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The Estimation Error Covariance Matrix for the Ideal State Reconstructor With Measurement Noise

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The Estimation Error Covariance Matrix for the Ideal State Reconstructor With Measurement Noise

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TECHNICAL PAPER

THE ESTIMATION ERROR COVARIANCE MATRIX FOR THE IDEAL STATE RECONSTRUCTOR WITH MEASUREMENT NOISE

I. INTRODUCTION

Recently, Polites developed a new state reconstructor for deterministic digital control systems which he called the Ideal State Reconstructor (ISR) [1-5]. It was so named because, for a linear time invariant plant with noise-free measurements, measurable inputs, and known parameters, the ISR exactly reconstructs the state of the plant and does so without requiring any knowledge of the plant's initial state. In addition, the ISR adds no new states or eigenvalues to the system. Nor does it affect the plant equation for the system in anyway; it affects the measurement equation only. However, systems in the real world normally have some measurement noise, nonmeasurable inputs, and parameter uncertainty. Hence, the effect of these things on the ISR needs to be determined. This paper addresses the first.

The outline of the paper is as follows. First, a brief review of the ISR is given, in Section II. Then, an analysis of the ISR with measurement noise is presented, in Section III. Next, an example is given, in Section IV, which illustrates the effect of measurement noise on the ISR. Finally, some conclusions and recommendations for further study are presented, in Section V.

II. A REVIEW OF THE IDEAL STATE RECONSTRUCTOR

Consider the ubiquitous continuous-time plant driven by a zero-order-hold with a sampled output, as shown in Figure 1. The state vector $\underline{x}(t) \in R^n$; the control input vector $\underline{u}(kT) \in R^r$; the standard output or measurement vector $\underline{y}_s(kT) \in R^m$; the system matrix $F \in R^{n\times n}$; the control matrix $G \in R^{n\times r}$; the standard output matrix $C_s \in R^{m\times n}$. It is well known that this system can be modeled at the sampling instants by the discrete state equations [6]

$$\underline{x}[(k+1)T] = A\underline{x}(kT) + B\underline{u}(kT) \tag{1}$$

$$\underline{y}_s(kT) = C_s\underline{x}(kT) \tag{2}$$

where

$$\phi(t) = \mathcal{L}^{-1}[(sI-F)^{-1}] \in R^{n\times n} \quad , \tag{3}$$

(1) Choose N so that

$$N \geq n/p \quad . \tag{6}$$

(2) Form the matrix α where

$$\alpha = \begin{bmatrix} C_F\ \phi(0) \\ C_F\ \phi(-\frac{T}{N}) \\ \vdots \\ C_F\ \phi[-(N-1)\frac{T}{N}] \end{bmatrix} \in R^{(Np)xn} \tag{7}$$

and check to be certain it has full rank (i.e., rank n). If so, proceed by letting

$$\alpha^* = (\alpha^T\alpha)^{-1}\ \alpha^T \in R^{nx(Np)} \quad . \tag{8}$$

(3) Choose q so that

$$q \geq n - m \quad . \tag{9}$$

(4) Now choose a matrix $D_- \in R^{qxn}$ so that

$$C_T = \begin{bmatrix} C_S \\ \hdashline D_- \end{bmatrix} \in R^{(m+q)xn} \tag{10}$$

has full rank (i.e., rank n).

(5) Let

$$C_T^* = (C_T{}^T C_T)^{-1}\ C_T{}^T \in R^{nx(m+q)} \tag{11}$$

and

$$H = D_- \alpha^* \in R^{qx(Np)} \quad . \tag{12}$$

(6) Partition H as follows

$$H = [H_0 \mid H_1 \mid \dots \mid H_{N-1}]$$

to reveal H_j, $j = 0,1,\dots,N-1$.

