \geqslant \mu_0 > 0$ is a given viscosity. We assume that A has three real, nonvanishing eigenvalues (p_0 of them are positive and $3-p_0$ are negative): in particular, this implies that the system (1.1) is hyperbolic.

For the above problem we are going to specify the interface conditions obtainable by the arguments previously mentioned. By the first approach, which we will refer to as *variational*, we find the following interface conditions at $x = b$, for all $t > 0$:

$$T_n\,\mathbf{w}^+ = T_n\,\mathbf{w}^-, \tag{1.3}$$

$$-\,\mu\mathbf{w}_x^+ + A\,\mathbf{w}^+ = A\,\mathbf{w}^-: \tag{1.4}$$

the rectangular matrix T_n has $3-p_0$ rows given by the left eigenvectors corresponding to the negative eigenvalues of A (see section 3.1).

With the second approach, which we will refer to as *nonvariational*, the interface conditions at $x = b$ and for all $t > 0$ are:

$$\mathbf{w}^+ = \mathbf{w}^-, \tag{1.5}$$

$$T_n\,\mathbf{w}_x^+ = T_n\,\mathbf{w}_x^-. \tag{1.6}$$

In particular, note that (1.5) does imply continuity of all unknowns at the interface, while (1.4) gives continuity of the "flux" at the interface, allowing a discontinuity on the unknowns (actually, a mild discontinuity, as the jump has the same order of the viscosity coefficient μ at the interface).

The above results are presented in section 4, as a consequence of a procedure of "increasing difficulty" carried out throughout sections 2 and 3. Precisely, in section 3 we deal with the steady counterpart of (1.1), (1.2) and in section 2 we detail the coupling between two time independent equations, one of first order and the other of second order (the proofs of the abstract results are given in the Appendix). Although the problems of sections 2 and 3 might be regarded as autonomous problems, actually they are treated as intermediate steps toward the analysis of the main problem (1.1), (1.2). For each and every problem, we present the numerical approximation based on the spectral collocation method and show how the interface conditions are used in this frame. This could be done for numerical methods based on different approaches, as well. Here we just remark that, in the numerical scheme, we must supplement the above interface conditions suitable compatibility relations at the interface. These arise from the hyperbolic nature of the problem in Ω^-: a thorough discussion is made in sections 2.2, 3.2, 4.2.

We end this introduction by noticing that (1.1), (1.2) present some similarities with the coupling between Euler and Navier-Stokes equations we mentioned at the beginning as a driving motivation for our work. The relevant difference lies in that the viscous terms in Navier-Stokes equations do not enjoy the particular diagonal structure as in the right hand side of (1.2). Since our analysis relies heavily upon this feature, there is no immediate application of our results to the coupling between Euler and Navier-Stokes equations. Nevertheless, it seems that several elements of our approach can be useful in that problem, too. From this point of view, the present work is an intermediate step toward our goal.

2. HYPERBOLIC-ELLIPTIC INTERACTION: THE SCALAR CASE

In this section we consider a one dimensional, linear, scalar problem. The two subsections are devoted to the analysis of the continuous problem (with special concern to different elliptic regularizations) and to its numerical approximation, respectively.

2.1. The differential problem

We begin by stating the boundary value problem, as follows. Let

(i) a, b, c be real numbers, with $a < b < c$;

(ii) α, β, μ be functions defined in $[a,c]$, with $\alpha \neq 0$, $\mu(x) \geq \mu_0 > 0$ for $x \in [a,c]$;

(iii) f be a function defined in $[a,c]$.

Then, consider the problem

(P): find u defined in $[a,b]$, v defined in $[b,c]$ such that

$$\alpha u_x + \beta u = f \qquad in\ (a,b); \tag{2.1}$$

$$-(\mu v_x)_x + \alpha v_x + \beta v = f \qquad in\ (b,c); \tag{2.2}$$

$$v(c) = 0; \tag{2.3}$$

$$u(a) = 0, \qquad \text{if } \alpha > 0 \ in\ [a,c]. \tag{2.4}$$

Clearly, the formulation of problem **(P)** is incomplete: it needs one coupling condition between u and v at the interface b, when $\alpha > 0$ in $[a,c]$, while two coupling conditions are required if $\alpha < 0$ in $[a,c]$ (in this case, (2.4) does not hold). Moreover, we may allow (2.3) to be substituted by $v_x(c) = 0$, if $\alpha > 0$ in $[a,c]$.

Remark 2.1 Problem **(P)** may be regarded as a stationary problem (in this case β might vanish identically) or else as a time discretization of an evolution advection-diffusion problem (hyperbolic in (a,b) and parabolic in (b,c)) by an implicit method (in this case, β behaves essentially like the reciprocal of the time discretization step). For this reason, we will always refer to problem **(P)** as to a "hyperbolic-elliptic" problem, even if **(P)** is a purely steady problem. By the way, we just note that the characteristic lines of the evolution hyperbolic problem enter the domain $(a,b) \times (0,+\infty)$ across $\{a\} \times (0,+\infty)$, when $\alpha > 0$ and across $\{b\} \times (0,+\infty)$, when $\alpha < 0$. This is the reason why we choose to impose condition (2.4) among others, which are equally admissible for the time-independent problem. When $\alpha < 0$, the same argument suggests not to impose any boundary condition at $x = a$ (though admissible for the very equation (2.1)); on the contrary, we are led to consider a condition on u at $x = b$. In the frame of the global problem (2.1), (2.2), this condition reads as an interface condition.

Two different types of elliptic regularizations are possible for problem **(P)**, both acceptable for some reason. We will see that the two ways are essentially different as for the behavior at the interface.

The case $\alpha > 0$.

Given $\epsilon > 0$, consider the problem

($\mathbf{P}_\epsilon$): to find u_ϵ defined in $[a,b]$, v_ϵ defined in $[b,c]$ such that

$$-\epsilon u_{\epsilon,xx} + \alpha u_{\epsilon,x} + \beta u_\epsilon = f \qquad \text{in } (a,b); \tag{2.5}$$

$$-(\mu v_{\epsilon,x})_x + \alpha v_{\epsilon,x} + \beta v_\epsilon = f \qquad \text{in } (b,c); \tag{2.6}$$

$$u_\epsilon(a) = 0; \tag{2.7}$$

$$v_\epsilon(c) = 0; \tag{2.8}$$

$$\left.\begin{array}{r} (i)\ \ u_\epsilon = v_\epsilon \\ (ii)\ \ \epsilon u_{\epsilon,x} = \mu v_{\epsilon,x} \end{array}\right\} \quad \text{at } x = b. \tag{2.9}$$

($\mathbf{P}_\epsilon$) is equivalent to a *variational* problem on the whole of (a,c); condition (2.9) expresses that u_ϵ and v_ϵ join continuously at b and that the flux across b is continuous, too.

About the existence of solutions to problem ($\mathbf{P}_\epsilon$) and their behavior as $\epsilon \to 0$, the following result holds (see Appendix, where the appropriate choices of functional spaces are made and the regularity assumptions on the data are specified).

Proposition 2.1 *Assume the coerciveness condition*

$$2\beta - \alpha_x \geqslant 0 \qquad \text{in } [a,c]. \tag{2.10}$$

Then, problem ($\mathbf{P}_\epsilon$) *has a unique solution. Furthermore, as* $\epsilon \to 0$, u_ϵ *and* v_ϵ *converge to a pair of functions* u, v *which satisfy (2.1), (2.2), (2.3), (2.4) and the interface condition*

$$\alpha u = -\mu v_x + \alpha v \qquad \text{at } x = b. \tag{2.11}$$

□

Remark 2.2 (2.11) means that the flux across b is conserved, as $\epsilon \to 0$. On the contrary, analytical solution of ($\mathbf{P}_\epsilon$) shows that u and v *do not join continuously* at b, in general. Actually, the closed form of the solution (as well as numerical experiments, see subsection 2.2.3) shows that the jump between u and v at b has the same order as μ, when $\mu \to 0$.

A second approach is to consider the following problem

($\mathbf{Q}_\epsilon$): to find u_ϵ defined in $[a,b]$, v_ϵ defined in $[b,c]$ such that (2.5), (2.6), (2.7), (2.8) and (2.9i) hold, along with the condition

$$u_{\epsilon,x} = v_{\epsilon,x} \qquad at \;\; x = b. \tag{2.12}$$

($\mathbf{Q}_\epsilon$) is equivalent to a *nonvariational* elliptic problem on the whole of (a,c): now we look for a pair of functions u_ϵ, v_ϵ which have a $\mathbf{C}^1$ junction at b.

Proposition 2.2 *Assume the coerciveness condition (2.10). Let u_ϵ, v_ϵ solve problem ($\mathbf{Q}_\epsilon$). As $\epsilon \rightarrow 0$, u_ϵ and v_ϵ converge to a pair of functions u, v which satisfy (2.1), (2.2), (2.3), (2.4) and the continuity condition*

$$u(b) = v(b) \tag{2.13}$$

at the interface.

$\square$

We remark that (2.12) is not preserved, in general, as $\epsilon \rightarrow 0$: this can be checked on the closed form of the solutions to problem ($\mathbf{Q}_\epsilon$), in some particular cases. Moreover, this feature is clearly shown by the numerical results presented in subsection 2.2.3. Thus, we are approaching a solution to problem (**P**) which is continuous but not $\mathbf{C}^1$ at b.

The case $\alpha < 0$.

In this case, one can consider the same problems ($\mathbf{P}_\epsilon$) and ($\mathbf{Q}_\epsilon$) as before. However, for a reason which will be clear in section 3, we prefer to perform a slight change in the two problems, namely replacing the Dirichlet condition (2.7) with a Neumann one. Note that the original problem (**P**) has no condition at all for $x = a$. Thus, we are dealing with a new couple of problems, which we denote by $(\mathbf{P}_\epsilon)_N$ and $(\mathbf{Q}_\epsilon)_N$, respectively. For clarity, we state them in detail.

$(\mathbf{P}_\epsilon)_N$: to find u_ϵ defined in $[a,b]$, v_ϵ defined in $[b,c]$ such that (2.5), (2.6), (2.8) and (2.9) hold, along with the condition

$$u_{\epsilon,x}(a) = 0. \tag{2.14}$$

$(\mathbf{Q}_\epsilon)_N$: to find u_ϵ defined in $[a,b]$, v_ϵ defined in $[b,c]$ such that (2.5), (2.6), (2.8), (2.9i), (2.12) and (2.14) hold.

The difference with respect to the case $\alpha > 0$ lies in the asymptotic behavior and, more precisely, in the interface conditions (remind that the limit problem (P) needs *two* conditions at b, in this case). The abstract analysis shown in the Appendix yields the following results (again, we do not specify the regularity on the data and on the unknowns here).

Proposition 2.3 *Assume the coerciveness condition (2.10). Then, problem* $(\mathbf{P}_\epsilon)_N$ *has a unique solution. Furthermore, as* $\epsilon \to 0$, u_ϵ *and* v_ϵ *converge to a pair of functions* u, v *which satisfy (2.1), (2.2), (2.3) and the following interface conditions:*

$$u(b) = v(b), \tag{2.15}$$

$$v_x(b) = 0. \tag{2.16}$$

□

Proposition 2.4 *Assume the coerciveness condition (2.10); moreover, suppose that* $\beta \geqslant \beta_0 > 0$ *in* $[a,b]$. *Let* u_ϵ, v_ϵ *solve problem* $(\mathbf{Q}_\epsilon)_N$. *As* $\epsilon \to 0$, u_ϵ *and* v_ϵ *converge to a pair of functions* u, v *which satisfy (2.1), (2.2), (2.3) and the following interface conditions:*

$$u(b) = v(b), \tag{2.17}$$

$$u_x(b) = v_x(b). \tag{2.18}$$

□

We point out that the condition at a for both $(\mathbf{P}_\epsilon)_N$ and $(\mathbf{Q}_\epsilon)_N$ is lost in the limit, as it is natural for this kind of problems.

Remark 2.3 By means of both approaches, the limit functions u and v enjoy a continuous junction at b. But the derivatives behave in a very different way (see (2.16) and (2.18)). Indeed, the limit of the solution to $(\mathbf{P}_\epsilon)_N$ shows an angle at b, in general, while the limit of the solution to $(\mathbf{Q}_\epsilon)_N$ is $\mathbf{C}^1$ at b. Thus, as in the previous case, the nonvariational approach is able to preserve an order of regularity higher by one, with respect to the variational one.

Remark 2.4 The two regularized problems with the original Dirichlet condition

(2.7) have the same type of asymptotic behavior as the problems with the Neumann condition (2.14). The difference lies in that in the Dirichlet case the value $u_\epsilon(a)$ *does not* converge to the corresponding value $u(a)$, which is true for the Neumann case of problems $(\mathbf{P}_\epsilon)_N$ and $(\mathbf{Q}_\epsilon)_N$.

Remark 2.5 A comment is needed about (2.18). This condition calls into play the first derivative of the solution to (2.1) at b: but (2.1) is a first order equation, hence (2.18) involves a boundary operator *of the same order* as the interior equation. Thus, the left hand side of (2.18) must be compatible with the collocation of the equation (2.1) at b. Precisely, whenever the data are smooth, we expect equation (2.1) to hold at b, hence (2.17) and (2.18) imply

$$\alpha v_x + \beta v = f \qquad at\ x = b. \tag{2.19}$$

2.2. The numerical approximation

Set $\Omega^- =]a,b[$, $\Omega^+ =]b,c[$. On the reference interval $[-1,1]$, let us consider the Chebyshev collocation points

$$x_j^* = -\cos\frac{\pi j}{N}, \qquad j = 0, \cdots, N, \tag{2.20}$$

whose images in the interval $\overline{\Omega^\pm}$ are denoted by $\{x_j^\pm\}$. Note that $x_0^- = a$, $x_N^- = x_0^+ = b$, $x_N^+ = c$.

