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ON THE INSTABILITY OF HYPERSONIC FLOW PAST A WEDGE[1]

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ABSTRACT

The instability of a compressible flow past a wedge is investigated in the hypersonic limit. Particular attention is given to Tollmien-Schlichting waves governed by triple-deck theory though some discussion of inviscid modes is given. It is shown that the attached shock has a significant effect on the growth rates of Tollmien-Schlichting waves. Moreover, the presence of the shock allows for more than one unstable Tollmien-Schlichting wave. Indeed an infinite discrete spectrum of unstable waves is induced by the shock, but these modes are unstable over relatively small but high frequency ranges. The shock is shown to have little effect on the inviscid modes considered by previous authors and an asymptotic description of inviscid modes in the hypersonic limit is given.

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1. INTRODUCTION

Our concern is with the instability of hypersonic flow around a wedge of small angle. The work is motivated by recent interest in the development of hypersonic vehicles. We do not account for real gas effects even though some of the vehicles in question will certainly operate at speeds where such effects are important. Here we shall concentrate on Tollmien-Schlichting waves governed by triple-deck theory though some discussion of inviscid modes will be given. In a related calculation, the Görtler vortex instability mechanism at hypersonic speeds is considered. There it is shown that centrifugal instabilities have their structure simplified considerably in the hypersonic limit. Here we show that this is not the case for Tollmien-Schlichting waves though inviscid modes have a relatively simple structure. In fact the logarithmically small layer at the edge of the boundary layer which controls inviscid modes is precisely the same layer where Görtler vortices become trapped at hypersonic speeds. It would therefore seem that the nonlinear interaction between inviscid and centrifugal instabilities is an important problem to be considered in the hypersonic limit.

Before giving more details of the problem to be considered here we shall give a brief review of some relevant previous calculations on compressible stability problems. For an excellent review of viscous and inviscid stability properties of compressible boundary layers the reader is referred to the article by Mack (1987). Perhaps the main feature of compressible boundary layers of practical importance is that they are unstable to both inviscid and viscous instability waves. The inviscid modes have wavelengths comparable with the boundary layer thickness whilst the Tollmien-Schlichting waves have longer wavelengths. Either the successive approximation procedure of Gaster (1974) or the formal asymptotic description based on triple-deck theory, as used for incompressible flows by Smith (1979a,b), can be extended to the compressible viscous instability problem in a routine manner. Such a calculation was given by Gaponov (1981) using essentially Gaster's approach whilst more recently Smith (1988) has applied triple-deck theory to the viscous modes of compressible boundary layers. A significant result of Smith (1988) is that when the free stream Mach number is increased there is critical size of Mach number in terms of the (large) Reynolds number at which the Tollmien-Schlichting downstream development takes place on a lengthscale comparable with that over which the basic state develops. At this stage the waves cannot be described by any quasi-parallel theory and the evolution can only be described by numerical integration of the governing linear partial differential equations. This is precisely the situation in the incompressible Görtler stability problem, see Hall (1983); perversely in the hypersonic limit the most significant Görtler vortices lose this property and become only weakly dependent on nonparallel effects. A question of some importance raised by Smith's work is whether the failure of the quasi-parallel approach at high Mach numbers means that many of the parallel flow calculations in this regime at finite Reynolds numbers are in error. Because Smith's prediction is based on a double high Reynolds number and Mach number limit, the regime at which this failure occurs must be identified by numerical integration of the governing linear partial differential equations.

The inviscid modes of instability of a compressible boundary layer have been well-documented by Mack (1987). In fact, there can be unstable two- and three-dimensional modes and neutral modes associated with a generalized inflection point and non-inflectional neutral modes. At Mach numbers above three it is the inviscid modes which have the highest growth rates and therefore presumably dominate the transition problem in compressible boundary layers. However, previous calculations for viscous and inviscid modes have taken little account of the presence of shocks in the flow field (but see Petrov 1984), though at high Mach numbers there is no question that they play an important role in determining the basic state. A primary aim of the present calculation is to gain some insight into the role of shock waves in the formation of viscous and inviscid modes in a compressible boundary layer.

The particular configuration which we investigate is the hypersonic flow around an aligned wedge of semi-angle θ. The inviscid flow in this case is a uniform state either side of straight shocks which make an angle $\phi = \sigma - \theta$ with the wedge. At the wedge a boundary layer is set up and the fluid velocity is reduced to zero inside this layer. We restrict our attention to the case when the wedge is insulated though our calculations can readily be extended to the isothermal case.

In order that Tollmien-Schlichting wave disturbances to this flow can be treated in a quasi-parallel manner we make the Newtonian approximation and take the distance of the shock from the wall to be comparable with the upper deck scale over which Tollmien-Schlichting waves governed by triple-deck theory will decay. At the shock we derive linearized boundary conditions which the Tollmien-Schlichting wave must satisfy. This condition effectively changes the eigenrelation from that discussed in the absence of shocks by Smith (1988). We show that the shock has a significant effect on the growth rate of the mode corresponding to that discussed by Smith. In addition we show that the presence of the shock allows for the existence of much more amplified modes trapped between the wedge and the shock. These modes occur at relatively high frequencies but are unstable over very short ranges of frequency.

Some discussion of inviscid disturbances is also given. At high Mach numbers we give the appropriate asymptotic structure of the so-called acoustic modes, and show that they will be influenced little by the presence of the shock. The results which we give in the hypersonic limit are in remarkably good agreement with Mack's (1987) results even at relatively low Mach numbers.

The procedure adopted in the rest of this paper is as follows: in §2 we formulate the appropriate sealings for Tollmien-Schlichting instabilities of the hypersonic flow past a wedge. In §3 the triple-deck equations for such modes are derived and in §4 the dispersion relationship appropriate to the linearized form of these equations is given and discussed. In §5 and draw some conclusions about the results of §4 and show how the appropriate inviscid stability modes develop in a hypersonic boundary layer on a wedge. Finally in an Appendix we derive the shock conditions appropriate to a disturbance with arbitrary length scales and use them to find the simplified shock condition for a Tollmien-Schlichting wave.

2. FORMULATION

The basic flow whose stability we examine is illustrated in Figure 1. For simplicity, the wedge is taken to be symmetrically aligned with an oncoming supersonic flow with velocity magnitude $\hat{U}$. Shocks of semi-angle σ develop from the tip of the wedge (the acute angle between the shock and the wedge is $\phi = \sigma - \theta$). Quantities upstream of the shock are indicated by the subscript u, and quantities in the so-called 'shock-layer' between the shock and wedge by the subscript s. Cartesian axes are introduced with the $\hat{x}$ and $\hat{y}$ coordinates aligned and normal with the upper surface of the wedge, and the $\hat{z}$ coordinate in the spanwise direction. The corresponding velocity components are $\underline{\hat{u}} = (\hat{u}, \hat{v}, \hat{w})$, while $\hat{t}$, $\hat{\rho}$, $\hat{p}$, $\hat{T}$ and $\hat{h}$ are used to denote time, density, pressure, temperature and enthalpy respectively. We will assume that the fluid is a perfect gas with ratio of specific heats γ; then the upstream Mach number, M_u, is given by

$$M_u = \frac{\hat{U}}{a_u}, \tag{2.1a}$$

where the sound speed a_u is given by

$$a_u^2 = \frac{\gamma \hat{p}_u}{\hat{\rho}_u} = (\gamma - 1) h_u. \tag{2.1b}$$

The inviscid solution for this flow configuration consists of uniform quantities on either sides of straight shocks (e.g. see Hayes and Probstein, 1966). Specifically,

$$\varepsilon = \frac{\hat{\rho}_u}{\hat{\rho}_s} = \left(\frac{\gamma - 1}{\gamma + 1}\right)\left(1 + \frac{2}{(\gamma - 1) M_u^2 \sin^2 \sigma}\right), \tag{2.2a}$$

$$\frac{\hat{p}_s}{\hat{p}_u} = 1 + \gamma M_u^2 \sin^2 \sigma (1 - \varepsilon), \tag{2.2b}$$

$$\frac{\hat{h}_s}{\hat{h}_u} = 1 + \frac{1}{2}(\gamma - 1)(1 - \varepsilon^2)M_u^2 \sin^2\sigma, \tag{2.2c}$$

$$\tan\phi = \varepsilon \tan\sigma, \tag{2.2d}$$

$$(\hat{u}_\parallel)_u = (\hat{u}_\parallel)_s = \hat{U}\cos\sigma, \quad (\hat{u}_\perp)_u = -\hat{U}\sin\sigma, \quad (\hat{u}_\perp)_s = -\varepsilon\hat{U}\sin\sigma, \tag{2.2e}$$

where $\hat{u}_\parallel$ and $\hat{u}_\perp$ are the velocity components parallel and perpendicular to the shock. From (2.1) it follows that between the shock and the wedge the fluid velocity has magnitude

$$\hat{U}_s = \hat{U}\cos\sigma(1 + \varepsilon^2\tan^2\sigma)^{\frac{1}{2}}, \tag{2.3}$$

and the local Mach number M_s is given by

$$M_s^2 = \frac{M_u^2\cos^2\sigma(1 + \varepsilon^2\tan^2\sigma)}{1 + \frac{1}{2}(\gamma - 1)(1 - \varepsilon^2)M_u^2\sin^2\sigma}, \tag{2.4a}$$

or equivalently

$$M_u^2 = \frac{M_s^2}{\cos^2\sigma(1 + \varepsilon^2\tan^2\sigma) - \frac{1}{2}(\gamma - 1)(1 - \varepsilon^2)M_s^2\sin^2\sigma}. \tag{2.4b}$$

Note that (2.3) specifies the slip velocity along the wedge, which in viscous flow necessitates the existence of a boundary layer.