(7) Form the matrix β where

$$\beta = \begin{bmatrix} C_F \left[\int_0^0 \phi(\lambda)d\lambda\right] G \\ C_F \left[\int_0^{-(T/N)} \phi(\lambda)d\lambda\right] G \\ \vdots \\ C_F \left[\int_0^{-(N-1)(T/N)} \phi(\lambda)d\lambda\right] G \end{bmatrix} \in R^{(Np)xr}$$

and let

$$E_- = H\beta \in R^{qxr} \quad .$$

The ISR is now completely defined for exact state reconstruction.

A special form of the ISR is obtained when C_s is a null matrix (i.e., $m = 0$) and $q = n - m = n$. Then, $D_- \in R^{nxn}$ and, by virtue of equations (10) and (11),

$$C_T = D_- \in R^{nxn} \tag{13}$$

and

$$C_T^* = D_-^{-1} \in R^{nxn} \quad , \tag{14}$$

respectively.

III. ANALYSIS OF THE IDEAL STATE RECONSTRUCTOR WITH MEASUREMENT NOISE

Now consider the system in Figure 3, which is the system in Figure 2 with measurement noise added. The symbol ($\sim$) denotes a quantity corrupted by noise. The measurement noise sources are $\underline{\nu}_S(t) \in R^m$ and $\underline{\nu}_F(t) \in R^p$. They are assumed to have the following statistical properties after sampling:

$$\left.\begin{aligned}
&E\,[\underline{\nu}_S(kT)] = \underline{0} \in R^m \;, \\
&E\,[\underline{\nu}_S(kT)\, \underline{\nu}_S^T(jT)] = R_{SS}\, \delta(k\text{-}j) \in R^{m\times m} \;, \\
&E\,[\underline{\nu}_F(k\tfrac{T}{N})] = \underline{0} \in R^p \;, \\
&E\,[\underline{\nu}_F(k\tfrac{T}{N})\, \nu_F^T(j\tfrac{T}{N})] = R_{FF}\, \delta(k\text{-}j) \in R^{p\times p} \;, \\
&E\,[\underline{\nu}_S(kT)\, \underline{\nu}_F^T\,(kT - j\tfrac{T}{N})] = R_{SF}\, \delta(j) \in R^{m\times p} \;, \\
&E\,[\underline{\nu}_F(kT)\, \underline{\nu}_S^T(kT - j\tfrac{T}{N})] = R_{FS}\, \delta(j) = R_{SF}^T\, \delta(j) \in R^{p\times m} \;,
\end{aligned}\right\} \qquad (15)$$

for all integer values of k and j. It is straightforward to show that the output of the ISR in Figure 3 can be expressed as

$$\tilde{\underline{y}}_T'(kT) = \underline{y}_T'(kT) + C_T^* \left\{ \begin{array}{c} \underline{\nu}_S(kT) \\ \hline H\, \underline{V}_F(kT) \end{array} \right\}$$

where $\underline{y}_T'(kT)$ is the output for the noise-free case and

$$\underline{V}_F(kT) = \begin{Bmatrix} \underline{\nu}_F(kT) \\ \underline{\nu}_F(kT - \frac{T}{N}) \\ \vdots \\ \underline{\nu}_F[kT - (N\text{-}1)\frac{T}{N}] \end{Bmatrix} \in R^{Np}$$