As an initial step, we consider two separate boundary value problems: a first order problem in Ω^- and a second order elliptic problem in Ω^+. Next, we introduce their numerical approximations based on the spectral collocation method. This presentation has the aim of providing the reader a guideline to the numerical approach of the coupled problem $(\mathbf{P})$.

2.2.1. The split model problem

The two separate differential problems in Ω^- and Ω^+ are the following (we keep the same terminology as in section 2.1).

"Hyperbolic" problem in Ω^-:

$$\left\{\begin{array}{ll} \alpha u_x + \beta u = f & in\ \Omega^-, \\ u(a) = u_a & if\ \alpha > 0, \\ u(b) = u_b & if\ \alpha < 0, \end{array}\right. \tag{2.21}$$

where u_a and u_b are given. The motivation for the different choice of boundary conditions is given in Remark 2.1.

Elliptic problem in Ω^+:

$$\left\{\begin{aligned} -(\mu v_x)_x + \alpha v_x + \beta v &= f && \text{in } \Omega^+, \\ B_b v &= v_b && \text{at } x = b, \\ B_c v &= v_c && \text{at } x = c, \end{aligned}\right. \tag{2.22}$$

where v_b and v_c are given and $B_b v$ and $B_c v$ are suitable combinations of v and v_x leading to a well posed problem.

The *spectral collocation approximation* to (2.21) is as follows (see, e.g., [CHQZ], Ch. 10 and 11). We look for $u_N \in \mathbf{P}_N$ (the space of algebraic polynomials of degree $\leqslant N$) such that

$$\alpha u_{N,x} + \beta u_N = f \qquad \text{at } x_j^-, \quad j = 1, \cdots, N-1, \tag{2.23}$$

supplemented by the two boundary equations:

$$\text{if } \alpha > 0 \quad \left\{\begin{aligned} &(i) & u_N &= u_a && \text{at } x_0^-, \\ &(ii) & \alpha u_{N,x} + \beta u_N &= f && \text{at } x_N^-; \end{aligned}\right. \tag{2.24}$$

$$\text{if } \alpha < 0 \quad \left\{\begin{aligned} &(i) & \alpha u_{N,x} + \beta u_N &= f && \text{at } x_0^-, \\ &(ii) & u_N &= u_b && \text{at } x_N^-. \end{aligned}\right. \tag{2.25}$$

The numerical approximation to (2.22), based on the spectral collocation method, is as follows. We look for $v_N \in \mathbf{P}_N$ satisfying

$$-[I_N(\mu v_{N,x})]_x + \alpha v_{N,x} + \beta v_N = f \quad \text{at } x_j^+, \; j = 1, \cdots, N-1, \tag{2.26}$$

$$B_b v_N = v_b \quad \text{at } x_0^+, \tag{2.27}$$

$$B_c v_N = v_c \quad \text{at } x_N^+, \tag{2.28}$$

where I_N is the interpolation operator at the points x_j^+.

2.2.2. The original coupled problem

Now we are in a position to describe the numerical approximation to the original coupled problem (P), taking (2.23)-(2.28) into account.

1. At the *interior points* of Ω^- and Ω^+, we impose the set of equations (2.23) and (2.26), respectively.
2. At $x = a$, we impose either (2.24i) (with $u_a = 0$) or (2.25i), according to the sign of α.
3. At $x = c$, we always enforce $v_N = 0$ (which corresponds to (2.28) with $v_c = 0$ and $B_c v_N = v_N$).
4. At $x = b$, we need two equations, in order to close the algebraic system. These depend both on the sign of α and on the interface conditions provided by either elliptic regularization (see section 2.1). In particular:

   (a) if $\alpha > 0$, we impose (2.24ii), along with either

   $$-\mu v_{N,x} + \alpha v_N = \alpha u_N \qquad (\textit{variational approach}) \tag{2.29}$$

   or

   $$v_N = u_N \qquad (\textit{nonvariational approach}); \tag{2.30}$$

   (b) if $\alpha < 0$, we impose the condition

   $$u_N = v_N \tag{2.31}$$

   (i.e. (2.25ii), with $u_b = v_N(x_0^+)$); the remaining equation is given by either

   $$v_{N,x} = 0 \qquad (\textit{variational approach}) \tag{2.32}$$

   or

   $$v_{N,x} = u_{N,x} \qquad (\textit{nonvariational approach}). \tag{2.33}$$

We note that (2.29), (2.30), (2.32) and (2.33) are but special versions of (2.27), with suitable choices of B_b and v_b. These are specified in table 1, which summarizes the equations to be fulfilled by the numerical solution at each collocation point (including boundary and interface).

Collocation points	$\alpha>0$		$\alpha<0$	
	variational	nonvariational	variational	nonvariational
$x = a$	(2.24i)	(2.24i)	(2.23)	(2.23)
$x_j^{(1)}$, $1 \leqslant j \leqslant N-1$	(2.23)	(2.23)	(2.23)	(2.23)
$x = b$	$(2.27)^{(1)}$ (L) (2.23) (C)	$(2.27)^{(2)}$ (L) (2.23) (C)	(2.25ii) (L) $(2.27)^{(3)}$ (L)	(2.25ii) (L) $(2.27)^{(4)}$ (L)
$x_j^{(2)}$, $1 \leqslant j \leqslant N-1$	(2.26)	(2.26)	(2.26)	(2.26)
$x = c$	(2.28)	(2.28)	(2.28)	(2.28)

Table 1. *Numerical approximation to problem* **P** *by spectral collocation method:* (L) = *limit condition given by the asymptotic analysis;* (C) = *compatibility condition (the transport equation must be collocated at the outflow boundary of* Ω_1*);*

(1) *with* $B_- v_N = -\mu\, v_{N,x} + \alpha v_N$ *and* $v_b = \alpha u_N(x_N^{(1)})$;

(2) *with* $B_- v_N = v_N$ *and* $v_b = u_N(x_N^{(1)})$;

(3) *with* $B_- v_N = v_{N,x}$ *and* $v_b = 0$;

(4) *with* $B_- v_N = v_{N,x}$ *and* $v_b = u_{N,x}(x_N^{(1)})$.

2.2.3. Some numerical results

Now we present several numerical experiments which support the theoretical results obtained in the previous subsections. We deal with the elliptic regularizations of problem (P), taking $(a, c) = (-1,1)$, with $b = 0$. In all cases ($\alpha > 0$ or $\alpha < 0$, variational or nonvariational approach), the equations are

$$- \epsilon u_{\epsilon,xx} + \alpha u_{\epsilon,x} + \beta u_\epsilon = f \qquad in\ (-1,0); \tag{2.34}$$

$$- (\mu v_{\epsilon,x})_x + \alpha v_{\epsilon,x} + \beta v_\epsilon = f \qquad in\ (0,1). \tag{2.35}$$

The interface conditions change according to the regularization chosen:

$$(variational) \begin{cases} (i)\ u_\epsilon = v_\epsilon, \\ (ii)\ \epsilon u_{\epsilon,x} = \mu v_{\epsilon,x} \end{cases} \quad at\ x = 0 \tag{2.36}$$

or

$$(nonvariational) \begin{cases} (i)\ u_\epsilon = v_\epsilon, \\ (ii)\ u_{\epsilon,x} = v_{\epsilon,x} \end{cases} \quad at\ x = 0. \tag{2.37}$$

The boundary conditions will be distinguished later.

These problems are solved by the Chebyshev collocation method described in advance for fully elliptic problems of the form (2.22).

(To be more precise, we have implemented the collocation method in a domain decomposition framework, in order to achieve the highest precision. To this end, three subdomains are used; within each of them, we take 50 points; the middle subdomain includes the interface point $x = 0$. At each interface between subdomains the C^1 continuity is enforced directly (see [FQZ]).)

The data we have used are the following

$$a = -1,\ b = 0,\ c = 1,\ f \equiv 1,\ \mu \equiv 1,\ \beta \equiv 1 + x^2. \tag{2.38}$$

A homogeneous Dirichlet condition is enforced at $x = 1$. About the point $x = -1$, we consider the case of a homogeneous Dirichlet condition, to begin with.

In Figure 2.1 we graph the results obtained for the variational approach, with $\epsilon = 0.005$ and $\epsilon = 0.1$, when $\alpha = 1$. In agreement with our theoretical results (see Remark 2.2), the solution exhibits a discontinuity as $\epsilon \to 0$ at the interface point $x = 0$. The discontinuity is revealed by the presence of oscillations near the interface, due to the Gibbs phenomenon. However, the jump is of the same order as the viscosity coefficient μ, as shown in Figure 2.2.

Figure 2.3 displays the results obtained for the nonvariational case, using the same data as in Figure 2.1. Note that, as $\epsilon \to 0$, the solution is continuous (though not C^1) at the interface point, as predicted by (2.17).

The comparison between variational and nonvariational approaches is clearer in Figure 2.4, where we take $\epsilon = 0.005$.

In Figures 2.5 and 2.6 we present the results obtained using the two approaches, with the same data as before, but with $\alpha = -1$. As predicted by the theory (see Propositions 2.2 and 2.3), as $\epsilon \to 0$ the nonvariational solution remains $\mathbf{C}^1$, while the variational one is just $\mathbf{C}^0$.

Finally, Figure 2.7 reports the results obtained with $\alpha = -1$, $\epsilon = 0.005$ and a homogeneous Neumann condition at $x = -1$ (rather than the Dirichlet one), using the variational and nonvariational approaches.

Figure captions

Figure 2.1 *Results for the variational approach, with* $\alpha = 1$*:* $\epsilon = 0.1$ *(dashed line),* $\epsilon = 0.005$ *(solid line).*

Figure 2.2 *Results for the variational approach, with* $\alpha = 1$, $\epsilon = 0.005$*:* $\mu = 1$ *(solid line),* $\mu = 0.1$ *(dash-dot line),* $\mu = 0.01$ *(dash-dash line)*

Figure 2.3 *Results for the nonvariational approach, with* $\alpha = 1$*:* $\epsilon = 0.1$ *(dashed line),* $\epsilon = 0.005$ *(solid line).*

Figure 2.4 *Comparison between the two approaches, with* $\alpha = 1$, $\epsilon = 0.005$*: nonvariational approach (dashed line), variational approach (solid line).*

Figure 2.5 *Results for the variational approach, with* $\alpha = -1$*:* $\epsilon = 0.1$ *(dashed line),* $\epsilon = 0.005$ *(solid line).*

Figure 2.6 *Results for the nonvariational approach, with* $\alpha = -1$*:* $\epsilon = 0.1$ *(dashed line),* $\epsilon = 0.005$ *(solid line).*

Figure 2.7 *Comparison between the two approaches, with* $\alpha = -1$, $\epsilon = 0.005$, *with homogeneous Neumann boundary condition at* $x = -1$ *and homogeneous Dirichlet boundary condition at* $x = 1$*: nonvariational approach (dashed line), variational approach (solid line).*

