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SINGLE-GRID SPECTRAL COLLOCATION

FOR THE NAVIER-STOKES EQUATIONS

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Single-Grid Spectral

Collocation for the Navier-Stokes Equations

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Abstract : The aim of the paper is to study a collocation spectral method to approximate the Navier-Stokes equations : only one grid is used, which is built from the nodes of a Gauss-Lobatto quadrature formula, either of Legendre or of Chebyshev type. The convergence is proven for the Stokes problem provided with inhomogeneous Dirichlet conditions, then thoroughly analysed for the Navier-Stokes equations. The practical implementation algorithm is presented, together with numerical results.

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I. Introduction.

The paper is devoted to the analysis of a spectral collocation method for approximating stationary Navier-Stokes equations, governing the flow of a viscous incompressible fluid in a domain Ω of $\mathbb{R}^2$ or $\mathbb{R}^3$

$$(I.1) \quad \left| \begin{array}{l} - \nu \, \Delta \mathbf{u} + \mathbf{grad} \, p + (\mathbf{u}.\nabla)\mathbf{u} = \mathbf{f} \quad \text{in } \Omega \quad , \\ \text{div } \mathbf{u} = 0 \quad \text{in } \Omega \quad , \end{array} \right.$$

provided with Dirichlet boundary conditions

$$(I.2) \quad \mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega \quad .$$

In this system, the data are the density of body forces $\mathbf{f}$, the velocity on the boundary $\mathbf{g}$ and the kinematic viscosity $\nu > 0$; the unknowns we want to approximate are the velocity $\mathbf{u}$ and the pressure p inside the domain Ω. The discrete solution is sought in a space of polynomials of high degree on Ω, and the equations (I.1)(I.2) are verified in a finite number of points, called collocation nodes. We refer to [VGH] and to [CHQZ] for a detailed bibliography about such methods. As far as the theoretical analysis is concerned, we limit ourselves to the test domain $\Omega =]-1,1[^2$. In this square, the collocation points form a cartesian grid : their coordinates belong to the set of the nodes of a Gauss-Lobatto quadrature formula. However, the numerical results prove that there is no difficulty to extend the method to three-dimensional and/or curved domains.

The method we analyse in this paper has the following features :

1) Only one grid is involved in the algorithm : indeed, all the equations of (I.1) will be satisfied in the same nodes. We refer to [MMo] and [BM2] for other collocation methods for the Navier-Stokes equations, which are called staggered grid methods.

2) Due to this unique grid, the velocity and the pressure are approximated by polynomials of the same degree. Then, it turns out that there exist some spurious modes for the pressure, i.e. some polynomials the gradient of which cancels at the collocation nodes; this fact has first been pointed out in [Mo]. In order to obtain the convergence of the pressure, it is necessary to choose a suitable discrete pressure space which does not contain these modes but retains good approximation properties. We shall use the space already proposed in [BMM][Mé] or [BCM].

3) The grid can be built from the Gauss-Lobatto quadrature formula associated with any family of Jacobi polynomials. Here, we shall treat two special cases. The first one is the case of Legendre polynomials; it is the simpler one, since its analysis involves the standard

variational formulation of the Navier-Stokes equations in standard Sobolev spaces. However, we also consider the case of Chebyshev polynomials; indeed, the nodes are then images by the cosine function of equidistant points, so that the use of the Fast Fourier Transform allows for a less expensive computation of the derivatives or of the nonlinear terms. This last method is numerically cheaper, but its analysis involves a non trivial formulation of the equations in appropriate weighted Sobolev spaces with the Chebyshev weight, the properties of which are given in [BM1].

The method we present has already been studied in a simpler case : for the Stokes problem that one obtains by neglecting the nonlinear terms in (I.1)

$$(I.3) \quad \begin{cases} - \nu \, \Delta \mathbf{u} + \mathbf{grad} \, p = \mathbf{f} & \text{in } \Omega \quad , \\ \text{div } \mathbf{u} = 0 & \text{in } \Omega \quad , \end{cases}$$

when it is provided with homogeneous Dirichlet boundary conditions

$$(I.4) \quad \mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \quad .$$

We only recall the results of [BMM] in the Legendre case, of [BCM] in the Chebyshev case. Then we extend them to the Stokes problem with non homogeneous boundary conditions, by using the lifting of polynomial boundary data of [BM1]. Finally, the method is applied to the full Navier-Stokes equations : the nonlinear terms are handled in a pseudospectral way. The justification follows from the discrete implicit function theorem of [BRR], in a slightly different form due to [C] (see also [CR]). The error estimates we obtain between the discrete solution and the exact one, when it is assumed to be smooth enough, are the same as for the Stokes problem. This theoretical justification is completed by numerical experiments, achieved in the case of a three-dimensional domain with Chebyshev nodes, and a description of the algorithm involved in the code is given. We refer to [Mé] for more details on the numerical implementation.

An outline of the paper is as follows. Section II is devoted to the definition of the discrete approximation spaces and collocation problems, first in the homogeneous case, then in the inhomogeneous one. In Section III, we recall the convergence results of [BMM] and [BCM] for the Stokes problem in the homogeneous case, then we complete them for inhomogeneous boundary conditions. In Section IV, the analysis is extended to the Navier-Stokes equations, in both cases of homogeneous and inhomogeneous boundary conditions. Finally, in Section V, the techniques required by the numerical implementation of the method are presented and examples of numerical results are given.

Notation

The norm of any Banach space E is denoted by $\|.\|_E$. For any pair (E,F) of Banach spaces, $\mathscr{L}(E,F)$ represents the space of continuous linear mappings from E into F. We mean by A ⊗ B the tensorial product of any sets A and B in a Banach space, while $A^{\otimes 2}$ is the tensorial product of A by itself. In all that follows, c, c' ... are generic constants, independent of the discretization parameter.

In Sections II to IV, we shall work in the square $\Omega =]-1,1[^2$. Let us precise some notation about this domain. The generic point in Ω will be denoted by $\mathbf{x} = (x,y)$ (or sometimes by (x_1, x_2)). We introduce the corners $\mathbf{a}_J$, $J \in \mathbb{Z}/4\mathbb{Z}$, of $\overline{\Omega}$ (where $\mathbf{a}_{J+1}$ follows $\mathbf{a}_J$ counterclockwise), and call Γ_J the edge with vertices $\mathbf{a}_{J-1}$ and $\mathbf{a}_J$; for any edge Γ_J , $J \in \mathbb{Z}/4\mathbb{Z}$, $\mathbf{n}_J$ is the unit outward normal to Ω on Γ_J and τ_J the unit vector orthogonal to $\mathbf{n}_J$, directed counterclockwise.
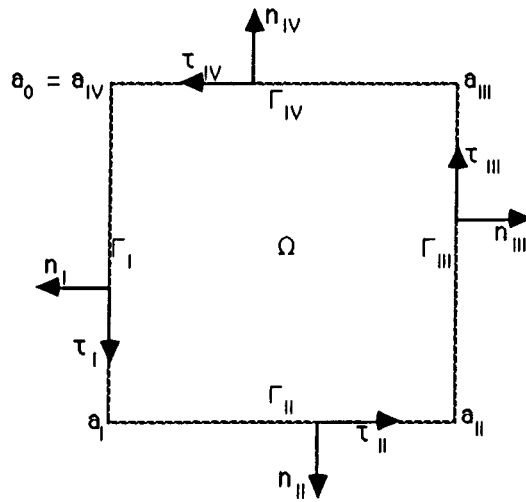
Figure I.1

Sobolev spaces

For any domain Δ in $\mathbb{R}^d$ and for any real number $s \geqslant 0$, we use the standard hilbertian Sobolev spaces $H^s(\Delta)$, the norm of which is denoted by $\|.\|_{s,\Delta}$. On the square Ω, we shall use the scalar product

(1.5) $(\varphi,\psi) = \int_\Omega \varphi(\mathbf{x})\,\psi(\mathbf{x})\,d\mathbf{x}$.

We also recall that, for any integer $m \geqslant 1$, the semi-norm

(1.6) $\quad |\varphi|_{m,\Omega} = (\sum_{j=0}^{m} \|\partial^m \varphi / \partial x^j \partial y^{m-j}\|_{0,\Omega}^2)^{1/2}$

is a norm on the space $H_0^m(\Omega)$ of the functions of $H^m(\Omega)$ which vanish on the boundary $\partial\Omega$ together with all their derivatives up to order $m-1$ (the traces being defined in the sense of [LM]). The dual space of $H_0^m(\Omega)$ will be denoted by $H^{-m}(\Omega)$, and it is standard to note that

(1.7) $\quad H^{-1}(\Omega) = \{ f + \partial g / \partial x + \partial h / \partial y, (f,g,h) \in L^2(\Omega)^3 \}$.

The space of functions in $L^2(\Omega)$ with a null integral is noted $L_0^2(\Omega)$.

Next, we recall some basic material about weighted spaces of Chebyshev type (for further details, see e.g. [CHQZ][BM1][M2]). If $\varrho(\zeta) = (1-\zeta^2)^{-1/2}$ denotes the Chebyshev weight on the interval $\Lambda =]-1,1[$, let

$$L_\varrho^2(\Lambda) = \{ \varphi : \Lambda \to \mathbb{R} ; \int_{-1}^{1} \varphi^2(\zeta) \varrho(\zeta) d\zeta < +\infty \}$$

be the Lebesgue space associated with the measure $\varrho(\zeta) d\zeta$, provided with the norm

(1.8) $\quad \|\varphi\|_{0,\varrho,\Lambda} = (\int_{-1}^{1} \varphi^2(\zeta) \varrho(\zeta) d\zeta)^{1/2}$.

The weighted Sobolev spaces are defined as follows : for any integer $m \geqslant 0$, $H_\varrho^m(\Lambda)$ is the subspace of $L_\varrho^2(\Lambda)$ of the functions such that their distributional derivatives of order $\leqslant m$ belong to $L_\varrho^2(\Lambda)$; it is a Hilbert space for the inner product associated with the norm

(1.9) $\quad \|\varphi\|_{m,\varrho,\Lambda} = (\sum_{k=0}^{m} |\varphi|_{k,\varrho,\Lambda}^2)^{1/2}$,

where

(1.10) $\quad |\varphi|_{k,\varrho,\Lambda} = \|d^k \varphi / d\zeta^k\|_{0,\varrho,\Lambda}$.

For a real number $s = m + \sigma$, $m \in \mathbb{N}$, $0 < \sigma < 1$, we define $H_\varrho^s(\Lambda)$ as the interpolation space between $H_\varrho^{m+1}(\Lambda)$ and $H_\varrho^m(\Lambda)$ of index $1-\sigma$ (cf. [LM]); we denote its norm by $\|.\|_{s,\varrho,\Lambda}$. Finally, we can apply a rotation and a translation to define similar Sobolev spaces on any segment of length 2 in $\mathbb{R}^2$. We use the same notation as before to indicate them, as well as their norms.

The Chebyshev weight on the square Ω is defined as $\omega(\mathbf{x}) = \varrho(x) \varrho(y)$. Let

$$L_\omega^2(\Omega) = \{ \varphi : \Omega \to \mathbb{R} ; \int_\Omega \varphi^2(\mathbf{x}) \omega(\mathbf{x}) d\mathbf{x} < +\infty \}$$

be the Lebesgue space associated with the measure $\omega(\mathbf{x}) d\mathbf{x}$, provided with the inner product

(1.11) $\quad (\varphi,\psi)_\omega = \int_\Omega \varphi(\mathbf{x}) \psi(\mathbf{x}) \omega(\mathbf{x}) d\mathbf{x}$

and the norm $\|.\|_{0,\omega,\Omega} = (.,.)_\omega^{1/2}$. Next, weighted Sobolev spaces are defined as follows : for any integer $m \geqslant 0$, $H_\omega^m(\Omega)$ is the subspace of $L_\omega^2(\Omega)$ of the functions such that their distributional derivatives of order $\leqslant m$ belong to $L_\omega^2(\Omega)$; it is a Hilbert space for the inner product associated with the norm

(1.12) $\quad \|\varphi\|_{m,\omega,\Omega} = (\sum_{k=0}^{m} |\varphi|_{k,\omega,\Omega}^2)^{1/2}$,

where

(I.13) $|\varphi|_{k,\omega,\Omega} = (\sum_{j=0}^{k} \|\partial^k\varphi/\partial x^j \partial y^{k-j}\|_{0,\omega,\Omega}^2)^{1/2}$.

For a real number $s = m + \sigma$, $m \in \mathbb{N}$, $0 < \sigma < 1$, we define $H_{\omega}^s(\Omega)$ as the interpolation space between $H_{\omega}^{m+1}(\Omega)$ and $H_{\omega}^m(\Omega)$ of index $1-\sigma$; we denote its norm by $\|.\|_{s,\omega,\Omega}$. Finally, for any integer $m \geq 1$, we consider the closed subspace of the functions of $H_{\omega}^m(\Omega)$ which vanish on the boundary $\partial\Omega$ together with all their derivatives up to order $m-1$ (the traces being defined in the sense of [LM]); this space, denoted by $H_{\omega,0}^m(\Omega)$, is the closure of $\mathscr{D}(\Omega)$ for the norm of $H_{\omega}^m(\Omega)$ (see [BM1, Prop. II.9]). Due to the Poincaré inequality, an equivalent norm on $H_{\omega,0}^m(\Omega)$ is the semi-norm $|.|_{m,\omega,\Omega}$. The dual space of $H_{\omega,0}^m(\Omega)$ will be denoted by $H_{\omega}^{-m}(\Omega)$; if the space $L_{\omega}^2(\Omega)$ is identified to its dual space, we have for instance

(I.14) $H_{\omega}^{-1}(\Omega) = \{ f + \partial g/\partial x + \partial h/\partial y, (f,g,h) \in L_{\omega}^2(\Omega)^3 \}$.

We also introduce the space $L_{\omega,0}^2(\Omega)$ of functions q in $L_{\omega}^2(\Omega)$ such that $\int_{\Omega} q(\mathbf{x}) \, \omega(\mathbf{x}) \, d\mathbf{x}$ is equal to 0.

Variational formulations

In order to treat the Legendre and Chebyshev approximations simultaneously, we introduce a letter A which is L in the Legendre case and C in the Chebyshev case, a parameter α equal to 0 in the Legendre case and to $-1/2$ in the Chebyshev case (this is the power of $(1-\zeta^2)$ involved in the corresponding weight). For instance, the symbol $H_A^s(\Omega)$ stands for the space $H^s(\Omega)$ in the Legendre case (A = L, α = 0) and for the space $H_{\omega}^s(\Omega)$ in the Chebyshev case (A = C, α = $-1/2$).

To write appropriate variational formulations of equations (I.1) and (I.3), we first consider the boundary condition (I.2). Let us assume that the function $\mathbf{g}$ is such that the $\mathbf{g}_J = \mathbf{g}_{|\Gamma_J}$, $J \in \mathbb{Z}/4\mathbb{Z}$, satisfy

(I.15) $\mathbf{g}_J \in H_A^{(1-\alpha)/2}(\Gamma_J)^2$, $J \in \mathbb{Z}/4\mathbb{Z}$,

(I.16) $\sum_{J \in \mathbb{Z}/4\mathbb{Z}} \int_{\Gamma_J} \mathbf{g}_J \cdot \mathbf{n}_J \, d\sigma = 0$.

Assume moreover, in the Legendre case,

(I.17)$_L$ $\int_0^2 [g_J(a_J - t\,\tau_J) - g_{J+1}(a_J + t\,\tau_{J+1})]^2 \, t^{-1} \, dt < +\infty$, $J \in \mathbb{Z}/4\mathbb{Z}$,

and, in the Chebyshev case,

(I.17)$_C$ $g_J(a_J) = g_{J+1}(a_J)$, $J \in \mathbb{Z}/4\mathbb{Z}$.

Then, there exists [G, Thm 1.5.2.3][BCM, Thm III.2] a function $\mathbf{u}_b$ in $H_A^1(\Omega)^2$ satisfying

(I.18) $\operatorname{div} \mathbf{u}_b = 0$ in Ω ,

(I.19) $\mathbf{u}_b = \mathbf{g}_J$ on Γ_J , $J \in \mathbb{Z}/4\mathbb{Z}$.

Next, we define the bilinear form a_A on $H_A^1(\Omega)^2 \times H_{A,0}^1(\Omega)^2$ by

$(1.20)_L \quad \forall\, \mathbf{u} \in H^1(\Omega)^2, \forall\, \mathbf{v} \in H_0^1(\Omega)^2, \quad a_L(\mathbf{u},\mathbf{v}) = \nu\,(\mathbf{grad\ u}\,,\mathbf{grad\ v}) \quad,$

$(1.20)_C \quad \forall\, \mathbf{u} \in H_\omega^1(\Omega)^2, \forall\, \mathbf{v} \in H_{\omega,0}^1(\Omega)^2,$

$$a_C(\mathbf{u},\mathbf{v}) = \nu \int_\Omega (\mathbf{grad\ u})(\mathbf{x})\,(\mathbf{grad}\,(\mathbf{v}\omega))(\mathbf{x})\,d\mathbf{x} \quad.$$

Clearly, for any $\mathbf{f}$ in $H_A^{-1}(\Omega)^2$ and any $\mathbf{g}$ satisfying (1.15), (1.16) and $(1.17)_A$, problem (1.3)(1.2) is equivalent to the following variational one : *Find a pair* $(\mathbf{u},p)$ *in* $H_A^1(\Omega)^2 \times L_{A,0}^2(\Omega)$, *with* $\mathbf{u}-\mathbf{u}_b$ *in* $H_{A,0}^1(\Omega)^2$, *such that*

$(1.21)_A \quad \begin{cases} \forall\, \mathbf{v} \in H_{A,0}^1(\Omega)^2, \quad a_A(\mathbf{u},\mathbf{v}) + (\mathbf{v}\,,\mathbf{grad}\,p)_A = (\mathbf{f},\mathbf{v})_A \quad, \\ \forall\, q \in L_A^2(\Omega), \quad (\mathrm{div}\,\mathbf{u}\,,q)_A = 0 \quad. \end{cases}$

In the Legendre case, it is well-known that problem $(1.21)_L$ admits a unique solution. In the Chebyshev case, it is also known [BCM, Thm III.2] (but may be not so well) that problem $(1.21)_C$ admits a unique solution. In both cases, the solution satisfies the stability estimate

$(1.22) \quad \|\mathbf{u}\|_{1,A,\Omega} + \|p\|_{0,A,\Omega} \leqslant c\,(\,\|\mathbf{f}\|_{H_A^{-1}(\Omega)^2} + \sum_{J\in\mathbb{Z}/4\mathbb{Z}} \|\mathbf{g}_J\|_{(1-\alpha)/2,A,\Gamma_J})$.

Of course, if the solution $(\mathbf{u},p)$ of $(1.21)_L$ belongs to $H_C^1(\Omega)^2 \times L_C^2(\Omega)$, then the pair $(\mathbf{u}\,,p - (1/\pi^2)\int_\Omega p(\mathbf{x})\,\omega(\mathbf{x})\,d\mathbf{x})$ is the solution of problem $(1.21)_C$.

As far as the Navier-Stokes equations (1.1)(1.2) are concerned, for any $\mathbf{f}$ in $H_A^{-1}(\Omega)^2$ and any $\mathbf{g}$ satisfying (1.15), (1.16) and $(1.17)_A$, they admit the following variational formulation : *Find a pair* $(\mathbf{u},p)$ *in* $H_A^1(\Omega)^2 \times L_{A,0}^2(\Omega)$, *with* $\mathbf{u}-\mathbf{u}_b$ *in* $H_{A,0}^1(\Omega)^2$, *such that*

$(1.23)_A \quad \begin{cases} \forall\, \mathbf{v} \in H_{A,0}^1(\Omega)^2, \quad a_A(\mathbf{u},\mathbf{v}) + (\mathbf{v}\,,\mathbf{grad}\,p)_A + ((\mathbf{u}.\nabla)\mathbf{u}\,,\mathbf{v}\,)_A = (\mathbf{f},\mathbf{v})_A \quad, \\ \forall\, q \in L_A^2(\Omega), \quad (\mathrm{div}\,\mathbf{u}\,,q)_A = 0 \quad. \end{cases}$

In the Legendre case, this problem admits at least one solution. If this solution belongs to $H_C^1(\Omega)^2 \times L_C^2(\Omega)$, it is also a solution of $(1.23)_C$, up to an additive constant on the pressure. More details will be given in Section IV.

II. The collocation problems.

We begin by introducing the collocation framework, especially the collocation grid. Then, we present the collocation discretization of the Stokes and Navier-Stokes equations provided with homogeneous boundary conditions. That leads us to define suitable discrete spaces of pressures. Finally, we can extend the collocation problems to the case of inhomogeneous boundary conditions.

II.1. The collocation framework.

Let us introduce some monodimensional notation. For any nonnegative integer n, $P_n(\Lambda)$ denotes the space of restrictions to $\Lambda =]-1,1[$ of polynomials of degree $\leqslant$ n. We shall use two families of orthogonal polynomials on Λ :

1) the Legendre polynomials $(L_n)_{n \geqslant 0}$, which are orthogonal for the measure $d\zeta$, normalized by the following condition : the Legendre polynomial L_n , $n \geqslant 0$, is of degree n and satisfies $L_n(\pm 1) = (\pm 1)^n$;

2) the Chebyshev polynomials $(T_n = \cos (n \operatorname{Arccos} \zeta))_{n \geqslant 0}$, which are orthogonal for the measure $\varrho(\zeta) \, d\zeta$; of course, the Chebyshev polynomial T_n , $n \geqslant 0$, is also of degree n and satisfies $T_n(\pm 1) = (\pm 1)^n$.

In order to have a unique notation in the Legendre and Chebyshev cases, we introduce, for each real number $\alpha > -1$, the Jacobi polynomials $(J_n^\alpha)_{n \geqslant 0}$ which are orthogonal for the measure $(1-\zeta^2)^\alpha \, d\zeta$. Since J_n^α is of degree n and such that

$$J_n^\alpha(\pm 1) = (\pm 1)^n \, \Gamma(n+\alpha+1)/n! \, \Gamma(\alpha+1) \quad ,$$

where Γ denotes the Euler's gamma-function, the Legendre polynomial L_n coincides with J_n^0 , while the Chebyshev polynomial T_n is equal to $4^n \, [(n!)^2 / (2n)!] \, J_n^{-1/2}$. Finally, we recall that the J_n^α , $n \geqslant 0$, are the eigenfunctions of a Sturm-Liouville operator, more precisely they satisfy

(II.1) $\quad ((1-\zeta^2)^{\alpha+1} \, J_n^{\alpha \prime})' + n(n+2\alpha+1) \, (1-\zeta^2)^\alpha \, J_n^\alpha = 0 \quad .$

We refer to [DR, § 1.13] for the properties of these orthogonal polynomials.