Before examining this boundary layer in detail we introduce a non-dimensionalization based on the flow quantities between the shock and the wedge, and a length scale L which is the distance of interest from the tip of the wedge. Specifically, we introduce coordinates $L\underline{x}$, velocities $\hat{U}_s\underline{u}$, time $Lt/\hat{U}_s$, pressure $\hat{\rho}_s\hat{U}_s^2 p$, density $\hat{\rho}_s\rho$, temperature $\hat{T}_s T$ and enthalpy $\hat{h}_s h$. The governing equations of the flow then become

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{u}) = 0, \tag{2.5a}$$

$$\rho\frac{D\underline{u}}{Dt} = -\underline{\nabla}p + \frac{1}{Re}[2\nabla\cdot(\mu\underline{\underline{e}}) + \underline{\nabla}((\mu' - \frac{2}{3}\mu)\nabla\cdot\underline{u})], \tag{2.5b}$$

$$\rho\frac{DT}{Dt} = (\gamma - 1)M_s^2\frac{Dp}{Dt} + \frac{1}{PrRe}\nabla\cdot(\mu\underline{\nabla}T) + \frac{(\gamma - 1)M_s^2}{Re}\Phi, \tag{2.5c}$$

$$\gamma M_s^2 p = \rho T, \quad h = T, \tag{2.5d, e}$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2.6a}$$

$$\Phi = 2\mu\underline{\underline{e}} : \underline{\underline{e}} + (\mu' - \frac{2}{3}\mu)(\nabla\cdot\underline{u})^2, \tag{2.6b}$$

μ and μ' are the shear and bulk viscosities respectively, both of which have been nondimensionalized by a typical viscosity $\hat{\mu}_s, P_r$ is the constant Prandtl number, and

$$Re = \frac{\hat{\rho}_s \hat{U}_s L}{\hat{\mu}_s} \tag{2.6c}$$

is the Reynolds number which we will assume is large. Henceforth, the subscript s will be omitted from M_s. We also introduce coordinates (ξ, η, z) where ξ is measured along the shock and η is normal to it. Then

$$\xi = x\cos\phi + y\sin\phi, \quad \eta = -x\sin\phi + y\cos\phi. \tag{2.7}$$

For simplicity we will assume (i) that the Prandtl number is unity ($Pr = 1$), (ii) that the shear viscosity is given by a Chapman law $\mu = CT$, and (iii) that the wedge walls are insulating, i.e. $\left(\frac{\partial T}{\partial y} = 0 \text{ on } y = 0\right)$. At large Reynolds numbers, there is then a simple solution to the boundary layer equations in terms of the Dorodnitsyn-Howarth variable (e.g. Stewartson 1964), in which the temperature at the wall, T_w, is given by

$$T_w = 1 + \frac{1}{2}(\gamma - 1)M^2. \tag{2.8}$$

The linear stability of this boundary layer flow has been extensively studied using the Orr-Sommerfeld quasi-parallel approximation (e.g. Mack 1969, 1984, 1987). Recently, Smith (1988) has shown how an asymptotic triple-deck description of lower branch Tollmien-Schlichting waves (i.e. 'first mode' waves) can be obtained for wave directions sufficiently oblique to be outside the local wave-Mach-cone direction. In the limit of large Mach number M, Smith (1988) found that the most rapidly growing waves have frequencies of order $Re^{\frac{1}{4}}T_w^{-1}M^{-\frac{1}{2}}$, and wavelengths in the x and z directions of order

$$Re^{-\frac{3}{8}}T_w^{\frac{3}{2}}M^{\frac{3}{4}} \quad \text{and} \quad Re^{-\frac{3}{8}}T_w^{\frac{3}{2}}M^{-\frac{1}{4}} \tag{2.9a, b}$$

respectively. As is conventional there are lower, middle, and upper decks in the y direction with scales

$$Re^{-\frac{5}{8}}T_w^{\frac{3}{2}}M^{\frac{1}{4}}, \quad Re^{-\frac{1}{2}}T_w \quad \text{and} \quad Re^{-\frac{3}{8}}T_w^{\frac{3}{2}}M^{-\frac{1}{4}} \tag{2.9c, d, e}$$

respectively.

One aim of this paper is to see how the growth rates of these oblique Tolllmien-Schlichting waves are modified by the presence of a shock. To this end, we attempt to scale the problem so that the shock occurs in the upper deck. Thus, since ϕ must be small, it follows from (2.2d) and (2.9e) that

$$\phi \sim \varepsilon\tan\sigma \sim Re^{-\frac{3}{8}}T_w^{\frac{3}{2}}M^{-\frac{1}{4}}, \tag{2.9f}$$

which from (2.9a) implies an x length scale of order $M\varepsilon \tan\sigma$. If the complications arising from non-parallel effects are to be excluded in order to concentrate on the effects of the shock interaction, we require

$$M\varepsilon\sigma << 1, \tag{2.10a}$$

i.e. that the wavelength of the Tollmien-Schlichting waves is much less than the distance from the apex of the wedge. From (2.2a) and (2.4a), $\varepsilon = 0(M^{-2}\sigma^{-2})$, so (2.10a) becomes

$$\sigma M >> 1. \tag{2.10b}$$

However, for $M_u^2 > 0$ in (2.4b) we require

$$(\gamma - 1)M^2\sigma^2 < 1. \tag{2.10c}$$

Hence, in order to consider the effect of the shock on the instability waves without the effects of nonparallelism we are *forced* to make the 'Newtonian' assumption

$$(\gamma - 1) << 1. \tag{2.10d}$$

It follows that when $(\gamma - 1)$ is not small there is no systematic asymptotic approach to this problem which allows the inclusion of shock effects without those associated with non-parallelism. Assuming that $(\gamma - 1)M^2$ is not small, i.e. that there is a significant temperature variation in the boundary layer, the interaction condition (2.9) becomes using (2.8)

$$(\gamma - 1)^{\frac{3}{2}} M^{\frac{19}{4}} \sigma \sim Re^{\frac{3}{8}}. \tag{2.11}$$

Before proceeding to asymptotic expansions based on (2.11), it is convenient to consider whether there are any further restrictions on the scales of $(\gamma - 1), M$ and σ. First, we note that a pressure/acoustic wave incident on a shock will produce entropy and shear waves as well as generating a reflected wave. The entropy/shear waves have a typical lengthscale normal to the shock of magnitude

$$\varepsilon\sigma\overline{\alpha}^{-1} \sim Re^{-\frac{3}{8}}(\gamma - 1)^{\frac{3}{2}} M^{\frac{7}{4}} \sigma^{-1}, \tag{2.12}$$

where $\overline{\alpha}$ is a typical wavenumber in the x-direction (see Section 3 below). For simplicity we choose to ignore viscous effects in these waves at leading order, in which case we require

$$\varepsilon\sigma\overline{\alpha}^{-1} >> (\overline{\alpha}Re)^{-\frac{1}{2}}, \quad \text{i.e. from (2.9a) and (2.11)} \quad \sigma^3 M^5 << Re. \tag{2.13}$$

If it is assumed that the shock has a viscous internal structure, then its thickness is order $(\varepsilon\sigma Re)^{-1}$ (e.g. Moore 1964), and (2.13) then ensures that the entropy/shear waves have a wavelength much larger than the thickness of the shock. It follows from (2.10a) that the waves have typical y scales much less than the width of the upper deck.

Since nonlinear effects are an important part in transition to turbulence, we will select a scaling which leads to a nonlinear problem, before linearizing to obtain an analytic solution. Following the scalings in Smith (1988), we conclude that a nonlinear lower-deck problem is recovered if

$$p \sim Re^{-\frac{1}{4}} M^{-\frac{3}{2}}. \tag{2.14}$$

In the upper deck this generates a velocity perturbation normal to the wedge (and normal to the shock) of order $Re^{-\frac{1}{4}} M^{-\frac{1}{2}}$. In order that linearized shock conditions are applicable, the undisturbed velocity normal to the shock should be much larger than this, i.e. from (2.2e)

$$\varepsilon\sigma >> Re^{-\frac{1}{4}} M^{-\frac{1}{2}}, \quad \text{i.e.} \quad \sigma^4 M^6 << Re. \tag{2.15}$$

If this condition is violated, nonlinear entropy waves result, a difficulty which is not tackled here. Also, note that if (2.15) is satisfied, then from (2.10b) so is (2.13).

In order to fix a scaling we will assume that (cf. (2.10c))

$$(\gamma - 1) M^2 \sigma^2 \sim 1, \tag{2.16a}$$

then (2.11) and the restrictions (2.10b) and (2.14) imply

$$M \sim \sigma^{\frac{8}{7}} Re^{\frac{3}{14}}, \quad Re^{-\frac{1}{10}} << \sigma << Re^{-\frac{1}{38}}. \tag{2.16b, c}$$

If the lower-deck is forced to remain linear then the upper bound on σ relaxes to $\sigma << Re^{-\frac{1}{122}}$. Note also that this scaling implies that the temperature in the undisturbed boundary is large. For convenience we introduce the scalings

$$(\gamma - 1) = \Gamma \sigma^{-\frac{30}{7}} Re^{-\frac{3}{7}}, \quad M = \sigma^{\frac{8}{7}} Re^{\frac{3}{14}} m, \tag{2.17a, b}$$

then

$$T_w \approx \frac{1}{2} \Gamma m \sigma^{-2} >> 1. \tag{2.17c}$$

Also, from (2.2d), (2.4b) and (2.7) the position of the shock is given by

$$y = x\varepsilon \tan\sigma \approx \frac{Re^{-\frac{3}{7}} \sigma^{-\frac{23}{7}} x}{m^2}, \tag{2.18}$$

since

$$\varepsilon \approx \frac{1}{\sigma^2 M^2}.$$

3. THE TRIPLE-DECK EQUATIONS

The scalings leading to these equations for compressible flow have been given elsewhere (e.g. Stewartson 1974), hence only a brief outline is given here. In all three decks the x, z and t scalings are

$$x = 1 + \sigma^{\frac{6}{7}} Re^{-\frac{3}{14}} C^{\frac{3}{8}} T_w^{\frac{3}{2}} m^{\frac{3}{4}} \lambda^{-\frac{5}{4}} X, \quad z = \sigma^{-\frac{2}{7}} Re^{-\frac{3}{7}} C^{\frac{3}{8}} T_w^{\frac{3}{2}} \lambda^{-\frac{5}{4}} m^{-\frac{1}{4}} Z,$$
$$t = \sigma^{\frac{4}{7}} Re^{-\frac{1}{7}} C^{\frac{1}{4}} T_w m^{\frac{1}{2}} \lambda^{-\frac{3}{2}} \tau, \tag{3.1}$$

where $\lambda = 0.470$ is the Blasius boundary layer skin-friction from the undisturbed middle deck solution.