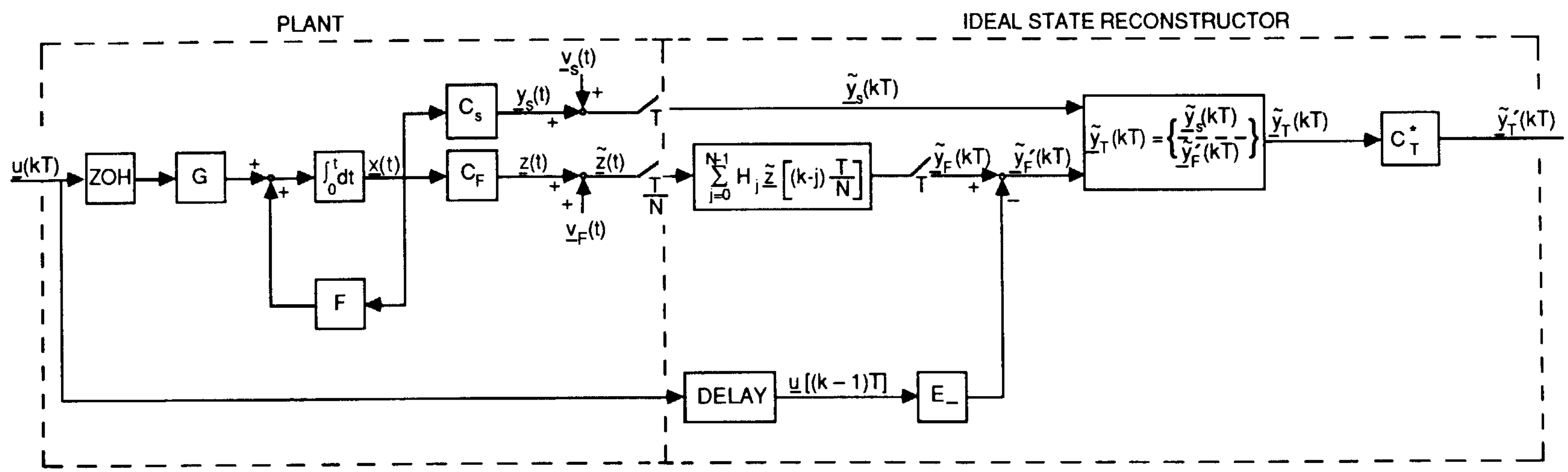

Figure 3. Block diagram of the plant with measurement noise and the Ideal State Reconstructor.

Assume the parameter matrices in the ISR are chosen by the procedure described in Section II and exact state reconstruction is achieved in the noise-free case. Then $\underline{y}_T'(kT)$ is given by equation (5). Obviously, with measurement noise, $\underline{\tilde{y}}_T'(kT)$ is an estimate of $\underline{x}(kT)$. Hence, the state estimation error for the ISR can be written as

$$\underline{\hat{x}}_e(kT) = \underline{\tilde{y}}_T'(kT) - \underline{x}(kT) = C_T^* \left\{ \frac{\underline{\nu}_S(kT)}{H\ \underline{V}_F(kT)} \right\} \in R^n \quad . \tag{16}$$

With equation (16) and the statistical properties of the measurement noise specified in equations (15), the statistical properties of $\underline{\hat{x}}_e(kT)$ are found to be

$$E\ [\underline{\hat{x}}_e(kT)] = \underline{0}$$

and

$$\mathrm{COV}\ [\underline{\hat{x}}_e(kT), \underline{\hat{x}}_e(jT)] = E\ [\underline{\hat{x}}_e(kT)\ \underline{\hat{x}}_e^T(jT)] - E\ [\underline{\hat{x}}_e(kT)]\ E\ [\underline{\hat{x}}_e^T(jT)] = R_{ee}\ \delta(k\text{-}j) \quad ,$$

for all integer values of k and j, where

$$\left.\begin{array}{l} R_{ee} = C_T^* \left[\begin{array}{c|c} R_{SS} & [R_{SF} 0 \ \dots \ 0]\ H^T \\ \hline H \begin{bmatrix} R_{SF}^T \\ 0 \\ \vdots \\ 0 \end{bmatrix} & H \begin{bmatrix} R_{FF} & & 0 \\ & \ddots & \\ 0 & & R_{FF} \end{bmatrix} H^T \end{array}\right] (C_T^*)^T \\ \\ \text{or} \\ \\ R_{ee} = C_T^* \left[\begin{array}{c|c} R_{SS} & R_{SF}\ H_0^T \\ \hline H_0\ R_{SF}^T & \sum_{i=0}^{N-1} (H_i\ R_{FF}\ H_i^T) \end{array}\right] (C_T^*)^T \quad . \end{array}\right\} \tag{17}$$