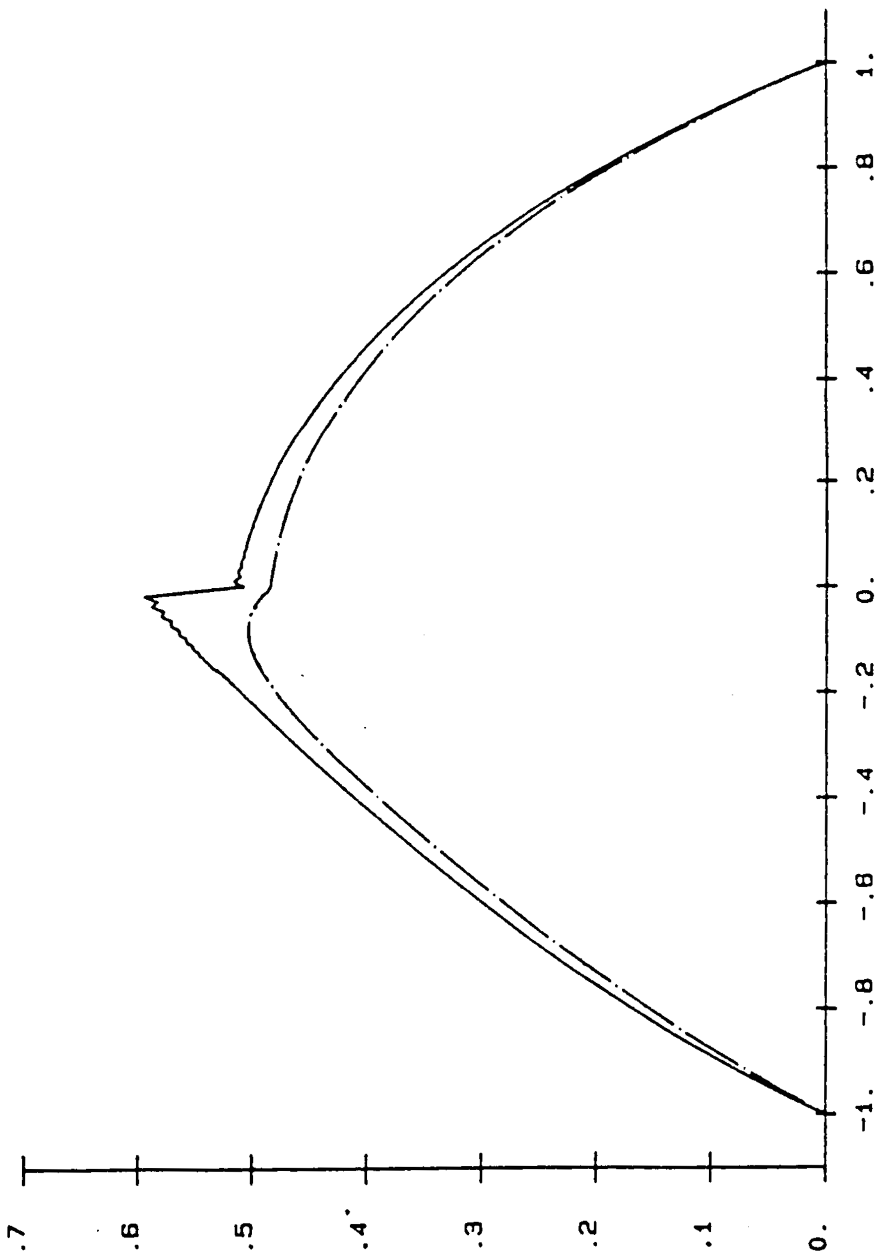


Fig. 2.1

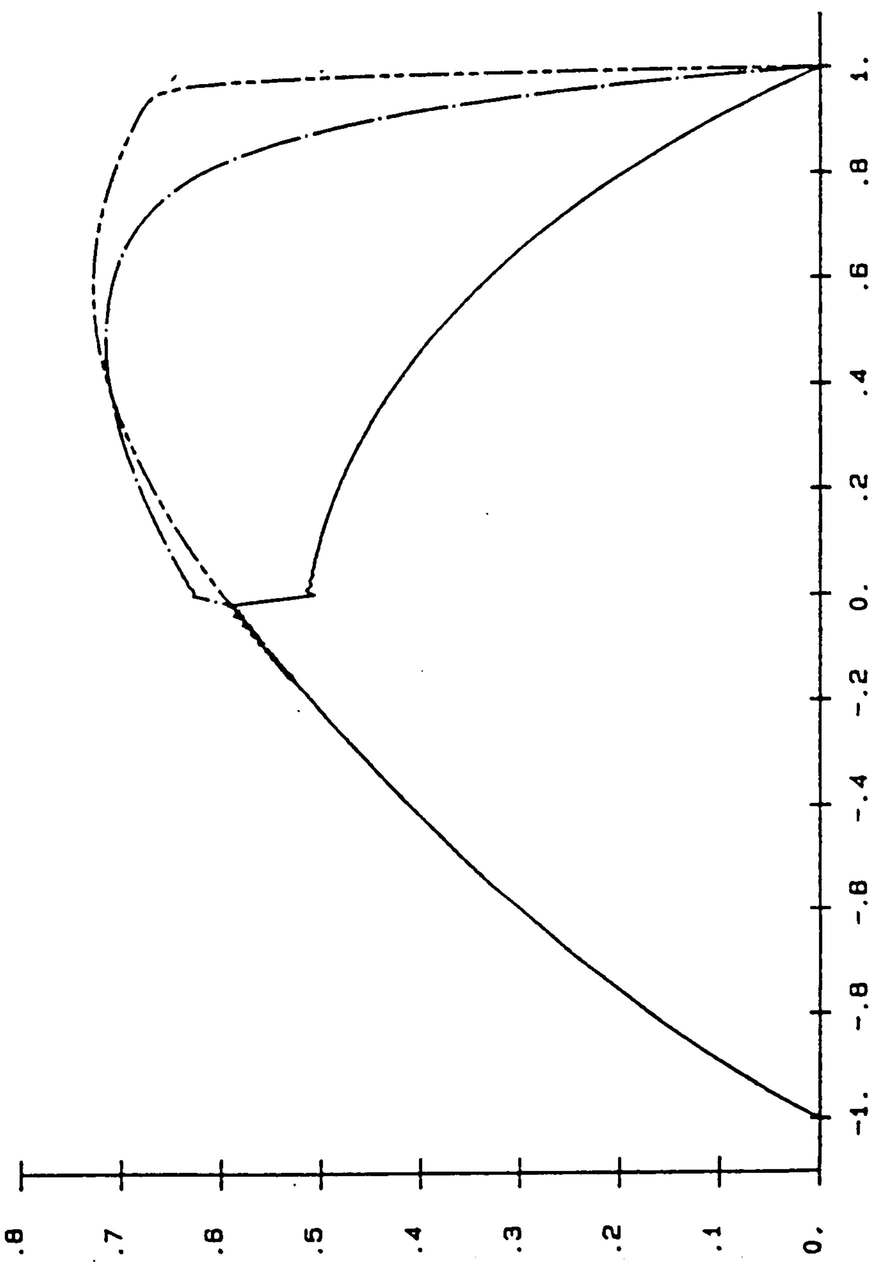
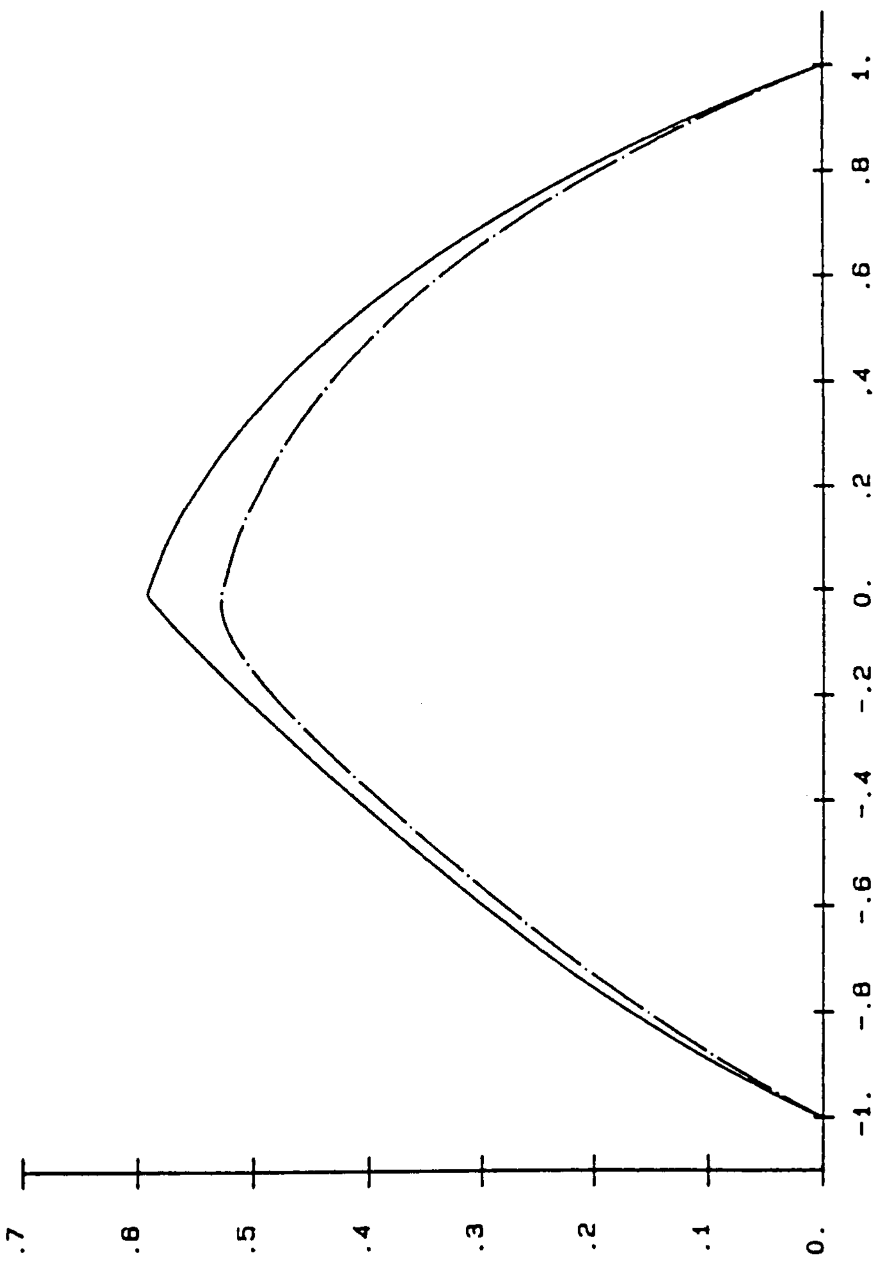


Fig. 2.2

Fig. 2.3



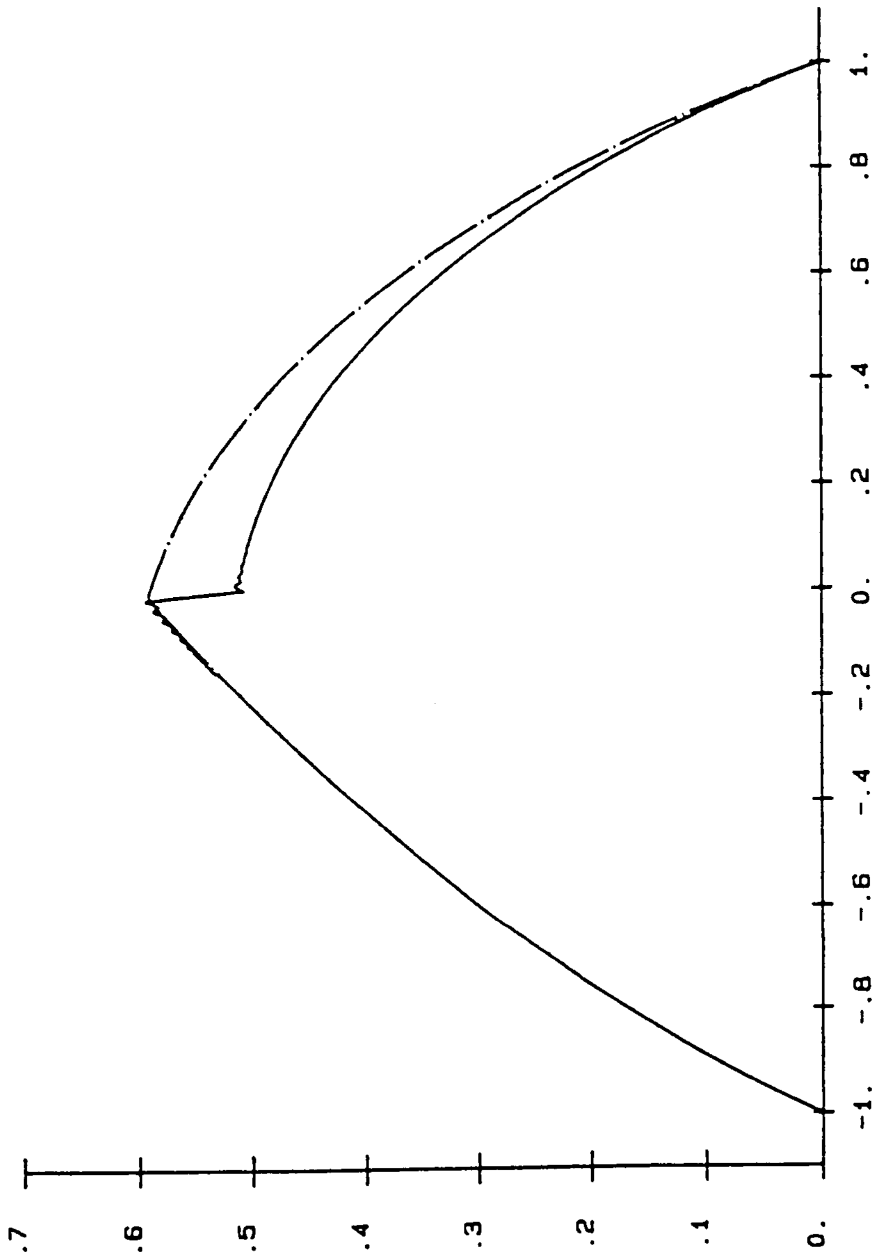
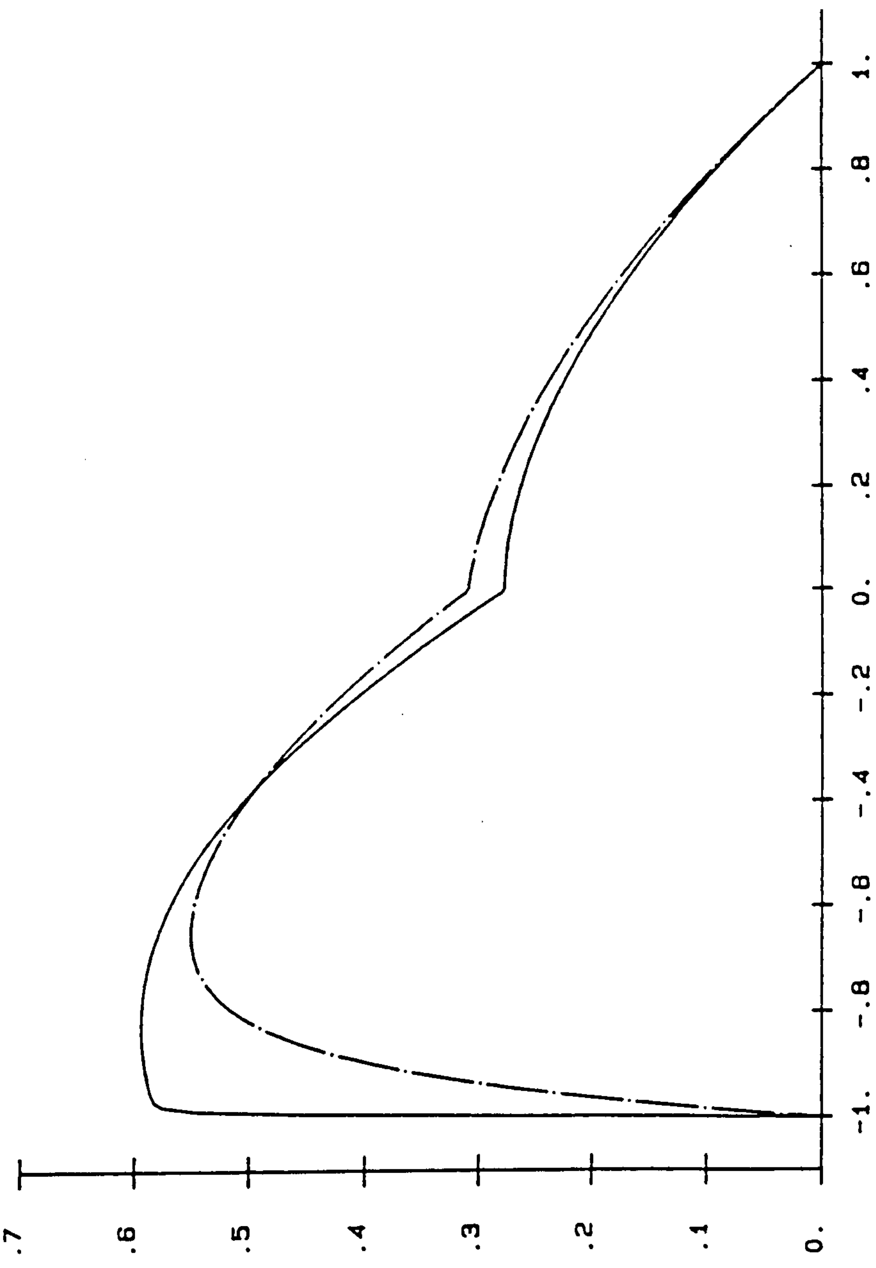