<u>Lower Deck</u>

In this layer, the scalings are

$$y = \sigma^{\frac{2}{7}} Re^{-\frac{4}{7}} C^{\frac{5}{8}} T_w^{\frac{3}{2}} m^{\frac{1}{4}} Y, \quad u \sim \sigma^{\frac{2}{7}} Re^{-\frac{1}{14}} C^{\frac{1}{8}} T_w^{\frac{1}{2}} m^{\frac{1}{4}} \lambda^{\frac{1}{4}} U,$$
$$v \sim \sigma^{-\frac{2}{7}} Re^{-\frac{3}{7}} C^{\frac{3}{8}} T_w^{\frac{1}{2}} \lambda^{\frac{3}{4}} m^{-\frac{1}{4}} V, \quad w \sim \sigma^{-\frac{6}{7}} Re^{-\frac{2}{7}} C^{\frac{1}{8}} T_w^{\frac{1}{2}} \lambda^{\frac{1}{4}} W, \tag{3.2}$$
$$p \sim \sigma^{-\frac{16}{7}} Re^{-\frac{3}{7}} m^{-2} + \sigma^{-\frac{12}{7}} Re^{-\frac{4}{7}} C^{\frac{1}{4}} \lambda^{\frac{1}{2}} m^{-\frac{3}{2}} P, \quad T \sim T_w, \quad \rho \sim T_w^{-1}.$$

On substituting into (2.5) we obtain at leading order

$$U_X + V_Y + W_Z = 0, \quad P_Y = 0$$
$$U_\tau + UU_X + VU_Y + WU_Z = U_{YY}, \tag{3.3}$$
$$W_\tau + UW_X + VW_Y + WW_Z = -P_Z + W_{YY}.$$

The boundary conditions are

$$U = V = W = 0 \quad \text{on } Y = 0,$$
$$U \to Y + A(X, Z, \tau), \quad W \to 0 \quad \text{as } Y \to \infty, \tag{3.4}$$

where the conditions as $Y \to \infty$ come from matching with the middle deck, and A is the so-called displacement function.

Middle Deck

The middle deck has the same thickness as the undisturbed boundary layer, which has a finite extent in y when $T_w >> 1$ (although an infinite extent in terms of the Dorodnitsyn-Howarth variable). Strictly this means that the middle deck should be divided into three regions (i) a boundary-layer region where $T >> 1$, (ii) a region where $T \sim 1$, and (iii) a logarithmically small transition region between the two.

In region (i) the standard scalings and solutions apply:

$$
\begin{aligned}
&y = Re^{-\frac{1}{2}}C^{\frac{1}{2}}T_w y^*,\\
&u \sim u_0^*(y^*) + \sigma^{\frac{2}{7}}Re^{-\frac{1}{14}}C^{\frac{1}{8}}T_w^{\frac{1}{2}}m^{\frac{1}{4}}\lambda^{-\frac{3}{4}}Au_{0y^*}^*,\\
&v \sim -\sigma^{-\frac{4}{7}}Re^{-\frac{5}{14}}C^{\frac{1}{4}}\lambda^{\frac{1}{2}}m^{-\frac{1}{2}}A_X u_0^*,\\
&w \sim \sigma^{-\frac{4}{7}}Re^{-\frac{5}{14}}C^{\frac{1}{4}}\lambda^{\frac{1}{2}}m^{-\frac{1}{2}}(R_0^*u_0^*)^{-1}D(X,Z,\tau),\\
&p \sim \sigma^{-\frac{16}{7}}Re^{-\frac{3}{7}}m^{-2} + \sigma^{-\frac{12}{7}}Re^{-\frac{4}{7}}C^{\frac{1}{4}}\lambda^{\frac{1}{2}}m^{-\frac{3}{2}}P,\\
&\rho \sim R_0^*(y^*) + Re^{-\frac{1}{14}}AR_{0y^*}^*,
\end{aligned}
\tag{3.5}
$$

where $u_0^*(y)$ and $R_0^*(y)$ are the undisturbed velocity and density profiles respectively (note that $R_0^* = 0(T_w^{-1})$), and $D_X = -P_Z$.

Essentially the same solution holds in regions (ii) and (iii), although minor rescalings are necessary. For example in region (ii) $R_0^* = u_0^* = 1$, which leads to simplifications in the expressions for u and ρ in particular. Also in region (iii), $u_{0y^*}^*$ scales with $(\log T_w)^{\frac{1}{2}}$ as a result of the kink in the velocity profile when $T_w >> 1$. This means that the largest velocity perturbations occur in this region. We note that as σ approaches $R^{-\frac{1}{10}}$ this may mean that the middle deck becomes nonlinear for smaller wave amplitudes than the lower deck.

Upper Deck

The scalings for the pressure/acoustic waves here are

$$y = \sigma^{-\frac{2}{7}}Re^{-\frac{3}{7}}C^{\frac{3}{8}}T_w^{\frac{3}{2}}\lambda^{-\frac{5}{4}}m^{-\frac{1}{4}}\overline{y}, \tag{3.6a}$$

$$u \sim 1 + \sigma^{-\frac{12}{7}}Re^{-\frac{4}{7}}C^{\frac{1}{4}}\lambda^{\frac{1}{2}}m^{-\frac{3}{2}}\overline{u}_1, \tag{3.6b}$$

$$(v,w) \sim \sigma^{-\frac{4}{7}}Re^{-\frac{5}{14}}C^{\frac{1}{4}}\lambda^{\frac{1}{2}}m^{-\frac{1}{2}}(\overline{v}_1,\overline{w}_1), \tag{3.6c}$$

$$p \sim \sigma^{-\frac{16}{7}} Re^{-\frac{3}{7}} m^{-2} + \sigma^{-\frac{12}{7}} Re^{-\frac{4}{7}} C^{\frac{1}{4}} \lambda^{\frac{1}{2}} m^{-\frac{3}{2}} \overline{p}_1, \tag{3.6d}$$

$$\rho \sim 1 + \sigma^{\frac{4}{7}} Re^{-\frac{1}{7}} C^{\frac{1}{4}} \lambda^{\frac{1}{2}} m^{\frac{1}{2}} \overline{\rho}_1, \tag{3.6e}$$

$$T \sim 1 + \sigma^{\frac{4}{7}} Re^{-\frac{4}{7}} C^{\frac{1}{4}} \lambda^{\frac{1}{2}} m^{\frac{1}{2}} \overline{T}_1. \tag{3.6f}$$

These yield the governing equation for the pressure

$$\overline{p}_{1XX} - \overline{p}_{1\overline{y}\overline{y}} - \overline{p}_{1ZZ} = 0. \tag{3.7}$$

One boundary condition comes from matching with the middle deck, i.e.

$$\overline{p}_{1\overline{y}} = A_{XX} \quad \text{on } \overline{y} = 0 \tag{3.8}$$

(note $\overline{p}_1 = P$ on $\overline{y} = 0$), while another boundary condition is to be applied at the shock at

$$\overline{y} = \overline{y}_s = \left(\frac{2}{\Gamma}\right)^{\frac{3}{2}} \frac{\lambda^{\frac{5}{4}}}{C^{\frac{3}{8}} m^{\frac{13}{4}}}. \tag{3.9}$$

As is conventional, solutions to (3.7) will be referred to as acoustic waves.

Shock Conditions

Wave transmission and reflection across shocks has been studied by Moore (1954), Ribner (1954), McKenzie and Westphal (1968) and others. In general, whenever an acoustic wave is incident on a shock, entropy and vorticity waves are generated in addition to a reflected/transmitted acoustic wave. The entropy/vorticity waves have the same frequency and the same wavelengths parallel to the shock as the acoustic waves. However, they propagate in the direction of the mean flow, which means that for the present scalings that their wavelengths normal to the shock are very much less than the acoustic wavelength given by (3.6a). The scaling (3.6) does not therefore describe them. In the appendix general jump conditions at a shock are given for incident linearized inviscid waves. The limiting process appropriate to the above scalings then yields the boundary condition

$$\overline{p}_1 = 0 \quad \text{at } \overline{y} = \overline{y}_s. \tag{3.10}$$

In deriving (3.10) from the exact solution to the general linear inviscid problem, we ensure that proper account is taken of the short wavelength entropy/vorticity waves. A multiple scales approach would be an alternative.

The entropy/vorticity waves propagate parallel to the direction of the basic flow, which is of course parallel to the wedge. It follows that in any situation where the forcing is over a distance comparable with the triple-deck scaling (3.1a), i.e. over a distance which although possibly large does not extend asymptotically far upstream, the entropy/vorticity waves will be concentrated in a narrow region close to the shock. Hence they cannot affect the solution in the middle and lower decks. [2]

4. THE DISPERSION RELATION

Solutions to the nonlinear system (3.3), (3.4), (3.7-10) can in general only be obtained numerically. However, analytic solutions can be found if the waves are of small amplitude so that the system linearizes. It is then convenient to focus attention on a single mode, so we write

$$U \sim Y + h\tilde{U}e^{i(\alpha X+\beta Z-\Omega t)} + c.c., \quad (V, W, P, A) \sim h(\tilde{V}, \tilde{W}, \tilde{P}, \tilde{A})e^{i(\alpha X+\beta Z-\Omega t)} + c.c..$$

Substituting into (3.3), and linearizing under the assumption that $h << 1$, we obtain (e.g. see Smith, Sykes & Brighton 1977)

$$\alpha\tilde{V} + \beta\tilde{W} = \frac{i\beta^2\tilde{P}}{(i\alpha)^{\frac{2}{3}}Ai'(\varsigma_0)}\int_{\varsigma_0}^{\varsigma} Ai(q)dq, \tag{4.1a}$$

$$i^{\frac{2}{3}}\alpha^{\frac{5}{3}}Ai'(\varsigma_0)\tilde{A} = i\beta^2\tilde{P}\int_{\varsigma_0}^{\infty} Ai(\varsigma)d\varsigma, \tag{4.1b}$$

where

$$\varsigma = (i\alpha)^{\frac{1}{3}}(Y - \Omega/\alpha), \quad \varsigma_0 = -i^{\frac{1}{3}}\Omega/\alpha^{\frac{2}{3}}. \tag{4.2}$$