Next, let N be a fixed integer $\geqslant 3$. We denote by ζ_j^A , $0 \leqslant j \leqslant N$, the zeros of the polynomial $(1-\zeta^2) \, J_N^{\alpha \prime}$, with $-1 = \zeta_0^A < \zeta_1^A < ... < \zeta_N^A = 1$. There exist weights ϱ_j^A , $0 \leqslant j \leqslant N$, such that the Gauss-Lobatto quadrature formula

(II.2) $\quad \int_{-1}^1 \Phi(\zeta) \, (1-\zeta^2)^\alpha \, d\zeta \simeq \sum_{j=0}^N \Phi(\zeta_j^A) \, \varrho_j^A$

is exact for any polynomial in $P_{2N-1}(\Lambda)$. We shall need the interpolation operator ι_N^A associated with these nodes : for any function φ in $C^0(\overline{\Lambda})$, $\iota_N^A \varphi$ belongs to $P_N(\Lambda)$ and satisfies

$$(II.3) \quad \iota_N^A \varphi(\zeta_j^A) = \varphi(\zeta_j^A) \quad , 0 \leqslant j \leqslant N \quad .$$

<u>Remark II.1</u> : It is well-known that the zeros ζ_j^C and the weights ϱ_j^C satisfy

$$(II.4) \quad \left| \begin{array}{l} \zeta_j^C = \cos((N-j)\pi/N) \quad , 0 \leqslant j \leqslant N \quad , \\[2mm] \varrho_j^C = \pi/N \quad , 1 \leqslant j \leqslant N-1, \quad \text{and} \quad \varrho_0^C = \varrho_N^C = \pi/2N \quad . \end{array} \right.$$

Note that, although the Arccos of the ζ_j^L , $0 \leqslant j \leqslant N$, are not strictly equidistant, their distribution is coarsely the same.

Example II.1 : For $N = 15$, we give the values of $- \zeta_j^A = \zeta_{N-j}^A$, $1 \leqslant j \leqslant 7$.

	Legendre case	Chebyshev case
j = 1	0.9695680462702179	0.9781476007338056
j = 2	0.8992005330934721	0.9135454576426009
j = 3	0.7920082918618152	0.8090169943749474
j = 4	0.6523887028824931	0.6691306063588582
j = 5	0.4860594218871376	0.5000000000000000
j = 6	0.2998304689007632	0.3090169943749474
j = 7	0.1013262735219494	0.1045284632676535

Now, let us consider the two-dimensional domain $\Omega =]-1,1[^2$. For any nonnegative integer n, we denote by $P_n(\Omega)$ the space of restrictions to Ω of polynomials of degree $\leqslant n$ with respect to each variable, i.e. the space $P_n(\Lambda) \otimes P_n(\Lambda)$; we also introduce the space $P_n^*(\Omega)$ of polynomials of $P_n(\Omega)$ which are equal to 0 on the boundary $\partial\Omega$.

For the fixed integer N, we define the grid Ξ_N^A by

$$(II.5) \quad \Xi_N^A = \{ x_{jk}^A = (\zeta_j^A , \zeta_k^A) ; 0 \leqslant j,k \leqslant N \} \quad .$$

The idea of defining the grid from the nodes of a Gauss type quadrature formula was first presented in [Go].

To each point x_{jk}^A in Ξ_N^A , we associate the weight $\varrho_{jk}^A = \varrho_j^A \varrho_k^A$. That allows us to define the following bilinear form on $C^0(\overline{\Omega}) \times C^0(\overline{\Omega})$

$$(II.6) \quad (\varphi,\psi)_{A,N} = \sum_{j=0}^N \sum_{k=0}^N \varphi(x_{jk}^A) \psi(x_{jk}^A) \varrho_{jk}^A \quad .$$

Since the quadrature formula (II.2) is exact on $P_{2N-1}(\Lambda)$, it coincides with the scalar product $(.,.)_A$ on $P_{N-1}(\Omega)$; it is known [CQ1, §3] that, on $P_N(\Omega)$, it is still a scalar product, and the norm : $\varphi \to (\varphi,\varphi)_{A,N}^{1/2}$ is equivalent on $P_N(\Omega)$ to the norm : $\varphi \to \|\varphi\|_{0,A,\Omega}$, with equivalence constants independent of N. Finally, we define the interpolation operator

$\mathcal{J}_N^A$ on the grid Ξ_N^A in the following way : for any function φ in $\mathcal{C}^0(\overline{\Omega})$, $\mathcal{J}_N^A \varphi$ belongs to $P_N(\Omega)$ and satisfies

(II.7) $\mathcal{J}_N^A \varphi(\mathbf{x}) = \varphi(\mathbf{x})$, $\mathbf{x} \in \Xi_N^A$.

II.2. The collocation problem for homogeneous boundary conditions.

We choose the space X_N of discrete velocities equal to $P_N(\Omega)^2$ and the space M_N of discrete pressures equal to a subspace of $P_N(\Omega)$ which we will precise later.

Let us assume that the data $\mathbf{f}$ belong to $\mathcal{C}^0(\Omega)^2$. The collocation approximation of the homogeneous Stokes problem (I.3)(I.4) is the following : *Find a pair* $(\mathbf{u}_N, p_N)$ *in* $X_N \times M_N$ *satisfying*

(II.8)$_A$ $\left| \begin{array}{l} (-\nu \Delta \mathbf{u}_N + \mathbf{grad}\, p_N)(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \quad , \mathbf{x} \in \Xi_N^A \cap \Omega \quad , \\ (\mathrm{div}\, \mathbf{u}_N)(\mathbf{x}) = 0 \quad , \mathbf{x} \in \Xi_N^A \quad , \end{array} \right.$

together with the boundary conditions

(II.9)$_A$ $\mathbf{u}_N(\mathbf{x}) = \mathbf{0}$, $\mathbf{x} \in \Xi_N^A \cap \partial\Omega$.

In order to state a variational formulation of problem (II.8)$_A$ (II.9)$_A$, we define three bilinear forms respectively on $\mathcal{C}^2(\overline{\Omega})^2 \times \mathcal{C}^0(\overline{\Omega})^2$, on $\mathcal{C}^0(\overline{\Omega})^2 \times \mathcal{C}^1(\overline{\Omega})$ and on $\mathcal{C}^1(\overline{\Omega})^2 \times \mathcal{C}^0(\overline{\Omega})$ by

(II.10) $\forall\, \mathbf{u} \in \mathcal{C}^2(\overline{\Omega})^2, \forall\, \mathbf{v} \in \mathcal{C}^0(\overline{\Omega})^2, \quad a_{A,N}(\mathbf{u},\mathbf{v}) = -\nu\, (\Delta \mathbf{u},\mathbf{v})_{A,N}$,

(II.11) $\forall\, \mathbf{v} \in \mathcal{C}^0(\overline{\Omega})^2, \forall\, q \in \mathcal{C}^1(\overline{\Omega}), \quad b_{1A,N}(\mathbf{v},q) = (\mathbf{v}\, , \mathbf{grad}\, q)_{A,N}$,

(II.12) $\forall\, \mathbf{v} \in \mathcal{C}^1(\overline{\Omega})^2, \forall\, q \in \mathcal{C}^0(\overline{\Omega}), \quad b_{2A,N}(\mathbf{v},q) = -(\mathrm{div}\, \mathbf{v}\, , q)_{A,N}$.

Since the quadrature formula (II.2) is exact on polynomials of degree $\leqslant 2N-1$, we have

(II.13)$_L$ $\forall\, \mathbf{u} \in P_N(\Omega)^2, \forall\, \mathbf{v} \in P_N^{\bullet}(\Omega)^2, \quad a_{L,N}(\mathbf{u},\mathbf{v}) = \nu\, (\mathbf{grad}\, \mathbf{u}\, , \mathbf{grad}\, \mathbf{v})_{L,N}$,

(II.13)$_C$ $\forall\, \mathbf{u} \in P_N(\Omega)^2, \forall\, \mathbf{v} \in P_N^{\bullet}(\Omega)^2, \quad a_{C,N}(\mathbf{u},\mathbf{v}) = \nu\, (\mathbf{grad}\, \mathbf{u}\, , \mathbf{grad}\, (\mathbf{v}\omega)\, \omega^{-1})_{A,N}$.

Moreover, we note that, in the Legendre case, the two forms $b_{1L,N}$ and $b_{2L,N}$ coincide on $P_N^{\bullet}(\Omega)^2 \times P_N(\Omega)$ while, in the Chebyshev case, one has

(II.14) $\forall\, \mathbf{v} \in P_N^{\bullet}(\Omega)^2, \forall\, q \in P_N(\Omega), \quad b_{1C,N}(\mathbf{v},q) = -(\mathrm{div}\, (\mathbf{v}\omega)\, \omega^{-1}, q)_{C,N}$.

Now, it is clear that problem (II.8)$_A$ (II.9)$_A$ is equivalent to the following variational one : *Find a pair* $(\mathbf{u}_N, p_N)$ *in* $P_N^{\bullet}(\Omega)^2 \times M_N$ *such that*

(II.15)$_A$ $\left| \begin{array}{l} \forall\, \mathbf{v}_N \in P_N^{\bullet}(\Omega)^2, \quad a_{A,N}(\mathbf{u}_N\, , \mathbf{v}_N) + b_{1A,N}(\mathbf{v}_N\, , p_N) = (\mathbf{f},\mathbf{v}_N)_{A,N} \quad , \\ \forall\, q_N \in P_N(\Omega), \quad b_{2A,N}(\mathbf{u}_N\, , q_N) = 0 \quad . \end{array} \right.$

Finally, to discretize the Navier-Stokes equations (I.1)(I.4), let us consider the

nonlinear term. Since the solution $\mathbf{u}$ of (I.1) is divergence-free, the quantity $(\mathbf{u}.\nabla)\mathbf{u}$ is equal to $\sum_{i=1}^{2} \partial(u_i\mathbf{u})/\partial x_i$, where u_1 and u_2 are the components of the velocity $\mathbf{u}$. Though this property is no longer true for the discrete problem in the general case, it seems more convenient to choose the second form, for reasons of numerical stability. Moreover, if a continuous function $\mathbf{u}$ is known by its values at the nodes $\mathbf{x}$ of Ξ_N^A, it is easy to derive the values of $u_i\mathbf{u}$ at the same nodes, hence to compute $\mathcal{J}_N^A(u_i\mathbf{u})$. The pseudo-spectral approximation, as suggested in [O1], consists in differentiating this interpolant i.e., in replacing $\partial(u_i\mathbf{u})/\partial x_i$ by $\partial\mathcal{J}_N^A(u_i\mathbf{u})/\partial x_i$.

These two arguments lead us to the following discretization of the Navier-Stokes equations : *Find a pair* $(\mathbf{u}_N , p_N)$ *in* $X_N \times M_N$ *satisfying*

$$(\text{II.16})_A \quad \begin{cases} [-\nu\,\Delta\mathbf{u}_N + \mathbf{grad}\,p_N + \sum_{i=1}^{2}\partial\mathcal{J}_N^A(u_{Ni}\mathbf{u}_N)/\partial x_i](\mathbf{x}) = \mathbf{f}(\mathbf{x}) \quad , \mathbf{x} \in \Xi_N^A \cap \Omega \quad , \\ (\text{div }\mathbf{u}_N)(\mathbf{x}) = 0 \quad , \mathbf{x} \in \Xi_N^A \quad , \end{cases}$$

together with the boundary conditions $(\text{II.9})_A$.

Of course, this system is equivalent to the variational problem : *Find a pair* $(\mathbf{u}_N , p_N)$ *in* $P_N^\bullet(\Omega)^2 \times M_N$ *such that*

$$(\text{II.17})_A \quad \begin{cases} \forall\,\mathbf{v}_N \in P_N^\bullet(\Omega)^2, \quad a_{A,N}(\mathbf{u}_N , \mathbf{v}_N) + b_{1A,N}(\mathbf{v}_N , p_N) \\ \qquad\qquad\qquad + \sum_{i=1}^{2}(\partial\mathcal{J}_N^A(u_{Ni}\mathbf{u}_N)/\partial x_i , \mathbf{v}_N)_{A,N} = (\mathbf{f},\mathbf{v}_N)_{A,N} \quad , \\ \forall\,q_N \in P_N(\Omega), \quad b_{2A,N}(\mathbf{u}_N , q_N) = 0 \quad . \end{cases}$$

Our purpose is to choose appropriate discrete pressure spaces M_N , such that problems $(\text{II.8})_A$ $(\text{II.9})_A$ and $(\text{II.16})_A$ $(\text{II.9})_A$ are well-posed.

II.3. The discrete pressure spaces.

It is known [Mo][Mé][BMM][BCM] that the space $P_N(\Omega)$ contains "spurious" modes for the pressure, i.e., polynomials the gradient of which vanishes at the collocation nodes of $\Xi_N^A \cap \Omega$; of course, even if they can be solved numerically (see Section V), the collocation problems cannot be well-posed if any of these modes belongs to M_N , hence we have to characterize them. More precisely, for $i = 1$ or 2, we define the subspaces

$$(\text{II.18}) \quad Z_{iA,N} = \{ q_N \in P_N(\Omega); \forall\,\mathbf{v}_N \in P_N^\bullet(\Omega)^2, b_{iA,N}(\mathbf{v}_N , q_N) = 0 \} \quad .$$

Let us also introduce, in the Chebyshev case, the polynomial s_N of $P_N(\Lambda)$ which satisfies

$$(\text{II.19}) \quad \forall\,\varphi_N \in P_N(\Lambda), \quad \sum_{j=0}^{N} s_N(\zeta_j^C)\,\varphi_N(\zeta_j^C)\,\varrho_j^C = \int_{-1}^{1}\varphi_N(\zeta)\,d\zeta \quad .$$

Finally, we need the Lagrange polynomial r_j^A associated with each node ζ_j^A , $0 \leqslant j \leqslant N : r_j^A$ belongs to $P_N(\Lambda)$, is equal to 1 in ζ_j^A and to 0 in any other node ζ_k^A , $0 \leqslant k \leqslant N, k \neq j$.

We have [BMM, Lemma V.1][BCM, Prop. V.2 and V.3]

__Lemma II.1__ : *The subspace* $Z_{iA,N}$, $i = 1$ *or* 2 , *is of dimension* 8. *It is spanned*

1) *in the Legendre case, for* $i = 1$ *or* 2 , *by* $\{ L_0 , L_N \}^{\otimes 2}$ *and* $\{ r_0^L , r_N^L \}^{\otimes 2}$;

2) *in the Chebyshev case, for* $i = 1$, *by* $\{ T_0 , T_N \}^{\otimes 2}$ *and* $\{ r_0^C , r_N^C \}^{\otimes 2}$, *for* $i = 2$, *by*
$\{ s_N , T_N \}^{\otimes 2}$ *and* $\{ r_0^C , r_N^C \}^{\otimes 2}$.

Let $M_{iA,N}^{\perp}$ denote the orthogonal subspace of $Z_{iA,N}$ in $P_N(\Omega)$ with respect to the scalar product $(.,.)_{A,N}$. In the sequel, we shall always choose the space of discrete pressures M_N such that the orthogonal projection operator $\pi_N : M_N \to M_{1A,N}^{\perp}$ with respect to the scalar product $(.,.)_{A,N}$ satisfies

(II.20) $\quad \forall q_N \in M_N , \quad \| q_N \|_{0,A,\Omega} \leq c \| \pi_N q_N \|_{0,A,\Omega}$.

__Remark II.2__ : Of course, the choice $M_N = M_{1A,N}^{\perp}$ is the most natural one. However, this space has not good approximation properties since it can be checked that all its elements vanish in the corners of the domain Ω (which is a priori not the case for the exact pressure). On the opposite, for any real number λ, $0 < \lambda < 1$, it is possible to build subspaces M_N which satisfy (II.20) and such that the following inclusion holds

(II.21) $\quad P_{[\lambda N]}(\Omega) \subset M_N \quad$ ([λN] denotes the integral part of λN) ,

which implies that these M_N have good approximation properties. Examples of such spaces are given in [BMM, Prop. V.3] in the Legendre case and in [BCM, (IV.61) and (IV.49)] in the Chebyshev case.

Now, problem $(II.8)_A$ $(II.9)_A$ seems overspecified, since there are eight equations more than unknowns. Due to the definition (II.18) of $Z_{2A,N}$, it turns out that problem $(II.15)_A$ is equivalent to : *Find a pair* (u_N , p_N) *in* $P_N^*(\Omega)^2 \times M_N$ *such that*

$(II.22)_A$ $\quad \begin{cases} \forall v_N \in P_N^*(\Omega)^2, \quad a_{A,N}(u_N , v_N) + b_{1A,N}(v_N , p_N) = (f,v_N)_{A,N} , \\ \forall q_N \in M_{2A,N}^{\perp} , \quad b_{2A,N}(u_N , q_N) = 0 . \end{cases}$

Clearly, the continuity equation in $(II.8)_A$ is redundant. However, let us denote by S_c the set of the four corners of the square Ω, and introduce a set $\hat{S}^A$ of four collocation points in $\Xi_N^A \setminus S_c$ satisfying the following property :

(II.23) $\quad \det(q_K(x_J)) \neq 0 \quad , \quad 1 \leq J, K \leq 4 \quad ,$

where x_J runs through $\hat{S}^A$ and q_K runs through $\{ L_0 , L_N \}^{\otimes 2}$ in the Legendre case and through $\{ s_N , T_N \}^{\otimes 2}$ in the Chebyshev case. The following result is proven in [BMM, Prop. V.1] and [BCM, Prop. V.7].

<u>Proposition II.1</u> : *Assume that hypothesis* (II.23) *holds. Problem* (II.8)$_A$ (II.9)$_A$ *is equivalent to the following one : Find* (u_N, p_N) *in* $X_N \times M_N$ *satisfying*

(II.24)$_A$

$$\begin{cases} (-\nu \Delta u_N + \mathbf{grad}\, p_N)(x) = f(x) & , x \in \Xi_N^A \cap \Omega \quad , \\ (\mathrm{div}\, u_N)(x) = 0 & , x \in \Xi_N^A \setminus \{ S_c \cup \hat{S}^A \} \quad . \end{cases}$$

together with the boundary conditions (II.9)$_A$.

In the same way, we have

<u>Proposition II.2</u> : *Assume that hypothesis* (II.23) *holds. Problem* (II.16)$_A$ (II.9)$_A$ *is equivalent to the following one : Find* (u_N, p_N) *in* $X_N \times M_N$ *satisfying*

(II.25)$_A$

$$\begin{cases} [-\nu \Delta u_N + \mathbf{grad}\, p_N + \sum_{i=1}^{2} \partial \mathcal{J}_N^A(u_{Ni} u_N)/\partial x_i](x) = f(x) & , x \in \Xi_N^A \cap \Omega \quad , \\ (\mathrm{div}\, u_N)(x) = 0 & , x \in \Xi_N^A \setminus \{ S_c \cup \hat{S}^A \} \quad . \end{cases}$$

together with the boundary conditions (II.9)$_A$.

In Section V, the reader will find practical ideas for solving this system, in particular how to choose a convenient set of degrees of freedom for the pressure.

II.4. The collocation problem for inhomogeneous boundary conditions.

In this paragraph we assume that f belongs to $\mathcal{C}^0(\Omega)^2$ and that the boundary data $\mathbf{g}$ are such that the $\mathbf{g}_J = \mathbf{g}_{|\Gamma_J}$, $J \in \mathbb{Z}/4\mathbb{Z}$, satisfy (I.15) and (I.16) but also belong to $\mathcal{C}^0(\overline{\Gamma}_J)^2$ and satisfy

(II.26) $\mathbf{g}_J(a_J) = \mathbf{g}_{J+1}(a_J)$, $J \in \mathbb{Z}/4\mathbb{Z}$.

We are now interested in the approximation of problem (I.3)(I.2). The first idea is to use the same discrete problem (II.8)$_A$ as in the homogeneous case and simply replace the boundary equation (II.9)$_A$ by

(II.27)$_A$ $u_N(x) = \mathbf{g}_J(x)$, $x \in \Xi_N^A \cap \overline{\Gamma}_J$, $J \in \mathbb{Z}/4\mathbb{Z}$.

But it turns out that this problem has no solution in the general case. Indeed, if the equation div u_N was satisfied in any point of Ξ_N^A , we would derive div $u_N = 0$ exactly. In particular this would imply five conditions for u_N at the boundary :

(II.28) $(\mathrm{div}\, u_N)(a_J) = 0$, $J \in \mathbb{Z}/4\mathbb{Z}$,

(II.29) $\sum_{J \in \mathbb{Z}/4\mathbb{Z}} \int_{\Gamma_J} u_N \cdot n_J \, d\sigma = 0$.

These equations solely depend upon the values of u_N at the boundary, hence upon the $\iota_N^A \mathbf{g}_J$, $J \in \mathbb{Z}/4\mathbb{Z}$. In general they are not verified, even if (I.16) holds (examples of functions $\mathbf{g}_J$ satisfying our assumptions but violating (II.28) and (II.29) are given in [BCM, (V.6)] and

in [Mé, Chap. 4]).

That is why we propose the following discrete problem : *Find* $(\mathbf{u_N}, p_N)$ *in* $X_N \times M_N$ *satisfying* $(II.22)_A$ *together with the boundary conditions* $(II.27)_A$.

Note however that this last problem is not so far from a collocation one, as the following proposition states it.

<u>Proposition II.3</u> : *Any solution* $(\mathbf{u_N}, p_N)$ *of problem* $(II.22)_A$ $(II.27)_A$ *in* $X_N \times M_N$ *satisfies the collocation equation*

$(II.30)_A$ $(-\nu \Delta \mathbf{u_N} + \mathbf{grad}\, p_N)(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ $, \mathbf{x} \in \Xi_N^A \cap \Omega$.

<u>Remark II.3</u> : By noting that the space $M_{2A,N}^{\perp}$ is exactly the image of $P_N^*(\Omega)^2$ by the divergence operator, it can be seen that solving the equation

$$\forall\, q \in M_{2A,N}^{\perp} \, , \quad b_{2A,N}(\mathbf{u_N}, q) = 0$$

in $(II.22)_A$ is actually equivalent to the minimization of $\|div\, \mathbf{u_N}\|_{A,N}$; this condition is implemented in practice, as will be seen in Section V (cf. also [Mé]).