Fig. 2.4

Fig. 2.5



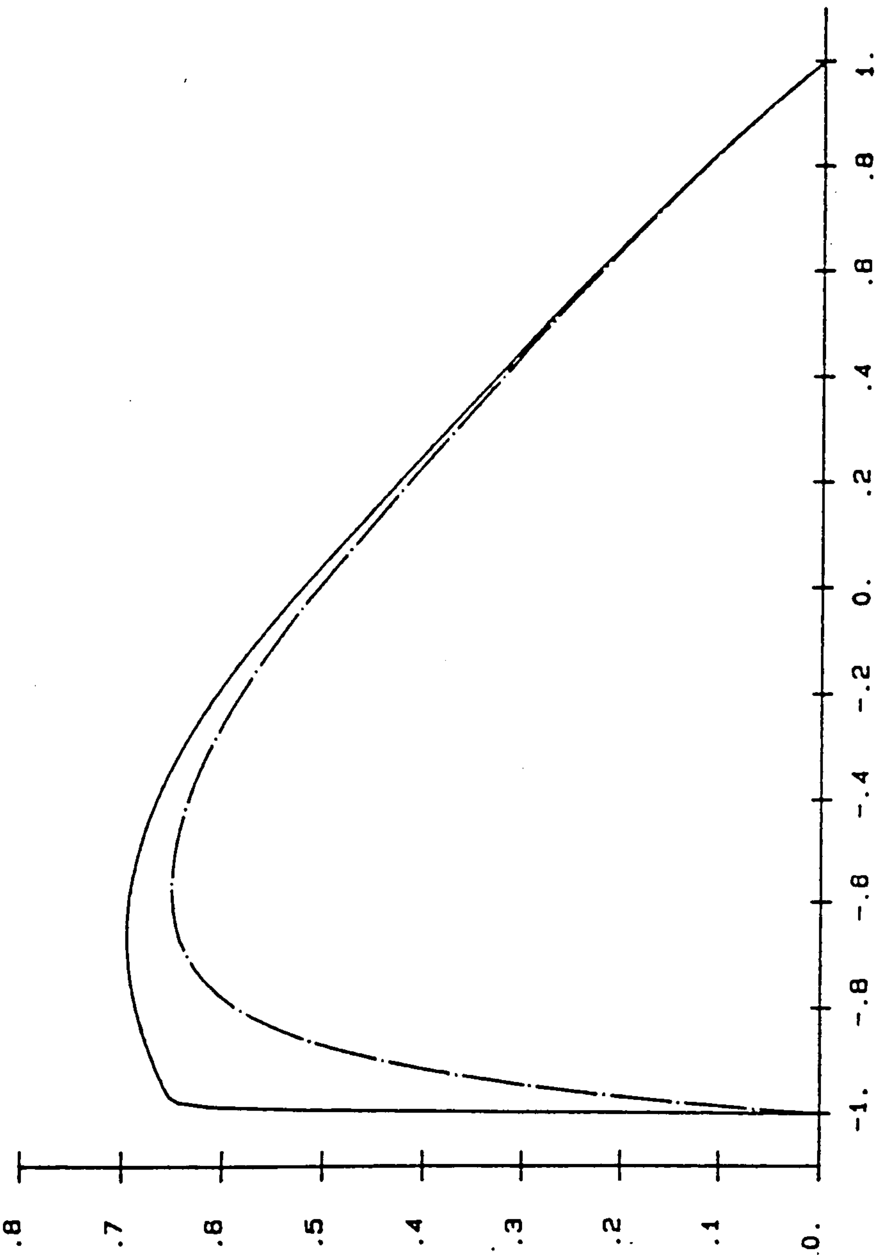


Fig. 2.6

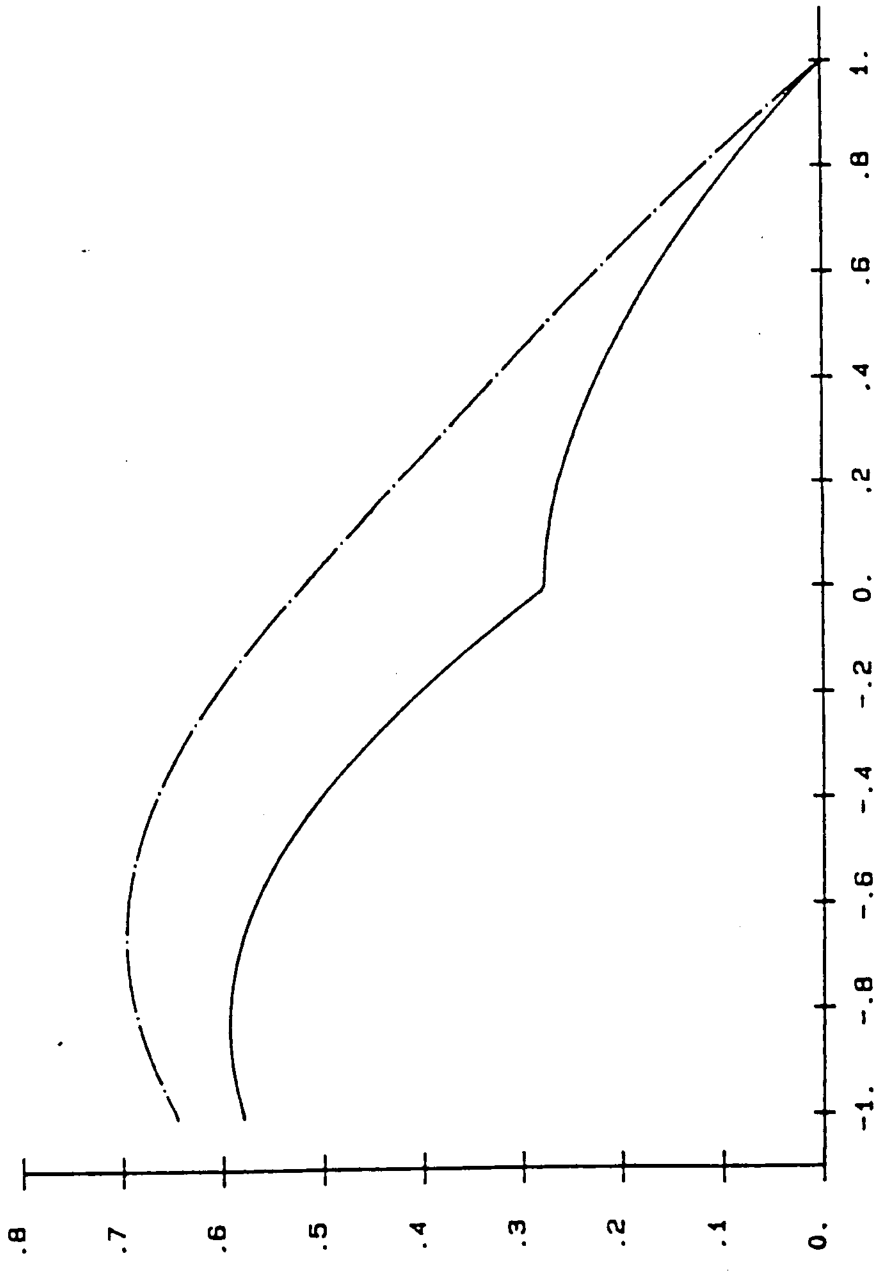


Fig. 2.7

3. HYPERBOLIC-ELLIPTIC INTERACTION: THE (TIME INDEPENDENT) VECTOR CASE

In this section we consider a boundary value problem for a system of three linear equations. Precisely, we deal with the stationary problem associated to (1.1), (1.2).

3.1. The differential problem

With the notations of the introduction, we seek for a pair of three dimensional vector functions $\mathbf{w}^-$ and $\mathbf{w}^+$ such that

$$A\,\mathbf{w}_x^- + B\mathbf{w}^- = \mathbf{F} \qquad in\ \Omega^-, \tag{3.1}$$

$$-(\mu\mathbf{w}_x^+)_x + A\,\mathbf{w}_x^+ + B\,\mathbf{w}^+ = \mathbf{F} \qquad in\ \Omega^+. \tag{3.2}$$

About boundary conditions, we must distinguish between the points a and c.

At $x = a$, we prescribe exactly p_0 conditions on $\mathbf{w}^-$, where p_0 is the number of positive eigenvalues of the matrix A. These conditions take the form

$$G^-\mathbf{w}^- = \mathbf{q}^- \qquad at\ x = a, \tag{3.3}$$

where $\mathbf{q}^-$ is a given vector with p_0 components, while G^- is a $p_0 \times 3$ matrix with rank p_0. The choice of G^- is subject to some restrictions that will be specified later.

At $x = c$, the boundary conditions can be written in the general form

$$G^+\mathbf{w}^+ + H^+\mathbf{w}_x^+ = \mathbf{q}^+ \qquad at\ x = c, \tag{3.4}$$

where G^+ and H^+ are 3×3 matrices and $\mathbf{q}^+$ is a given vector with three components. (3.4) must provide 3 independent equations, which are admissible for the elliptic system (3.2). In general, (3.4) yields a coupling between the three components of $\mathbf{w}^+$ and their derivatives. However, in some special circumstances, (3.4) might lead to three equations, each of them containing only one component and/or its derivative.

Problem (3.1)-(3.4) needs $(6 - p_0)$ further conditions at the interface point $x = b$. Essentially, three of them are requested by the elliptic system (3.2), while $(3 - p_0)$ (the number of negative eigenvalues of A) pertain to the hyperbolic system (3.1).

To write down the interface conditions, let us introduce the matrix T which diagonalizes A and denote by

$$\Lambda = T\,A\,T^{-1} \tag{3.5}$$

the (diagonal) eigenvalue matrix. We write Λ as

$$\Lambda = \begin{bmatrix} \Lambda_p & 0 \\ 0 & \Lambda_n \end{bmatrix}, \tag{3.6}$$

where Λ_p is the diagonal matrix of the p_0 positive eigenvalues of A, while Λ_n is made by the remaining $3-p_0$ negative eigenvalues. Correspondingly, we write T as

$$T = \begin{bmatrix} T_p \\ T_n \end{bmatrix}, \tag{3.7}$$

where T_p is the submatrix of the first p_0 rows of T and T_n is the rest (note that the rows of T are made by the left eigenvectors of A).

The interface conditions we consider here are of two types: either

$$\begin{cases} (i) \ -\mu \mathbf{w}_x^+ + A\,\mathbf{w}^+ = A\,\mathbf{w}^- & (3\ cond.s) \\ (ii) \qquad\qquad T_n\,\mathbf{w}^+ = T_n\,\mathbf{w}^- & (3-p_0\ cond.s) \end{cases} \tag{3.8}$$

(variational approach) or

$$\begin{cases} (i) \qquad \mathbf{w}^+ = \mathbf{w}^- & (3\ cond.s) \\ (ii)\ T_n\,\mathbf{w}_x^+ = T_n\,\mathbf{w}_x^- & (3-p_0\ cond.s) \end{cases} \tag{3.9}$$

(nonvariational approach).

In both cases, we impose as many conditions as requested.

The rest of this subsection is devoted to a mathematical justification of (3.8) and (3.9) by means of the asymptotic procedure on elliptic regularizations, in analogy with the scalar case.

Precisely, for a given $\epsilon > 0$, we consider the regularized problem

$$-\epsilon\,\mathbf{w}_{\epsilon,xx}^- + A\,\mathbf{w}_{\epsilon,x}^- + B\,\mathbf{w}_\epsilon^- = \mathbf{F} \qquad in\ \Omega^-, \tag{3.10}$$

$$-(\mu\,\mathbf{w}_{\epsilon,x}^+)_x + A\,\mathbf{w}_{\epsilon,x}^+ + B\,\mathbf{w}_\epsilon^+ = \mathbf{F} \qquad in\ \Omega^+, \tag{3.11}$$

with boundary conditions

$$\begin{cases} G^-\mathbf{w}_\epsilon^- = \mathbf{q}^- \\ T_n\,\mathbf{w}_{\epsilon,x}^- = 0 \end{cases} \quad at\ x = a, \tag{3.12}$$

$$G^+\mathbf{w}_\epsilon^+ + H^+\mathbf{w}_{\epsilon,x}^+ = \mathbf{q}^+ \quad at\ x = c \tag{3.13}$$

and interface conditions at the point $x = b$

$$(\textit{variational approach}) \quad \left| \begin{array}{l} \mathbf{w}_\epsilon^- = \mathbf{w}_\epsilon^+ \\ \epsilon\, \mathbf{w}_{\epsilon,x}^- = \mu\, \mathbf{w}_{\epsilon,x}^+ \end{array} \right. \tag{3.14}$$

or

$$(\textit{nonvariational approach}) \quad \left| \begin{array}{l} \mathbf{w}_\epsilon^- = \mathbf{w}_\epsilon^+ \\ \mathbf{w}_{\epsilon,x}^- = \mathbf{w}_{\epsilon,x}^+ . \end{array} \right. \tag{3.15}$$

In (3.12), the original boundary condition has been added a homogeneous Neumann condition on $T_n \mathbf{w}_\epsilon^-$: this is not the only possibility, but it is optimal, in some sense (see Remark 3.1).