The solution to (3.7) subject to (3.8) and (3.10) is

$$\begin{aligned} \tilde{\bar{p}}_1 &= \frac{\alpha^2}{(\beta^2-\alpha^2)^{\frac{1}{2}}}\frac{\sinh((\beta^2-\alpha^2)^{\frac{1}{2}}(\bar{y}_s-\bar{y}))}{\cosh((\beta^2-\alpha^2)^{\frac{1}{2}}\bar{y}_s)}\tilde{A} \qquad \beta^2 > \alpha^2, \\ \tilde{\bar{p}}_1 &= \frac{\alpha^2}{(\alpha^2-\beta^2)^{\frac{1}{2}}}\frac{\sin((\alpha^2-\beta^2)^{\frac{1}{2}}(\bar{y}_s-\bar{y}))}{\cos((\alpha^2-\beta^2)^{\frac{1}{2}}\bar{y}_s)}\tilde{A} \qquad \beta^2 < \alpha^2. \end{aligned} \tag{4.3}$$

Hence, from (4.1), (4.3) and the fact that $\tilde{P} = \tilde{\bar{p}}_1$ on $\bar{y} = 0$, it follows that

$$\frac{(i\alpha)^{\frac{1}{3}}\beta^2\int_{\varsigma_0}^{\infty}Ai(\varsigma)d\varsigma}{Ai'(\varsigma_0)} = \begin{cases} \dfrac{(\beta^2-\alpha^2)^{\frac{1}{2}}}{\tanh((\beta^2-\alpha^2)^{\frac{1}{2}}\bar{y}_s)} & \beta^2 > \alpha^2 \\ \dfrac{(\alpha^2-\beta^2)^{\frac{1}{2}}}{\tan((\alpha^2-\beta^2)^{\frac{1}{2}}\bar{y}_s)} & \beta^2 < \alpha^2 \end{cases} \tag{4.4}$$

Note that for $\beta^2 > \alpha^2$, the dispersion relation for a flow without a shock is recovered in the limit $\bar{y}_s \to \infty$ (Smith 1988).

[2] We note that this is doubly so if $\sigma >> Re^{-\frac{1}{38}}$ since second order viscous effects ensure that the entropy/vorticity waves decay over a distance much less than the width of the upper deck - although it follows from (2.16c) that the lower deck must be linear for our analysis to be valid for such values of σ.

The dispersion relation (4.4) admits both growing and decaying modes. Neutral curves are given by

$$\varsigma_0 = -c_1 i^{\frac{1}{3}}, \quad (\beta^2 - \alpha^2)^{\frac{1}{2}} = c_2 \alpha^{\frac{1}{3}} \beta^2 \tanh((\beta^2 - \alpha^2)^{\frac{1}{2}} \overline{y}_s), \tag{4.5}$$

where $c_1 \approx 2.3$ and $c_2 \approx 1.0$ (a similar expression exists for $\beta^2 < \alpha^2$). These are plotted as solid curves in Figure 2 for three different values of $\overline{y}_s$. The diagonal dashed curve defines the wave-Mach-cone. Above this line the acoustic waves in the upper deck are purely sinusoidal, beneath it they either grow or decay in $\overline{y}$. When there is no shock in the upper deck, waves described by triple-deck theory are constrained to lie in the region of parameter space below the diagonal (Smith 1988); with shocks present there is no such limitation.

Asymptotic formulae for these neutral curves can be derived that agree well with the numerical results:

$$\alpha \sim \frac{(n+\frac{1}{2})\pi}{\overline{y}_s} + \left(\frac{\overline{y}_s}{2(n+\frac{1}{2})\pi} - \frac{c_2}{\overline{y}_s} \left(\frac{(n+\frac{1}{2})\pi}{\overline{y}_s} \right)^{\frac{2}{3}} \right) \beta^2, \quad \alpha = O(1), \ \beta << 1, \ n = 1,2,\dots, \tag{4.6a}$$

$$\alpha \sim \beta + \left(\frac{n\pi}{\overline{y}_s}\right)^2 \frac{1}{2\beta} - \left(\frac{n\pi}{\overline{y}_s}\right)^4 \frac{1}{8\beta^3} + \frac{1}{c_2 \overline{y}_s} \left(\frac{n\pi}{\overline{y}_s}\right)^2 \frac{1}{\beta^{\frac{10}{3}}}, \quad \alpha, \ \beta >> 1, \ n = 1,2,\dots, \tag{4.6b}$$

$$\alpha \sim \left(\frac{1}{c_2 \beta} \right)^3, \quad \alpha << 1, \ \beta >> 1. \tag{4.6c}$$

These formulae confirm that there are an infinite number of neutral waves, and that with the exception of the subsonic mode, they all asymptote to the line $\beta = \alpha$ for α, β large.

The short dashed lines in Figure 2 also represent waves with zero growth rate, but they have an infinite frequency and hence our asymptotic analysis breaks down in their vicinity. These lines correspond to the points in the (α, β) plane where

$$\tan((\alpha^2 - \beta^2)^{\frac{1}{2}} \overline{y}_s) \to \infty.$$

More precisely if we write

$$(\alpha^2 - \beta^2)^{\frac{1}{2}} \overline{y}_s = (n + \frac{1}{2})\pi - \delta$$

for $n = 0, 1, \dots$ where δ is small and positive, then it follows

$$\Omega_r = \frac{\overline{y}_s}{\delta (n+\frac{1}{2})\pi} \beta^2 \sqrt{\beta^2 + (n+\tfrac{1}{2})^2 \frac{\pi^2}{\overline{y}_s^2}} + \dots,$$

$$\Omega_i = \mathrm{sgn}(\delta) \left(\beta^2 + (n + \tfrac{1}{2})^2 \frac{\pi^2}{\overline{y}_s^2} \right)^{\frac{1}{2}} |2\Omega_R|^{-\frac{1}{2}} + \dots. \tag{4.7a, b}$$

Hence, if β is held fixed, then when α is within distance $O(\delta)$ of a dotted curve, the disturbance has frequency $O(\delta^{-1})$ and a growth rate $O(\delta^{\frac{1}{2}})$. The wave grows or decays depending on whether it is below or above the dotted curve respectively.

It also follows from (4.7b) that the growth rates in the unstable regions above $\alpha = \beta$ increase with n. In fact if we hold β fixed and let $\alpha \to \infty$, analysis of (4.4) shows that the unstable intervals are of range $0(\alpha^{-\frac{2}{3}})$ and that the growth rates in the unstable intervals are typically $0(\alpha^{\frac{2}{3}})$. The frequency in the unstable intervals also increases like $\alpha^{\frac{2}{3}}$, and so the unstable intervals for $n >> 1$ correspond to high frequency, short wavelength modes with increasingly high growth rates. However, the fact that these modes occur over decreasingly short ranges in α as n inceases means that they might not be excited naturally in a physically realistic flow situation.

In Figure 3 we show the dependence of Ω_r and Ω_i on α for the case $\overline{y}_s = 4$, $\beta = 1$. We observe the predicted monotonic increase in the growth rates and the shortening of the range of unstable wavenumbers as α increases. Results at other values of $\overline{y}_s$ and β are similar with the range over which the oscillations in α occur decreasing as either $\overline{y}_s$ or β are increased.

It is perhaps significant that the modes which reduce to those discussed by Smith (1988) when $\overline{y}_s \to \infty$, i.e. the subsonic modes, have growth rates less than those with $\beta < \alpha$. As a measure of the growth rate of the subsonic mode we can take its value close to the line $\alpha = \beta$. In fact in all the cases calculated we found that this mode had its maximum value in this neighbourhood. In Figure 4, we show the dependence of Ω_i on $\overline{y}_s$ for different values of the spanwise wavenumber β with $\alpha = .999\beta$. We see that Ω_i increases to a maximum and then decreases. Thus, for a given (α, β) close to the line $\beta = \alpha$ there is an optimum value of $\overline{y}_s$ which maximizes the growth rate. In fact it follows from (4.4) that if

$$\alpha^2 - \beta^2 = \Delta \quad \text{with} \quad |\Delta| << 1, \quad \overline{y}_s|\Delta|^{\frac{1}{2}} << 1,$$

then close to $\alpha = \beta$

$$\overline{y}_s i^{\frac{1}{3}} \alpha^{\frac{7}{3}} \simeq \frac{Ai'(\varsigma_0)}{\int_{\varsigma_0}^{\infty} Ai(\varsigma)d\varsigma}.$$

Asymptotic analysis for this expression for small and large $\overline{y}_s$ demonstrates that the growth rate has a maximum for intermediate $\overline{y}_s$, e.g.

$$\Omega \sim \alpha^3 \overline{y}_s + \left(\frac{i}{\alpha \overline{y}_s}\right)^{\frac{1}{2}} \quad \text{as} \quad \overline{y}_s \to \infty.$$

We note however that if $\overline{y}_s|\Delta|^{\frac{1}{2}}$ is not small then the above simplification does not hold. Further for $\overline{y}_s|\Delta|^{\frac{1}{2}}$ large the asymptotic behaviour depends on the sign of Δ. If $\Delta < 0$, then the eigenrelation (4.4) reduces to

$$(i\alpha)^{\frac{1}{3}} \beta^2 \int_{\varsigma_0}^{\infty} Ai(\varsigma)d\varsigma = (\beta^2 - \alpha^2)^{\frac{1}{2}} Ai'(\varsigma_0), \tag{4.8}$$

which is valid for $\overline{y}_s >> 1$, α, $\beta = 0(1)$. As expected this is the eigenrelation of Smith (1988). In Figure 5 we show the dependence of Ω_r and Ω_i on $\overline{y}_s$ for different values of β with $\alpha = 1$. For these subsonic modes the growth rates approach constant values as $\overline{y}_s \to \infty$, in agreement with Smith's (1988) shock-free analysis. Hence subsonic modes of fixed wavelength do not feel presence of the shock when $\overline{y}_s$ is large (nor do short-wavelength subsonic modes when $\overline{y}_s = O(1)$).

However, if $\overline{y}_s \to \infty$ with $\Delta > 0$ there is no simplification similar to (4.8) since the growth rate oscillates, with the regions of stability and instability becoming increasingly thin as $\overline{y}_s$ is increased.