In the same way, we define the approximation of the inhomogeneous Navier–Stokes equations (I.1)(I.2) as : *Find* $(\mathbf{u_N}, p_N)$ *in* $X_N \times M_N$ *such that*

$(II.31)_A$
$$\left\{ \begin{array}{l} \forall\, \mathbf{v_N} \in P_N^*(\Omega)^2, \quad a_{A,N}(\mathbf{u_N}, \mathbf{v_N}) + b_{1A,N}(\mathbf{v_N}, p_N) \\ \qquad\qquad\qquad + \sum_{i=1}^{2} (\partial \mathcal{J}_N^A(u_{Ni}\mathbf{u_N})/\partial x_i , \mathbf{v_N})_{A,N} = (\mathbf{f},\mathbf{v_N})_{A,N} \quad , \\ \forall\, q_N \in M_{2A,N}^{\perp} , \quad b_{2A,N}(\mathbf{u_N}, q_N) = 0 \quad . \end{array} \right.$$

together with the boundary conditions $(II.27)_A$. We also have the

<u>Proposition II.4</u> : *Any solution* $(\mathbf{u_N}, p_N)$ *of problem* $(II.31)_A$ $(II.27)_A$ *in* $X_N \times M_N$ *satisfies the collocation equation*

$(II.32)_A$ $[-\nu \Delta \mathbf{u_N} + \mathbf{grad}\, p_N + \sum_{i=1}^{2} (\partial \mathcal{J}_N^A(u_{Ni}\mathbf{u_N})/\partial x_i](\mathbf{x}) = \mathbf{f}(\mathbf{x})$ $, \mathbf{x} \in \Xi_N^A \cap \Omega$.

In the following sections, it will be proven that the four discrete problems are well-posed.

III. Convergence results for the Stokes problem.

The convergence of the method in the case of homogeneous boundary conditions has already been thoroughly analysed [BMM][BCM], hence we only recall the results. Then, we extend them to the nonhomogeneous case.

III.1. The case of homogeneous boundary conditions.

Problem $(II.8)_A$ $(II.9)_A$ is actually analysed through its variational formulation $(II.22)_A$. We recall the main properties of the bilinear forms involved in this formulation, which are the corner-stone of the study. For $i = 1$ or 2, let us define the kernels

$(III.1)$ $\qquad K_{iA,N} = \{ v_N \in P_N^\circ(\Omega)^2 \; ; \; \forall q_N \in P_N(\Omega), b_{iA,N}(v_N, q_N) = 0 \}$.

Clearly, $K_{1A,N}$ in the Legendre case and $K_{2A,N}$ in both cases coincide with the subspace of divergence-free polynomials in $P_N^\circ(\Omega)^2$, while $K_{1A,N}$ in the Chebyshev case is the subspace of polynomials v_N in $P_N^\circ(\Omega)^2$ such that $\text{div}(v_N \omega)$ is equal to 0.

Proposition III.1 : *There exist constants γ, δ_1 and δ_2 independent of N such that the forms $a_{A,N}$, $b_{1A,N}$ and $b_{2A,N}$ satisfy the following continuity properties*

$(III.2)$
$$\begin{cases} \forall u_N \in P_N(\Omega)^2, \forall v_N \in P_N^\circ(\Omega)^2, \; |a_{A,N}(u_N, v_N)| \leqslant \gamma \, \|u_N\|_{1,A,\Omega} \, \|v_N\|_{1,A,\Omega} \; , \\ \forall v_N \in P_N^\circ(\Omega)^2, \forall q_N \in P_N(\Omega), \; |b_{1A,N}(v_N, q_N)| \leqslant \delta_1 \, \|v_N\|_{1,A,\Omega} \, \|q_N\|_{0,A,\Omega} \; , \\ \forall v_N \in P_N(\Omega)^2, \forall q_N \in P_N(\Omega), \; |b_{2A,N}(v_N, q_N)| \leqslant \delta_2 \, \|v_N\|_{1,A,\Omega} \, \|q_N\|_{0,A,\Omega} \; . \end{cases}$$

In the Legendre case, there exists a constant $\alpha_L > 0$ independent of N such that

$(III.3)_L \quad \forall w_N \in P_N^\circ(\Omega)^2, \; a_{L,N}(w_N, w_N) \geqslant \alpha_L \|w_N\|_{1,\Omega}^2 \quad ;$

in the Chebyshev case, there exists a constant $\alpha_C > 0$ independent of N such that

$(III.3)_C \quad \forall w_N \in K_{2A,N}, \exists v_N \in K_{1A,N}, v_N \neq 0 \, / \, a_{C,N}(w_N, v_N) \geqslant \alpha_C \|w_N\|_{1,\omega,\Omega} \|v_N\|_{1,\omega,\Omega} \quad ;$

There exists a constant $\beta_i > 0$ independent of N such that

$(III.4) \quad \forall q_N \in M_{iA,N}^\perp, \exists v_N \in P_N^\circ(\Omega)^2, v_N \neq 0 \, /$
$$b_{iA,N}(v_N, q_N) \geqslant \beta_i \, N^{-2} \, \|v_N\|_{1,A,\Omega} \|q_N\|_{0,A,\Omega} \quad ;$$

if hypothesis (II.20) holds, there exists a constant $\beta_i' > 0$ independent of N such that

$(III.5) \quad \forall q_N \in M_N, \exists v_N \in P_N^\circ(\Omega)^2, v_N \neq 0 \, /$
$$b_{1A,N}(v_N, q_N) \geqslant \beta_i' \, N^{-2} \, \|v_N\|_{1,A,\Omega} \|q_N\|_{0,A,\Omega} \quad .$$

Using a well-known theorem for saddle-point problems [B][GR, Chapter I, Corollary 4.1][T, Chap. I, Th. 2.1] in the Legendre case and its generalization to nonsymmetric

problems [N][BCM, Corollary II.2], we derive from this proposition the following theorems [BMM, Thm V.1][BCM,Thm V.1].

Theorem III.1 : *Assume that hypothesis* (II.20) *holds. For any function* f *in* $\mathcal{C}^0(\Omega)^2$, *the collocation approximation* (II.8)$_A$ (II.9)$_A$ *to the Stokes problem* (I.3)(I.4) *has a unique solution* $(\mathbf{u}_N , p_N)$ *in* $P_N^\bullet(\Omega)^2 \times M_N$.

The error estimates have been proven respectively in [BMM, Thm V.1 and V.2] in the Legendre case and in [BCM, Thm V.2 and V.3] in the Chebyshev case. Note that the main arguments of the proofs will be recalled in the following subsection, in order to study the inhomogeneous case.

Theorem III.2 : *Assume that hypothesis* (II.20) *holds, that the solution* $(\mathbf{u},p)$ *of the Stokes problem* (I.3)(I.4) *is such that* $\mathbf{u}$ *belongs to* $H_A^s(\Omega)^2$ *for a real number* $s \geqslant 1$, *and the data* f *belong to* $H_A^\sigma(\Omega)^2$ *for a real number* $\sigma > 1$. *Then the solution* $(\mathbf{u}_N , p_N)$ *of problem* (II.8)$_A$ (II.9)$_A$ *satisfies*

(III.6) $\|\mathbf{u}-\mathbf{u}_N\|_{1,A,\Omega} \leqslant c \, (\, N^{1-s} \|\mathbf{u}\|_{s,A,\Omega} + N^{1+2\alpha-\sigma} \|f\|_{\sigma,A,\Omega} \,)$

for a constant c *independent of* N.

Theorem III.3 : *Assume that hypotheses* (II.20) *and* (II.21) *hold and that the solution* $(\mathbf{u},p)$ *of the Stokes problem* (I.3)(I.4) *belongs to* $H_A^s(\Omega)^2 \times H_A^{s-1}(\Omega)$ *for a real number* $s \geqslant 1$, *and the data* f *belong to* $H_A^\sigma(\Omega)^2$ *for a real number* $\sigma > 1$. *Then the solution* $(\mathbf{u}_N , p_N)$ *of problem* (II.8)$_A$ (II.9)$_A$ *satisfies*

(III.7) $\|p-p_N\|_{0,A,\Omega} \leqslant c \, \{ \, N^{3-s} \, (\, \|\mathbf{u}\|_{s,A,\Omega} + \|p\|_{s-1,A,\Omega} \,) + N^{3+2\alpha-\sigma} \|f\|_{\sigma,A,\Omega} \, \}$

for a constant c *independent of* N.

Remark III.1 : Let us consider for a while the problem : *Find* $(\mathbf{u}_N , p_N)$ *in* $P_N^\bullet(\Omega)^2 \times M_N$ *such that*

(III.8)$_A$
$$\begin{cases} \forall \, \mathbf{v}_N \in P_N^\bullet(\Omega)^2, \quad a_{A,N}(\mathbf{u}_N , \mathbf{v}_N) + b_{1A,N}(\mathbf{v}_N , p_N) = (f,\mathbf{v}_N)_A \ , \\ \forall \, q_N \in P_N(\Omega), \quad b_{2A,N}(\mathbf{u}_N , q_N) = 0 \ , \end{cases}$$

which is problem (II.15)$_A$ with $(f,\mathbf{v}_N)_{A,N}$ replaced by $(f,\mathbf{v}_N)_A$. Then, Theorems III.1 to III.3 are still valid. Furthermore, by reading the proof of [BMM][BCM], it is easy to see that the estimates (III.6) and (III.7) can be replaced respectively by

(III.9) $\|\mathbf{u}-\mathbf{u}_N\|_{1,A,\Omega} \leqslant c \, N^{1-s} \|\mathbf{u}\|_{s,A,\Omega}$,

and

(III.10) $\|p-p_N\|_{0,A,\Omega} \leqslant c \, N^{3-s} \, (\, \|\mathbf{u}\|_{s,A,\Omega} + \|p\|_{s-1,A,\Omega} \,)$.

This will be used in the following section.

III.2. The case of inhomogeneous boundary conditions.

Our purpose is now to study the discrete problem $(II.22)_A$ $(II.27)_A$. Since we need an element in the space of trial functions X_N that satisfies, in a discrete sense, the boundary condition $(I.2)$, we state the following lifting result that can be derived from [BM1, Prop. V.1].

Lemma III.1 : *There exists an operator* Q_N^A *from the subspace of all polynomials* $\Phi_N = (\varphi_{NJ})_{J \in \mathbb{Z}/4\mathbb{Z}}$ *in* $\prod_{J \in \mathbb{Z}/4\mathbb{Z}} P_N(\Gamma_J)$ *satisfying*

$(III.11)$ $\quad \varphi_{NJ}(a_J) = \varphi_{N,J+1}(a_J)$, $J \in \mathbb{Z}/4\mathbb{Z}$,

into $P_N(\Omega)$ *such that, for any such polynomial* Φ_N ,

$(III.12)$ $\quad Q_N^A(\Phi_N) = \varphi_{NJ}$ *on* Γ_J , $J \in \mathbb{Z}/4\mathbb{Z}$.

Moreover the following estimate is satisfied

$(III.13)$ $\quad \|Q_N^A(\Phi_N)\|_{1,A,\Omega} \leqslant c\, N^{1-\alpha} \sum_{J \in \mathbb{Z}/4\mathbb{Z}} \|\varphi_{NJ}\|_{0,A,\Gamma_J}$.

Sketch of proof : From [BM1, Prop. V.1], we derive that there exists an operator Q_N^A which satisfies $(III.12)$ and such that one has

$$\|Q_N^A(\Phi_N)\|_{1,A,\Omega} \leqslant c\, (\, N^{1-\alpha} \sum_{J \in \mathbb{Z}/4\mathbb{Z}} \|\varphi_{NJ}\|_{0,A,\Gamma_J} + N^{-2\alpha} \sum_{J \in \mathbb{Z}/4\mathbb{Z}} |\varphi_{NJ}(a_J)|)$$.

Then, estimate $(III.13)$ follows from the previous line and the inverse inequality [Q, (2.4) and (3.2)], valid for any polynomial φ_N in $P_N(\Lambda)$,

$$\|\varphi_N\|_{L^\infty(\Lambda)} \leqslant c\, N^{1+\alpha} \|\varphi_N\|_{0,A,\Lambda}$$.

The previous result allows us to check that the discrete problem $(II.22)_A$ $(II.27)_A$ is well-posed.

Theorem III.4 : *Assume that hypothesis* $(II.20)$ *holds. For any function* f *in* $\mathcal{C}^0(\Omega)^2$, *the collocation approximation* $(II.22)_A$ $(II.27)_A$ *of the Stokes problem* $(I.3)(I.2)$ *has a unique solution* (u_N , p_N) *in* $X_N \times M_N$.

Proof : If we choose $u_{N,b}$ equal to the image of $(\iota_N^A g_J)_{J \in \mathbb{Z}/4\mathbb{Z}}$ by the operator Q_N^A , the polynomial $\tilde{u}_N = u_N - u_{N,b}$ belongs to $P_N^\bullet(\Omega)^2$. Then the pair (u_N , p_N) is a solution of $(II.22)_A$ $(II.27)_A$ if and only if the pair $(\tilde{u}_N , p_N)$ satisfies :

$$(\text{III.14})_A \quad \left| \begin{array}{l} \forall \ v_N \in P_N^*(\Omega)^2, \quad a_{A,N}(\tilde{u}_N, v_N) + b_{1A,N}(v_N, p_N) \\ \qquad\qquad\qquad\qquad\qquad = (f, v_N)_{A,N} - a_{A,N}(u_{N,b}, v_N) \quad, \\ \forall \ q_N \in M_{2A,N}^\perp, \quad b_{2A,N}(\tilde{u}_N, q_N) = -b_{2A,N}(u_{N,b}, q_N) \quad. \end{array} \right.$$

Due to Proposition III.1, we derive the result from [BCM, Corollary II.2].

Next, we will study the approximation of divergence-free functions by divergence-free polynomials, thus generalizing the result of [SV] to the case of non homogeneous boundary conditions. Let us set

(III.15) $\quad K_A(\Omega) = \{ \ w \in H_A^3(\Omega)^2 \ ; \text{div } w = 0 \text{ in } \Omega \ \}$.

Lemma III.2 : *There exists an operator* R_N^A *from* $K_A(\Omega)$ *into* $P_N(\Omega)^2 \cap K_A(\Omega)$ *such that the following estimate is satisfied for any real number* $s \geqslant 3$ *: for any function* w *in* $K_A(\Omega) \cap H_A^s(\Omega)^2$,

(III.16) $\quad \|w - R_N^A w\|_{1,A,\Omega} \leqslant c \, N^{1-s} \|w\|_{s,A,\Omega}$.

Proof : Let us recall [M1, Remark II.3 and Lemma IV.2] that there exists an operator $\pi_N^{A,2}$ from $H_A^2(\Lambda)$ into $P_N(\Lambda)$ which satisfies for any function φ in $H_A^2(\Lambda)$

$$(\text{III.17}) \quad \left| \begin{array}{l} \pi_N^{A,2}\varphi(-1) = \varphi(-1) \quad \text{and} \quad \pi_N^{A,2}\varphi(+1) = \varphi(+1) \quad, \\ (d\pi_N^{A,2}\varphi/d\zeta)(-1) = (d\varphi/d\zeta)(-1) \quad \text{and} \quad (d\pi_N^{A,2}\varphi/d\zeta)(+1) = (d\varphi/d\zeta)(+1) \ . \end{array} \right.$$

Moreover, we have for any function φ in $H_A^s(\Lambda)$, $s \geqslant 2$,

(III.18) $\quad \|\varphi - \pi_N^{A,2}\varphi\|_{2,A,\Lambda} + N \|\varphi - \pi_N^{A,2}\varphi\|_{1,A,\Lambda} + N^2 \|\varphi - \pi_N^{A,2}\varphi\|_{0,A,\Lambda} \leqslant c \, N^{2-s} \|\varphi\|_{s,A,\Lambda}$.

Next, for any function w in $K_A(\Omega)$, there exists a unique function ψ in $H_A^4(\Omega) \cap L_{A,0}^2(\Omega)$ such that w is equal to **curl** ψ in Ω; moreover, if w belongs to $H_A^s(\Omega)^2$, $s \geqslant 3$, it satisfies

(III.19) $\quad \|\psi\|_{s+1,A,\Omega} \leqslant c \, \|w\|_{s,A,\Omega}$.

Setting $\psi_N = (\pi_N^{A,2} \otimes \pi_N^{A,2})\psi$, we define $R_N^A w$ as being equal to **curl** ψ_N . It remains to estimate

$$\|w - R_N^A w\|_{1,A,\Omega} \leqslant c \, \|\psi - \psi_N\|_{2,A,\Omega}$$
$$\leqslant c \, (\|\psi - \psi_N\|_{H_A^2(\Lambda, L_A^2(\Lambda))} + \|\psi - \psi_N\|_{H_A^1(\Lambda, H_A^1(\Lambda))} + \|\psi - \psi_N\|_{L_A^2(\Lambda, H_A^2(\Lambda))}).$$

From (III.18), we infer

$$\|\psi - (\pi_N^{A,2} \otimes \pi_N^{A,2})\psi\|_{H_A^2(\Lambda, L_A^2(\Lambda))}$$
$$\leqslant \|\psi - (\pi_N^{A,2} \otimes id)\psi\|_{H_A^2(\Lambda, L_A^2(\Lambda))} + \|\pi_N^{A,2} \otimes (id - \pi_N^{A,2})\psi\|_{H_A^2(\Lambda, L_A^2(\Lambda))}$$
$$\leqslant c \, N^{1-s} \|\psi\|_{H_A^{s+1}(\Lambda, L_A^2(\Lambda))} + \|id \otimes (id - \pi_N^{A,2})\psi\|_{H_A^2(\Lambda, L_A^2(\Lambda))}$$
$$\leqslant c \, N^{1-s} (\|\psi\|_{H_A^{s+1}(\Lambda, L_A^2(\Lambda))} + \|\psi\|_{H_A^2(\Lambda, H_A^{s-1}(\Lambda))})$$

$$\leqslant c\, N^{1-s}\,(\,\|\psi\|_{H_A^{s+1}(\Lambda,L_A^2(\Lambda))} + \|\psi\|_{H_A^2(\Lambda,H_A^{s-1}(\Lambda))}\,)$$

$$\leqslant c\, N^{1-s}\,\|\psi\|_{s+1,A,\Omega}\quad.$$

The quantities $\|\psi-\psi_N\|_{H_A^1(\Lambda,H_A^1(\Lambda))}$ and $\|\psi-\psi_N\|_{L_A^2(\Lambda,H_A^2(\Lambda))}$ are estimated in the same way.

Finally, we obtain

$$\|w-R_N^A w\|_{1,A,\Omega}\ \leqslant c\, N^{1-s}\,\|\psi\|_{s+1,A,\Omega}\quad,$$

which together with (III.19) gives the result.

Of course, other divergence-free polynomials approximations of divergence-free functions in $H_A^1(\Omega)^2$ can be built. However, note that, for any function w in $K_A(\Omega)$, the operator R_N^A satisfies

(III.20) $(R_N^A w)(a_J) = w(a_J)$, $J \in \mathbb{Z}/4\mathbb{Z}$.

Moreover, it has the following useful property.

Corollary III.1 : *The operator* R_N^A *satisfies the following estimate for any real number* $\tau \geqslant 2$ *: for any function* w *in* $K_A(\Omega)$ *such that the trace* $w_{|\Gamma_J}$ *belongs to* $H_A^\tau(\Gamma_J)^2$,

(III.21) $\|w_{|\Gamma_J} - R_N^A w_{|\Gamma_J}\|_{0,A,\Gamma_J} \leqslant c\, N^{-\tau}\,\|w_{|\Gamma_J}\|_{\tau,A,\Gamma_J}$.

Proof : We write

$$\|w_{|\Gamma_J}-R_N^A w_{|\Gamma_J}\|_{0,A,\Gamma_J} \leqslant c\,(\,\|\partial\psi/\partial x - \partial\psi_N/\partial x\|_{0,A,\Gamma_J} + \|\partial\psi/\partial y - \partial\psi_N/\partial y\|_{0,A,\Gamma_J}\,)\quad.$$

In the case $J = I$ or III for instance, using (III.17) and noting that the operators $\partial/\partial x$ and $id \otimes \pi_N^{A,2}$ commute, we have

$$(\partial(\pi_N^{A,2}\otimes\pi_N^{A,2})\psi/\partial x)(\pm 1,y) = (\partial(id\otimes\pi_N^{A,2})\psi/\partial x)(\pm 1,y) = (id\otimes\pi_N^{A,2})(\partial\psi/\partial x)(\pm 1,y)$$

and, similarly,

$$(\partial(\pi_N^{A,2}\otimes\pi_N^{A,2})\psi/\partial y)(\pm 1,y) = (\partial(id\otimes\pi_N^{A,2})\psi/\partial y)(\pm 1,y)\quad,$$

so that, by (III.18),

$$\|w_{|\Gamma_J}-R_N^A w_{|\Gamma_J}\|_{0,A,\Gamma_J} \leqslant c\,(\,\|(\partial\psi/\partial x)-(id\otimes\pi_N^{A,2})(\partial\psi/\partial x)\|_{0,A,\Gamma_J}$$
$$+ \|\psi-(id\otimes\pi_N^{A,2})\psi\|_{1,A,\Gamma_J}\,)$$
$$\leqslant c\, N^{-\tau}\,(\,\|\partial\psi/\partial x\|_{\tau,A,\Gamma_J} + \|\psi\|_{\tau+1,A,\Gamma_J}\,) \leqslant c\, N^{-\tau}\,\|w_{|\Gamma_J}\|_{\tau,A,\Gamma_J}\quad.$$

The cases $J = II$ and IV are studied in the same way by exchanging the variables x and y.

We are going to introduce a slightly different approximation to the Stokes problem (I.3)(I.2), that satisfies conditions (I.16) and (II.26). In all that follows, we assume that, if (u,p) is the solution of (I.3)(I.2), the function u belongs to $H_A^3(\Omega)^2$; then, we set

(III.22) $z_{N-1} = R_{N-1}^A u$.