In order to exploit the results of section 2.1, it is natural to diagonalize the system (3.10), (3.11). This is done by introducing the characteristic variables associated to the system, namely

$$\mathbf{z}_\epsilon^\pm = T\, \mathbf{w}_\epsilon^\pm :$$

denote by $(\mathbf{z}_\epsilon^\pm)_p$ the first p_0 components of $\mathbf{z}_\epsilon^\pm$ and by $(\mathbf{z}_\epsilon^\pm)_n$ the remaining $3 - p_0$ components. Thus, (3.10)-(3.15) imply that $\mathbf{z}_\epsilon^\pm$ satisfy the equations:

$$-\,\epsilon\, \mathbf{z}_{\epsilon,xx}^- + \Lambda \mathbf{z}_{\epsilon,x}^- + B_T \mathbf{z}_\epsilon^- = \mathbf{F}_T \qquad in\ \Omega^-, \tag{3.16}$$

$$-\,(\mu\, \mathbf{z}_{\epsilon,x}^+)_x + \Lambda \mathbf{z}_{\epsilon,x}^+ + B_T \mathbf{z}_\epsilon^+ = \mathbf{F}_T \qquad in\ \Omega^+ \tag{3.17}$$

(where $B_T = T\, B\, T^{-1}$ and $\mathbf{F}_T = T\, \mathbf{F}$), boundary conditions

$$\left| \begin{array}{l} G_T^- \mathbf{z}_\epsilon^- = \mathbf{q}^- \\ (\mathbf{z}_{\epsilon,x}^-)_n = 0 \end{array} \right. \quad at\ \ x = a\,, \tag{3.18}$$

$$G_T^+ \mathbf{z}_\epsilon^+ + H_T^+ \mathbf{z}_{\epsilon,x}^+ = \mathbf{q}^+ \qquad at\ \ x = c \tag{3.19}$$

(where $G_T^\pm = G^\pm T^{-1}$ and $H_T^+ = H^+ T^{-1}$) and interface conditions at the point $x = b$

$$(\textit{variational approach}) \quad \left| \begin{array}{l} \mathbf{z}_\epsilon^- = \mathbf{z}_\epsilon^+ \\ \epsilon\, \mathbf{z}_{\epsilon,x}^- = \mu\, \mathbf{z}_{\epsilon,x}^+ \end{array} \right. \tag{3.20}$$

or

$$(\textit{nonvariational approach}) \quad \left| \begin{array}{l} \mathbf{z}_\epsilon^- = \mathbf{z}_\epsilon^+ \\ \mathbf{z}_{\epsilon,x}^- = \mathbf{z}_{\epsilon,x}^+ . \end{array} \right. \tag{3.21}$$

In (3.18), the matrix G_T^- must satisfy the following assumption:

$$\begin{array}{c} \textit{the submatrix given by the first} \\ p_0 \textit{ columns of } G_T^- \textit{ is nonsingular.} \end{array} \tag{3.22}$$

(3.22) poses restrictions on the choice of G^- in (3.3), depending on A.

We are now in a position to use Propositions 2.1-2.4, whence we get the following convergence results (we suppose existence of solutions to the regularized problems).

Proposition 3.1 *(Variational approach) As* $\epsilon \to 0$, *the solution* $\mathbf{z}_\epsilon^-$, $\mathbf{z}_\epsilon^+$ *to (3.16)-(3.20) converges to a pair of functions* $\mathbf{z}^-$, $\mathbf{z}^+$ *which satisfy:*

$$\Lambda \mathbf{z}_x^- + B_T \mathbf{z}^- = \mathbf{F}_T \quad \text{in } \Omega^-, \tag{3.23}$$

$$-(\mu\, \mathbf{z}_x^+)_x + \Lambda \mathbf{z}_x^+ + B_T \mathbf{z}^+ = \mathbf{F}_T \quad \text{in } \Omega^+, \tag{3.24}$$

$$G_T^- \mathbf{z}^- = \mathbf{q}^- \quad \text{at } x = a, \tag{3.25}$$

$$G_T^+ \mathbf{z}^+ + H_T^+ \mathbf{z}_x^+ = \mathbf{q}^+ \quad \text{at } x = c, \tag{3.26}$$

$$-\mu\, \mathbf{z}_{p,x}^+ + \Lambda_p \mathbf{z}_p^+ = \Lambda_p \mathbf{z}_p^- \quad \text{at } x = b, \tag{3.27}$$

$$\mathbf{z}_n^+ = \mathbf{z}_n^- \quad \text{at } x = b, \tag{3.28}$$

$$\mathbf{z}_{n,x}^+ = 0 \quad \text{at } x = b \tag{3.29}$$

(in (3.27)-(3.29), $\mathbf{z}_p^\pm$ *denotes the first* p_0 *components of* $\mathbf{z}^\pm$ *and* $\mathbf{z}_n^\pm$ *denotes the remaining* $3 - p_0$ *components).*

□

Proposition 3.2 *(Nonvariational approach) As* $\epsilon \to 0$, *the solution* $\mathbf{z}_\epsilon^-$, $\mathbf{z}_\epsilon^+$ *to (3.16)-(3.19) and (3.21) converges to a pair of functions* $\mathbf{z}^-$, $\mathbf{z}^+$ *which satisfy (3.23)-(3.26) and*

$$\mathbf{z}^+ = \mathbf{z}^- \quad \text{at } x = b, \tag{3.30}$$

$$\mathbf{z}_{n,x}^+ = \mathbf{z}_{n,x}^- \quad \text{at } x = b. \tag{3.31}$$

□

By re-transforming these results in terms of the physical variables $\mathbf{w}^\pm$, we find (3.1)-(3.4) and either

$$\left\{ \begin{array}{llll} (i) & -\mu T_p \mathbf{w}_x^+ + \Lambda_p T_p \mathbf{w}^+ = & \Lambda_p T_p \mathbf{w}^- & (p_0 \textit{ cond.s}) \\ (ii) & T_n \mathbf{w}^+ = & T_n \mathbf{w}^- & (3 - p_0 \textit{ cond.s}) \\ (iii) & T_n \mathbf{w}_x^+ = & 0 & (3 - p_0 \textit{ cond.s}) \end{array} \right. \tag{3.32}$$

or (3.9), according to the regularization chosen. It remains to show that (3.32) is equivalent to (3.8). Actually, taking (3.32ii) into account, (3.32iii) can be written as

$$-\mu T_n \mathbf{w}_x^+ + \Lambda_n T_n \mathbf{w}^+ = \Lambda_n T_n \mathbf{w}^- .$$

Together with (3.32i), this last condition gives

$$-\mu T \mathbf{w}_x^+ + \Lambda T \mathbf{w}^+ = \Lambda T \mathbf{w}^- .$$

Multiplying by T^{-1} and recalling that μ is a scalar function, we get

$$-\mu \mathbf{w}_x^+ + A\,\mathbf{w}^+ = A\,\mathbf{w}^- ,$$

whence (3.8) follows.

Remark 3.1 A Dirichlet condition on $T_n \mathbf{w}_\epsilon^-$ in (3.12) (i.e. a Dirichlet condition on $(\mathbf{z}_\epsilon^-)_n$ in (3.18)) is as good as the Neumann condition we considered in (3.12), provided (3.25) involves the p_0 characteristic variables corresponding to positive eigenvalues only. This means that the last $3-p_0$ columns of G_T^- ought to vanish identically. Essentially, the reason of this drawback is that the Dirichlet condition cannot guarantee the convergence of $(\mathbf{z}_\epsilon^-)_n(a)$ to $\mathbf{z}_n^-(a)$. Thus, the strategy of reducing the analysis of the system to that of the scalar case cannot deal with a condition of type (3.3) involving the value of $\mathbf{z}_n^-(a)$. However, a more sophisticated vector approach could be performed, capable of overcoming this difficulty (see [BBB], [BR], [L]).

3.2. The numerical approximation

We adopt the notations of section 2.2 for the collocation points.

The spectral collocation approximation to problem (3.1)-(3.4) reads as follows. We look for $\mathbf{w}_N^- \in (\mathbf{P}_N)^3$ and $\mathbf{w}_N^+ \in (\mathbf{P}_N)^3$ satisfying:

$$A\,\mathbf{w}_{N,x}^- + B\,\mathbf{w}_N^- = \mathbf{F} \quad at\ x_j^-,\ j=1,\cdots,N-1, \tag{3.33}$$

$$-[I_N(\mu \mathbf{w}_{N,x}^+)]_x + A\,\mathbf{w}_{N,x}^+ + B\,\mathbf{w}_N^+ = \mathbf{F} \quad at\ x_j^+,\ j=1,\cdots,N-1. \tag{3.34}$$

The conditions at $x=a$ are of two types:

(i) p_0 prescribed boundary conditions (see (3.3)):

$$G^- \mathbf{w}_N^- = \mathbf{q}^- \qquad at\ x_0^- , \tag{3.35}$$

(ii) $(3-p_0)$ compatibility conditions:

$$T_n\,[A\,\mathbf{w}_{N,x}^- + B\,\mathbf{w}_N^- - \mathbf{F}] = 0 \qquad at\ x_0^- . \tag{3.36}$$

Note that (3.36) are nothing but the collocation at x_0^- of the equations on the characteristic variables corresponding to negative eigenvalues: they generalize to the vector case the compatibility condition (2.25i) for the scalar case, yielding a stable and consistent scheme (see, e.g., [CQ]).

At the right boundary point c, we enforce the prescribed boundary conditions (3.4) on the discrete solution, namely

$$G^+\mathbf{w}_N^+ + H^+\mathbf{w}_{N,x}^+ = \mathbf{q}^+ \qquad at\ x_N^+ . \tag{3.37}$$

Now, we come to the conditions at the interface point b. As usual, we distinguish between the variational and the nonvariational approaches which have been used. The results of the analysis presented in section 3.1 (see Propositions 3.1 and 3.2) suggest the proper continuity conditions to be enforced at the interface point.

(a) Variational approach.

(i) p_0 compatibility conditions on the equations corresponding to the positive characteristic variables:

$$T_p\,[A\,\mathbf{w}_{N,x}^- + B\,\mathbf{w}_N^- - \mathbf{F}] = 0 \quad at\ b\ (= x_N^-); \tag{3.38}$$

(ii) $(3-p_0)$ conditions of continuity on the negative characteristic variables:

$$T_n\,\mathbf{w}_N^- = T_n\,\mathbf{w}_N^+ \qquad at\ b\ (= x_N^-), \tag{3.39}$$

obtainable from (3.8ii);

(iii) 3 conditions of continuity of the "flux" on the physical variables:

$$-\mu\mathbf{w}_{N,x}^+ + A\,\mathbf{w}_N^+ = A\,\mathbf{w}_N^- \quad at\ b\ (= x_0^+). \tag{3.40}$$

obtainable from (3.8i).

Remark 3.2 Notice that the hyperbolic system (3.33) has been supplemented three conditions at the interface point $b\ (= x_N^-)$ in (i) and (ii) (see (3.38) and (3.39)). Similarly, the elliptic system (3.34) has been given three Newton-like conditions at the interface point $b\ (= x_0^+)$ in (iii) (see (3.40)).

(b) Nonvariational approach.

(i) p_0 compatibility conditions on the equations corresponding to the positive characteristic variables, given by (3.38);

(ii) $(3-p_0)$ conditions of continuity on the negative characteristic variables, given by (3.39);

(iii) p_0 conditions of continuity on the positive characteristic variables:

$$T_p \mathbf{w}_N^+ = T_p \mathbf{w}_N^- \qquad \text{at } b\ (= x_0^+\ ): \tag{3.41}$$

both (ii) and (iii) are obtainable from (3.9i);

(iv) $(3-p_0)$ conditions of continuity of first derivatives on the negative characteristic variables:

$$T_n\ \mathbf{w}_{N,x}^+ = T_n \mathbf{w}_{N,x}^- \qquad \text{at } b\ (= x_0^+\ ), \tag{3.42}$$

obtainable from (3.9ii).

The same kind of considerations as in Remark 3.2 can be made in this case, too.

We note that (ii) and (iii) amount to require that $\mathbf{w}_N^+ = \mathbf{w}_N^-$ at b.

Remark 3.3 An efficient (and quite natural) method to solve problems of the form (3.33)-(3.37), supplemented with the interface conditions (3.38)-(3.40) (or (3.38), (3.39), (3.41) and (3.42)), relies upon an iterative procedure alternating the solution of a hyperbolic problem in Ω^- and of an elliptic one in Ω^+.

At each step, the iterative method entails within Ω^- the solution of the hyperbolic problem (3.33) with the boundary conditions (3.35) and (3.36) at the left hand boundary x_0^- , and (3.38), (3.39) at the right hand boundary x_N^- . Next, in Ω^+ we solve the elliptic problem (3.34) with the boundary condition (3.37) at the right hand boundary x_N^+ and the conditions (3.40) (or (3.41), (3.42)) at the left hand boundary x_0^+ . Finally, a relaxation procedure on the interface variables is generally needed, in order to ensure the convergence of the above process.