5. CONCLUSIONS

We have shown that if shock effects are to be included in a Tollmien-Schlichting stability analysis of flow past a wedge, then the Newtonian approximation, $(\gamma - 1) << 1$, must be made if the complicating effects of non-parallelism are to be avoided. With this assumption we have seen that the viscous modes have their dispersion relationship crucially altered by the presence of a shock at a distance $\overline{y}_s$ from the wedge scaled on the upper deck thickness. In the limit $\overline{y}_s \to \infty$ the modes of Smith (1988) are recovered and the shock has no zeroth order effect on the growth rate of these modes. In addition at large values of $\overline{y}_s$ there is an infinite discrete spectrum of disturbances which persist to the shock. These modes have large growth rates at high frequencies but occur only over small ranges of wavenumber. This suggests that the frequency might have to be tuned to produce instability. It is therefore an open question as to whether these additional modes play a critical role in the transition process in hypersonic flows. We also point out that where our analysis predicts infinite frequencies, i.e. close to the dotted curves in Figure 2a,b,c, then our asymptotic expansions will fail and a new structure must be set up in order to account for the faster disturbance response. This problem has not been investigated in this paper.

The structure of the eigenfunctions associated with the different modes shown in Figure 2 depends on which side of the line $\alpha = \beta$ the given mode occurs. If $\beta < \alpha$ then the disturbances are described by a combination of exponentially growing and decaying functions. In any of the limits where the shockless eigenrelation is recovered, e.g. $\overline{y}_s \to \infty$ or $\beta \to \infty$, the exponentially decaying function dominates and the eigenfunctions also tend to those of the shockless problem. The modes which have $\beta > \alpha$ are described by trigonometric functions and therefore in any limit involving $\overline{y}_s, \beta_1$ or α they are oscillatory and 0(1) between the wedge and the shock. This class of the mode is perhaps best thought of as sound waves trapped between the wedge and shock and amplified by the boundary layer.

So far we have only discussed the effect of the shock on viscous modes of instability yet it is well-known that compressible shear flows can also support unstable inviscid disturbances. The inviscid modes of instability for a compressible boundary layer have wavelengths comparable with the boundary layer thickness so we consider perturbations to the flow described in §2 but with wavelengths scaled on the main deck thickness. In order to be consistent with previous investigations we drop the re-scaling (3.1) introduced in order to simplify the triple-deck analysis of §3.

Following for example Mack (1987), we scale wavelengths on the main deck thickness and the wavespeed on the free-stream speed. The pressure perturbation P for an inviscid mode then satisfies the compressible Rayleigh equation

$$P'' - \frac{2\overline{u}'}{\overline{u}-c}P' + \frac{\overline{T}'}{\overline{T}}P' - (\alpha^2+\beta^2)(1 - \frac{\alpha^2(\overline{u}-c)^2M^2}{(\alpha^2+\beta^2)\overline{T}})P = 0. \tag{5.1}$$

Here a dash denotes a derivative with respect to boundary layer variable y^*, whilst the basic velocity field $\overline{u}$ and temperature $\overline{T}$ for an insulated wall are given by

$$\overline{u} = f'(\eta),\ \overline{T} = 1 + \frac{\gamma-1}{2}M^2(1-f'^2), \tag{5.2a, b}$$

where f is the Blasius function and η is the Dorodnitsy-Howarth variable defined by

$$\eta = \int_0^{y^*} \frac{dy^*}{\overline{T}}. \tag{5.2c}$$

The quantities α and β are downstream and spanwise wavenumbers, whilst c is the wavespeed. We confine our attention to modes which satisfy

$$P' = 0 \text{ on } y^* = 0, \quad P \to 0 \text{ as } y^* \to \infty, \tag{5.3a, b}$$

which together with (5.1) specify an eigenrelation $c = c(\alpha, \beta)$. The point at issue here is whether the decay of P when $y \to \infty$ is sufficiently rapid to mean that the shock located outside the boundary layer has a negligible effect on the disturbance. In order to see whether this is the case we discuss the structure of (5.1) in the limit $M \to \infty$.

Since this work was completed we learned of an independent investigation of (5.1) for $M >> 1$ by S. N. Brown and F. T. Smith. They have concentrated on the so-called vorticity mode (Mack, 1987), while we will study the acoustic modes. We restrict our discussion of (5.1) to the minimum which explains the large M structure of P and shows that in the wedge problem considered in this paper the shock generally has only an exponentially small effect on the eigenrelation. Further, we concentrate on the generalized inflection point neutral modes of (5.1) which have $\beta = 0$. The discussion we give can be extended to unstable two and three-dimensional modes, although an extra critical-layer region then needs to be included.

The generalized inflection point occurs where $\overline{u}_{y^* y^*}\overline{T} = \overline{u}_{y^*}\overline{T}_{y^*}$, and if we write

$$\eta = B + \sqrt{2\log M^2} - \frac{\log\sqrt{2\log M^2} + \log\tilde{Y}}{\sqrt{2\log M^2}}, \tag{5.4}$$

where B comes from the large η asymptotic form of f

$$f = \eta - B + \frac{de^{-(\eta-B)^2/2}}{(\eta - B)^2} + \dots, \tag{5.5a}$$

then we can show that it occurs at

$$\tilde{Y} = \frac{1}{(\gamma - 1)d}.$$

The wavespeed is then given by $\overline{u}$ evaluated at the inflection point so that (5.1) is not singular there. Thus, we have

$$c = 1 - \frac{\tilde{c}}{M^2} + \dots, \tag{5.5b}$$

with $\tilde{c} = \frac{1}{\gamma-1}$.

Next we anticipate the change of scale of the boundary layer for $M >> 1$ and write

$$\alpha = \frac{A}{M^2} + \dots .$$

It follows that the zeroth order approximation to (5.1) for $0(1)$ values of η is

$$\frac{d^2P}{d\eta^2} - \frac{2f''}{f'-1}\frac{dP}{d\eta} - \frac{A^2(\gamma-1)^2(1-f'^2)^2}{4}\left(1 - \frac{2(1-f')}{(\gamma-1)(1+f')}\right)P = 0. \tag{5.6}$$

For large values of η it is found that the solutions of (5.6) are such that

$$P \sim \frac{De^{-(\eta-B)^2}}{\eta - B} + E, \tag{5.7}$$

where D and E are arbitrary constants, one of which can be fixed as a normalisation condition. Anticipating the result we choose $D = 1$ without loss of generality. E is then determined by investigating the solution of (5.1) in the region where $\tilde{Y} = 0(1)$, and also in the region above this logarithmically thin layer. In passing it is of interest to note that this layer also controls the Görtler vortex mechanism in hypersonic boundary layers (Hall, 1988).

For $\tilde{Y} = 0(1)$ we seek a solution of (5.1) by writing

$$P = \tilde{P}(\tilde{Y}) + \dots,$$

and after some manipulation we find that $\tilde{P}$ satisfies

$$\tilde{Y}^2\frac{d^2\tilde{P}}{d\tilde{Y}^2} + \tilde{Y}\frac{d\tilde{P}}{d\tilde{Y}} + \frac{2d\tilde{Y}^2}{\tilde{c} - d\tilde{Y}}\frac{d\tilde{P}}{d\tilde{Y}} = 0, \tag{5.8a}$$

which is to be solved in the range $0 < \tilde{Y} < \infty$. The solution of this equation is

$$\tilde{P} = \tilde{D}(\frac{d^2\tilde{Y}^2}{2} - 2\tilde{c}d\tilde{Y} + \tilde{c}^2 \log \tilde{Y}) + \tilde{E}, \tag{5.8b}$$

where $\tilde{D}$ and $\tilde{E}$ are constants. This solution is to be matched with (5.7) when $\tilde{Y} \to \infty$ and a solution of (5.1) valid for $\eta - B - \sqrt{2\log M^2} >> (\sqrt{2\log M^2})^{-1}$. The former condition yields

$$\tilde{D} = \frac{2\sqrt{2\log M^2}}{d^2 M^4} + \ldots.$$

Above this logarithmically thin region there is an outer region with the scaled coordinate $\overline{\eta} = M^{-2}\eta$ (note from (5.2c) that $y^* = O(1)$ in this region). The solution in this region is $P \sim \overline{E}\exp(-A\overline{\eta}) = \overline{E}\exp(-\alpha\eta)$. Matching this solution to the logarithmic term in (5.8b) for $\tilde{Y}$ small we find that

$$\overline{E} \sim -\frac{4\tilde{c}^2 \log M^2}{ad^2 M^2},$$

and matching to the constant yields

$$\tilde{E} \sim \overline{E}.$$

Note that this means that the constant term dominates the $\tilde{Y}$-dependent part of the complementary function in (5.8b). However, on substituting (5.8b) into (5.1) we confirm that (5.8a) is still the leading order equation for $\tilde{P}'(\tilde{Y})$. An alternative way to see this is to work with the equation for the velocity fluctuation normal to the wall, rather than that for the pressure fluctuation.

Finally, matching back to the lower layer adjacent to the wall we conclude that

$$E \sim \tilde{E} \sim \overline{E} << 1.$$

Therefore, A is determined by the eigenvalue problem specified by (5.6) together with

$$P' = 0 \text{ on } \eta = 0, \quad P \to 0 \text{ as } \eta \to \infty. \tag{5.9}$$

This problem was solved numerically to yield the sequence of eigenvalues $A = 4.81$, 13.84, 23.34, 33.03, 42.82, 52.67, 62.55, 72.45, 82.37, 92.29, ...

In Figure 6 we have shown a comparison of our one term asymptotic form for α with the values given by Mack (1987). The agreement is satisfactory except our analysis needs modification to explain the 'kinks' in the eigencurves found by Mack (but see also the work of S.N. Brown and F.T. Smith, private communication). The eigenfunctions associated with the above set of eigenvalues are shown in Figure 7. Finally, we note that the higher order modes of the eigenvalue problem can be derived by applying the WKB method to (5.6). In fact we anticipate that the WKB desription of the modes discussed above coupled to the vorticity mode description of Brown and Smith could be used to explain the kinks in the neutral curves given by Mack. Thus we expect that these eigenvalues are split apart by an exponentially small amount in the manner discussed by for example DiPrima and Hall (1984) for the Taylor problem.

The main result of the above analysis is that the high Mach number structure of the inviscid modes does not lead to a reduced rate of decay at infinity. Hence the shock cannot have anything other than an exponentially small effect on them, at least until it is within an order one distance of the boundary-layer. Then the steady flow changes and the effect of the shock is felt within the outer region. We conclude that the viscous modes are those which are most likely to be influenced by the presence of shocks in high Mach number flows.

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APPENDIX: SHOCK CONDITIONS

In this appendix we give the conditions that must hold at the shock – see Moore (1954), Ribner (1954) and McKenzie and Westphal (1968) for similar derivations.