Let us remark that the data $z_{N-1|\Gamma_J}$, $J \in \mathbb{Z}/4\mathbb{Z}$, define a boundary condition that satisfies (I.16) and (II.26). Hence we can define two auxiliary problems

a) *Find* $(\hat{u},\hat{p})$ *in* $H_A^1(\Omega)^2 \times L_{A,0}^2(\Omega)$ *such that*

$(III.23)_A$ $\quad \begin{cases} \forall \ v \in H_{A,0}^1(\Omega)^2, \quad a_A(\hat{u},v) + (v , \mathbf{grad} \ \hat{p})_A = (f,v)_A \quad , \\ \forall \ q \in L_A^2(\Omega), \quad (\text{div} \ \hat{u} , q)_A = 0 \quad , \end{cases}$

and satisfying the boundary conditions

$(III.24)_A$ $\quad \hat{u} = z_{N-1|\Gamma_J} \quad$ on $\Gamma_J \quad$, $J \in \mathbb{Z}/4\mathbb{Z}$;

b) *Find* $(\hat{u}_N , \hat{p}_N)$ *in* $P_N(\Omega)^2 \times M_N$ *such that*

$(III.25)_A$ $\quad \begin{cases} \forall \ v_N \in P_N^*(\Omega)^2, \quad a_{A,N}(\hat{u}_N , v_N) + b_{1A,N}(v_N , \hat{p}_N) = (f,v_N)_{A,N} \quad , \\ \forall \ q_N \in M_{2A,N}^\perp , \quad b_{2A,N}(\hat{u}_N , q_N) = 0 \quad , \end{cases}$

and satisfying the boundary conditions

$(III.26)_A$ $\quad \hat{u}_N(x) = z_{N-1|\Gamma_J}(x) \quad$, $x \in \Xi_N^A \cap \Gamma_J$, $J \in \mathbb{Z}/4\mathbb{Z}$.

The error bound between the solutions u and u_N of problem (I.3)(I.2) and $(II.22)_A$ $(II.27)_A$ will be obtained by studying the differences between u and $\hat{u}$, $\hat{u}$ and $\hat{u}_N$, $\hat{u}_N$ and u_N .

Lemma III.3 : *Assume that the solution* (u,p) *of the Stokes problem* (I.3)(I.2) *is such that* u *belongs to* $H_A^s(\Omega)^2$ *for a real number* $s \geqslant 3$. *The following estimate is satisfied*

$(III.27) \quad \|u-\hat{u}\|_{1,A,\Omega} \leqslant c \ N^{1-s} \|u\|_{s,A,\Omega}$

for a constant $c > 0$ *independent of* N.

Proof : Since the pair $u-\hat{u}$ is the solution of a Stokes problem with null body forces and boundary data equal to $u_{|\Gamma_J} - z_{N-1|\Gamma_J}$, $J \in \mathbb{Z}/4\mathbb{Z}$, it follows from the stability estimate (I.22) that

$\|u-\hat{u}\|_{1,A,\Omega} \leqslant c \sum_{J \in \mathbb{Z}/4\mathbb{Z}} \|u_{|\Gamma_J} - z_{N-1|\Gamma_J}\|_{(1-\alpha)/2,A,\Gamma_J}$.

Due to the trace theorem [LM, Chap.1, Th. 8.3][BM1, Thm II.2], that implies

$\|u-\hat{u}\|_{1,A,\Omega} \leqslant c \|u-z_{N-1}\|_{1,A,\Omega}$.

Then, we deduce the lemma from Lemma III.2.

Similarly we can obtain an error bound between u_N and $\hat{u}_N$.

Lemma III.4 : *Assume that the boundary data* g_J , $J \in \mathbb{Z}/4\mathbb{Z}$, *belong to* $H_\varrho^\tau(\Gamma_J)^2$ *for a real number* $\tau \geqslant 2$. *The following estimate is satisfied*

$(III.28) \quad \|u_N-\hat{u}_N\|_{1,A,\Omega} \leqslant c \ N^{7/2-\tau} \sum_{J \in \mathbb{Z}/4\mathbb{Z}} \|g_J\|_{\tau,A,\Gamma_J}$

for a constant $c > 0$ *independent of* N.

Proof : It follows from problems $(II.22)_A$ $(II.27)_A$ and $(III.25)_A$ $(III.26)_A$ that the polynomial $\mathbf{u}_N - \hat{\mathbf{u}}_N$ is the collocation approximation of a Stokes problem with null body forces and boundary data equal to $\iota_N^A \mathbf{g}_J - \mathbf{z}_{N-1|\Gamma_J}$, $J \in \mathbb{Z}/4\mathbb{Z}$. Let $\mathbf{w}_{N,b}$ denote the image of $(\iota_N^A \mathbf{g}_J - \mathbf{z}_{N-1|\Gamma_J})_{J \in \mathbb{Z}/4\mathbb{Z}}$ by the operator Q_N^A . Setting $\mathbf{w}_N = \mathbf{u}_N - \hat{\mathbf{u}}_N - \mathbf{w}_{N,b}$ and $r_N = p_N - \hat{p}_N$, we see that the pair $(\mathbf{w}_N , r_N)$ is the only solution in $P_N^{\bullet}(\Omega)^2 \times M_N$ of

$(III.29)_A$
$$\begin{cases} \forall \; \mathbf{v}_N \in P_N^{\bullet}(\Omega)^2, \quad a_{A,N}(\mathbf{w}_N , \mathbf{v}_N) + b_{1A,N}(\mathbf{v}_N , r_N) = - a_{A,N}(\mathbf{w}_{N,b} , \mathbf{v}_N) \quad, \\ \forall \; q_N \in M_{2A,N}^{\perp}, \quad b_{2A,N}(\mathbf{w}_N , q_N) = - b_{2A,N}(\mathbf{w}_{N,b} , q_N) \quad. \end{cases}$$

Using [BCM, Corollary II.2] together with Proposition III.1, we obtain

$$\|\mathbf{w}_N\|_{1,A,\Omega} \leqslant c \, N^2 \, \|\mathbf{w}_{N,b}\|_{1,A,\Omega} \quad,$$

so that, using Lemma III.1,

$$\|\mathbf{u}_N - \hat{\mathbf{u}}_N\|_{1,A,\Omega} \leqslant c \, N^2 \, N^{1-\alpha} \sum_{J \in \mathbb{Z}/4\mathbb{Z}} (\|\mathbf{g}_J - \iota_N^A \mathbf{g}_J\|_{0,A,\Gamma_J} + \|\mathbf{u}_{|\Gamma_J} - \mathbf{z}_{N-1|\Gamma_J}\|_{0,A,\Gamma_J}) \quad.$$

The lemma follows from Corollary III.1 and from the following estimate for the interpolation error [CQ1 Thms 3.1 and 3.2], valid for any real number $s > 1/2$:

$(III.30)$ $\quad \|\varphi - \iota_N^A \varphi\|_{0,A,\Lambda} \leqslant c \, N^{1/2+\alpha-s} \|\varphi\|_{s,A,\Lambda}$.

Finally, in order to get now an error bound between $\hat{\mathbf{u}}$ and $\hat{\mathbf{u}}_N$, we note that problem $(III.25)_A$ $(III.26)_A$ is a discrete approximation of problem $(III.23)_A$ $(III.24)_A$. That allows us to derive the following estimate.

Lemma III.5 : *Assume that the solution* $(\mathbf{u},p)$ *of the Stokes problem* $(1.3)(1.2)$ *belongs to* $H_A^s(\Omega)^2 \times H_A^{s-1}(\Omega)$ *for a real number* $s \geqslant 3$, *that the data* $\mathbf{f}$ *belong to* $H_A^\sigma(\Omega)^2$ *for a real number* $\sigma > 1$ *and that the boundary data* $\mathbf{g}_J$, $J \in \mathbb{Z}/4\mathbb{Z}$, *belong to* $H_A^\tau(\Gamma_J)^2$ *for a real number* $\tau \geqslant 2$. *The following estimate is satisfied*

$(III.31)$ $\quad \|\hat{\mathbf{u}} - \hat{\mathbf{u}}_N\|_{1,A,\Omega} \leqslant c \, (N^{1-s} \|\mathbf{u}\|_{s,A,\Omega} + N^{1+2\alpha-\sigma} \|\mathbf{f}\|_{\sigma,A,\Omega} + N^{-\tau} \sum_{J \in \mathbb{Z}/4\mathbb{Z}} \|\mathbf{g}_J\|_{\tau,A,\Gamma_J})$

for a constant $c > 0$ *independent of* N.

Proof : Let us set $\mathbf{u}^* = \hat{\mathbf{u}} - \mathbf{z}_{N-1}$ and $\mathbf{u}_N^* = \hat{\mathbf{u}}_N - \mathbf{z}_{N-1}$. Thus, $(\mathbf{u}^*, \hat{p})$ is the solution in $H_{A,0}^1(\Omega)^2 \times L_{A,0}^2(\Omega)$ of

$(III.32)_A$
$$\begin{cases} \forall \; \mathbf{v} \in H_{A,0}^1(\Omega)^2, \quad a_A(\mathbf{u}^*, \mathbf{v}) + (\mathbf{v} , \mathbf{grad} \, \hat{p})_A = (\mathbf{f}, \mathbf{v})_A - a_A(\mathbf{z}_{N-1} , \mathbf{v}) \quad, \\ \forall \; q \in L_A^2(\Omega), \quad (\mathrm{div} \, \mathbf{u}^* , q)_A = 0 \quad, \end{cases}$$

and $(\mathbf{u}_N^* , \hat{p}_N)$ is the solution in $P_N^{\bullet}(\Omega)^2 \times M_N$ of

$(III.33)_A$ $\left|\begin{array}{l} \forall \, v_N \in P_N^\bullet(\Omega)^2, \quad a_{A,N}(u_N^*, v_N) + b_{1A,N}(v_N, \hat{p}_N) = (f, v_N)_{A,N} - a_{A,N}(z_{N-1}, v_N), \\ \forall \, q_N \in M_{2A,N}^\perp, \quad b_{2A,N}(u_N^*, q_N) = 0 \; . \end{array}\right.$

Next, we use an abstract error estimate due to [BCM, Corollary II.3] : since the forms $a_A(.,.)$ and $a_{A,N}(.,.)$ coincide on $P_{N-1}(\Omega)^2 \times P_N^\bullet(\Omega)^2$ and since z_{N-1} belongs to $P_{N-1}(\Omega)^2$, for any polynomial v_{N-1} in $P_{N-1}^\bullet(\Omega)^2$ and any polynomial w_N in $P_N^\bullet(\Omega)^2$ such that

$$\forall \, q_N \in M_{2A,N}^\perp, \quad b_{2A,N}(w_N, q_N) = 0 \quad ,$$

the following estimate holds (note that $u^* - u_N^* = \hat{u} - \hat{u}_N$)

$(III.34)$ $\left|\begin{array}{l} \|\hat{u} - \hat{u}_N\|_{1,A,\Omega} \leqslant c \, \{ \|u^* - v_{N-1}\|_{1,A,\Omega} + \|u^* - w_N\|_{1,A,\Omega} \\ \\ \qquad\qquad + \displaystyle\sup_{z_N \in P_N^\bullet(\Omega)^2} \frac{(f, z_N)_A - (f, z_N)_{A,N}}{\|z_N\|_{1,A,\Omega}} \quad . \end{array}\right.$

The more convenient here is to choose $v_{N-1} = w_N = 0$. So it remains to estimate $\|u^*\|_{1,A,\Omega}$ and the last term.

1) We have

$$\|u^*\|_{1,A,\Omega} \leqslant \|u - \hat{u}\|_{1,A,\Omega} + \|u - z_{N-1}\|_{1,A,\Omega} \quad ,$$

so that, by Lemmas III.2 and III.3,

$(III.35)$ $\quad \|u^*\|_{1,A,\Omega} \leqslant c \, (N^{1-s} \|u\|_{s,A,\Omega} + N^{-\tau} \sum_{J \in \mathbb{Z}/4\mathbb{Z}} \|g_J\|_{\tau,A,\Gamma_J}) \quad .$

2) We recall [CQ1, §3] that the scalar product $(.,.)_A$ induces a norm on $P_N(\Omega)$ which is equivalent to $\|.\|_{0,A,\Omega}$. Hence, choosing f_{N-1} in $P_{N-1}(\Omega)^2$, we obtain for any z_N in $P_N^\bullet(\Omega)^2$

$$(f, z_N)_A - (f, z_N)_{A,N} = (f - f_{N-1}, z_N)_A - (\mathcal{J}_N^A f - f_{N-1}, z_N)_{A,N}$$
$$\leqslant c \, (\|f - f_{N-1}\|_{0,A,\Omega} + \|f - \mathcal{J}_N^A f\|_{0,A,\Omega}) \|z_N\|_{0,A,\Omega} \quad .$$

Let us recall that the orthogonal projection Π_N^A from $L^2(\Omega)$ onto $P_N(\Omega)$ satisfies the following estimate for any φ in $H_A^s(\Omega)$, $s \geqslant 0$,

$(III.36)$ $\quad \|\varphi - \Pi_N^A \varphi\|_{0,A,\Omega} \leqslant c \, N^{-s} \|\varphi\|_{s,A,\Omega} \quad .$

Taking for instance $f_{N-1} = \Pi_{N-1}^A f$ and noting that $\mathcal{J}_N^A$ is equal to $\iota_N^A \otimes \iota_N^A$, we derive from $(III.30)$ and $(III.36)$

$(III.37)$ $\quad (f, z_N)_A - (f, z_N)_{A,N} \leqslant c \, N^{1+2\alpha-\sigma} \|f\|_{\sigma,A,\Omega} \|z_N\|_{0,A,\Omega} \quad .$

Finally, estimate $(III.31)$ follows from $(III.34)$, $(III.35)$ and $(III.37)$.

From Lemmas III.3 to III.5, we derive the main error estimate.

<u>Theorem III.5</u> : *Assume that hypothesis* $(II.20)$ *holds and that the solution* (u,p) *of the Stokes problem* $(1.3)(1.2)$ *is such that* u *belongs to* $H_A^s(\Omega)^2$ *for a real number* $s \geqslant 3$ *, that the data* f *belong to* $H_A^\sigma(\Omega)^2$ *for a real number* $\sigma > 1$ *and that the boundary data* g_J *,*

$J \in \mathbb{Z}/4\mathbb{Z}$, *belong to* $H_A^\tau(\Gamma_J)^2$ *for a real number* $\tau \geqslant 2$. *Then, the solution* $(\mathbf{u}_N , p_N)$ *of problem* $(II.22)_A$ $(II.27)_A$ *satisfies*

$$(III.38) \quad \|\mathbf{u}-\mathbf{u}_N\|_{1,A,\Omega} \leqslant c\, (N^{1-s} \|\mathbf{u}\|_{s,A,\Omega} + N^{1+2\alpha-\sigma} \|\mathbf{f}\|_{\sigma,A,\Omega}$$
$$+ N^{7/2-\tau} \sum_{J \in \mathbb{Z}/4\mathbb{Z}} \|\mathbf{g}_J\|_{\tau,A,\Gamma_J})$$

for a constant $c > 0$ *independent of* N.

We conclude with an estimate for the pressure.

Theorem III.6 : *Assume that hypotheses* $(II.20)$ *and* $(II.21)$ *hold and that the solution* $(\mathbf{u},p)$ *of the Stokes problem* $(1.3)(1.2)$ *belongs to* $H_A^s(\Omega)^2 \times H_A^{s-1}(\Omega)$ *for a real number* $s \geqslant 3$, *that the data* $\mathbf{f}$ *belong to* $H_A^\sigma(\Omega)^2$ *for a real number* $\sigma > 1$ *and that the boundary data* $\mathbf{g}_J$, $J \in \mathbb{Z}/4\mathbb{Z}$, *belong to* $H_A^\tau(\Gamma_J)^2$ *for a real number* $\tau \geqslant 2$. *Then, the solution* $(\mathbf{u}_N , p_N)$ *of problem* $(II.22)_A$ $(II.27)_A$ *satisfies*

$$(III.39) \quad \|p-p_N\|_{0,A,\Omega} \leqslant c\, \{ N^{3-s} (\|\mathbf{u}\|_{s,A,\Omega} + \|p\|_{s-1,A,\Omega})$$
$$+ N^{3+2\alpha-\sigma} \|\mathbf{f}\|_{\sigma,A,\Omega} + N^{11/2-\tau} \sum_{J \in \mathbb{Z}/4\mathbb{Z}} \|\mathbf{g}_J\|_{\tau,A,\Gamma_J} \}$$

for a constant $c > 0$ *independent of* N.

Proof : Using the Inf-Sup condition of Lemma III.1, we derive from $(1.21)_A$ and $(II.22)_A$ that, for any q_N in M_N ,

$$\|p-p_N\|_{0,A,\Omega} \leqslant c\, N^2 \, \{ \|\mathbf{u}-\mathbf{u}_N\|_{1,A,\Omega} + \|p-q_N\|_{0,A,\Omega}$$

$$+ \sup_{\mathbf{z}_N \in P_N^\bullet(\Omega)^2} [\frac{(\mathbf{z}_N , \operatorname{grad} q_N)_A - b_{1A,N}(\mathbf{z}_N , q_N)}{\|\mathbf{z}_N\|_{1,A,\Omega}} + \frac{(\mathbf{f},\mathbf{z}_N)_A - (\mathbf{f},\mathbf{z}_N)_{A,N}}{\|\mathbf{z}_N\|_{1,A,\Omega}}] \} \quad .$$

Owing to $(II.21)$, taking for instance $q_N = \Pi_{[\lambda N]}^A p$ and using $(III.36),(III.38)$ and $(III.37)$, we obtain easily $(III.39)$.

Remark III.3 : As in the case of homogeneous boundary conditions, if we consider the problem : *Find* $(\mathbf{u}_N , p_N)$ *in* $X_N \times M_N$ *such that*

$$(III.40)_A \quad \begin{cases} \forall\, \mathbf{v}_N \in P_N^\bullet(\Omega)^2, \quad a_{A,N}(\mathbf{u}_N , \mathbf{v}_N) + b_{1A,N}(\mathbf{v}_N , p_N) = (\mathbf{f},\mathbf{v}_N)_A \quad , \\ \forall\, q_N \in M_{2A,N}^\perp, \quad b_{2A,N}(\mathbf{u}_N , q_N) = 0 \end{cases}$$

and satisfying the boundary conditions $(II.27)_A$, Theorems III.4 to III.6 are still valid, and it is easy to see that the estimates $(III.38)$ and $(III.39)$ can be replaced respectively by

$$(III.41) \quad \|\mathbf{u}-\mathbf{u}_N\|_{1,A,\Omega} \leqslant c\, (N^{1-s} \|\mathbf{u}\|_{s,A,\Omega} + N^{7/2-\tau} \sum_{J \in \mathbb{Z}/4\mathbb{Z}} \|\mathbf{g}_J\|_{\tau,A,\Gamma_J}) \quad ,$$

and

$$(III.42) \quad \|p-p_N\|_{0,A,\Omega} \leqslant c\, \{ N^{3-s} (\|\mathbf{u}\|_{s,A,\Omega} + \|p\|_{s-1,A,\Omega}) + N^{11/2-\tau} \sum_{J \in \mathbb{Z}/4\mathbb{Z}} \|\mathbf{g}_J\|_{\tau,A,\Gamma_J} \} \quad .$$

IV. Convergence analysis for the Navier-Stokes equations.

The aim of this section is to obtain, for the discrete problems $(II.16)_A$ $(II.9)_A$ and $(II.31)_A$ $(II.27)_A$, convergence results similar to those which were proven in the linear case. We begin by describing the main tools of the analysis, together with some properties of the exact equations. Then we establish some technical lemmas. This allows us to prove the convergence and to give error estimates for the velocity in both the homogeneous and inhomogeneous cases. Finally, error bounds are also derived for the pressure.

IV.1. The main tools.

To study the discrete problems, we shall use a fixed point theorem due to M. CROUZEIX [C, Th. 2.2], which is a refined form of the discrete implicit function theorem of [BRR]. For the reader's convenience, let us recall this theorem : we consider a C^1-mapping F_N from a Banach space Z_N into itself and we assume that u_N^* is a point in Z_N such that $DF_N(u_N^*)$ is an isomorphism of Z_N . We denote by ε_N , γ_N and $\wedge_N(\eta)$, $\eta \geqslant 0$, the quantities

(IV.1)
$$\varepsilon_N = \|F_N(u_N^*)\|_{Z_N} \qquad \gamma_N = \|(DF_N(u_N^*))^{-1}\|_{\mathscr{L}(Z_N,Z_N)}$$
$$\wedge_N(\eta) = \sup \{ \|DF_N(w_N)-DF_N(u_N^*)\|_{\mathscr{L}(Z_N,Z_N)} ; w_N \in Z_N \text{ and } \|w_N-u_N^*\|_{Z_N} \leqslant \eta \} .$$

Theorem IV.1 : *Let us assume that* $2\gamma_N \wedge_N(2\gamma_N\varepsilon_N) < 1$, *then for each* $\eta \geqslant 2\gamma_N\varepsilon_N$ *such that* $\gamma_N \wedge_N(\eta) < 1$, *there exists a unique solution* u_N *of the equation* $F_N(u_N) = 0$ *in the ball* $S_N = \{ w_N \in Z_N ; \|w_N-u_N^*\|_{Z_N} \leqslant \eta \}$. *This solution satisfies*

(IV.2) $\qquad \forall w_N \in S_N , \quad \|u_N-w_N\|_{Z_N} \leqslant [\gamma_N/(1-\gamma_N \wedge_N(\eta))] . \|F_N(w_N)\|_{Z_N}$.

Let us precise in what framework we shall apply this theorem to the Navier-Stokes equations. We begin with the continuous problem. Let B_A denote the subspace of all functions g in $H_A^{(1-\alpha)/2}(\partial\Omega)^2$ satisfying (I.16) and (II.26). With the Stokes problem, we associate the operators T_A and $\tilde{T}_A$ respectively from $H_A^{-1}(\Omega)^2$ into $H_{A,0}^1(\Omega)^2$ and from $H_A^{-1}(\Omega)^2 \times B_A$ into $H_A^1(\Omega)^2$: for any f in $H_A^{-1}(\Omega)^2$, $T_A f$ is equal to the function u, where (u,p) is the solution of (I.3)(I.4) in $H_{A,0}^1(\Omega)^2 \times L_{A,0}^2(\Omega)$; for any (f,g) in $H_A^{-1}(\Omega)^2 \times B_A$, $\tilde{T}_A(f,g)$ is equal to the function u, where (u,p) is the solution of (I.3)(I.2) in $H_A^1(\Omega)^2 \times L_{A,0}^2(\Omega)$. Of course, for any f in $H_A^{-1}(\Omega)^2$, $\tilde{T}_A(f,0)$ coincides with $T_A f$.