The details and the convergence analysis will be presented in a forthcoming paper.

4. HYPERBOLIC-PARABOLIC SYSTEMS FOR TIME DEPENDENT PROBLEMS

In this section we consider the problem (1.1), (1.2) presented in the introduction, endowed with its boundary, initial and interface conditions.

4.1. The differential problem

With a, b, c chosen in the usual way, we look for a three dimensional vector valued function $\mathbf{w}^{\pm}$ defined for $x \in \Omega^{\pm}$, $t > 0$, satisfying

$$\mathbf{w}_t^- + A\,\mathbf{w}_x^- + B\,\mathbf{w}^- = \mathbf{F} \qquad \text{for } x \in \Omega^- = (a,b),\ t > 0, \tag{4.1}$$

$$\mathbf{w}_t^+ - (\mu \mathbf{w}_x^+)_x + A\,\mathbf{w}_x^+ + B\,\mathbf{w}^+ = \mathbf{F} \qquad \text{for } x \in \Omega^+ = (b,c),\ t > 0, \tag{4.2}$$

where A, B, $\mathbf{F}$ and μ are given as like as in the introduction.

The system (4.1), (4.2) must be given an initial condition

$$\mathbf{w}^{\pm}(x,0) = \mathbf{w}_0^{\pm}(x), \qquad x \in \Omega^{\pm} \tag{4.3}$$

and boundary conditions, which we take again of the form (3.3) and (3.4), namely

$$G^-\mathbf{w}^- = \mathbf{q}^- \qquad at\ \ x = a,\ t > 0, \tag{4.4}$$

$$G^+\mathbf{w}^+ + H^+\mathbf{w}_x^+ = \mathbf{q}^+ \qquad at\ \ x = c,\ t > 0, \tag{4.5}$$

where G^-, G^+, H^+, $\mathbf{q}^-$ and $\mathbf{q}^+$ may depend on t.

Analogously, at the interface line $\{b\} \times (0,+\infty)$ we impose conditions which are the natural extension of (3.8) and (3.9) to the evolution case: either

$$\text{(variational approach)} \quad \left\{ \begin{array}{ll} (i) & T_n \mathbf{w}^+ = T_n \mathbf{w}^- \\ (ii) & -\mu \mathbf{w}_x^+ + A\,\mathbf{w}^+ = A\,\mathbf{w}^- \end{array} \right. \tag{4.6}$$

or

$$\text{(nonvariational approach)} \quad \left\{ \begin{array}{ll} (i) & \mathbf{w}^+ = \mathbf{w}^- \\ (ii) & T_n \mathbf{w}_x^+ = T_n \mathbf{w}_x^- \end{array} \right. \tag{4.7}$$

for $x = b$ and for $t > 0$.

The interface conditions (4.6) or (4.7) might be derived directly by means of regularized parabolic problems, in analogy to the procedure presented in section 3.1.

On the other hand, several heuristic justifications of these conditions may be given. For instance, one may take the Laplace transform of (4.1), (4.2), at least formally: the new unknowns satisfy a problem similar to (3.1)-(3.4). This means that the interface conditions for the new unknowns are precisely (3.8) or (3.9): by anti-transforming these conditions one gets exactly (4.6) or (4.7).

Furthermore, problem (3.1)-(3.4) can be viewed as a (possible) steady state for the time-dependent problem (4.1)-(4.5), or else as the time-discretization (at any time level) of problem (4.1)-(4.5), using an implicit time-stepping scheme. In both cases, in section 3.1 we have seen that the interface conditions (3.8) or (3.9) are appropriate for problem (3.1)-(3.4). Thus, (4.6) or (4.7) turn out to be appropriate for problem (4.1)-(4.5).

4.2. The numerical approximation

First, we consider a *semidiscrete (continuous in time)* approximation of problem (4.1)-(4.5), endowed either with (4.6) or with (4.7). Keeping the same notations of the preceding sections 2.2, 3.2, we apply the spectral collocation method in space, that is, we look for two mappings

$$t \to \mathbf{w}_N^{\pm}(t) \in (\mathbf{P}_N)^3$$

satisfying, for all $t > 0$ and all $j = 1, \cdots, N-1$,

$$\mathbf{w}_{N,t}^- + A\,\mathbf{w}_{N,x}^- + B\,\mathbf{w}_N^- = \mathbf{F} \quad at\ x_j^-\,, \tag{4.8}$$

$$\mathbf{w}_{N,t}^+ - [I_N(\mu \mathbf{w}_{N,x}^+)]_x + A\,\mathbf{w}_{N,x}^+ + B\,\mathbf{w}_N^+ = \mathbf{F} \quad at\ x_j^+\,, \tag{4.9}$$

At the left boundary we impose the conditions

$$G^-\mathbf{w}_N^- = \mathbf{q}^-, \quad T_n\,[\mathbf{w}_{N,t}^- + A\,\mathbf{w}_{N,x}^- + B\,\mathbf{w}_N^- - \mathbf{F}] = 0 \tag{4.10}$$

for $x = x_0^-$ and $t > 0$, while at the right boundary the conditions are

$$G^+\mathbf{w}_N^+ + H^+\mathbf{w}_{N,x}^+ = \mathbf{q}^+ \tag{4.11}$$

for $x = x_N^+$ and $t > 0$. Eventually, the two alternative sets of interface conditions to be requested for $x = x_N^- = x_0^+$ and $t > 0$ are the following:

(a) Variational approach,

$$T_p[\mathbf{w}_{N,t}^- + A\,\mathbf{w}_{N,x}^- + B\,\mathbf{w}_N^- - \mathbf{F}] = 0, \tag{4.12}$$

$$T_n\,\mathbf{w}_N^- = T_n\,\mathbf{w}_N^+, \tag{4.13}$$

$$-\mu \mathbf{w}_{N,x}^+ + A\,\mathbf{w}_N^+ = A\,\mathbf{w}_N^-. \tag{4.14}$$

(b) Nonvariational approach.

$$T_p\,[\mathbf{w}^-_{N,t} + A\,\mathbf{w}^-_{N,x} + B\,\mathbf{w}^-_N - \mathbf{F}] = 0, \tag{4.15}$$

$$\mathbf{w}^-_N = \mathbf{w}^+_N, \tag{4.16}$$

$$T_n\,\mathbf{w}^+_{N,x} = T_n\,\mathbf{w}^-_{N,x}. \tag{4.17}$$

A *fully discrete approximation* to problem (4.1)-(4.5), endowed either with (4.6) or with (4.7) can be achieved by applying a time-stepping procedure to (4.8), (4.9). Whatever scheme (either implicit or explicit) one uses to advance from a known time level t^k to a new one t^{k+1}, the interface conditions, as well as the boundary conditions , need to be imposed at the new time t^{k+1}.

If an *explicit scheme* is used in this regard, at the time t^{k+1} the unknown vectors $\{\mathbf{w}^-_N(x_j^-)\}$ and $\{\mathbf{w}^+_N(x_j^+)\}$, $j = 1,\cdots,N-1$, can be computed independently of the boundary and interface values. Once these internal values are available, the boundary equations (4.10) and (4.11), together with the interface conditions (4.12)-(4.14) (or (4.15)-(4.17)), can be solved to provide the remaining values at boundary and interface points. Actually, we note that the presence of derivatives in space among boundary and interface conditions relates boundary and interface values to each other. We also note that the differential equations between brackets in (4.10) and in (4.12) (or (4.15)) ought to be advanced by the same explicit scheme which was used for the equations at the internal points.

When an *implicit time marching scheme* is used, the internal unknowns are not decoupled from the remaining ones any more. As an example, we detail the case of the simplest implicit scheme, namely the first order forward Euler scheme.

Denoting by Δt the time step, by $t^k = k\ \Delta t$ the k-th time level and by $(\mathbf{w}^{\mp}_N)^k$ the spectral solutions at the time t^k, the corresponding problem reads:

$$(\mathbf{w}^-_N)^{k+1} + \Delta t\ [A\,\mathbf{w}^-_{N,x} + B\,\mathbf{w}^-_N - \mathbf{F}]^{k+1} - (\mathbf{w}^-_N)^k = 0, \tag{4.18}$$

$$(\mathbf{w}^+_N)^{k+1} + \Delta t\ \{ -[I_N(\mu\mathbf{w}^+_{N,x})]_x + $$
$$+ A\,\mathbf{w}^+_{N,x} + B\,\mathbf{w}^+_N - \mathbf{F}\,\}^{k+1} - (\mathbf{w}^+_N)^k = 0. \tag{4.19}$$

The boundary equations (4.10) and (4.11) are discretized as follows:

$$\left\{\begin{aligned} &[G^-\mathbf{w}^-_N - \mathbf{q}^-]^{k+1} = 0, \\ &T_n\,\{(\mathbf{w}^-_N)^{k+1} + \Delta t\ [A\,\mathbf{w}^-_{N,x} + B\,\mathbf{w}^-_N - \mathbf{F}]^{k+1} - (\mathbf{w}^-_N)^k\} = 0 \end{aligned}\right. \tag{4.20}$$

at x_0^- ;

$$[G^+ \mathbf{w}_N^+ + H^+ \mathbf{w}_{N,x}^+ - \mathbf{q}^+]^{k+1} = 0 \qquad (4.21)$$

at x_N^+. Analogously, the interface conditions (4.12)-(4.14) give:

$$T_p \{ (\mathbf{w}_N^-)^{k+1} + \Delta t \, [A \, \mathbf{w}_{N,x}^- + B \, \mathbf{w}_N^- - \mathbf{F}]^{k+1} - (\mathbf{w}_N^-)^k \} = 0, \qquad (4.22)$$

$$T_n \, [(\mathbf{w}_N^-)^{k+1} - (\mathbf{w}_N^+)^{k+1}] = 0, \qquad (4.23)$$

$$-\mu^{k+1} (\mathbf{w}_{N,x}^+)^{k+1} + A \, (\mathbf{w}_N^+)^{k+1} = A \, (\mathbf{w}_N^-)^{k+1}, \qquad (4.24)$$

at x_0^+. The alternative interface equations (4.15)-(4.17) read:

$$T_p \{ (\mathbf{w}_N^-)^{k+1} + \Delta t \, [A \, \mathbf{w}_{N,x}^- + B \, \mathbf{w}_N^- - \mathbf{F}]^{k+1} - (\mathbf{w}_N^-)^k \} = 0, \qquad (4.25)$$

$$(\mathbf{w}_N^-)^{k+1} = (\mathbf{w}_N^+)^{k+1}, \qquad (4.26)$$

$$T_n \, [(\mathbf{w}_{N,x}^-)^{k+1} - (\mathbf{w}_{N,x}^+)^{k+1}] = 0. \qquad (4.27)$$

We notice that the structure of the system would be the same when using other implicit time-marching schemes (such as, for instance, the second order Beam & Warming scheme).

Remark 4.1 We note that (4.18)-(4.21) with the interface conditions (4.22)-(4.24) (or (4.25)-(4.27)) have the same shape as the time independent problem (3.33)-(3.42) considered in the previous section. Clearly, in (3.33)-(3.42) we must replace $\mathbf{w}$ by $\mathbf{w}_N^{k+1}$, B by $B + (\Delta t)^{-1} I$ and $\mathbf{F}$ by $\mathbf{F}^{k+1} + (\Delta t)^{-1} \mathbf{w}_N^k$, respectively. Therefore, the same iterative procedure can be used in order to decouple the hyperbolic problem in Ω^- and the elliptic one in Ω^+.

APPENDIX: abstract analysis of the regularizing problems presented in section 2

In this Appendix, we detail the existence and asymptotic convergence results stated in Propositions 2.1-2.4 for problems $(\mathbf{P}_\epsilon)$, $(\mathbf{Q}_\epsilon)$, $(\mathbf{P}_\epsilon)_N$ and $(\mathbf{Q}_\epsilon)_N$.

As a standard notation, whenever Ω is an open interval and k is a positive integer we introduce the Sobolev space (see [A])

$$\mathbf{H}^k(\Omega) = \{ v \in \mathbf{L}^2(\Omega) : D^m v \in \mathbf{L}^2(\Omega), m = 1, \cdots, k \}. \qquad (A.1)$$

$\mathbf{H}^k(\Omega)$ is a Hilbert space with norm

$$\| v \|_{\mathbf{H}^k(\Omega)} = \left[\| v \|^2_{\mathbf{L}^2(\Omega)} + \sum_{m=1}^{k} \| D^m v \|^2_{\mathbf{L}^2(\Omega)} \right]^{1/2}.$$

Since Ω is one dimensional, we have that

$$\mathbf{H}^k(\Omega) \subset \mathbf{C}^{k-1}(\overline{\Omega}), \tag{A.2}$$

for all positive integer k, the embedding being compact. In particular, $\mathbf{H}^1(\Omega)$ is made by functions continuous up to the boundary. Therefore, the following (usual) notation is meaningful:

$$\mathbf{H}_0^1(\Omega) = \{ v \in \mathbf{H}^1(\Omega) : v = 0 \textit{ at the endpoints of } \Omega \}. \tag{A.3}$$

Unless otherwise stated, we will make the following assumptions on the data of problem (**P**):

$$\mu \in \mathbf{L}^\infty(b,c), \quad \alpha \in \mathbf{H}^1(a,c), \quad \beta \in \mathbf{L}^2(a,c), \quad f \in \mathbf{L}^2(a,c). \tag{A.4}$$

Problem ($\mathbf{P}_\epsilon$).

Recall that $\alpha > 0$ in this case.