For a shock given by $\eta = f(\xi, z, t)$, the jump conditions across a shock are (e.g. Majda 1983)

$$
\begin{aligned}
&\tfrac{\partial f}{\partial t}[p] + \tfrac{\partial f}{\partial \xi}[\rho u] - [\rho v] + \tfrac{\partial f}{\partial z}[\rho w] = 0, \\
&\tfrac{\partial f}{\partial t}[\rho u] + \tfrac{\partial f}{\partial \xi}[\rho u^2 + p] - [\rho u v] + \tfrac{\partial f}{\partial z}[\rho u w] = 0, \\
&\tfrac{\partial f}{\partial t}[\rho v] + \tfrac{\partial f}{\partial \xi}[\rho u v] - [\rho v^2 + p] + \tfrac{\partial f}{\partial z}[\rho v w] = 0, \\
&\tfrac{\partial f}{\partial t}[\rho w] + \tfrac{\partial f}{\partial \xi}[\rho u w] - [\rho v w] + \tfrac{\partial f}{\partial z}[\rho w^2 + p] = 0, \\
&\tfrac{\partial f}{\partial t}[\rho \mathcal{E}] + \tfrac{\partial f}{\partial \xi}[u(\rho \mathcal{E} + p)] - [v(\rho \mathcal{E} + p)] + \tfrac{\partial f}{\partial z}[w(\rho \mathcal{E} + p)] = 0,
\end{aligned}
\tag{A.1}
$$

where $\mathcal{E} = \frac{p}{(\gamma-1)\rho} + \frac{1}{2}(u^2 + v^2 + w^2)$. The basic solution specific by (2.2) can be shown to satisfy these jump conditions with $f \equiv 0$.

We assume that there is a small disturbance beneath the shock and write

$$
(\rho, u, v, w, p, \mathcal{E}) = (R, U, V, W, P, E) + (\tilde{r}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}, \tilde{e}). \tag{A.2a}
$$

For our flow any waves above the shock propagate towards the shock. Hence the disturbance cannot extend above the shock, where we write

$$
(\rho, u, v, w, p, \mathcal{E}) = (\overline{R}, \overline{U}, \overline{V}, \overline{W}, \overline{P}, \overline{E}). \tag{A.2b}
$$

The linearized shock conditions are obtained by substituting (A.2) into (A.1) and neglecting all nonlinear disturbance terms. The position of the shock will vary by only a small amount from $\eta = 0$, so we write $f = \tilde{f}$. Also, since the undisturbed flow is a constant above and below the shock, to a consistent approximation the jump conditions can be evaluated at $\eta = 0$ instead of $\eta = \tilde{f}$. Finally, we assume that the linear disturbance can be expressed as a superposition of Fourier modes, so that it is sufficient to study a single mode, i.e. we assume

$$
\tilde{p} \propto \exp(i(\alpha\xi + \beta z - \Omega t)), \text{ etc..} \tag{A.3}
$$

Note that here α, β and Ω are not the scaled quantites of section 4. After a little manipulation it follows from (A.1) that

$$
\begin{aligned}
&-i(\Omega-\alpha U)(\overline{R}-R)\tilde{f}+R\tilde{v}+V\tilde{r}=0,\\
&i\alpha(\overline{P}-P)\tilde{f}+RV\tilde{u}=0,\\
&2RV\tilde{v}+V^2\tilde{r}+\tilde{p}=0,\\
&RV\tilde{w}+i\beta(\overline{P}-P)\tilde{f}=0,
\end{aligned}
\qquad (A.4)
$$

$$
-i(\Omega-\alpha U)(\overline{RE}-RE)\tilde{f}+i\alpha U(\overline{P}-P)\tilde{f}+(RE+P)\tilde{v}+VR\tilde{e}+VE\tilde{r}+V\tilde{p}=0.
$$

$\tilde{f}$ and $\tilde{r}$ can be eliminated from the above to obtain

$$
\begin{aligned}
&\beta\tilde{u}-\alpha\tilde{w}=0,\\
&RV\tilde{v}+\tilde{p}-\tfrac{(\Omega-\alpha U)RV}{\alpha V}\tilde{u}=0,\\
&\left[\tfrac{\gamma}{\gamma-1}P+\tfrac{1}{2}R(V^2-U^2)\right]\tilde{v}+\left[\tfrac{\gamma+1}{2(\gamma-1)}V-\tfrac{U^2}{2V}\right]\tilde{p}+\tfrac{(\Omega-\alpha V)(\overline{RE}-RE)RV}{\alpha(\overline{P}-P)}\tilde{u}=0,
\end{aligned}
\qquad (A.5)
$$

where for our basic state and nondimensionalization

$$
R=1,\quad P=\frac{1}{\gamma M^2},\quad U=\frac{1}{(1+\varepsilon^2\mathcal{T}^2)^{\frac{1}{2}}},\quad V=\varepsilon\overline{V},\quad \overline{V}=-\frac{\mathcal{T}}{(1+\varepsilon^2\mathcal{T}^2)^{\frac{1}{2}}},
$$

$$
\overline{P}=\frac{\varepsilon}{\gamma M^2}-\frac{\varepsilon(\gamma-1)(1-\varepsilon^2)\mathcal{T}^2}{2\gamma(1+\varepsilon^2\mathcal{T}^2)},\quad \mathcal{T}=\tan\sigma.
$$

Beneath the shock the linear waves have solutions proportional to $e^{i\underline{k}\cdot\underline{\xi}-i\Omega t}$, where $\underline{k}=(\alpha,\nu,\beta)$ and $\underline{\xi}=(\xi,\eta,z)$. The solutions for the different types of wave have the following forms:

Acoustic:

$$
\begin{aligned}
&(\Omega-\underline{U}.\underline{k})^2=k^2a^2\quad\text{where}\quad U=(U,V,0),\ a^2=\tfrac{\gamma P}{R}\\
&(\tilde{p},\tilde{r})=(1,\tfrac{1}{a})e^{i\underline{k}.\underline{\xi}-i\Omega t},\quad (\tilde{u},\tilde{v},\tilde{w})=\tfrac{\underline{k}}{R(\Omega-\underline{U}.\underline{k})}e^{i\underline{k}.\underline{\xi}-i\Omega t}
\end{aligned}
\qquad (A.6)
$$

Entropy:

$$
\Omega-\underline{U}.\underline{k}=0,\quad \tilde{r}=\tilde{r}_s e^{i\underline{k}.\underline{\xi}-i\Omega t},\quad \tilde{p}=\tilde{u}=\tilde{v}=\tilde{w}=0 \qquad (A.7)
$$

Vorticity:

$$
\begin{aligned}
&\Omega-\underline{U}.\underline{k}=0,\quad (\tilde{u},\tilde{v},\tilde{w})=(\tilde{u}_s,-\nu^{-1}(\alpha\tilde{u}_s+\beta\tilde{w}_s),\tilde{w}_s)e^{i\underline{k}.\underline{\xi}-i\Omega t}\\
&\tilde{p}=\tilde{r}=0.
\end{aligned}
\qquad (A.8)
$$

We denote the pressure amplitudes of the incident and reflected acoustic waves by p_1 and p_2 respectively, and their corresponding η-wavenumbers by ν_1 and ν_2 respectively. Then substituting (A.6-8) into (A.5), and using the nondimensional forms of (2.2), we obtain after some manipulation

$$\left(\frac{K_1 + K_2\nu_1}{\Omega - \alpha V - \nu_1 V}\right) p_1 + \left(\frac{K_1 + K_2\nu_2}{\Omega - \alpha U - \nu_2 V}\right) p_2 = 0,$$

where

$$K_1 = \frac{-(\Omega - \alpha U)((\Omega - \alpha U)^2(1 + \varepsilon^2\mathcal{T}^2)(3 - \gamma + (\gamma + 1)\varepsilon) - (\alpha^2 + \beta^2)\varepsilon\mathcal{T}^2((\gamma + 1) - (\gamma + 5)\varepsilon))}{(\Omega - \alpha U)^2(1 + \varepsilon^2\mathcal{T}^2) + (\alpha^2 + \beta^2)\varepsilon\mathcal{T}^2},$$

$$K_2 = \frac{(\gamma + 1)(1 - \varepsilon)\varepsilon\mathcal{T}}{(1 + \varepsilon^2\mathcal{T}^2)^{\frac{1}{2}}}.$$

On substituting the scalings (2.16) into the above, we find that at leading order $|K_1| >> |K_2|$, $\alpha U >> |\Omega - \nu_1 V|$, and hence

$$p_1 + p_2 = 0,$$

i.e. condition (3.10). Expressions for other quantities can be obtained similarly. For $\sigma >> Re^{-\frac{1}{10}}$ the $\tilde{u}, \tilde{w}$ and $\tilde{r}$ perturbations are dominated by the entropy/shear wave contributions; in particular for our scalings

$$\tilde{u} \sim \frac{2\tilde{\alpha} m^2 \sigma^{\frac{15}{7}} Re^{\frac{3}{14}}}{(\tilde{\alpha}^2 m^2 - \tilde{\beta}^2)^{\frac{1}{2}}} p_1,$$

$$\tilde{w} \sim \frac{2\tilde{\beta} m^2 \sigma^{\frac{23}{7}} Re^{\frac{3}{7}}}{(\tilde{\alpha}^2 m^2 - \tilde{\beta}^2)^{\frac{1}{2}}} p_1,$$

$$\tilde{r} \sim \frac{-2\tilde{\alpha} m^4 \sigma^{\frac{31}{7}} Re^{\frac{9}{14}}}{(\tilde{\alpha}^2 m^2 - \tilde{\beta}^2)^{\frac{1}{2}}} p_1,$$

where

$$\alpha = \sigma^{\frac{15}{7}} Re^{\frac{3}{14}} \tilde{\alpha}, \quad \beta = \sigma^{\frac{23}{7}} Re^{\frac{3}{7}} \tilde{\beta}.$$

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FIGURE CAPTIONS

Figure 1. The geometry of the wedge and shock for a high Mach number flow.

Figure 2a. The neutral curve $\alpha = \alpha(\beta)$ for $\overline{y}_s = 1$.

Figure 2b. The neutral curve $\alpha = \alpha(\beta)$ for $\overline{y}_s = 4$.