Next, we consider the nonlinear term. We fix a function f in $H_A^{-1}(\Omega)^2$ and a function g in B_A , and we define the following mappings

(IV.3) $\qquad G(w) = \sum_{i=1}^2 \partial(w_i w)/\partial x_i - f$ and $\tilde{G}(w) = (G(w) , -g)$.

Clearly, the Navier-Stokes equations (I.1)(I.4) have the following equivalent formulation : *Find a function* **u** *in* $H^1_{A,0}(\Omega)^2$ *such that*

$(IV.4)_A$ $\mathbf{u} + T_A G(\mathbf{u}) = 0$.

The Navier-Stokes equations (I.1)(I.2) have the following equivalent formulation : *Find a function* **u** *in* $H^1_A(\Omega)^2$ *such that*

$(IV.5)_A$ $\mathbf{u} + \tilde{T}_A \tilde{G}(\mathbf{u}) = 0$.

To check that these problems are well-posed, we need the

<u>Lemma IV.1</u> : *For any* f *in* $H^{-1}_A(\Omega)^2$, *the mapping* **G** *is of class* $\mathcal{C}^\infty$ *from* $H^1(\Omega)^2$ *into* $H^{-1}_A(\Omega)^2$ *and from* $H^1_\omega(\Omega)^2$ *into* $H^{-1}_A(\Omega)^2$. *Furthermore, for any* **w** *in* $H^1_A(\Omega)^2$, *the operator* $DG(\mathbf{w})$ *is compact from* $H^1_A(\Omega)^2$ *into* $H^{-1}_A(\Omega)^2$.

Proof : Since the space $H^1_\omega(\Omega)$ is contained in $H^1(\Omega)$, due to (I.7) and (I.14), it suffices to prove that the mapping **G** is of class $\mathcal{C}^\infty$ from $H^1(\Omega)^2$ into $H^{-1}_\omega(\Omega)^2$. For any **u** and **w** in $H^1(\Omega)^2$, we have

$$\forall \mathbf{v} \in H^1_{\omega,0}(\Omega)^2, \quad |\sum^2_{i=1} \int_\Omega (\partial(u_i\mathbf{w})/\partial x_i) \, \mathbf{v}\omega \, d\mathbf{x}| = |\sum^2_{i=1} \int_\Omega (u_i\mathbf{w})(\partial(\mathbf{v}\omega)/\partial x_i) \, d\mathbf{x}|.$$

Since the mapping : $\varphi \to (\int_\Omega |\mathbf{grad}\,(\varphi\omega)|^2 \, \omega^{-1} \, d\mathbf{x})^{1/2}$ is a norm on $H^1_{\omega,0}(\Omega)$ equivalent to the usual one [BM1, Lemma III.2], we derive

$(IV.6)$ $\forall \mathbf{v} \in H^1_{\omega,0}(\Omega)^2, \quad |\sum^2_{i=1} \int_\Omega (\partial(u_i\mathbf{w})/\partial x_i) \, \mathbf{v}\omega \, d\mathbf{x}| \leqslant c \sum^2_{i=1} \|u_i w_j\|_{0,\omega,\Omega} \|\mathbf{v}\|_{1,\omega,\Omega}.$

We recall [LM, Thm 4.1][BM1, Lemma III.1] the imbedding of $H^{1/2}(\Omega)$ into $L^2_\omega(\Omega)$. Moreover, using the Calderón extension theorem [A, Thm 4.32] together with [G, Thm 1.4.4.2], we know that the mapping : $(\varphi,\psi) \to \varphi\psi$ is bilinear continuous from $H^1(\Omega) \times H^1(\Omega)$ into $H^{1-\varepsilon}(\Omega)$ for any $\varepsilon > 0$. Hence we have for $0 < \varepsilon \leqslant 1/2$

$(IV.7)$ $\|u_i w_j\|_{0,\omega,\Omega} \leqslant c \|u_i w_j\|_{1/2,\Omega} \leqslant c' \|u_i w_j\|_{1-\varepsilon,\Omega} \leqslant c'' \|u_i\|_{1,\Omega} \|w_j\|_{1,\Omega}$.

From (IV.6) and (IV.7), we obtain

$$\forall \mathbf{v} \in H^1_{\omega,0}(\Omega)^2, \quad |\sum^2_{i=1} \int_\Omega (\partial(u_i\mathbf{w})/\partial x_i) \, \mathbf{v}\omega \, d\mathbf{x}| \leqslant c \|\mathbf{u}\|_{1,\Omega} \|\mathbf{w}\|_{1,\Omega} \|\mathbf{v}\|_{1,\omega,\Omega}$$

Then, it is an easy matter to derive from (IV.3) that **G** is of class $\mathcal{C}^\infty$ from $H^1(\Omega)^2$ into $H^{-1}_\omega(\Omega)^2$. The compactness of $DG(\mathbf{w})$ from $H^1(\Omega)^2$ into $H^{-1}_\omega(\Omega)^2$, follows from the previous lines and from the compactness of the imbedding $H^{1-\varepsilon}(\Omega) \subset H^{1/2}(\Omega)$, $0 \leqslant \varepsilon < 1/2$.

<u>Corollary IV.1</u> : *For any* f *in* $H^{-1}_A(\Omega)^2$, *problem* (I.1)(I.4) *has at least a solution* (**u**,p) *in* $H^1_{A,0}(\Omega)^2 \times L^2_{A,0}(\Omega)$. *For any* (**f**,**g**) *in* $H^{-1}_A(\Omega)^2 \times B_A$, *problem* (I.1)(I.2) *has at least a solution* (**u**,p) *in* $H^1_A(\Omega)^2 \times L^2_{A,0}(\Omega)$.

Proof : In the Legendre case, the corollary states a well-known result [GR, Chapter IV, Thms 2.1 and 2.3]. Next, in the Chebyshev case, since the space $H_\omega^{-1}(\Omega)$ is contained in $H^{-1}(\Omega)$ and the space B_C is contained in B_L, there exists at least a pair $(\mathbf{u},p)$ in $H^1(\Omega)^2 \times L_0^2(\Omega)$ solution of problem (I.1)(I.4) (resp. (I.1)(I.2)). From Lemma IV.1, $\mathbf{G}(\mathbf{u})$ is an element of $H_\omega^{-1}(\Omega)^2$. Let $(\mathbf{u}',p')$ be the solution in $H_\omega^1(\Omega)^2 \times L_{\omega,0}^2(\Omega)$ of the Stokes problem with data $- \mathbf{G}(\mathbf{u})$. Then, both $(\mathbf{u},p)$ and $(\mathbf{u}',p')$ are solutions in $H_0^1(\Omega)^2 \times L^2(\Omega)$ of the Stokes problem with data $- \mathbf{G}(\mathbf{u})$; the uniqueness of the solution of the Stokes problem implies that $\mathbf{u}$ and $\mathbf{u}'$ coincide, and that $p-p'$ is constant, equal to $(1/\pi) \int_\Omega p(\mathbf{x})\, \omega(\mathbf{x})\, d\mathbf{x}$. We see that $(\mathbf{u},p')$ belongs in fact to $H_\omega^1(\Omega)^2 \times L_{\omega,0}^2(\Omega)$ and is a solution of (I.1)(I.4) (resp. (I.1)(I.2)).

We state a last property of the continuous problem. It is interesting here to note that, since the second argument in $\tilde{\mathbf{G}}$ is constant, the operators $1 + T_A D\mathbf{G}(\mathbf{u})$ and $1 + \tilde{T}_A D\tilde{\mathbf{G}}(\mathbf{u})$ coincide on $H_{A,0}^1(\Omega)^2$.

Lemma IV.2 : *For any real number $q > 2$, there exists a constant $c(q,\nu)$ such that, if a solution $(\mathbf{u},p)$ of problem (I.1)(I.4) (resp. (I.1)(I.2)) satisfies*

(IV.8) $\|\mathbf{u}\|_{L^q(\Omega)} \leqslant c(q,\nu)$,

the operator $1 + T_A D\mathbf{G}(\mathbf{u})$ is an isomorphism of $H_{A,0}^1(\Omega)^2$ (resp. the operator $1 + \tilde{T}_A D\tilde{\mathbf{G}}(\mathbf{u})$ is an isomorphism of $H_A^1(\Omega)^2$).

Proof : By the compactness result of Lemma IV.1, the operator $1 + T_A D\mathbf{G}(\mathbf{u})$ is an isomorphism of $H_{A,0}^1(\Omega)^2$ and the operator $1 + \tilde{T}_A D\tilde{\mathbf{G}}(\mathbf{u})$ is an isomorphism of $H_A^1(\Omega)^2$ if and only if they are injective, i.e. the only solution $(\mathbf{w},r)$ in $H_{A,0}^1(\Omega)^2 \times L_{A,0}^2(\Omega)$ of the following linearized Stokes problem

$$\begin{cases} \forall\, \mathbf{v} \in H_{A,0}^1(\Omega)^2, \quad a_A(\mathbf{w},\mathbf{v}) + (\mathbf{v}, \mathbf{grad}\, r)_A + (D\mathbf{G}(\mathbf{u}).\mathbf{w}, \mathbf{v})_A = 0 \quad, \\ \forall\, q \in L_A^2(\Omega), \quad (\mathrm{div}\, \mathbf{w}, q)_A = 0 \quad, \end{cases}$$

is $(\mathbf{0},0)$. In the Legendre case, the form a_L is clearly elliptic on $H_0^1(\Omega)^2$; in the Chebyshev case, it is proven [BCM, Prop. III.2] that, for any divergence-free function $\mathbf{w}$ in $H_{\omega,0}^1(\Omega)^2$, there exists $\mathbf{v}$ in $H_{\omega,0}^1(\Omega)^2$, satisfying div $(\mathbf{v}\omega) = 0$, such that

$a_C(\mathbf{w},\mathbf{v}) \geqslant c \|\mathbf{w}\|_{1,\omega,\Omega} \|\mathbf{v}\|_{1,\omega,\Omega}$.

These properties, together with (IV.6) in the Chebyshev case, give

$\nu \|\mathbf{w}\|_{1,A,\Omega} \leqslant c \sum_{i=1}^{2} \| u_i w_j \|_{0,A,\Omega}$.

Next, in the Legendre case, using the imbedding of $H^1(\Omega)$ into any $L^s(\Omega)$, $s < +\infty$, we have at once

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$$\|u_i\,w_j\|_{0,\Omega} \leqslant c(q)\,\|u_i\|_{L^{q}(\Omega)}\,\|w_j\|_{1,\Omega} \quad ;$$

in the Chebyshev case, a similar argument leads to

$$\|u_i\,w_j\|_{0,\omega,\Omega} = \|u_i\,w_j\,\omega^{1/2}\|_{0,\Omega} \leqslant c(q)\,\|u_i\|_{L^{q}(\Omega)}\,\|w_j\,\omega^{1/2}\|_{1,\Omega} \leqslant c'(q)\,\|u_i\|_{L^{q}(\Omega)}\,\|w_j\|_{1,\omega,\Omega} \quad .$$

In both cases, we obtain

$$\nu\,\|w\|_{1,A,\Omega} \leqslant c(q)\,\|u\|_{L^{q}(\Omega)}\,\|w\|_{1,A,\Omega} \quad ,$$

and the lemma is proven for an appropriate constant $c(q,\nu)$.

In the sequel, we shall always assume that the data f belong to a space $H_A^{\sigma}(\Omega)^2$ for a real number $\sigma > 1$ and that the boundary data g_J, $J \in \mathbb{Z}/4\mathbb{Z}$, belong to a space $H_A^{\tau}(\Gamma_J)^2$ for a real number $\tau \geqslant 2$ and satisfy (I.16) and (II.26). We consider a solution u of the Navier-Stokes equations (I.1)(I.4) (resp. (I.1)(I.2)) which is nonsingular in the following sense: the operator $1 + T_A DG(u)$ is an isomorphism of $H_{A,0}^1(\Omega)^2$ (resp. the operator $1 + \tilde{T}_A D\tilde{G}(u)$ is an isomorphism of $H_A^1(\Omega)^2$); by virtue of Lemma IV.2, such solutions exist for f and g small enough ! Even in the standard Sobolev spaces, regularity results of the solution (u,p) as a consequence of the regularity of f are not easy to derive [G, §7.3], whence we shall assume in the sequel that there exists a real number $s \geqslant 1$ ($s \geqslant 3$ in the case of non-homogeneous boundary conditions) such that the velocity u belongs to $H_A^s(\Omega)^2$.

We turn now to the discrete problems $(II.16)_A$ $(II.9)_A$ and $(II.31)_A$ $(II.27)_A$. As for the exact Navier-Stokes equations, we must define the operators $T_{A,N}$ and $\tilde{T}_{A,N}$ respectively from $H_A^{-1}(\Omega)^2$ into $P_N^{\bullet}(\Omega)^2$ and from $H_A^{-1}(\Omega)^2 \times B_A$ into $P_N(\Omega)^2$: for any f in $H_A^{-1}(\Omega)^2$, $T_{A,N}f$ is equal to the function u_N, where $(u_N\,,\,p_N)$ is the solution of problem $(III.8)_A$ in $P_N^{\bullet}(\Omega)^2 \times M_N$; for any (f,g) in $H_A^{-1}(\Omega)^2 \times B_A$, $\tilde{T}_{A,N}(f,g)$ is equal to the function u_N, where $(u_N\,,\,p_N)$ is the solution of problem $(III.40)_A$ $(II.27)_A$ in $P_N(\Omega)^2 \times M_N$. As in the continuous case, for f in $H_A^{-1}(\Omega)^2$, $T_{A,N}f$ and $\tilde{T}_{A,N}(f,0)$ coincide.

Next, we consider the nonlinear term. Due to $(II.17)_A$, we need the following operator S_N^A, defined from $\mathcal{C}^0(\overline{\Omega})$ into $P_N(\Omega)$ by : for any function f in $\mathcal{C}^0(\overline{\Omega})$, $S_N^A f$ satisfies

(IV.9) $\quad \forall\,\varphi \in P_N(\Omega), \quad (S_N^A f,\varphi)_A = (f,\varphi)_{A,N} \quad .$

Then, we set for any function w in $\mathcal{C}^0(\overline{\Omega})^2$

(IV.10) $\quad G_{A,N}(w) = \sum_{i=1}^{2} S_N^A(\partial(\mathcal{J}_N^A(w_i w))/\partial x_i - f)$ and $\tilde{G}_{A,N}(w) = (\,G_{A,N}(w)\,,\,-g\,) \quad .$

This definition is equivalent to

(IV.11) $\quad \forall\,v_N \in X_N, \quad (G_{A,N}(w),v_N)_A = (\sum_{i=1}^{2} \partial(\mathcal{J}_N^A(w_i w))/\partial x_i\,,\,v_N)_{A,N} - (f,v_N)_{A,N} \quad .$

Finally, problem $(II.16)_A$ $(II.9)_A$ has the following equivalent formulation : *Find a polynomial u_N in $P_N^{\bullet}(\Omega)^2$ such that*

$(IV.12)_A$ $\quad u_N + T_{A,N} G_{A,N}(u_N) = 0$.

Problem $(II.31)_A$ $(II.27)_A$ has the following equivalent formulation : *Find a polynomial u_N in $P_N(\Omega)^2$ such that*

$(IV.13)_A$ $\quad u_N + \tilde{T}_{A,N} \tilde{G}_{A,N}(u_N) = 0$.

Our purpose is now to study these two discrete problems by applying Theorem IV.1 respectively to the mappings $F_N = 1 + T_{A,N} G_{A,N}$ defined from $Z_N = P_N^*(\Omega)^2$ into itself and $\tilde{F}_N = 1 + \tilde{T}_{A,N} \tilde{G}_{A,N}$ defined from $\tilde{Z}_N = P_N(\Omega)^2$ into itself.

We know from [M2, Chap. 2, Thm III.2][BM, Thm IV.5] that there exist a projection operator $\overline{\Pi}_N^A$ from $H_A^1(\Omega)$ onto $P_N(\Omega)$ such that the following estimate is valid for any real number $s \geqslant 1$,

$(IV.14)$ $\quad \forall \varphi \in H_A^s(\Omega), \quad \|\varphi - \overline{\Pi}_N^A \varphi\|_{1,A,\Omega} + N \|\varphi - \overline{\Pi}_N^A \varphi\|_{0,A,\Omega} \leqslant c N^{1-s} \|\varphi\|_{s,A,\Omega}$,

and a projection operator $\overline{\Pi}_{N,0}^A$ from $H_{A,0}^1(\Omega)$ onto $P_N^*(\Omega)$ such that the following estimate is valid for any real number $s \geqslant 1$,

$(IV.15)$ $\quad \left| \begin{array}{l} \forall \varphi \in H_{A,0}^1(\Omega) \cap H_A^s(\Omega), \\ \\ \|\varphi - \overline{\Pi}_{N,0}^A \varphi\|_{1,A,\Omega} + N \|\varphi - \overline{\Pi}_{N,0}^A \varphi\|_{0,A,\Omega} \leqslant c N^{1-s} \|\varphi\|_{s,A,\Omega} \end{array} \right.$.

Let us denote by N' the integral part of $(N-1)/2$. We choose u_N^* equal to $\overline{\Pi}_{N',0}^A u$ in the case of homogeneous boundary conditions and to $\overline{\Pi}_{N'}^A \cdot u$ in the case of inhomogeneous ones (this definition of u_N^* seems very complicated, but the fact that $u_{N,i} u_N^*$ belongs to $P_{N-1}(\Omega)^2$ will make the estimates more straightforward, as it will appear later). Due to $(IV.14)$ and $(IV.15)$, we note that

$(IV.16)$ $\quad \|u - u_N^*\|_{1,A,\Omega} + N \|u - u_N^*\|_{0,A,\Omega} \leqslant c N^{1-s} \|u\|_{s,A,\Omega}$.

The computation of the corresponding constants ε_N and $\tilde{\varepsilon}_N$, γ_N and $\tilde{\gamma}_N$, $\Lambda_N(\eta)$ and $\tilde{\Lambda}_N(\eta)$ will be achieved in the next paragraph.

IV.2. Technical results.

We begin by stating some results about the linear operators $T_{A,N}$ and $\tilde{T}_{A,N}$.

Proposition IV.1 : *For any f in $H_A^{-1}(\Omega)^2$, the operator $T_{A,N}$ satisfies*

$(IV.17)$ $\quad \|T_{A,N} f\|_{1,A,\Omega} \leqslant c \sup_{v_N \in P_N^*(\Omega)^2} \dfrac{(f, v_N)_A}{\|v_N\|_{1,A,\Omega}}$,

and

$(IV.18)$ $\quad \lim_{N \to \infty} \|(T_A - T_{A,N}) f\|_{1,A,\Omega} = 0$.

Moreover, if the solution $T_A f$ belongs to $H_A^s(\Omega)^2$ for a real number $s \geqslant 1$, it satisfies the estimate

(IV.19) $\|(T_A - T_{A,N})f\|_{1,A,\Omega} \leq c\,N^{1-s}\|T_A f\|_{s,A,\Omega}$;

If the boundary data g_J , $J \in \mathbb{Z}/4\mathbb{Z}$, belong to $H_A^\tau(\Gamma_J)^2$ for a real number $\tau \geq 2$ and if the solution $\tilde{T}_A(f,g)$ belongs to $H_A^s(\Omega)^2$ for a real number $s \geq 3$, it satisfies the estimate

(IV.20) $\|(\tilde{T}_A - \tilde{T}_{A,N})(f,g)\|_{1,A,\Omega} \leq c\,(\,N^{1-s}\|\tilde{T}_A(f,g)\|_{s,A,\Omega} + N^{7/2-\tau}\sum_{J \in \mathbb{Z}/4\mathbb{Z}}\|g_J\|_{\tau,A,\Gamma_J}\,)$.

Proof : By Proposition III.1, we obtain at once

$$\|T_{A,N}f\|_{1,A,\Omega} \leq c \sup_{v_N \in K_{1A,N}} \frac{a_{A,N}(T_{A,N}f\,,\,v_N)}{\|v_N\|_{1,A,\Omega}} \leq c' \sup_{v_N \in P_N^\bullet(\Omega)^2} \frac{(f,v_N)_A}{\|v_N\|_{1,A,\Omega}} \quad ,$$

which is (IV.17). Next, due to the definition of the operators $T_{A,N}$ and $\tilde{T}_{A,N}$, the estimates. (IV.19) and (IV.20) have already been stated in (III.9) and (III.41) respectively. Finally, (IV.18) holds by classical arguments using (IV.19) and the density of $\mathcal{D}(\Omega)$ into $H_{A,0}^1(\Omega)$.

In order to estimate the nonlinear term, we need the following lemma.

Lemma IV.3 : For any real number $\varepsilon > 0$, there exists a constant c such that , for any φ_N and ψ_N in $P_N(\Omega)$, the following estimate is satisfied

(IV.21) $\|(Id - \mathcal{J}_N^A)(\varphi_N\psi_N)\|_{0,A,\Omega} + \|(Id - \pi_{N-1}^A)(\varphi_N\psi_N)\|_{0,A,\Omega} \leq c\,N^{\varepsilon-1}\|\varphi_N\|_{1,A,\Omega}\|\psi_N\|_{1,A,\Omega}$.