If the performance index J for the ISR is chosen to be the trace of COV[$\hat{\underline{x}}_e(kT)$, $\hat{\underline{x}}_e(kT)$], then

$$J = \text{trace}\left\{\text{COV}\left[\hat{\underline{x}}_e(kT), \hat{\underline{x}}_e(kT)\right]\right\} = \text{trace}\,(R_{ee}) \quad , \tag{18}$$

which is invariant with time.

Consider the special form of the ISR where C_s is a null matrix (i.e., m = 0) and q = n-m = n. In this case, there are no standard measurements and, hence, no associated measurement noise. Therefore, R_{SS} and R_{SF} are null matrices. With this and equations (12), (14), and (17),

$$R_{ee} = \alpha^* \begin{bmatrix} R_{FF} & & 0 \\ & \ddots & \\ 0 & & R_{FF} \end{bmatrix} (\alpha^*)^T \quad . \tag{19}$$

By virtue of equations (7), (8), (15), (18), and (19), once the plant is fixed and the measurement noise is specified, the performance index J is only a function of N, the number of measurements used in estimating the state every T seconds. The sensitivity of J to N will be illustrated by means of an example, presented in Section IV.

IV. AN EXAMPLE

Consider the system in Figure 4. It is a double integrator plant with measurement noise and the special form of the ISR described in Sections II and III. It is assumed that the statistical properties of the measurement noise after sampling are given by equations (15) with R_{FF} = 1; R_{SS} and R_{SF} are null matrices. Furthermore, it is assumed that the time interval between state estimates is T = 1 sec. The number of measurements N, used in estimating the state at a given time, is a parameter which will be varied to see the affect of N on the estimator performance index J.