Figure 2c. The neutral curve $\alpha = \alpha(\beta)$ for $\overline{y}_s = 16$.

Figure 3a. The growth rate Ω_i as a function of α for $\overline{y}_s = 4, \beta = 1$.

Figure 3b. The wavespeed Ω_r as a function of α for $\overline{y}_s = 4, \beta = 1$.

Figure 4a. Ω_r and Ω_i as functions of $\overline{y}_s$ for $\beta = 1,\ \alpha = 0.999\beta$.

Figure 4b. Ω_r and Ω_i as functions of $\overline{y}_s$ for $\beta = 2,\ \alpha = 0.999\beta$.

Figure 5. The growth rates as a function of $\overline{y}_s$ for $\alpha = 1, \beta = 2, 3, 4$.

Figure 6. The neutral curves for the generalized inflection point modes with $\beta = 0$. Mack's (1987) results. High M asymptotic prediction.

Figure 7a. The eigenfunction $P(y)$ for $A = 4.81$.

Figure 7b. The eigenfunction $P(y)$ for $A = 13.84$.

Figure 7c. The eigenfunction $P(y)$ for $A = 23.34$.

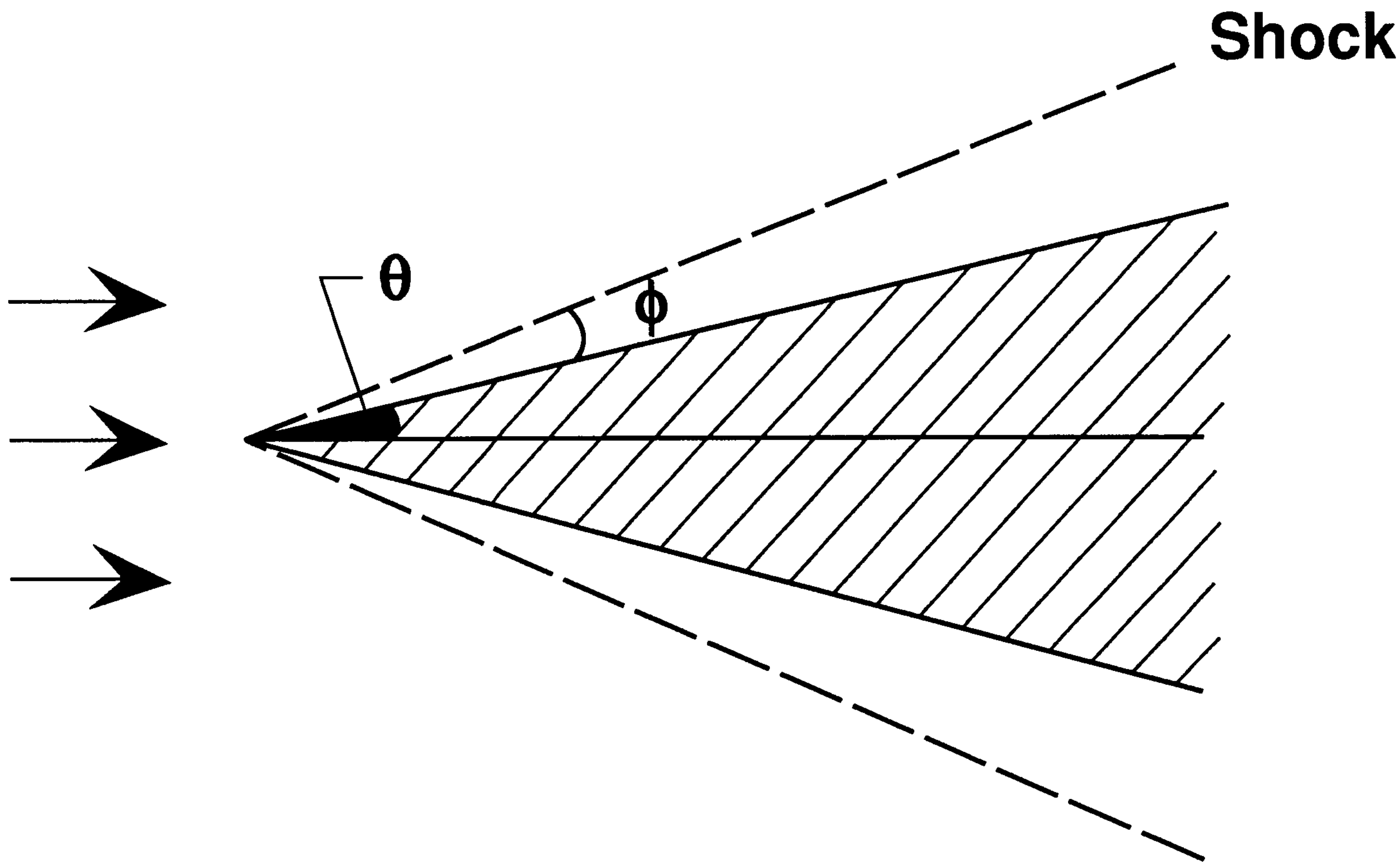
Figure 7d. The eigenfunction $P(y)$ for $A = 33.03$.

Figure 7e. The eigenfunction $P(y)$ for $A = 42.82$.

Figure 7f. The eigenfunction $P(y)$ for $A = 52.67$.

Figure 7g. The eigenfunction $P(y)$ for $A = 62.55$.

Figure 7h. The eigenfunction $P(y)$ for $A = 72.45$.



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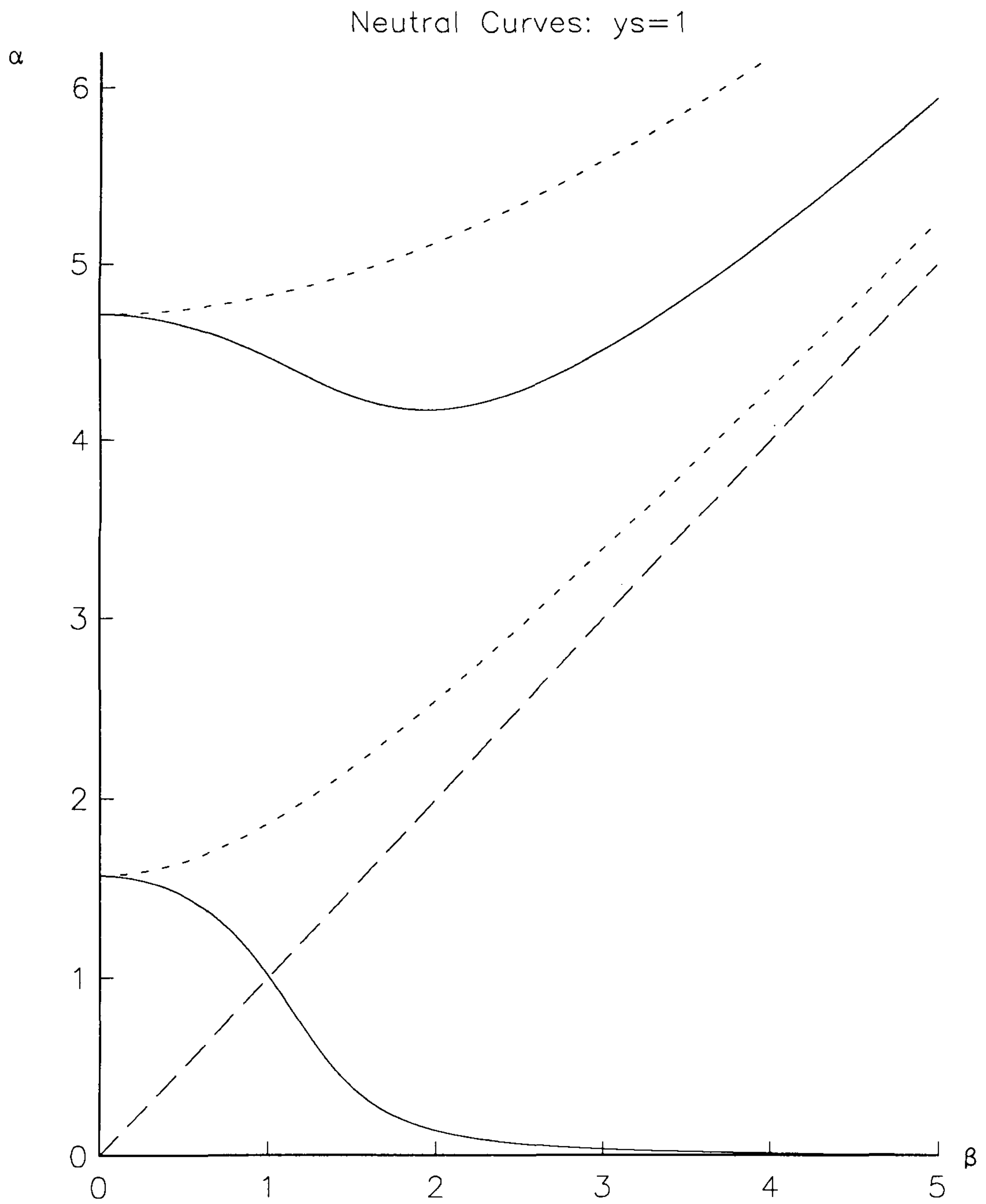
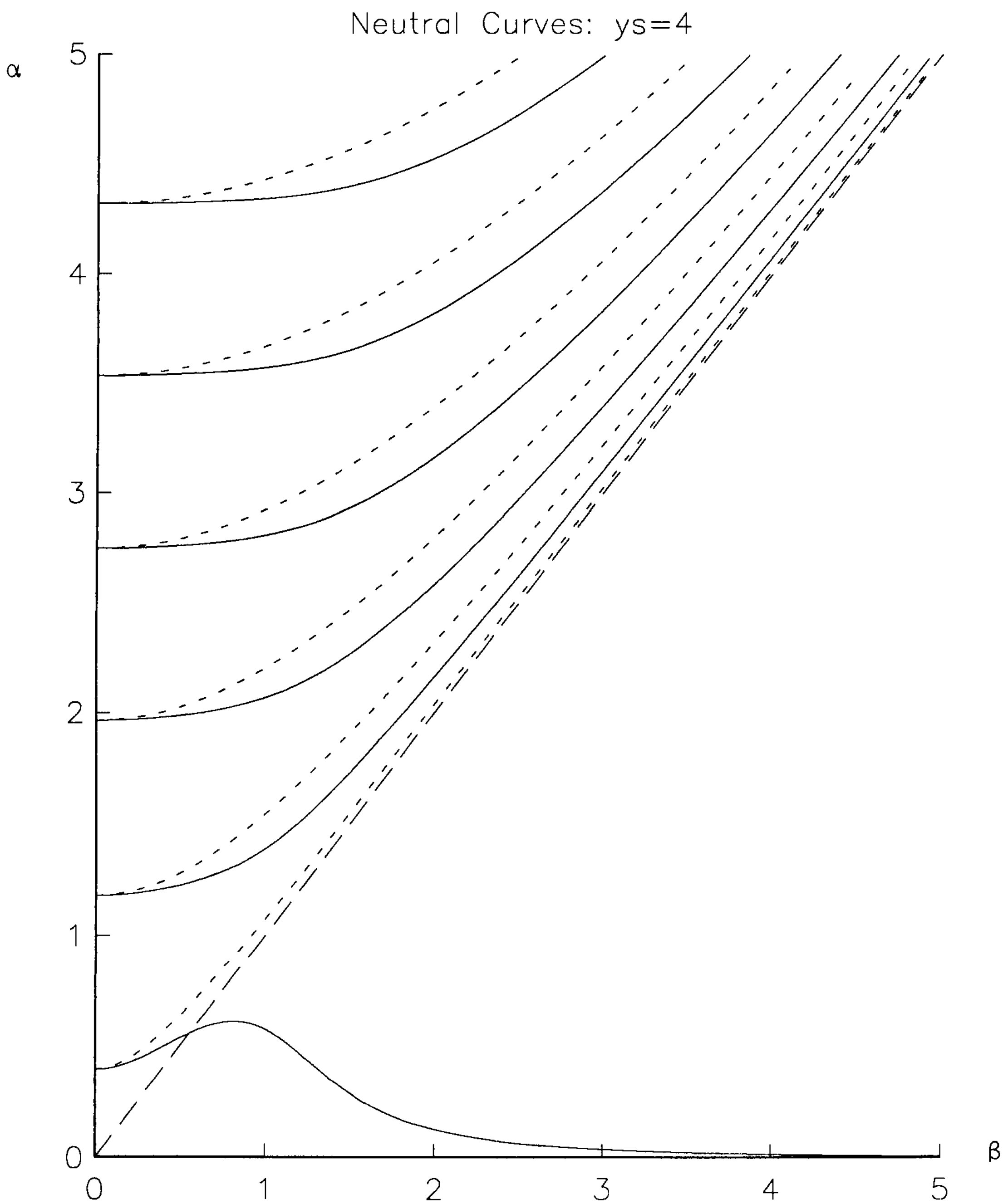