Proof : Recalling that N' stands for the integral part of $(N-1)/2$, we write

$$(Id - \mathcal{J}_N^A)(\varphi_N\psi_N) = (Id - \mathcal{J}_N^A)\,[\,(\varphi_N - \bar{\pi}_{N'}^A\cdot\varphi_N)(\psi_N - \bar{\pi}_{N'}^A\cdot\psi_N) + \bar{\pi}_{N'}^A\cdot\varphi_N(\psi_N - \bar{\pi}_{N'}^A\cdot\psi_N)$$
$$+ (\varphi_N - \bar{\pi}_{N'}^A\cdot\varphi_N)\bar{\pi}_{N'}^A\cdot\psi_N\,] \quad ,$$

so that

$$\|(Id - \mathcal{J}_N^A)(\varphi_N\psi_N)\|_{0,A,\Omega} \leq \|(\varphi_N - \bar{\pi}_{N'}^A\cdot\varphi_N)(\psi_N - \bar{\pi}_{N'}^A\cdot\psi_N)\|_{0,A,\Omega}$$
$$+ \|\bar{\pi}_{N'}^A\cdot\varphi_N(\psi_N - \bar{\pi}_{N'}^A\cdot\psi_N)\|_{0,A,\Omega} + \|(\varphi_N - \bar{\pi}_{N'}^A\cdot\varphi_N)\bar{\pi}_{N'}^A\cdot\psi_N\|_{0,A,\Omega}$$
$$+ \|\mathcal{J}_N^A[(\varphi_N - \bar{\pi}_{N'}^A\cdot\varphi_N)(\psi_N - \bar{\pi}_{N'}^A\cdot\psi_N)]\|_{0,A,\Omega} + \|\mathcal{J}_N^A[\bar{\pi}_{N'}^A\cdot\varphi_N(\psi_N - \bar{\pi}_{N'}^A\cdot\psi_N)]\|_{0,A,\Omega}$$
$$+ \|\mathcal{J}_N^A[(\varphi_N - \bar{\pi}_{N'}^A\cdot\varphi_N)\bar{\pi}_{N'}^A\cdot\psi_N]\|_{0,A,\Omega} \quad .$$

This implies

$$\|(Id - \mathcal{J}_N^A)(\varphi_N\psi_N)\|_{0,A,\Omega} \leq c\,(\,\|\varphi_N - \bar{\pi}_{N'}^A\cdot\varphi_N\|_{L^\infty(\Omega)}\|\psi_N - \bar{\pi}_{N'}^A\cdot\psi_N\|_{0,A,\Omega}$$
$$+ \|\bar{\pi}_{N'}^A\cdot\varphi_N\|_{L^\infty(\Omega)}\|\psi_N - \bar{\pi}_{N'}^A\cdot\psi_N\|_{0,A,\Omega} + \|\varphi_N - \bar{\pi}_{N'}^A\cdot\varphi_N\|_{0,A,\Omega}\|\bar{\pi}_{N'}^A\cdot\psi_N\|_{L^\infty(\Omega)}\,)$$
$$\leq c\,(\,\|\varphi_N\|_{L^\infty(\Omega)} + \|\bar{\pi}_{N'}^A\cdot\varphi_N\|_{L^\infty(\Omega)})\|\psi_N - \bar{\pi}_{N'}^A\cdot\psi_N\|_{0,A,\Omega}$$
$$+ \|\varphi_N - \bar{\pi}_{N'}^A\cdot\varphi_N\|_{0,A,\Omega}\|\bar{\pi}_{N'}^A\cdot\psi_N\|_{L^\infty(\Omega)} \quad .$$

Using the imbedding of $H_A^{1+\varepsilon/2}(\Omega)$ into $L^\infty(\Omega)$ for any $\varepsilon > 0$ and (IV.14), we obtain for any $s \geq 1$

(IV.22) $\|(Id - \mathcal{J}_N^A)(\varphi_N\psi_N)\|_{0,A,\Omega} \leq c\,N^{-s}\,[(\,\|\varphi_N\|_{1+\varepsilon/2,A,\Omega} + \|\bar{\pi}_{N'}^A\cdot\varphi_N\|_{1+\varepsilon/2,A,\Omega})\|\psi_N\|_{s,A,\Omega}$

$$+ \|\varphi_N\|_{s,A,\Omega} \|\overline{\pi}_N^A \cdot \psi_N\|_{1+\varepsilon/2,A,\Omega}] \quad .$$

Consequently, we derive the estimate of $\|(\mathrm{Id}-\mathcal{J}_N^A)(\varphi_N\psi_N)\|_{0,A,\Omega}$ as an easy consequence of the inverse inequality [CQ1, Lemmas 2.1 and 2.4], valid for any integer m and any real number $r, 0 \leqslant m \leqslant r$,

(IV.23) $\quad \forall \varphi_N \in P_N(\Omega), \quad \|\varphi_N\|_{r,A,\Omega} \leqslant c \, N^{2(r-m)} \|\varphi_N\|_{m,A,\Omega} \quad .$

The term $\|(\mathrm{Id}-\pi_{N-1}^A)(\varphi_N\psi_N)\|_{0,A,\Omega}$ is estimated exactly in the same way.

We can now state the following result.

Proposition IV.2 : *For N large enough , the operator* $DF_N(u_N^*) = 1 + T_{A,N}DG_{A,N}(u_N^*)$ *is an isomorphism of* $P_N^\bullet(\Omega)^2$, *and* γ_N *is bounded by a constant* γ *independent of N. For N large enough , the operator* $D\tilde{F}_N(u_N^*) = 1 + \tilde{T}_{A,N}D\tilde{G}_{A,N}(u_N^*)$ *is an isomorphism of* $P_N(\Omega)^2$, *and* $\tilde{\gamma}_N$ *is bounded by a constant* $\tilde{\gamma}$ *independent of N.*

Proof : We write $DF_N(u_N^*)$ and $D\tilde{F}_N(u_N^*)$ in the form

(IV.24) $\quad DF_N(u_N^*) = [1 + T_A DG(u)] - (T_A - T_{A,N})DG(u) - T_{A,N}(DG(u)-DG(u_N^*))$
$$- T_{A,N}(DG-DG_{A,N})(u_N^*) \quad ,$$

and (since the second arguments in $\tilde{G}$ and $\tilde{G}_{A,N}$ are constant)

(IV.25) $\quad D\tilde{F}_N(u_N^*) = [1 + \tilde{T}_A D\tilde{G}(u)] - (T_A - T_{A,N})DG(u) - T_{A,N}(DG(u)-DG(u_N^*))$
$$- T_{A,N}(DG-DG_{A,N})(u_N^*) \quad .$$

Since the operator $1 + T_A DG(u)$ is an isomorphism of $H_{A,0}^1(\Omega)^2$ and the operator $1 + \tilde{T}_A D\tilde{G}(u)$ is an isomorphism of $H_A^1(\Omega)^2$, there exists a constant c_0 independent of N such that, for any w_N in $P_N^\bullet(\Omega)^2$,

(IV.26) $\quad \|[1 + T_A DG(u)].w_N\|_{1,A,\Omega} \geqslant c_0 \|w_N\|_{1,A,\Omega} \quad ,$

and, for any w_N in $P_N(\Omega)^2$,

(IV.27) $\quad \|[1 + \tilde{T}_A D\tilde{G}(u)].w_N\|_{1,A,\Omega} \geqslant c_0 \|w_N\|_{1,A,\Omega} \quad .$

It remains to bound the three other terms in (IV.24) and (IV.25). Let w_N be any polynomial in X_N .

1) It follows from (IV.18) and from the compactness of the operator $DG(u)$ (see Lemma IV.1) that

$$\lim_{N \to \infty} \|(T_A - T_{A,N})DG(u)\|_{\mathcal{L}(H_A^1(\Omega)^2, H_A^1(\Omega)^2)} = 0 \quad .$$

Hence, for N large enough, one has

(IV.28) $\quad \|(T_A - T_{A,N})DG(u).w_N\|_{1,A,\Omega} \leqslant (c_0/4) \|w_N\|_{1,A,\Omega} \quad .$

2) It follows from (IV.17) and from the continuity of the operator DG (see Lemma IV.1)

that

$$\|T_{A,N}(DG(u)-DG(u_N^*))\|_{\mathcal{L}(H_A^1(\Omega)^2, H_A^1(\Omega)^2)} \leqslant c \, \|DG(u)-DG(u_N^*)\|_{\mathcal{L}(H_A^1(\Omega)^2, H_A^{-1}(\Omega)^2)}$$

$$\leqslant c' \, \|u-u_N^*\|_{1,A,\Omega} \quad .$$

From (IV.16) together with a density argument, we infer the convergence of u_N^* to u, whence, for N large enough,

(IV.29) $\quad \|T_{A,N}(DG(u)-DG(u_N^*)).w_N\|_{1,A,\Omega} \leqslant (c_0/4) \, \|w_N\|_{1,A,\Omega}$.

3) We recall that Π_N^A denotes the orthogonal projection operator from $L_A^2(\Omega)$ onto $P_N(\Omega)$ and we note that, for any φ_N in $P_{N-1}(\Omega)$, $S_N^A \varphi_N$ is equal to φ_N (see (IV.9)). Thus, by (IV.3) and (IV.10), we know that, for any v_N in $P_N^\bullet(\Omega)^2$,

$$((DG-DG_{A,N})(u_N^*).w_N \,, v_N)$$

$$= \sum_{i=1}^2 ((\partial/\partial x_i)(Id-\Pi_{N-1}^A)(w_{Ni}u_N^* + u_{Ni}^* w_N), v_N)_A$$

$$- \sum_{i=1}^2 ((\partial/\partial x_i)(\mathcal{J}_N^A - \Pi_{N-1}^A)(w_{Ni}u_N^* + u_{Ni}^* w_N), v_N)_{A,N} \quad ,$$

whence, by (IV.17),

$$\| T_{A,N}(DG-DG_{A,N})(u_N^*).w_N \|_{1,A,\Omega}$$

$$\leqslant c \sum_{i=1}^2 (\, \|(Id-\mathcal{J}_N^A)(w_{Ni}u_N^*) \|_{0,A,\Omega} + \|(Id-\Pi_{N-1}^A)(w_{Ni}u_N^*) \|_{0,A,\Omega}$$

$$+ \|(Id-\mathcal{J}_N^A)(u_{Ni}^* w_N) \|_{0,A,\Omega} + \|(Id-\Pi_{N-1}^A)(u_{Ni}^* w_N) \|_{0,A,\Omega}) \quad .$$

From this estimate together with Lemma IV.3, we derive

$$\| T_{A,N}(DG-DG_{A,N})(u_N^*).w_N \|_{1,A,\Omega} \leqslant c \, N^{\epsilon-1} \, \|w_N\|_{1,A,\Omega} \, \|u_N^*\|_{1,A,\Omega} \quad ,$$

whence

(IV.30) $\quad \| T_{A,N}(DG-DG_{A,N})(u_N^*).w_N \|_{1,A,\Omega} \leqslant c \, N^{\epsilon-1} \, \|u\|_{1,A,\Omega} \, \|w_N\|_{1,A,\Omega}$.

Finally, we conclude from (IV.24) to (IV.30) that, for N large enough,

$$\forall \, w_N \in P_N^\bullet(\Omega)^2, \quad \|DF_N(u_N^*).w_N\|_{1,A,\Omega} \geqslant (c_0/4) \, \|w_N\|_{1,A,\Omega} \quad ,$$

and

$$\forall \, w_N \in P_N(\Omega)^2, \quad \|D\tilde{F}_N(u_N^*).w_N\|_{1,A,\Omega} \geqslant (c_0/4) \, \|w_N\|_{1,A,\Omega} \quad ,$$

which proves the proposition.

<u>Lemma IV.4</u> : *The constants* $\Lambda_N(\eta)$ *and* $\tilde{\Lambda}_N(\eta)$ *satisfy*

(IV.31) $\quad \Lambda_N(\eta) \leqslant c \, \eta \quad$ *and* $\quad \tilde{\Lambda}_N(\eta) \leqslant c \, \eta \quad .$

Proof : Let w_N be any element in X_N . We have

$$\|T_{A,N}(DG_{A,N}(w_N-u_N^*))\|_{\mathcal{L}(X_N, X_N)}$$

$$\leqslant \|T_{A,N}(DG(w_N-u_N^*))\|_{\mathcal{L}(X_N, X_N)} + \|T_{A,N}(DG-DG_{A,N})(w_N-u_N^*)\|_{\mathcal{L}(X_N, X_N)} \quad .$$

Using (IV.17) and the continuity of the operator DG (see Lemma IV.1) yields that

$$\|T_{A,N}(DG(w_N-u_N^*))\|_{\mathscr{L}(X_N,X_N)} \leqslant c\,\|DG(w_N-u_N^*)\|_{\mathscr{L}(X_N,H_A^{-1}(\Omega)^2)} \leqslant c'\,\|w_N-u_N^*\|_{1,A,\Omega} \quad .$$

On the other hand, we know that, for any z_N in $P_N(\Omega)^2$,

$$\|T_{A,N}(DG-DG_{A,N})(w_N-u_N^*).z_N\|_{1,A,\Omega}$$
$$\leqslant c\sum_{i=1}^{2}[\|(Id-\mathcal{J}_N^A)[(w_{Ni}-u_{Ni}^*)z_N]\|_{0,A,\Omega} + \|(Id-\pi_{N-1}^A)[(w_{Ni}-u_{Ni}^*)z_N]\|_{0,A,\Omega}$$
$$+ \|(Id-\mathcal{J}_N^A)[z_{Ni}(w_N-u_N^*)]\|_{0,A,\Omega} + \|(Id-\pi_{N-1}^A)[z_{Ni}(w_N-u_N^*)]\|_{0,A,\Omega}] \quad ,$$

so that, by Lemma IV.3,

$$\|T_{A,N}(DG-DG_{A,N})(w_N-u_N^*)\|_{\mathscr{L}(X_N,X_N)} \leqslant c\,N^{\epsilon-1}\,\|w_N-u_N^*\|_{1,A,\Omega} \quad .$$

These two inequalities, together with the definition (IV.1) of $\Lambda_N(\eta)$ and $\tilde{\Lambda}_N(\eta)$, imply (IV.31).

Lemma IV.5 : *The constants ϵ_N and $\tilde{\epsilon}_N$ satisfy*

$$(IV.32) \quad \epsilon_N \leqslant c(u)\,N^{1-s} + c(f)\,N^{1+2\alpha-\sigma} \quad and \quad \tilde{\epsilon}_N \leqslant c(u)\,N^{1-s} + c(f)\,N^{1+2\alpha-\sigma} + c(g)\,N^{7/2-\tau} \quad .$$

Proof : Using $(IV.4)_A$, we write $F_N(u_N^*)$ in the form

$$F_N(u_N^*) = u_N^* + T_{A,N}\,G_{A,N}(u_N^*) - u - T_A\,G(u)$$
$$= (u_N^*-u) + (T_{A,N}-T_A)G(u) + T_{A,N}(G(u_N^*)-G(u)) + T_{A,N}(G_{A,N}(u_N^*)-G(u_N^*))$$

which gives

$$(IV.33) \quad \epsilon_N \leqslant \|u-u_N^*\|_{1,A,\Omega} + \|(T_A-T_{A,N})G(u)\|_{1,A,\Omega} + \|T_{A,N}(G(u)-G(u_N^*))\|_{1,A,\Omega}$$
$$+ \|T_{A,N}(G(u_N^*)-G_{A,N}(u_N^*))\|_{1,A,\Omega} \quad .$$

It remains to estimate these four terms.

1) Using (IV.16) yields

$$(IV.34) \quad \|u-u_N^*\|_{1,A,\Omega} \leqslant c\,N^{1-s}\,\|u\|_{s,A,\Omega} \quad .$$

2) It follows from (IV.19) that

$$\|(T_A-T_{A,N})G(u)\|_{1,A,\Omega} \leqslant c\,N^{1-s}\,\|T_A G(u)\|_{s,A,\Omega} \quad ,$$

whence, thanks to $(IV.4)_A$,

$$(IV.35) \quad \|(T_A-T_{A,N})G(u)\|_{1,A,\Omega} \leqslant c\,N^{1-s}\,\|u\|_{s,A,\Omega} \quad .$$

3) Due to (IV.17) and to the continuity of G (see Lemma IV.1), we have

$$\|T_{A,N}(G(u)-G(u_N^*))\|_{1,A,\Omega} \leqslant c\,\|u-u_N^*\|_{1,A,\Omega} \quad ,$$

so that

$$(IV.36) \quad \|T_{A,N}(G(u)-G(u_N^*))\|_{1,A,\Omega} \leqslant c\,N^{1-s}\,\|u\|_{s,A,\Omega} \quad .$$

4) From (IV.17), we derive

$$\|T_{A,N}(G(u_N^*)-G_{A,N}(u_N^*))\|_{1,A,\Omega} \leqslant c \sup_{v_N \in P_N^*(\Omega)^2} \frac{((G(u_N^*)-G_{A,N}(u_N^*),v_N)_A}{\|v_N\|_{1,A,\Omega}} .$$

Using the definitions (IV.3) and (IV.10) of G and $G_{A,N}$, we have for any v_N in $P_N^*(\Omega)^2$, in the Legendre case,

$$(G(u_N^*)-G_{L,N}(u_N^*),v_N)$$
$$= \sum_{i=1}^{2} [(u_{Ni}^* u_N^*, \partial v_N/\partial x_i)_N - (u_{Ni}^* u_N^*, \partial v_N/\partial x_i)] - (f,v_N) + (f,v_N)_N ,$$

and, in the Chebyshev case,

$$(G(u_N^*)-G_{C,N}(u_N^*),v_N)_\omega$$
$$= \sum_{i=1}^{2} [(u_{Ni}^* u_N^*, (\partial(\omega v_N)/\partial x_i)\, \omega^{-1})_{\omega,N} - (u_{Ni}^* u_N^*,(\partial(\omega v_N)/\partial x_i)\, \omega^{-1})_\omega]$$
$$- (f,v_N)_\omega + (f,v_N)_{\omega,N} ,$$

But, since $u_{Ni}^* u_N^*$ belongs to $P_{N-1}(\Omega)^2$, the exact and discrete scalar products coincide. Hence, we obtain from (III.37)

$$| (G(u_N^*)-G_{A,N}(u_N^*),v_N)_A | = | (f,v_N)_A - (f,v_N)_{A,N} | \leqslant c N^{1+2\alpha-\sigma} \|f\|_{\sigma,A,\Omega} \|v_N\|_{1,A,\Omega} ,$$

so that

(IV.37) $\quad \|T_{A,N}(G(u_N^*)-G_{A,N}(u_N^*))\|_{1,A,\Omega} \leqslant c N^{1+2\alpha-\sigma} \|f\|_{\sigma,A,\Omega} .$

Finally, we derive the desired estimate for ε_N from inequalities (IV.33) to (IV.37). The estimate for $\tilde{\varepsilon}_N$ can be obtained in a very similar way, by writing

$$\tilde{F}_N(u_N^*) = (u_N^*-u) + (\tilde{T}_{A,N}-\tilde{T}_A)\tilde{G}(u) + T_{A,N}(G(u_N^*)-G(u)) + T_{A,N}(G_{A,N}(u_N^*)-G(u_N^*))$$

and using (IV.20) instead of (IV.21).

IV.3. Existence result and error estimates for the velocity.

We can now prove the main results of this section.

Theorem IV.2 : *Assume that hypothesis (II.20) holds and that there exists a solution (u,p) of the Navier–Stokes equations (I.1)(I.4) such that the operator* $1 + T_A DG(u)$ *is an isomorphism of* $H_{A,0}^1(\Omega)^2$ *; assume moreover that u belongs to* $H^s(\Omega)^2$ *for a real number s > 1 and that the data f belong to* $H_\omega^\sigma(\Omega)^2$ *for a real number σ > 1. For N large enough, problem* (II.16)$_A$ (II.9)$_A$ *admits a solution* (u_N, p_N) *in* $P_N^*(\Omega)^2 \times M_N$. *Moreover, it satisfies*

(IV.38) $\quad \|u-u_N\|_{1,A,\Omega} \leqslant c(u)\, N^{1-s} + c(f)\, N^{1+2\alpha-\sigma}$

for constants c(u) and c(f) independent of N.

Theorem IV.3 : *Assume that hypothesis (II.20) holds and that there exists a solution (u,p) of the Navier–Stokes equations (I.1)(I.2) such that the operator* $1 + \tilde{T}_A D\tilde{G}(u)$ *is an isomorphism of* $H_A^1(\Omega)^2$ *; assume moreover that u belongs to* $H^s(\Omega)^2$ *for a real number*

$s \geqslant 3$, *that the data* f *belong to* $H_{\omega}^{\sigma}(\Omega)^2$ *for a real number* $\sigma > 1$ *and that the boundary data* g_J , $J \in \mathbb{Z}/4\mathbb{Z}$, *belong to* $H_A^{\tau}(\Gamma_J)^2$ *for a real number* $\tau > 7/2$. *For N large enough, problem* $(II.31)_A$ $(II.27)_A$ *admits a solution* (u_N, p_N) *in* $P_N(\Omega)^2 \times M_N$. *Moreover, it satisfies*

(IV.39) $\|u - u_N\|_{1,A,\Omega} \leqslant c(u) \, N^{1-s} + c(f) \, N^{1+2\alpha-\sigma} + c(g) \, N^{7/2-\tau}$.

for constants $c(u)$, $c(f)$ *and* $c(g)$ *independent of N.*

Proof : Using Proposition IV.2 and Lemmas IV.4 and IV.5, we notice that $2\gamma_N \, \Lambda_N(2\gamma_N \varepsilon_N)$ and $2\tilde{\gamma}_N \, \tilde{\Lambda}_N(2\tilde{\gamma}_N \tilde{\varepsilon}_N)$ are bounded respectively by $c \, \varepsilon_N$ and $c \, \tilde{\varepsilon}_N$; consequently, the assumptions of Theorem IV.1 are satisfied for N large enough. Hence, there exists a constant $c > 0$ independent of N such that, for each $\eta < c$, there is a unique solution u_N of $(IV.12)_A$ in the ball $S_N = \{ w_N \in P_N^{\bullet}(\Omega)^2 ; \|w_N - u_N^*\|_{1,A,\Omega} \leqslant \eta \}$ (resp. a unique solution u_N of $(IV.13)_A$ in the ball $\tilde{S}_N = \{ w_N \in P_N(\Omega)^2 ; \|w_N - u_N^*\|_{1,A,\Omega} \leqslant \eta \}$). Next, from (IV.2), we derive the estimate

$$\|u_N - u_N^*\|_{1,A,\Omega} \leqslant c \, \|F_N(u_N^*)\|_{1,A,\Omega} \quad (\text{resp.} \, \|u_N - u_N^*\|_{1,A,\Omega} \leqslant c \, \|\tilde{F}_N(u_N^*)\|_{1,A,\Omega}) \quad ,$$

which, together with Lemma IV.5, yields (IV.38) and (IV.39).

Next, by Proposition III.1 , there exists a unique p_N in M_N such that

$$\forall \, v_N \in P_N^{\bullet}(\Omega)^2, \quad b_{1A,N}(v_N, p_N) = - a_{A,N}(u_N, v_N) - (G_{A,N}(u_N), v_N)_A$$

and the pair (u_N, p_N) is a solution of the corresponding problem $(II.16)_A$ $(II.9)_A$ or $(II.31)_A$ $(II.27)_A$.

Remark IV.1 : The error bounds we obtain are exactly the same as for the Stokes problem ; in particular, the result is still optimal with respect to the regularity of the solution (and also of the data f when Chebyshev approximation is used).

IV.4. Error estimates for the pressure.

In order to state an error bound for the pressure, we need a lemma.