Manipulating the plant in Figure 4 into the format of Figure 3 yields

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad , \tag{20}$$

$$G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad ,$$

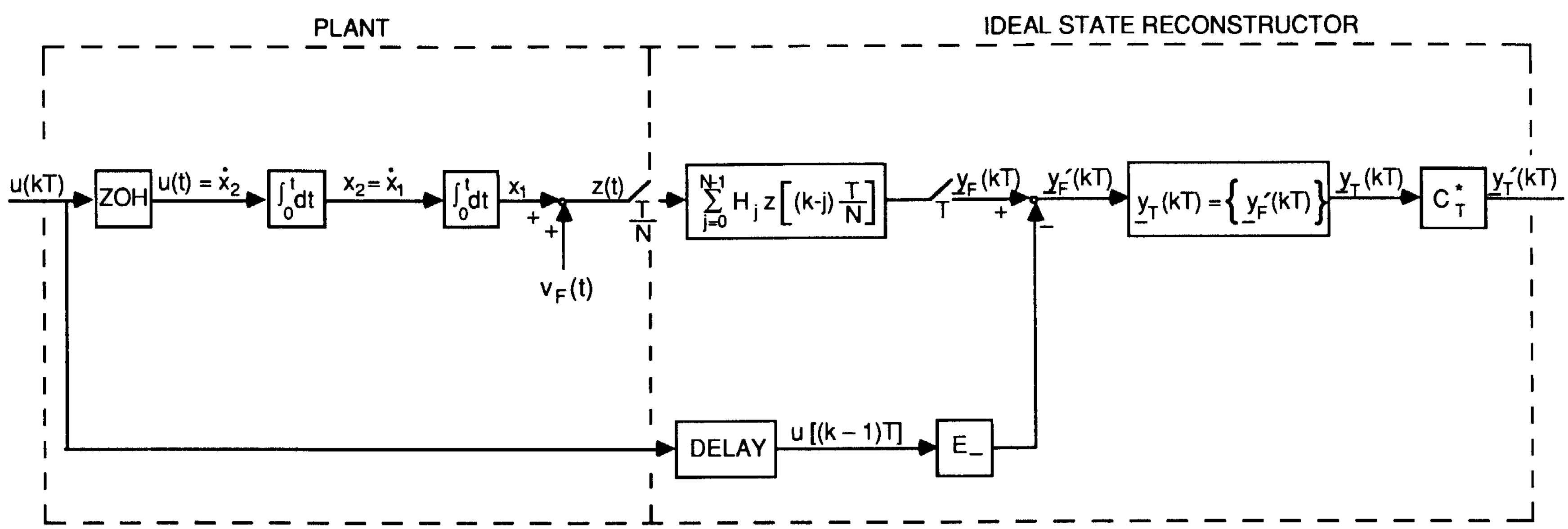

Figure 4. Double integrator plant with measurement noise and the special form of the Ideal State Reconstructor.

and

$$C_F = [1 \quad 0] \quad , \tag{21}$$

where C_s is a null matrix. It follows from the definitions of F, G, C_F, and C_s given in Section II that n = 2, r = p = 1, and m = 0. From equations (3) and (20),

$$\phi(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \quad . \tag{22}$$

Proceed now with the design of the ISR using the procedure outlined in Section II. From equation (6), the requirement on N is that $N \geqslant 2$. From equations (7), (21), and (22),

$$\alpha = \begin{bmatrix} 1 & 0 \\ 1 & -1/N \\ 1 & -2/N \\ \vdots & \vdots \\ 1 & -(N-1)/N \end{bmatrix}$$

when T = 1 sec. For $N \geqslant 2$, α is seen to have full rank [i.e., rank $(\alpha) = 2$] as required. Since N is a variable, determination of α^* according to equation (8) is best done numerically. From equation (9), the requirement on q is that $q \geqslant 2$. For the special form of the ISR, q = 2. Choose

$$D\_ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so that, by virtue of equation (13),

$$C_T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad ,$$

which has full rank as required.

The remainder of the calculations to define the ISR are straightforward and will be omitted here, except for the calculation of the performance index J, defined by equations (18) and (19) for the special form of the ISR. This was done numerically for various values of N. The results are shown graphically in Figure 5. The graph indicates that the more measurements used in estimating the state at a given time, the better the estimate, on a statistical basis. It also shows that for $N \geqslant 4$, J is essentially proportional to 1/N, ala the variance for the average of N independent random variables with the same statistical properties [7].