Under the assumption (A.4) (actually, under *milder* assumptions), ($\mathbf{P}_\epsilon$) can be written in a rigorous variational form:

find $w_\epsilon \in \mathbf{W}$ *such that, for all* $\phi \in \mathbf{W}$,

$$\int_a^c a_\epsilon w_{\epsilon,x} \phi_x \, dx + \int_a^c \alpha w_{\epsilon,x} \phi dx + \int_a^c \beta w_\epsilon \phi dx = \int_a^c f \, \phi dx, \tag{A.5}$$

where

$$\mathbf{W} = \mathbf{H}_0^1(a,c), \qquad a_\epsilon = \begin{cases} \epsilon & in \ (a,b) \\ \mu & in \ (b,c). \end{cases} \tag{A.6}$$

If w_ϵ solves (A.5), then the functions

$$u_\epsilon = w_{\epsilon|(a,b)}, \quad v_\epsilon = w_{\epsilon|(b,c)} \tag{A.7}$$

solve (2.5)-(2.9): this is easily checked by means of suitable choices of ϕ in (A.5). In particular, (A.5) entails the equation (in the distribution sense)

$$-(a_\epsilon w_{\epsilon,x})_x + \alpha w_{\epsilon,x} + \beta w_\epsilon = f \qquad in \ (a,c), \tag{A.8}$$

whence

$$a_\epsilon w_{\epsilon,x} \in \mathbf{H}^1(a,c). \tag{A.9}$$

By (A.2) it follows that both w_ϵ and $a_\epsilon w_{\epsilon,x}$ are continuous in $[a,c]$, hence (2.7), (2.8) and (2.9) have the classical meaning.

In order to achieve an existence result for ($\mathbf{P}_\epsilon$), from now on we make the following requests:

$$\mu \geqslant \mu_0 \quad in\ (b,c), \tag{A.10}$$

for a suitable strictly positive constant μ_0 and

$$2\beta - \alpha_x \geqslant 0 \quad in\ (a,c). \tag{A.11}$$

Lemma A.1 *Under the assumptions (A.4), (A.10), (A.11),* $(\mathbf{P}_\epsilon)$ *has a unique solution.*

Proof. It is possible to apply Lax-Milgram lemma, because (A.10), (A.11) and Poincaré inequality imply that problem $(\mathbf{P}_\epsilon)$ is coercive. It goes without saying that coerciveness fails as $\epsilon \to 0$.

□

Now, let us discuss the asymptotic behavior of w_ϵ as $\epsilon \to 0$. We recall the notations (A.7) and the assumptions (A.4), (A.10) and (A.11), which still hold.

Lemma A.2 *There is a constant* $C > 0$ *such that*

$$\| w_\epsilon \|_{\mathrm{L}^2(a,c)} \leqslant C, \tag{A.12}$$

$$\| v_{\epsilon,x} \|_{\mathrm{L}^2(b,c)} \leqslant C, \tag{A.13}$$

$$\| \sqrt{\epsilon}\, u_{\epsilon,x} \|_{\mathrm{L}^2(a,b)} \leqslant C. \tag{A.14}$$

Proof. Plug the function ϕ in (A.5), with $\phi = e^{-x} w_\epsilon$ in (a,b), $\phi = e^{-b} w_\epsilon$ in (b,c), then integrate by parts. The assumptions and Poincaré inequality give the results.

□

Lemma A.3 *The* L^2 *norm of* $u_{\epsilon,x}$ *is bounded in a right neighborhood of the left boundary* $x = a$.

Proof. Let ψ be a smooth function in (a,c), vanishing outside a right neighborhood of a. Take the $\mathrm{L}^2(a,c)$ scalar product of (A.8) by $\psi w_{\epsilon,x}$: the assertion follows by (A.12) and (A.14).

□

Now, let us introduce the function

$$\Phi_\epsilon = a_\epsilon w_{\epsilon,x} - \alpha w_\epsilon : \tag{A.15}$$

We already know that $\Phi_\epsilon \in \mathbf{H}^1(a,c)$ (see (A.9)).

Lemma A.4 *The* $\mathbf{H}^1$ *norm of* Φ_ϵ *is bounded in* (a,c).

Proof. Lemma A.2 gives the boundedness of Φ_ϵ in $\mathbf{L}^2(a,c)$; (A.8) and (A.12) give the boundedness of $\Phi_{\epsilon,x}$ in $\mathbf{L}^2(a,c)$.

□

Now, we are in a position to give the following result, which completes and refines the statement of Proposition 2.1.

Proposition A.1 *Assume (A.4), (A.10), (A.11). There are* $u \in \mathbf{L}^2(a,b)$ *and* $v \in \mathbf{L}^2(b,c)$ *which satisfy*

$$\alpha u_x + \beta u = f \qquad in\ \mathbf{L}^2(a,b); \tag{A.16}$$

$$-(\mu v_x)_x + \alpha v_x + \beta v = f \qquad in\ \mathbf{L}^2(b,c); \tag{A.17}$$

$$u(a) = 0; \tag{A.18}$$

$$v(c) = 0; \tag{A.19}$$

$$\alpha u = -\mu v_x + \alpha v \qquad at\ x = b. \tag{A.20}$$

Proof. As a consequence of Lemmas A.2-A.4, of Banach-Alaoglu-Bourbaki theorem and of (A.2), we can find $u \in \mathbf{L}^2(a,b)$, $v \in \mathbf{L}^2(b,c)$ and $\Phi \in \mathbf{H}^1(a,c)$ such that (upon extracting a subfamily)

(i) $u_\epsilon \to u$ weakly in $\mathbf{L}^2(a,b)$;

(ii) $v_\epsilon \to v$ weakly in $\mathbf{H}^1(b,c)$;

(iii) $\Phi_\epsilon \to \Phi$ weakly in $\mathbf{H}^1(a,c)$;

(iv) $u_\epsilon(a) \to u(a)$;

(v) $v_\epsilon(c) \to v(c)$;

(vi) $v_\epsilon(b) \to v(b)$;

(vii) $\epsilon u_{\epsilon,x} \to 0$ strongly in $\mathbf{L}^2(a,b)$.

Note that the value $u(a)$ is well defined, because of Lemma A.3. (i)-(iii) and (vii) permit to pass to the limit in (A.5): this gives (A.16) and (A.17). (A.18) and (A.19) follow by (iv) and (v), respectively, since $u_\epsilon(a) = v_\epsilon(c) = 0$. Finally, (i)-(iii) and (vii) entail that $\Phi = -\alpha u$ in (a,b) and $\Phi = \mu v_x - \alpha v$ in (b,c), whence (A.20) follows, by (vi).

□

Remark A.1 Analogous results could be proved when replacing the homogeneous Dirichlet condition at c by a Neumann condition or by a Newton-type condition.

Problem $(\mathbf{P}_\epsilon)_N$.

Recall that $\alpha < 0$ in this case.

For this problem, the variational formulation is still (A.5), just changing the function space: now we take

$$\mathbf{W} = \{ v \in \mathbf{H}^1(a,c) : v(c) = 0 \}. \tag{A.21}$$

The existence holds the same way as in the previous case and the asymptotic analysis is analogous. We detail the main steps, under the assumption:

$$\beta \in \mathbf{L}^\infty(a,b). \tag{A.22}$$

Moreover, we still assume (A.4), (A.10), (A.11) and use the notations (A.7).

Lemma A.5 *There is a constant $C > 0$ such that*

$$\| w_\epsilon \|_{\mathbf{L}^2(a,c)} \leqslant C, \tag{A.23}$$

$$\| v_{\epsilon,x} \|_{\mathbf{L}^2(b,c)} \leqslant C, \tag{A.24}$$

$$\| u_{\epsilon,x} \|_{\mathbf{L}^2(a,b)} \leqslant C, \tag{A.25}$$

$$\sqrt{\epsilon}\, | u_{\epsilon,x}(b) | \leqslant C, \tag{A.26}$$

$$\| (\mu v_{\epsilon,x})_x \|_{\mathbf{L}^2(b,c)} \leqslant C. \tag{A.27}$$

Proof. (A.23) and (A.24) follow by plugging the function ϕ in (A.5), with $\phi = e^x\, w_\epsilon$ in (a,b), $\phi = e^b\, w_\epsilon$ in (b,c), then integrating by parts. The assumptions and Poincaré inequality give the results.
To prove (A.25) and (A.26), take the $\mathbf{L}^2(a,c)$ scalar product of (A.8) by $\psi w_{\epsilon,x}$, where ψ is any smooth function, vanishing in (b,c), then integrate by parts.
Finally, (A.27) follows by (A.8), (A.23) and (A.24).

□

Thus, we are in a position to prove the main result, which was summarized in Proposition 2.2.

Proposition A.2 *Assume (A.4), (A.10), (A.11), (A.22). Moreover, assume that* μ

is continuous at $x = b$. Then, there are $u \in \mathbf{H}^1(a,b)$ and $v \in \mathbf{H}^1(b,c)$ which satisfy (A.16), (A.17), (A.19) and the interface conditions

$$u(b) = v(b), \tag{A.28}$$

$$v_x(b) = 0. \tag{A.29}$$

Proof. As a consequence of the previous Lemma and of (A.2), we can show the existence of $u \in \mathbf{H}^1(a,b)$ and $v \in \mathbf{H}^1(b,c)$ such that (upon extracting a subfamily)

(i) $u_\epsilon \rightarrow u$ weakly in $\mathbf{H}^1(a,b)$;

(ii) $v_\epsilon \rightarrow v$ weakly in $\mathbf{H}^1(b,c)$;

(iii) $\mu v_{\epsilon,x} \rightarrow \mu v_x$ weakly in $\mathbf{H}^1(a,c)$;

(iv) $v_\epsilon(c) \rightarrow v(c)$;

(v) $u_\epsilon(b) \rightarrow u(b)$ and $v_\epsilon(b) \rightarrow v(b)$;

(vi) $(\mu v_{\epsilon,x})(b) \rightarrow (\mu v_x)(b)$;

(vii) $\epsilon u_{\epsilon,x}(b) \rightarrow 0$.

(i)-(iii) permit to pass to the limit in (A.5). The conditions at $x = c$ and $x = b$ follow by (iv)-(vii), noting that $\mu(b) > 0$ (see (A.10)).

□

Remark A.2 If we take a homogeneous Dirichlet condition at $x = a$ instead of the Neumann one, then $(\mathbf{P}_\epsilon)_N$ coincides with $(\mathbf{P}_\epsilon)$; so does its variational formulation. But now we are assuming $\alpha < 0$, hence the asymptotic behavior is different from that of the case $\alpha > 0$. It is easy to see that the final Proposition A.2 still holds, with u found in $\mathbf{L}^2(a,b)$: actually, the convergence of u_ϵ to u is only $\mathbf{L}^2(a,b)$ (weak), whence we cannot have a convergence of $u_\epsilon(a)$ to $u(a)$, in general. Actually, Figure 2.5 shows a numerical evidence of a boundary layer for u_ϵ at $x = a$, although the limit function u is obviously continuous in $[a,b]$ (see (A.16) and (A.2)). This feature makes $(\mathbf{P}_\epsilon)_N$ preferable, especially in view of the applications to systems (sections 3 and 4).

Problems $(\mathbf{Q}_\epsilon)$ and $(\mathbf{Q}_\epsilon)_N$.

Now, the two problems do not admit a "natural" global variational formulation and the question of existence and the asymptotic behavior are somewhat more complicate. Nevertheless if we assume that (A.4) holds and that

$$\mu \textit{ is continuous at } x = b, \tag{A.30}$$

then the equations and the boundary and interface conditions defining $(\mathbf{Q}_\epsilon)$ and $(\mathbf{Q}_\epsilon)_N$ make sense, provided the solutions are sought for in $\mathbf{H}^1(a,b)$ and $\mathbf{H}^1(b,c)$, respectively.

We begin with problem $(\mathbf{Q}_\epsilon)$, recalling that $\alpha > 0$. It can be shown that $(\mathbf{Q}_\epsilon)$ has a unique solution u_ϵ, v_ϵ, at least for ϵ small, under the assumptions (A.4), (A.10), (A.11), (A.30). The asymptotic behavior is being investigated now, under the same assumptions.

Lemma A.6 *There is a constant* $C > 0$ *such that*

$$\| u_\epsilon \|_{\mathbf{L}^2(a,b)} \leqslant C, \tag{A.31}$$

$$\| \sqrt{\epsilon} u_{\epsilon,x} \|_{\mathbf{L}^2(a,b)} \leqslant C, \tag{A.32}$$

$$| u_\epsilon(b) | \leqslant C. \tag{A.33}$$

Proof. (i) Take the $\mathbf{L}^2(a,b)$ scalar product of (2.5) by $\mu(b) e^{-x} u_\epsilon$, then integrate by parts.

(ii) Take the $\mathbf{L}^2(b,c)$ scalar product of (2.6) by $\epsilon e^{-b} v_\epsilon$, then integrate by parts.

(iii) Add the two equations provided by (i) and (ii), term by term: the conclusion follows by Poincaré inequality.