Lemma IV.6 : *The approximate velocity* u_N, *as defined in Theorem IV.2, satisfies*

(IV.40) $\displaystyle \sup_{v_N \in P_N^{\bullet}(\Omega)^2} \frac{(G(u) - G_{A,N}(u_N), v_N)_A}{\|v_N\|_{1,A,\Omega}} \leqslant c(u) \, N^{1-s} + c(f) \, N^{1+2\alpha-\sigma}$.

The approximate velocity u_N, *as defined in Theorem IV.3, satisfies*

(IV.41) $\displaystyle \sup_{v_N \in P_N^{\bullet}(\Omega)^2} \frac{(G(u) - G_{A,N}(u_N), v_N)_A}{\|v_N\|_{1,A,\Omega}} \leqslant c(u) \, N^{1-s} + c(f) \, N^{1+2\alpha-\sigma} + c(g) \, N^{7/2-\tau}$.

Proof : Let v_N be any element in $P_N^{\bullet}(\Omega)^2$. We consider only the case of homogeneous

boundary conditions, since the proof in the general case is strictly the same. We compute

$$(G(u)-G_{A,N}(u_N),v_N)_A = (G(u)-G(u_N),v_N)_A + (G(u_N)-G_{A,N}(u_N),v_N)_A \quad .$$

Lemma IV.1 and (IV.38) give at once

$$|(G(u)-G(u_N),v_N)_A| \leqslant (\,c(u)\,N^{1-s} + c(f)\,N^{1+2\alpha-\sigma}\,)\,\|v_N\|_{1,A,\Omega} \quad .$$

From the definitions (IV.3) and (IV.10) of G and $G_{A,N}$, we obtain

$$|(G(u_N)-G_{A,N}(u_N),v_N)_A|$$
$$\leqslant c\,(\,\textstyle\sum_{i=1}^{2}\,\|(Id-\mathcal{J}_N^A)(u_{Ni}u_N)\|_{0,A,\Omega} + \|(Id-\Pi_{N-1}^A)(u_{Ni}u_N)\|_{0,A,\Omega}\,)\,\|v_N\|_{1,A,\Omega}$$
$$+ |(f,v_N)_A - (f,v_N)_{A,N}| \quad .$$

Using Lemma IV.3, we know that, for $\varepsilon > 0$,

$$\|(Id-\mathcal{J}_N^A)(u_{Ni}u_N)\|_{0,A,\Omega} + \|(Id-\Pi_{N-1}^A)(u_{Ni}u_N)\|_{0,A,\Omega}$$
$$= \|(Id-\mathcal{J}_N^A)(u_{Ni}u_N-u_{Ni}^*u_N^*)\|_{0,A,\Omega} + \|(Id-\Pi_{N-1}^A)(u_{Ni}u_N-u_{Ni}^*u_N^*)\|_{0,A,\Omega}$$
$$= \|(Id-\mathcal{J}_N^A)[u_{Ni}(u_N-u_N^*) + u_N^*(u_{Ni}-u_{Ni}^*)]\|_{0,A,\Omega}$$
$$+ \|(Id-\Pi_{N-1}^A)[u_{Ni}(u_N-u_N^*) + u_N^*(u_{Ni}-u_{Ni}^*)]\|_{0,A,\Omega}$$
$$\leqslant c\,N^{\varepsilon-1}\,(\,\|u_N\|_{1,A,\Omega} + \|u_N^*\|_{1,A,\Omega}\,)\,\|u_N-u_N^*\|_{1,A,\Omega}$$
$$\leqslant c(u)\,N^{\varepsilon-1}\,(\,\|u-u_N\|_{1,A,\Omega} + \|u-u_N^*\|_{1,A,\Omega}\,) \quad .$$

Then, Theorem IV.2, (IV.16) and (III.37) yield for $\varepsilon > 0$

$$|G(u_N)-G_{A,N}(u_N),v_N)_A| \leqslant (\,c(u)\,N^{\varepsilon-s} + c\,N^{1+2\alpha-\sigma}\,\|f\|_{\sigma,A,\Omega}\,)\,\|v_N\|_{1,A,\Omega} \quad .$$

Finally, these two bounds imply (IV.40).

Theorem IV.4 : *Assume that hypotheses (II.20) and (II.21) hold and that there exists a solution* (u,p) *of the Navier-Stokes equations (I.1)(I.4) such that the operator* 1 + TDG(u) *is an isomorphism of* $H_{A,0}^1(\Omega)^2$; *assume moreover that it belongs to* $H_A^s(\Omega)^2 \times H_A^{s-1}(\Omega)$ *for a real number* s > 1 *and that the data* f *belong to* $H_A^\sigma(\Omega)^2$ *for a real number* $\sigma > 1$. *Then, the solution* (u_N,p_N) *of problem* (II.16)$_A$ (II.9)$_A$ *satisfies*

(IV.42) $\|p-p_N\|_{0,A,\Omega} \leqslant c(u,p)\,N^{3-s} + c(f)\,N^{3+2\alpha-\sigma}$

for constants c(u,p) *and* c(f) *independent of* N.

Theorem IV.5 : *Assume that hypotheses (II.20) and (II.21) hold and that there exists a solution* (u,p) *of the Navier-Stokes equations (I.1)(I.2) such that the operator* 1 + $\tilde{T}D\tilde{G}$(u) *is an isomorphism of* $H_A^1(\Omega)^2$; *assume moreover that it belongs to* $H_A^s(\Omega)^2 \times H_A^{s-1}(\Omega)$ *for a real number* $s \geqslant 3$, *that the data* f *belong to* $H_A^\sigma(\Omega)^2$ *for a real number* $\sigma > 1$ *and that the boundary data* g_J, $J \in \mathbb{Z}/4\mathbb{Z}$, *belong to* $H_A^\tau(\Gamma_J)^2$ *for a real number* $\tau > 7/2$. *Then, the solution* (u_N,p_N) *of problem* (II.31)$_A$ (II.27)$_A$ *satisfies*

(IV.43) $\|p-p_N\|_{0,A,\Omega} \leqslant c(u,p)\,N^{3-s} + c(f)\,N^{3+2\alpha-\sigma} + c(g)\,N^{11/2-\tau}$

for constants $c(\mathbf{u},p)$, $c(\mathbf{f})$ *and* $c(\mathbf{g})$ *independent of* N.

Proof : Let us introduce the solution $(\tilde{\mathbf{u}}_N, \tilde{p}_N)$ in $X_N \times M_N$ of the following problem :

$(\text{IV.44})_A$ $\left|\begin{array}{l} \forall\, \mathbf{v}_N \in P_N^\bullet(\Omega)^2, \quad a_{A,N}(\tilde{\mathbf{u}}_N, \mathbf{v}_N) + b_{1A,N}(\mathbf{v}_N, \tilde{p}_N) + (G(\mathbf{u}), \mathbf{v}_N)_A = 0 \quad, \\[2mm] \forall\, q_N \in M_{2A,N}^\perp, \quad b_{2A,N}(\tilde{\mathbf{u}}_N, q_N) = 0 \quad, \end{array}\right.$

together with the boundary conditions $(\text{II}.9)_A$ (resp. $(\text{II}.27)_A$). Since $\tilde{\mathbf{u}}_N$ is equal to $-\,T_{A,N}G(\mathbf{u})$ (resp. $-\,\tilde{T}_{A,N}\tilde{G}(\mathbf{u})$), we deduce from (IV.19) (resp. (IV.20)) that

$(\text{IV.45})\quad \|\mathbf{u}-\tilde{\mathbf{u}}_N\|_{1,A,\Omega} \leqslant c(\mathbf{u})\, N^{1-s}\quad (\text{resp.}\ \|\mathbf{u}-\tilde{\mathbf{u}}_N\|_{1,A,\Omega} \leqslant c(\mathbf{u})\, N^{1-s} + c(\mathbf{g})\, N^{7/2-\tau})\quad;$

moreover, we obtain from (III.10) and (III.42)

$(\text{IV.46})\quad \|p-\tilde{p}_N\|_{0,A,\Omega} \leqslant c(\mathbf{u},p)\, N^{3-s}\quad (\text{resp.}\ \|p-\tilde{p}_N\|_{0,A,\Omega} \leqslant c(\mathbf{u},p)\, N^{3-s} + c(\mathbf{g})\, N^{11/2-\tau})\quad.$

Next, due to $(\text{IV.44})_A$ and $(\text{II}.17)_A$ or $(\text{II}.31)_A$, we notice that, for any $\mathbf{v}_N$ in $P_N^\bullet(\Omega)^2$,

$$b_{1A,N}(\mathbf{v}_N, p_N-\tilde{p}_N) = a_{A,N}(\tilde{\mathbf{u}}_N, \mathbf{v}_N) + (G(\mathbf{u}),\mathbf{v}_N)_A - a_{A,N}(\mathbf{u}_N, \mathbf{v}_N) - (G_{A,N}(\mathbf{u}_N),\mathbf{v}_N)_A\quad,$$

so that, from Proposition III.1 , we deduce

$(\text{IV.47})\quad \|p_N-\tilde{p}_N\|_{0,A,\Omega} \leqslant c\, N^2 \displaystyle\sup_{\mathbf{v}_N \in P_N^\bullet(\Omega)^2} \frac{a_{A,N}(\mathbf{u}_N-\tilde{\mathbf{u}}_N, \mathbf{v}_N) + (G(\mathbf{u})-G_{A,N}(\mathbf{u}_N),\mathbf{v}_N)_A}{\|\mathbf{v}_N\|_{1,A,\Omega}}\quad.$

Let $\mathbf{v}_N$ be any element in $P_N^\bullet(\Omega)^2$. By the uniform continuity of $a_{A,N}$, we have

$$a_{A,N}(\mathbf{u}_N-\tilde{\mathbf{u}}_N, \mathbf{v}_N) \leqslant c\, \|\mathbf{u}_N-\tilde{\mathbf{u}}_N\|_{1,A,\Omega}\, \|\mathbf{v}_N\|_{1,A,\Omega}$$

$$\leqslant c\, (\,\|\mathbf{u}-\tilde{\mathbf{u}}_N\|_{1,A,\Omega} + \|\mathbf{u}-\mathbf{u}_N\|_{1,A,\Omega}\,)\, \|\mathbf{v}_N\|_{1,A,\Omega}\quad,$$

so that one can bound this term from (IV.38) or (IV.39) and (IV.45). Using this estimate and Lemma IV.6 in (IV.47) yields

$(\text{IV.48})\quad \|p_N-\tilde{p}_N\|_{0,A,\Omega} \leqslant c(\mathbf{u})\, N^{3-s} + c(\mathbf{f})\, N^{3+2\alpha-\sigma}$

$\qquad\qquad (\text{resp.}\ \|p_N-\tilde{p}_N\|_{0,A,\Omega} \leqslant c(\mathbf{u})\, N^{3-s} + c(\mathbf{f})\, N^{3+2\alpha-\sigma} + c(\mathbf{g})\, N^{11/2-\tau})\quad,$

which, together with (IV.46), gives (IV.42) and (IV.43).

That ends the theoretical results which can be proven for both the Legendre and Chebyshev approximations of the Navier–Stokes equations. It remains to apply this method to real problems, as will be done in the next section.

V. Resolution algorithm and numerical results.

In this section, we describe the resolution algorithm we use for numerical applications. It has been proposed first by Y. MORCHOISNE [Mo] and is aimed at solving the time-dependent Navier-Stokes equations

(V.1) $\begin{cases} \partial u/\partial t - \nu \Delta u + \textbf{grad}\,p + (u.\nabla)u = f & \text{in } \Omega \times (0,T) \quad, T > 0 \quad, \\ \text{div } u = 0 \quad \text{in } \Omega \times (0,T) \quad, \end{cases}$

with initial condition $u(0) = u^0$ in Ω. But it can also be used to compute stationary cases as it will be shown in the following. As far as time-dependent problems are concerned, time discretization is achieved with the help of a finite difference scheme. While the convection term is handled explicitly by an Adams-Bashforth approximation, the diffusive term is implicitly treated in order to ensure stability.

Let us introduce a fixed time step $\delta t > 0$. At each time $(n+1)\,\delta t$, $n \geqslant 0$, we compute an approximation u_N^{n+1} in X_N of the velocity $u((n+1)\,\delta t)$. Furthermore, in order to make the numerical computation easier, we first compute a scalar quantity q_N^{n+1} in $P_N(\Omega)$, that we call the pseudo-pressure, such that $\textbf{grad}\,q_N^{n+1}$ is an approximation of the pressure gradient $(\textbf{grad}\,p)((n+1)\,\delta t)$. When the convergence is reached, the discrete pressure p_N is then obtained by a post-treatment which is performed by solving a Poisson problem.

Numerical applications (see § V.4) have been made with a Chebyshev spectral discretization. Thanks to this choice, we can employ the Fast Fourier Transform (FFT) in the computation of the derivatives (see [CLW][CT][O2]).

V.1. The discrete problem for velocity and pseudo-pressure.

We consider the Navier-Stokes equations (V.1)(I.2) with null right-hand side f. For a given function g satisfying the assumptions (I.15), (I.16) and (II.26), we introduce the subspace $X_N(g)$ of all polynomials in X_N satisfying the boundary conditions $(II.27)_A$.

Let $(u_N^0\,,\,q_N^0)$ be any initial quantities in $X_N(g) \times P_N(\Omega)$. We assume that $(u_N^n\,,\,q_N^n)$ is known in $X_N(g) \times P_N(\Omega)$, and we seek $(u_N^{n+1}\,,\,q_N^{n+1})$ in $X_N(g) \times P_N(\Omega)$ such that

(V.2) $\begin{cases} [L\,\dfrac{u_N^{n+1}-u_N^n}{\delta t} - \nu \Delta u_N^{n*} + \textbf{grad}\,q_N^{n+1} + (u_N^{n*}.\nabla)u_N^{n*}](x) = 0 \quad, x \in \Xi_N^A \cap \Omega \quad, \\ (\text{div } u_N^{n+1}\,,\,\text{div } u_N^{n+1})_{A,N} = \displaystyle\inf_{w_N \in X_N(g)} (\text{div } w_N\,,\,\text{div } w_N)_{A,N} \quad. \end{cases}$

In the equations (V.2), we use the following notation : for any integer $n \geqslant 1$,

$$\mathbf{u}_N^{n*} = (3/2)\ \mathbf{u}_N^n - (1/2)\ \mathbf{u}_N^{n-1} \quad .$$

Moreover, the operator L is a finite difference approximation of $\mathrm{Id} - \eta\Delta$, where η is a positive parameter. More precisely, we set $L = L_1 L_2$, where

(V.3) $L_i = \mathrm{Id} - \eta\partial_i$, $i = 1$ or 2,

and ∂_i is the second-order finite difference operator : for example, if i is equal to 1, for any function w in $\mathcal{C}^0(\overline{\Omega})$, we define for any node $\mathbf{x}_{jk}^A = (\zeta_j^A, \zeta_k^A)$ in $\Xi_N^A \cap \Omega$

$$\partial_1 w(\mathbf{x}_{jk}^A) = \frac{2}{\zeta_{j+1}^A - \zeta_{j-1}^A} \times$$

$$[\ \frac{w(\zeta_{j+1}^A, \zeta_k^A) - w(\zeta_j^A, \zeta_k^A)}{\zeta_{j+1}^A - \zeta_j^A} - \frac{w(\zeta_j^A, \zeta_k^A) - w(\zeta_{j-1}^A, \zeta_k^A)}{\zeta_j^A - \zeta_{j-1}^A}\ ] \quad .$$

The parameter η verifies

$$\eta = \beta\ \nu\ \delta t + \gamma\ V\ \delta t^2 \quad ,$$

where β and γ are two nonnegative constants, and V is an estimate of the velocity norm. Note that, if L is chosen equal to the identity (i.e., $\eta = 0$), we have an explicit Adams–Bashforth scheme. In fact, however, we choose β and γ large enough to ensure a good stability of the scheme. Indeed, this scheme, when applied to the one-dimensional Bürgers' equation, has been analysed in the periodic case; it has been proven [Mé, Chap. 1] that for η large enough, it is unconditionally stable and has a precision upper-bounded by $c(\sigma)\ (\nu\ \delta t + \delta t^2 + N^{-\sigma})$ for all real numbers $\sigma > 0$.

V.2. Velocity and pseudo-pressure computation.

Problem (V.2) is solved in two steps. First, we compute a predictor $\mathbf{u}_{N,\,P}^{n+1}$ of the velocity in $X_N^A(\mathbf{g})$. Then a corrector $(\mathbf{v}_N, q_N)$ in $P_N^*(\Omega)^2 \times P_N(\Omega)$ is computed, so that the pair $(\mathbf{u}_N^{n+1}, q_N^{n+1})$ defined by

(V.4) $\begin{cases} \mathbf{u}_N^{n+1} = \mathbf{u}_{N,\,P}^{n+1} + \mathbf{v}_N \quad , \\ q_N^{n+1} = q_N^n + q_N \end{cases}$

verifies the equations (V.2).

(i) Velocity predictor computation.

We first solve the following problem : *Find a polynomial* $\mathbf{u}_{N,\,P}^{n+1}$ *in* $X_N^A(\mathbf{g})$ *such that*

(V.5) $[L\ \dfrac{\mathbf{u}_{N,\,P}^{n+1} - \mathbf{u}_N^n}{\delta t} - \nu\ \Delta\mathbf{u}_N^{n*} + \mathbf{grad}\ q_N^n + (\mathbf{u}_N^{n*}.\nabla)\mathbf{u}_N^{n*}](\mathbf{x}) = 0 \quad , \mathbf{x} \in \Xi_N^A \cap \Omega$.

This problem can be handled with standard linear system algorithms. Indeed we can associate with each operator L_i , $i = 1$ or 2, the operator $\tilde{L}_i$ defined from $\mathcal{C}^0(\overline{\Omega})$ into $P_N(\Omega)$

by : for any function w in $\mathbb{C}^0(\overline{\Omega})$, $\tilde{L}_i w$ belongs to $P_N(\Omega)$ and satisfies

$$(V.6) \quad \begin{cases} \tilde{L}_i w(x) = L_i w(x) \quad , x \in \Xi_N^A \cap \Omega \quad , \\ \tilde{L}_i w(x) = w(x) \quad , x \in \Xi_N^A \cap \partial\Omega \quad . \end{cases}$$

From [V, Thm VI.3], we deduce that the operators $\tilde{L}_i$, $i = 1$ and 2, are (easily) invertible in $P_N(\Omega)$ for any $\eta \geqslant 0$. We set $\tilde{L} = \tilde{L}_1 \tilde{L}_2$.

Let s be in X_N, the solution w in X_N of the problem $\tilde{L}w = s$ is obtained by solving successively the two following problems : find w_1 in X_N such that $\tilde{L}_1 w_1 = s$ and find w_2 in X_N such that $\tilde{L}_2 w_2 = w_1$; thus we have $w = w_2$.

(ii) Velocity corrector and pseudo-pressure computation.

Thanks to (V.4), the pair (u_N^{n+1}, q_N^{n+1}) is the solution of (V.2) if and only if the pair (v_N, q_N) of $P_N^*(\Omega)^2 \times P_N(\Omega)$ satisfies

$$(V.7) \quad \begin{cases} (L \, \dfrac{v_N}{\delta t} + \mathbf{grad} \, q_N)(x) = 0 \quad , x \in \Xi_N^A \cap \Omega \quad , \\ (\text{div } u_{N,P}^{n+1} + \text{div } v_N , \text{div } u_{N,P}^{n+1} + \text{div } v_N)_{A,N} = \inf_{w_N \in X_N^A(g)} (\text{div } w_N , \text{div } w_N)_{A,N} \quad . \end{cases}$$

In order to solve the problem (V.7), we introduce an operator $\mathcal{A}$ from $P_N(\Omega)$ into $P_N(\Omega)$ which connects the pseudo-pressure q to div v. We first define the operator $\overline{\mathbf{grad}}$ in the following way : for any r_N in $P_N(\Omega)$, $\overline{\mathbf{grad}} \, r_N$ belongs to $P_N(\Omega)^2$ and satisfies

$$(V.8) \quad \begin{cases} (\overline{\mathbf{grad}} \, r_N)(x) = (\mathbf{grad} \, r_N)(x) \quad , x \in \Xi_N^A \cap \Omega \quad , \\ (\overline{\mathbf{grad}} \, r_N)(x) = 0 \quad , x \in \Xi_N^A \cap \partial\Omega \quad . \end{cases}$$

Then, we set, for any r_N in $P_N(\Omega)$,

$$(V.9) \quad \mathcal{A} r_N = - \delta t \, \text{div} \, (\tilde{L}^{-1} \, \overline{\mathbf{grad}} \, r_N) \quad .$$

Thus, we can consider the two following problems : Find q_N in $P_N(\Omega)$ such that

$$(V.10) \quad (\mathcal{A} q_N + \text{div } u_{N,P}^{n+1} , \mathcal{A} q_N + \text{div } u_{N,P}^{n+1})_{A,N}$$
$$= \inf_{r_N \in P_N(\Omega)} (\mathcal{A} r_N + \text{div } u_{N,P}^{n+1} , \mathcal{A} r_N + \text{div } u_{N,P}^{n+1})_{A,N}$$

and, secondly : Find v_N in $P_N^*(\Omega)^2$ such that

$$(V.11) \quad (L \, \dfrac{v_N}{\delta t} + \mathbf{grad} \, q_N)(x) = 0 \quad , x \in \Xi_N^A \cap \Omega \quad .$$

In the computations, we handle the resolution of (V.7) by solving the system (V.10)(V.11). Hence, we are going to prove that the equations (V.10) and (V.11) are equivalent to (V.7).

First, we need some results in order to prove that the minimization problem (V.10)

is equivalent to the minimization problem in (V.7).

Lemma V.1 : *If the parameter* η *is small enough, the kernel of the operator* $\mathscr{A}$ *is equal to* $Z_{1A,N}$ *and its range is equal to* $M_{2A,N}^{\perp}$.

Proof : Clearly, $Z_{1A,N}$ is contained in the kernel of $\mathscr{A}$ and the range of $\mathscr{A}$ is contained in the range of the divergence operator, hence in $M_{2A,N}^{\perp}$. Consequently, it suffices to prove that the kernel of $\mathscr{A}$ is contained in $Z_{1A,N}$, since that this would imply that it is equal to it and of dimension 8, and that the image of $\mathscr{A}$ and $M_{2A,N}^{\perp}$ have the same codimension 8 in $P_N^{\bullet}(\Omega)^2$.