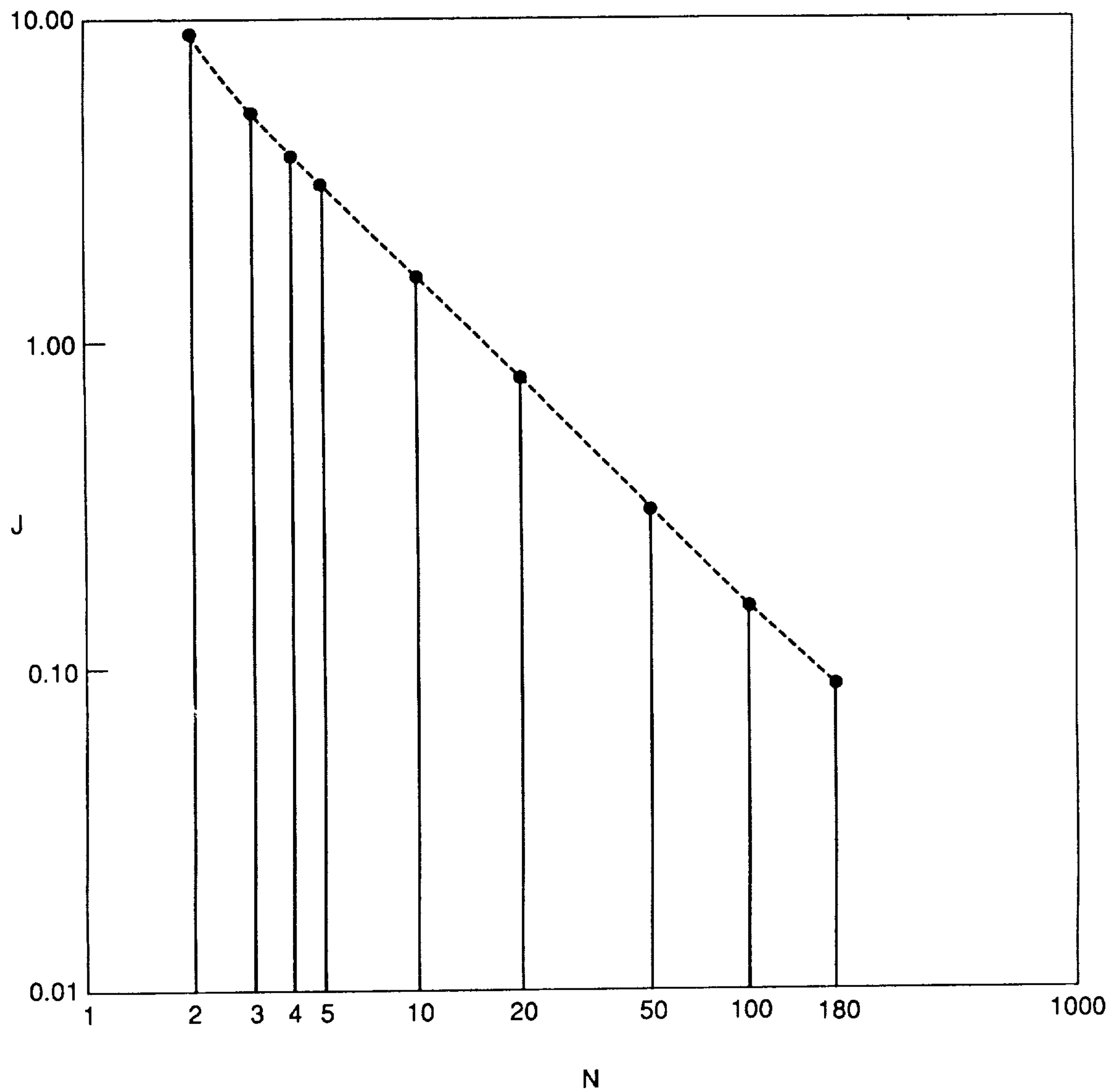

Figure 5. J versus N for the example.

V. CONCLUSIONS AND RECOMMENDATIONS

This paper derived a general expression for the state estimation error covariance matrix for the Ideal State Reconstructor (ISR) when the input measurements are corrupted by measurement noise. An example was presented which showed that the more measurements used in estimating the state of the plant at a given time, the better the estimate, on a statistical basis. Intuitively, one would expect this to be true in general. Assuming this is the case, then the ISR is ideally suited for systems where measurements become available more frequently than the state needs to be estimated and control law equations need to be solved. A good implementation of it would be to have a microprocessor dedicated to multiplying the measurements made every T/N seconds by the weighting matrices H_j, $j = 0,1,\ldots,N-1$, and summing these results, recursively. After N repetitions, the result could be transferred to a central processor along with the measurements made every T seconds. In the central processor, the remainder of the calculations to estimate the state of the system could be made and the control law equations solved. Before the control law equations are solved, the estimated state from the ISR could be passed through a Kalman filter to further improve it. This is an area for future study. Determining the effects of process noise and modeling errors on the ISR is also.

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16. ABSTRACT

This paper derives a general expression for the state estimation error covariance matrix for the Ideal State Reconstructor when the input measurements are corrupted by measurement noise. An example is presented which shows that the more measurements used in estimating the state at a given time, the better the estimate.

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