□

Lemma A.7 *There is a constant* $C > 0$ *such that*

$$\| v_\epsilon \|_{\mathbf{H}^1(b,c)} \leqslant C, \tag{A.34}$$

$$| v_{\epsilon,x}(b) | \leqslant C, \tag{A.35}$$

$$\| u_\epsilon \|_{\mathbf{H}^1(a,b)} \leqslant C. \tag{A.36}$$

Proof. Let $\zeta_\epsilon \in \mathbf{H}^1(b,c)$ be the solution of

$$-(\mu \zeta_{\epsilon,x})_x = 0 \;\; in \;\; (b,c), \;\; \zeta_\epsilon(b) = v_\epsilon(b), \;\; \zeta_\epsilon(c) = 0.$$

By (A.33), the $\mathbf{H}^1(b,c)$ norm of ζ_ϵ is bounded, as well as the value of $\zeta_{\epsilon,x}(b)$. Moreover, the function $d_\epsilon \equiv v_\epsilon - \zeta_\epsilon$ belongs to $\mathbf{H}_0^1(b,c)$ and satisfies

$$-(\mu d_{\epsilon,x})_x + \alpha d_{\epsilon,x} + \beta d_\epsilon = g_\epsilon, \tag{A.37}$$

where $g_\epsilon = f - \alpha\zeta_{\epsilon,x} - \beta\zeta_\epsilon$ is bounded in $\mathbf{L}^2(b,c)$. Multiplying (A.37) in $\mathbf{L}^2(b,c)$ by d_ϵ, it follows that the $\mathbf{H}^1(b,c)$ norm of d_ϵ is bounded, whence (A.34).

Next, we multiply (A.37) by $\psi\mu d_{\epsilon,x}$, where ψ is a smooth function vanishing outside a right neighborhood of b: (A.35) follows easily.

Finally, (A.36) can be proved by taking the $\mathbf{L}^2(a,b)$ scalar product of (2.5) by $u_{\epsilon,x}$ and using (A.31), (A.35).

□

From Lemma A.7 we get the following proposition (see Proposition 2.3).

Proposition A.3 *Assume (A.4), (A.10), (A.11), (A.30). There are $u \in \mathbf{H}^1(a,b)$ and $v \in \mathbf{H}^1(b,c)$ which satisfy (A.16), (A.17), (A.18), (A.19) and (A.28).*

Proof. Let u_ϵ, v_ϵ solve $(\mathbf{Q}_\epsilon)$. By Lemma A.7, there are $u \in \mathbf{H}^1(a,b)$ and $v \in \mathbf{H}^1(b,c)$ such that (upon extracting a subfamily)

(i) $u_\epsilon \to u$ weakly in $\mathbf{H}^1(a,b)$;

(ii) $v_\epsilon \to v$ weakly in $\mathbf{H}^1(b,c)$;

(iii) $u_\epsilon(a) \to u(a)$;

(iv) $v_\epsilon(c) \to v(c)$;

(v) $u_\epsilon(b) \to u(b)$ and $v_\epsilon(b) \to v(b)$.

All of these properties permit to pass to the limit in the regularized problem $(\mathbf{Q}_\epsilon)$. Thus, the proof follows easily.

□

Now we come to problem $(\mathbf{Q}_\epsilon)_N$: recall that $\alpha < 0$.

This case looks somewhat trickier than the previous one and the natural choices for test functions do not seem to be appropriate, in proving the a priori estimates. Even more, it can be shown that problem $(\mathbf{Q}_\epsilon)_N$ may fail to have a solution under the assumptions (A.4), (A.10), (A.11), (A.30) (which were sufficient for existence in the previous case).

This trouble seems to be motivated by the lack of a maximum principle under the sole coerciveness condition (A.11) on β. For this reason, we discuss problem $(\mathbf{Q}_\epsilon)_N$ under the further hypothesis:

$$\beta(x) \geqslant 0 \quad \text{for } x \text{ a.e. in } (a,b). \tag{A.38}$$

We just note that such an assumption is not strongly restrictive if the problem we are dealing with is regarded as a time discretization of an evolution problem by an implicit method (see section 4.2).

It can be shown that $(\mathbf{Q}_\epsilon)_N$ has a unique solution u_ϵ , v_ϵ , under the assumptions (A.4), (A.10), (A.11), (A.30), (A.38).

The asymptotic behavior is being investigated now, under the same assumptions: for technical reasons, we will confine the situation a bit more, making the further hypothesis:

$$f \in \mathbf{L}^\infty(a,b), \quad \beta(x) \geqslant \beta_0 > 0 \textit{ for } x \textit{ a.e. in } (a,b), \tag{A.39}$$

for some β_0. This allows us to get low order estimates on u_ϵ and v_ϵ. Later on, we will make further assumptions in order to find higher order estimates.

Lemma A.8 *There is a constant* $C > 0$ *such that*

$$\| v_\epsilon \|_{\mathbf{H}^1(b,c)} \leqslant C, \tag{A.40}$$

$$| v_{\epsilon,x}(b) | \leqslant C, \tag{A.41}$$

$$\| (\mu v_{\epsilon,x})_x \|_{\mathbf{L}^2(b,c)} \leqslant C. \tag{A.42}$$

Proof. (i) Take the $\mathbf{L}^2(a,b)$ scalar product of (2.5) by $\mu(b)\phi u_\epsilon$, where

$$\phi(x) = \frac{1}{\epsilon} \exp\{\frac{1}{\epsilon} \int_x^b \alpha(t)dt\}.$$

Then, integrate by parts.

(ii) Take the $\mathbf{L}^2(b,c)$ scalar product of (2.6) by v_ϵ, then integrate by parts.

(iii) Add the two equations provided by (i) and (ii), term by term. Recalling (A.10), (A.11), (A.39), we find that

$$\epsilon\,\mu(b)\int_a^b \phi u_{\epsilon,x}{}^2dx + \mu_0\int_b^c v_{\epsilon,x}{}^2dx + \mu(b)\,\beta_0\int_a^b \phi u_\epsilon{}^2dx - \frac{1}{2}\,\alpha(b)v_\epsilon{}^2(b) \leqslant$$

$$\leqslant \mu(b)\int_a^b f\ \phi u_\epsilon dx + \int_b^c f\, v_\epsilon dx\ . \tag{A.43}$$

Now, by (A.39) we have

$$\int_a^b f\ \phi u_\epsilon dx \leqslant \| f \|_{\mathbf{L}^\infty(a,b)} \int_a^b \phi\, | u_\epsilon | dx,$$

so that Poincaré inequality in (A.43) gives

$$\epsilon \int_a^b \phi u_{\epsilon,x}{}^2dx + k_1\int_b^c v_{\epsilon,x}{}^2dx + \beta_0\int_a^b \phi u_\epsilon{}^2dx + k_2 v_\epsilon{}^2(b) \leqslant$$

$$\leqslant \|f\|_{\mathbf{L}^\infty(a,b)} \int_a^b \phi\, |u_\epsilon|\, dx + k_3, \tag{A.44}$$

where k_i are positive constants, $i = 1,2,3$. In particular, it follows that

$$\beta_0 \int_a^b \phi u_\epsilon^2 dx \leqslant \|f\|_{\mathbf{L}^\infty(a,b)} \int_a^b \phi\, |u_\epsilon|\, dx + k_3$$

and an elementary computation shows that the integral $\int_a^b \phi\, |u_\epsilon|\, dx$ is bounded. Thus, (A.44) and Poincaré inequality imply (A.40) and the boundedness of $v_\epsilon(b)$.

To show (A.41), take the $\mathbf{L}^2(b,c)$ scalar product of (2.6) by $\mu \psi v_{\epsilon,x}$, where ψ is a nonnegative, smooth function, vanishing near c, with $\psi(b) = 1$. After integration by parts, (A.41) follows by (A.40).

Finally, (A.42) follows by (A.40) and by the very equation (2.6).

□

Lemma A.9 *There is a constant* $C > 0$ *such that*

$$\|u_\epsilon\|_{\mathbf{H}^1(a,b)} \leqslant C. \tag{A.45}$$

Proof. Take the $\mathbf{L}^2(a,b)$ scalar product of (2.5) by $e^x u_\epsilon$, then integrate by parts. By Lemma A.8, it follows that the $\mathbf{L}^2(a,b)$ norm of u_ϵ is bounded, as well as the value $u_\epsilon(a)$.

Next, take the $\mathbf{L}^2(a,b)$ scalar product of (2.5) by $u_{\epsilon,x}$, then integrate by parts. The conclusion follows by Lemma A.8 and by the first part of this proof.

□

Now, we are in a position to prove part of the results stated in Proposition 2.4.

Proposition A.4 *Assume (A.4), (A.10), (A.11), (A.30), (A.39). There are* $u \in \mathbf{H}^1(a,b)$ *and* $v \in \mathbf{H}^1(b,c)$ *which satisfy (A.16), (A.17), (A.19), (A.28).*

Proof. Let u_ϵ, v_ϵ solve $(\mathbf{Q}_\epsilon)_N$. By Lemmas A.8, A.9, there are $u \in \mathbf{H}^1(a,b)$ and $v \in \mathbf{H}^1(b,c)$ such that (upon extracting a subfamily)

(i) $u_\epsilon \rightarrow u$ weakly in $\mathbf{H}^1(a,b)$;

(ii) $v_\epsilon \to v$ weakly in $\mathbf{H}^1(b,c)$;

(iii) $v_\epsilon(c) \to v(c)$;

(iv) $u_\epsilon(b) \to u(b)$ and $v_\epsilon(b) \to v(b)$.

All of these properties permit to pass to the limit in the regularized problem $(\mathbf{Q}_\epsilon)_N$. Thus, the proof follows easily.

□

To complete Proposition 2.4, it remains to show that the derivatives of the limit functions u, v of the preceding Proposition join continuously. To this end, we assume that the data α, β, f are more regular than it was until now, precisely:

$$\begin{gathered} \alpha \textit{ is Lipschitz continuous in } [a,b], \\ \beta \in \mathbf{H}^1(a,b),\ f \in \mathbf{H}^1(a,b). \end{gathered} \tag{A.46}$$

Lemma A.10 *Assume (A.4); (A.10), (A.11), (A.30), (A.39), (A.46). The* $\mathbf{L}^2$ *norm of* $u_{\epsilon,xx}$ *is bounded in a left neighborhood of the interface point* $x = b$.

Proof. Take the $\mathbf{L}^2(a,b)$ scalar product of (2.5) by $u_{\epsilon,xx}$ (which lies in $\mathbf{L}^2(a,b)$ because of the equation itself). Next, integrate by parts in all terms except in the first. Recalling (A.41), (A.45) and (A.46), we get

$$\| \sqrt{\epsilon} u_{\epsilon,xx} \|_{\mathbf{L}^2(a,b)} \leqslant C, \tag{A.47}$$

for some $C > 0$. Finally, take the derivative of (2.5) and multiply it in $\mathbf{L}^2(a,b)$ by $\phi u_{\epsilon,xx}$, where ϕ is smooth, nonnegative, with $\phi(a)=0$. By (A.45), (A.47) and recalling that $\alpha < 0$, the assertion follows.

□

Proposition A.5 *Assume (A.4), (A.10), (A.11), (A.30), (A.39), (A.46). The functions* u, v *considered in Proposition A.4 satisfy*

$$u_x = v_x \quad \textit{at } x = b. \tag{A.48}$$

Proof. Since the property holds for u_ϵ and v_ϵ (see (2.12)), it is enough to prove that:

(i) $u_{\epsilon,x}(b) \to u_x(b)$;

(ii) $v_{\epsilon,x}(b) \to v_x(b)$.

(i) follows by Lemma A.10 and by (A.2); (ii) follows by (A.2) and by (A.42), recalling (A.10) and (A.30).

□

Thus, the proof of Propositions 2.1-2.4 is complete.

REFERENCES

[A] Adams, R. *Sobolev Spaces.* Academic Press, New York (1975).

[BBB] Bardos, C., Brézis, D. and Brézis, H. *Perturbations singulières et prolongements maximaux d'opérateurs positifs*, Arch. Rational Mech. Anal., **53** (1973), 69-100.

[BR] Bardos, C. and Rauch, J. *Maximal positive boundary value problems as limits of singular perturbation problems*, Trans. Amer Math. Soc., **270** (1982), 377-408.

[CQ] Canuto, C. and Quarteroni, A. *The boundary treatment for spectral approximations to hyperbolic systems*, J. Comput. Phys., **71** (1987), 100-110.

[CHQZ] Canuto, C., Hussaini, M.Y., Quarteroni, A. and Zang, T.A. *Spectral Methods in Fluid Dynamics.* Springer-Verlag, New York Heidelberg Berlin (1988).

[FQZ] Funaro, D., Quarteroni, A. and Zanolli, P. *An iterative procedure with interface relaxation for domain decomposition methods*, SIAM J. Numer. Anal., to appear.

[CM] Chorin, A.J. and Marsden, J.E. *A Mathematical Introduction to Fluid Mechanics.* Springer-Verlag, New York Heidelberg Berlin (1979).

[L] Lions, J.L. *Perturbations Singulières dans les Problèmes aux Limites et en Controle Optimal.* Springer-Verlag, Berlin Heidelberg New York (1973).

[S] Saad, M.A. *Compressible Fluid Flow.* Prentice Hall, Englewood Cliffs (1985).

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16. Abstract We deal with the coupling of hyperbolic and parabolic systems in a domain Ω divided into two disjoint subdomains Ω^+ and Ω^-. Our main concern is to find out the proper interface conditions to be fulfilled at the surface separating the two domains. Next, we will use them in the numerical approximation of the problem. The justification of the interface conditions is based on a singular perturbation analysis, that is, the hyperbolic system is rendered parabolic by adding a small artificial "viscosity." As this goes to zero, the coupled parabolic-parabolic problem degenerates into the original one, yielding some conditions at the interface. These we take as interface conditions for the hyperbolic-parabolic problem. Actually, we discuss two alternative sets of interface conditions according to whether the regularization procedure is variational or nonvariational. We show how these conditions can be used in the frame of a numerical approximation to the given problem. Furthermore, we discuss a method of resolution which alternates the resolution of the hyperbolic problem within Ω^- and of the parabolic one within Ω^+. The spectral collocation method is proposed, as an example of space discretization (different methods could be used as well); both explicit and implicit time-advancing schemes are considered. The present study is a preliminary step toward the analysis of the coupling between Euler and Navier-Stokes equations for compressible flows.

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