Thus, let q_N be any polynomial in the kernel of $\mathscr{A}$. That implies that L^{-1} $\overline{\mathrm{grad}}\, q_N$ is divergence-free in Ω; since it belongs to $P_N^{\bullet}(\Omega)^2$, there exists a unique polynomial ψ_N in $P_{N+1}(\Omega) \cap H_{\alpha,0}^2(\Omega)$ such that

(V.12) $\quad L^{-1}\, \overline{\mathrm{grad}}\, q_N = \mathbf{curl}\, \psi_N \quad$ in Ω .

Writing the expansion of ψ_N in the form

$$\psi_N(x) = (1-x^2) \sum_{n=1}^{N} \hat{\psi}_n(y)\, (J_n^{\alpha})'(x) \quad ,$$

we obtain $(\partial(\psi_N\,(1-x^2)^{\alpha})/\partial x)(x) = \sum_{n=1}^{N} \hat{\psi}_n(y)\, n(n+2\alpha+1)\, J_n^{\alpha}(x)\, (1-x^2)^{\alpha}$, so that the degree of ψ_N with respect to y is $\leq N$. Using a similar argument for the variable x, we deduce that ψ_N belongs in fact to $P_N(\Omega)$. Next, we compute

$$(\overline{\mathrm{grad}}\, q_N \; , \mathbf{curl}\, (\psi_N(1-x^2)^{\alpha}(1-y^2)^{\alpha})\, (1-x^2)^{-\alpha}(1-y^2)^{-\alpha})_{A,N}$$

$$= (\mathrm{grad}\, q_N \; , \mathbf{curl}\cdot(\psi_N(1-x^2)^{\alpha}(1-y^2)^{\alpha})\, (1-x^2)^{-\alpha}(1-y^2)^{-\alpha})_{A,N}$$

$$= \int_{\Omega} \mathbf{grad}\, q_N \cdot \mathbf{curl}\, (\psi_N(1-x^2)^{\alpha}(1-y^2)^{\alpha})\, dx = 0 \quad .$$

That implies by (V.12)

$$(L(\mathbf{curl}\, \psi_N) \; , \mathbf{curl}\, (\psi_N(1-x^2)^{\alpha}(1-y^2)^{\alpha})\, (1-x^2)^{-\alpha}(1-y^2)^{-\alpha})_{A,N} = 0 \quad ,$$

or equivalently

$$(L(\mathbf{curl}\, \psi_N) \; , \mathbf{curl}\, (\psi_N(1-x^2)^{\alpha}(1-y^2)^{\alpha})\, (1-x^2)^{-\alpha}(1-y^2)^{-\alpha})_{A,N} = 0 \quad .$$

1) In the case $\eta = 0$, we have proven that

$$((\mathbf{curl}\, \psi_N) \; , \mathbf{curl}\, (\psi_N(1-x^2)^{\alpha}(1-y^2)^{\alpha})\, (1-x^2)^{-\alpha}(1-y^2)^{-\alpha})_{A,N} = 0 \quad .$$

From the ellipticity of this form on $P_N^{\bullet}(\Omega)$, we deduce at once that ψ_N is equal to 0, hence that $\overline{\mathrm{grad}}\, q_N$ is equal to $\mathbf{0}$, and q_N belongs to $Z_{1A,N}$.

2) In the case $\eta > 0$, denoting by $c(N)$ the norm of the operators ∂_1 and ∂_2 on the space $P_N(\Omega)$ provided with the norm $\|.\|_{1,A,\Omega}$ and writing $L - \mathrm{id} = - \eta\, \partial_1 - \eta\, \partial_2 + \eta^2\, \partial_1\partial_2$, we obtain

$$0 \geq c\, \|\psi_N\|_{1,A,\Omega}^2 - \eta\, [2\, c(N) + \eta\, c(N)^2]\, \|\psi_N\|_{1,A,\Omega}^2 \quad ,$$

whence the result for η small enough.

From the two results of this lemma, we derive respectively the two following propositions.

<u>Proposition V.1</u> : *If the parameter* η *is small enough, the system* (V.10)(V.11) *is equivalent to problem* (V.7).

<u>Proposition V.2</u> : *If the parameter* η *is small enough, the set of values* {(**grad** q_N)(**x**), **x** $\in \Xi_N^A \cap \Omega$} , *where* q_N *is a solution of problem* (V.10) , *is uniquely defined and the solution* $\mathbf{v}_N$ *of problem* (V.11) *is uniquely defined.*

In both Legendre and Chebyshev cases, the minimization problem (V.10) can be solved thanks to the Axelsson's minimization algorithm, which was aimed to problems associated with symmetrical nonnegative operators or with operators the symmetrical part of which is positive definite [Ax][J][Mé]. In our case, even if the operator does not satisfy these assumptions, the algorithm turns out to be efficient when appropriate re-initializations are used [Mé, Chap. 2, § VII.3 and Chap. 4, § IV.1].

<u>Remark V.1</u> : Note that it is rather standard to set up problems concerning pseudo-pressure, as in (V.10), by eliminating the velocity of the continuity equation. The basic idea of this procedure relies upon the Uzawa's algorithm [G1], since the pseudo-pressure plays the role of the Lagrange multiplier.

V.3. Pressure post-treatment.

Once we have reached the stationary state, i.e., the time $(n+1)$ δt when the velocity becomes independent of the integer n, we can compute the pressure.

We set

$$(V.13) \quad \mathbf{s}^{n+1} = L \frac{\mathbf{u}_N^{n+1} - \mathbf{u}_N^n}{\delta t} - \nu \Delta \mathbf{u}_N^{n*} + (\mathbf{u}_N^{n*}.\nabla)\mathbf{u}_N^{n*} \quad ,$$

and we seek the pressure p_N in $P_N(\Omega)$ as the solution of

$$(V.14) \quad \left| \begin{array}{l} \Delta p_N(\mathbf{x}) = - (\text{div } \mathbf{s}^{n+1})(\mathbf{x}) \quad , \mathbf{x} \in \Xi_N^A \cap \Omega \quad , \\ (\partial p_N/\partial n)(\mathbf{x}) = - (\mathbf{s}^{n+1}.\mathbf{n})(\mathbf{x}) \quad , \mathbf{x} \in \Xi_N^A \cap \partial\Omega \quad . \end{array} \right.$$

Here, the vector **n** denotes the unit outward normal vector to Ω on $\partial\Omega$.

Problem (V.14) is solved through a finite difference preconditioning method which involves the operator L defined in (V.3). Thus, p_N is computed as the limit of the following

sequence $(p_{N,k})_{k \geqslant 0}$. We set $p_{N,0} = q_N^{n+1}$; then, assuming that $p_{N,k}$ is known, we compute $p_{N,k+1}$ as the solution in $P_N(\Omega)$ of

$$(V.15) \quad \begin{cases} (L(p_{N,k+1}-p_{N,k}))(x) = \lambda \, (\Delta p_{N,k} + \text{div } S^{n+1})(x) \quad , x \in \Xi_N^A \cap \Omega \quad , \\ (\partial(p_{N,k+1}-p_{N,k})/\partial n_{|DF})(x) = - \mu \, (\partial p_{N,k}/\partial n + S^{n+1}.n)(x) \quad , x \in \Xi_N^A \cap \partial\Omega \quad , \end{cases}$$

where λ and μ are suitable parameters and the operator $\partial/\partial n_{|DF}$ is defined as follows : for any polynomial r_N in $P_N(\Omega)$, we set for each point x of $\Xi_N^A \cap \partial\Omega$,

1) if x is not a corner, assuming for instance that $x = x_{0k} = (\zeta_0^A, \zeta_k^A)$ belongs to Γ_I,

$$(\partial r_N/\partial n_{|DF})(x_{0k}) = \frac{r_N(\zeta_0^A, \zeta_k^A) - r_N(\zeta_1^A, \zeta_k^A)}{\zeta_0^A - \zeta_1^A} \quad ,$$

2) if x is a corner, assuming for instance that x is equal to $a_I = (\zeta_0^A, \zeta_0^A)$,

$$(\partial r_N/\partial n_{|DF})(a_I) = \frac{2 \, r_N(\zeta_0^A, \zeta_0^A) - r_N(\zeta_1^A, \zeta_0^A) - r_N(\zeta_0^A, \zeta_1^A)}{2 \, (\zeta_0^A - \zeta_1^A)} \quad ;$$

the operator $\partial/\partial n_{|DF}$ is defined similarly on the three other edges Γ_{II}, Γ_{III} and Γ_{IV} of the square.

The parameters λ and μ are chosen experimentally, in order to ensure the convergence of the sequence $(p_{N,k})_{k \geqslant 0}$. In our computation, they are respectively equal to 0.01 and 0.75.

The finite difference preconditioning method is well-known for spectral computations. It allows one to avoid direct inversion of spectral operators (e.q. the operator Δ in our case), which is expensive because the corresponding matrices are full (see [02], for instance).

Finally, let us remark that we did not look for the pressure in the space M_N as is suggested in the theory of § III.3. This approach is now under consideration.

V.4. Numerical results.

The numerical experiments were performed in domains of $\mathbb{R}^3$ with curved geometries, in which we generalized the previous algorithm. Indeed, we can use spectral techniques in a curved connected open set $\tilde{\Omega} \subset \mathbb{R}^3$, if there exists a one-to-one function $\mathcal{F}$ which is sufficiently smooth and maps the reference cube $\Omega =]-1,1[^3$ onto $\tilde{\Omega}$. Thanks to the function $\mathcal{F}$, a problem initially set in $\tilde{\Omega}$ is brought back to the cube Ω.

The set of spurious modes for pressure can be identified for three-dimensional problems [BMM, Lemme V.1][BCM, Remark IV.3], and we can obtain a well-posed problem for the Navier-Stokes equations in a cube as has been done in the two-dimensional case.

Moreover, we refer to [Mé, Chap. 2 and 3] for details about discretization of Navier–Stokes equations in curved geometries.

Let $\mathbf{x} = (x_1, x_2, x_3)$ and $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ denote the generic points of Ω and $\tilde{\Omega}$, respectively. In our numerical applications, we have considered the set $\tilde{\Omega}$ to be defined as a curved hexahedron of $\mathbb{R}^3$, two opposite sides of which are plane and parallel. Without restriction, we assume that these two sides are parallel to the plane $\tilde{x}_2 = 0$. The nozzle $\tilde{\Omega}$ is then defined from its boundary : let $f_i : \overline{\Lambda} = [-1,1]^2 \to \mathbb{R}^3$, $1 \leqslant i \leqslant 4$, be the parametrizations of the four other sides (see Figure V.1).

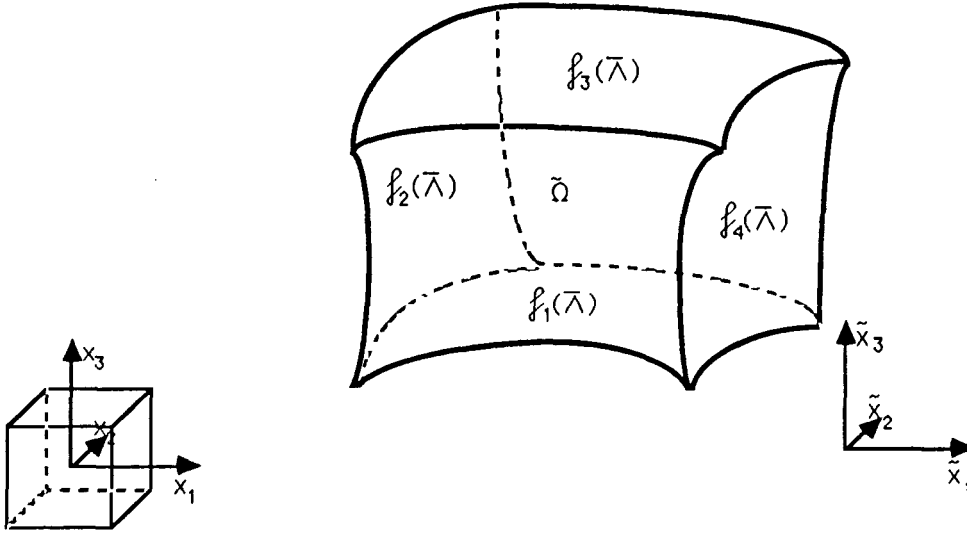

<u>Figure V.1</u>

The function $\mathcal{F}$ can be defined as follows : for $\mathbf{x} = (x_1, x_2, x_3)$, we set

$$\mathcal{F}(\mathbf{x}) = ((1+x_3)/2)\, f_3(x_1, x_2) + ((1-x_3)/2)\, f_1(x_1, x_2)$$
$$+ ((1+x_1)/2)\, [f_4(x_2, x_3) - ((1+x_3)/2)\, f_4(x_2, +1) - ((1-x_3)/2)\, f_4(x_2, -1)]$$
$$+ ((1-x_1)/2)\, [f_2(x_2, x_3) - ((1+x_3)/2)\, f_2(x_2, +1) - ((1-x_3)/2)\, f_2(x_2, -1)] \quad .$$

Clearly, we have $\mathcal{F}(\partial\Omega) = \partial\tilde{\Omega}$. Moreover, we assume that the function $\mathcal{F}$ is one-to-one and that $\mathcal{F}(\Omega) = \tilde{\Omega}$. This last property can be deduced from hypotheses of smoothness concerning the function $\mathcal{F}$ (see [Mé, Chap. 3, Th. V.1]).

In the example below, the open set $\tilde{\Omega}$ is a nozzle, the cross section of which in any plane $x_1 = $ constant is a rectangle. The function $\mathcal{F}$ is simply defined by

$$\mathcal{F}(\mathbf{x}) = \begin{vmatrix} \tilde{x}_1 = \alpha\, x_1 + \beta \\ \tilde{x}_2 = x_2 \\ \tilde{x}_3 = x_3\, f(x_1) \end{vmatrix}$$

where f is a fourth-degree polynomial with maxima at $x_1 = \pm 1$ and a minimum at a point $x_1 = x_1^0$ of $]-1,1[$. The boundary conditions $\mathbf{g}$ are given by

$$\mathbf{g} \circ \mathcal{F}(\mathbf{x}) = \begin{vmatrix} 3\,(1-x_3^2)\,/\,4\,f(x_1) \\ 0 \\ 0 \end{vmatrix} \quad , x_1 = \pm 1 \quad ,$$

$$\begin{vmatrix} 3\,(1-x_3^2)\,/\,4\,f(x_1) \\ 0 \\ 3\,(1-x_3^2)\,x_3\,f'(x_1)\,/\,4\,\alpha\,f(x_1) \end{vmatrix} \quad , x_2 = \pm 1 \quad ,$$

$$\begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} \quad , x_3 = \pm 1 \quad .$$

Note that we can extend the boundary data $\mathbf{g}$ into a divergence-free function u_b on $\tilde{\Omega}$, by setting

$$u_b \circ \mathcal{F}(\mathbf{x}) = \begin{vmatrix} 3\,(1-x_3^2)\,/\,4\,f(x_1) \\ 0 \\ 3\,(1-x_3^2)\,x_3\,f'(x_1)\,/\,4\,\alpha\,f(x_1) \end{vmatrix} \quad .$$

The viscosity coefficient ν is equal to 10^{-2}. The discretization is performed in the space of polynomials of degree $\leqslant N$, with $N = 16$. Thus the mesh is made up of 17^3 collocation nodes associated with the Chebyshev polynomials. The time step δt is equal to 10^{-3}.

Figure V.2 shows the mesh. In Figure V.3, the velocity iso-norms and the iso-pressure lines are presented in the plane $x_2 = 0$, at the time $10\,\delta t$ (when the stationary state is already reached).

Figure V.4 shows the iso-pseudo-pressure lines. The spurious mode $T_N(x_1)$ (the extrema of which coincide with the vertical lines of the mesh) appears clearly and totally hides the pressure behavior.

Figure V.5 shows the convergence of the algorithm (V.13) for the pressure post-treatment. Due to Neumann boundary conditions, the convergence is rather slow, so that the technique must be improved in order to obtain an efficient pressure solver.

Nevertheless, we obtain good results concerning the velocity. Note that we have

$$((\operatorname{div} u_N^{n+1}) \circ \mathcal{F}, (\operatorname{div} u_N^{n+1}) \circ \mathcal{F})_{C,N}^{1/2} = 6.4 \cdot 10^{-4} \quad .$$

This quantity can of course be reduced by increasing the number of collocation points. We refer to [Mé, Chap. 4, § IV.3] for another way to reduce $((\operatorname{div} u_N^{n+1}) \circ \mathcal{F}, (\operatorname{div} u_N^{n+1}) \circ \mathcal{F})_{C,N}$.
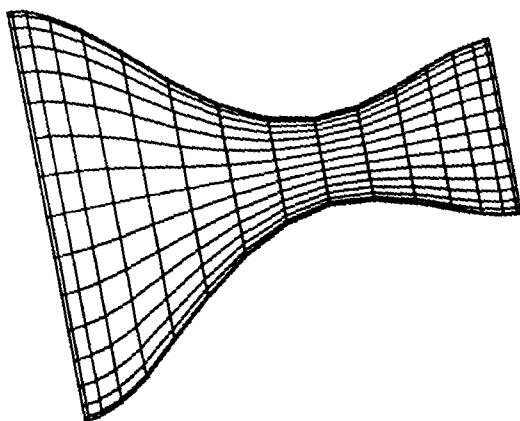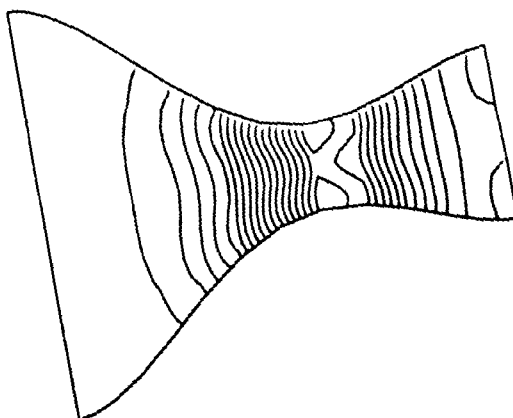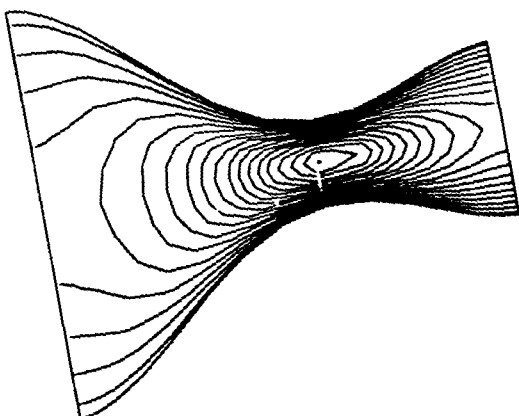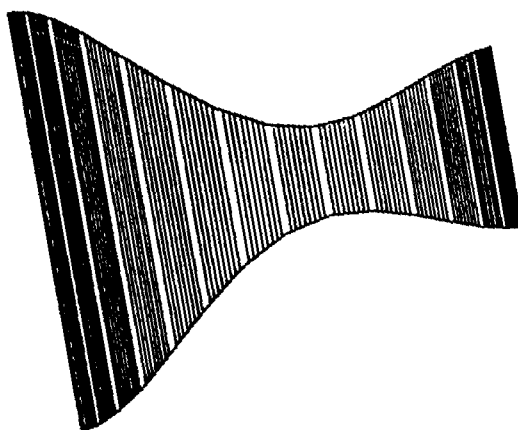
Figure V.2

Figure V.3

Figure V.4
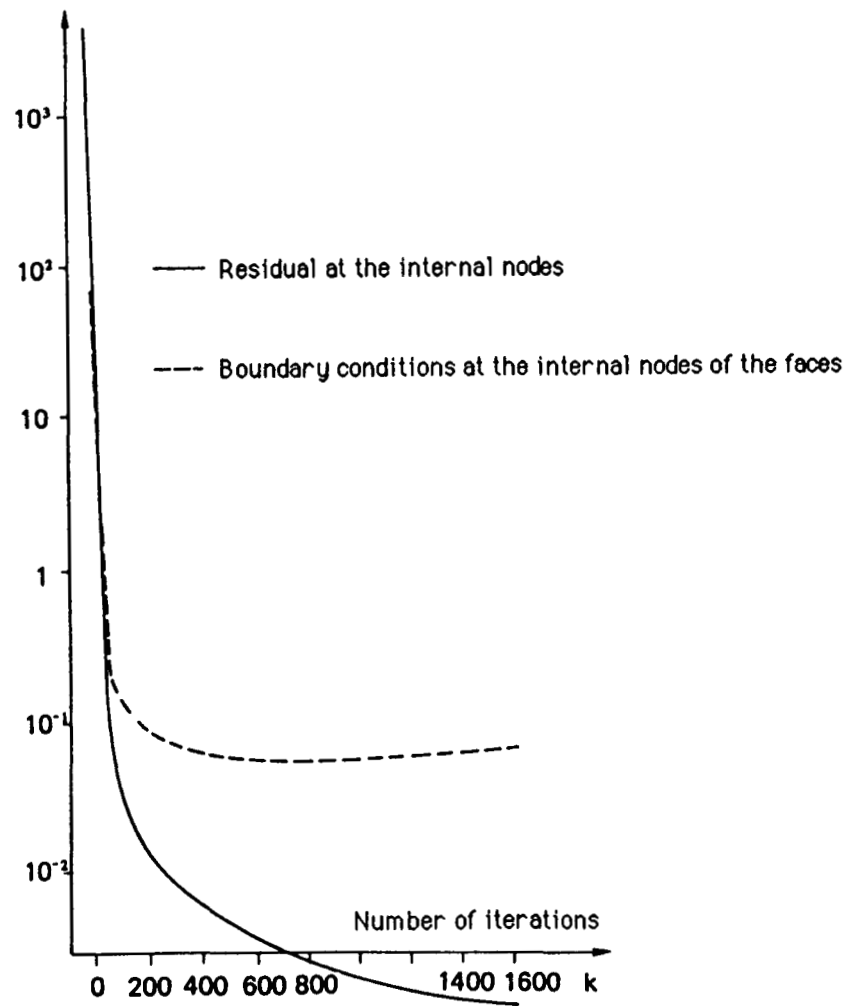
Figure V.5

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16. Abstract

    The aim of the paper is to study a collocation spectral method to approximate the Navier-Stokes equations: only one grid is used, which is built from the nodes of a Gauss-Lobatto quadrature formula, either of Legendre or of Chebyshev type. The convergence is proven for the Stokes problem provided with inhomogeneous Dirichlet conditions, then thoroughly analyzed for the Navier-Stokes equations. The practial implementation algorithm is presented, together with numerical results.

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