Figure 2a.



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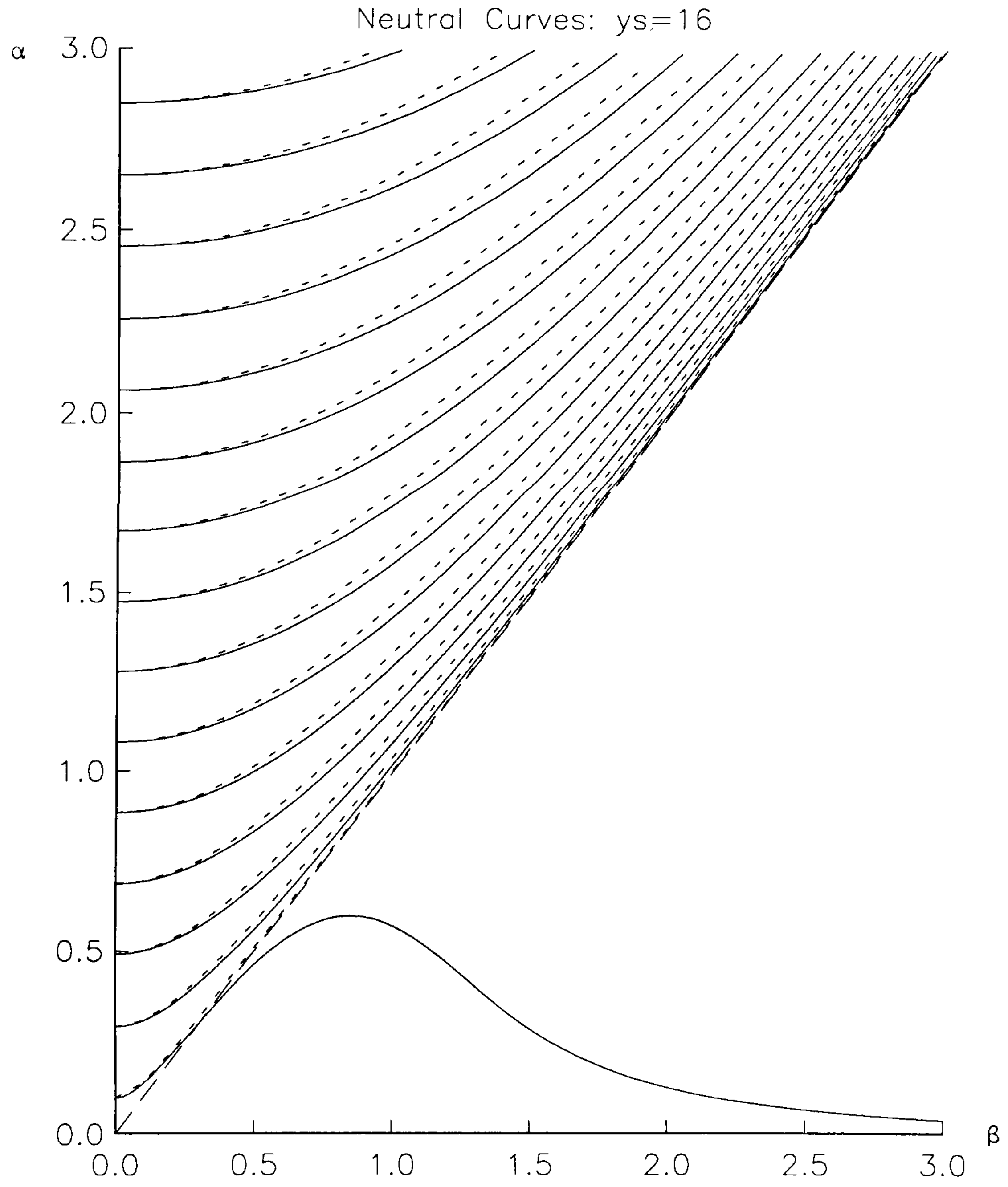

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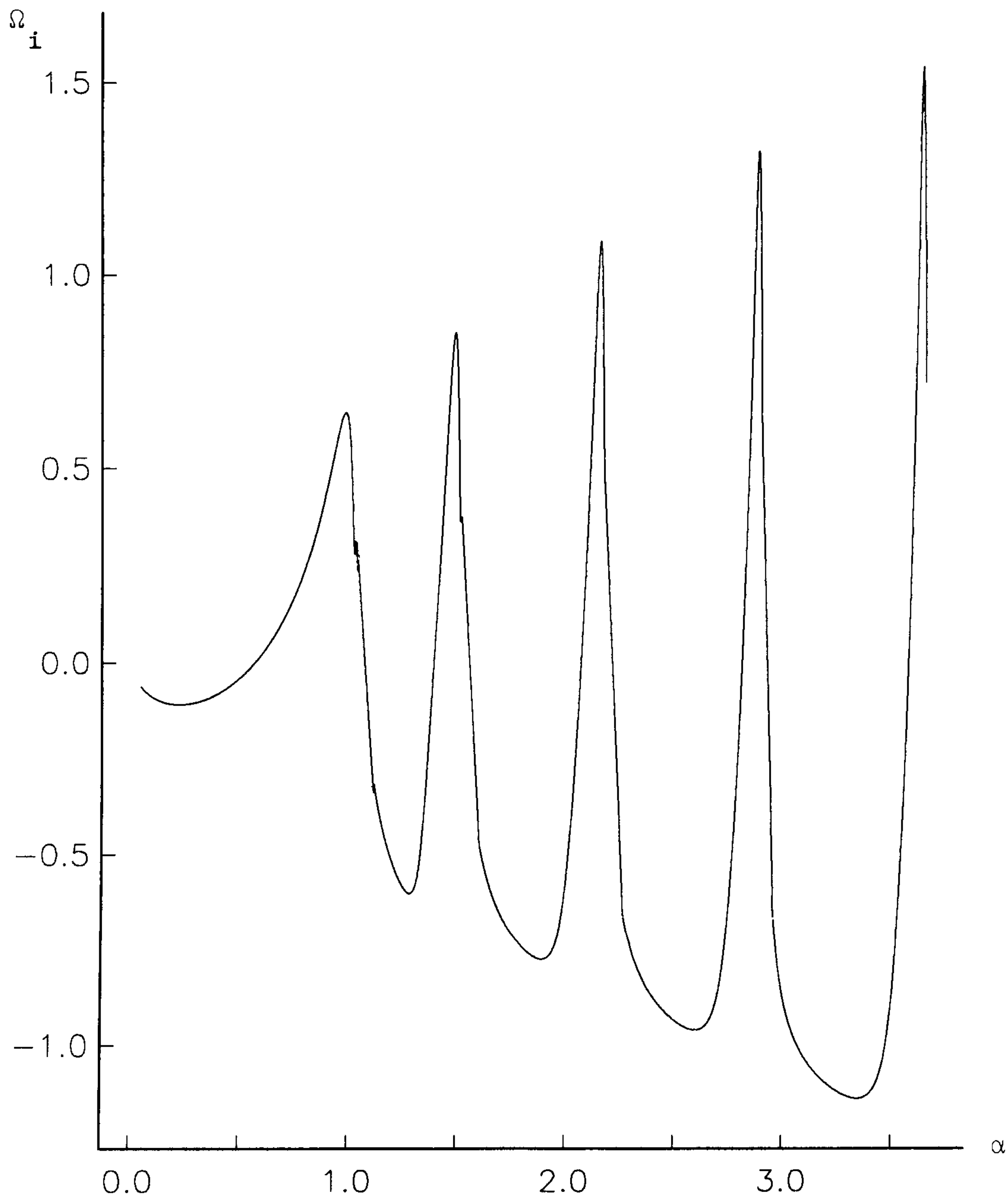

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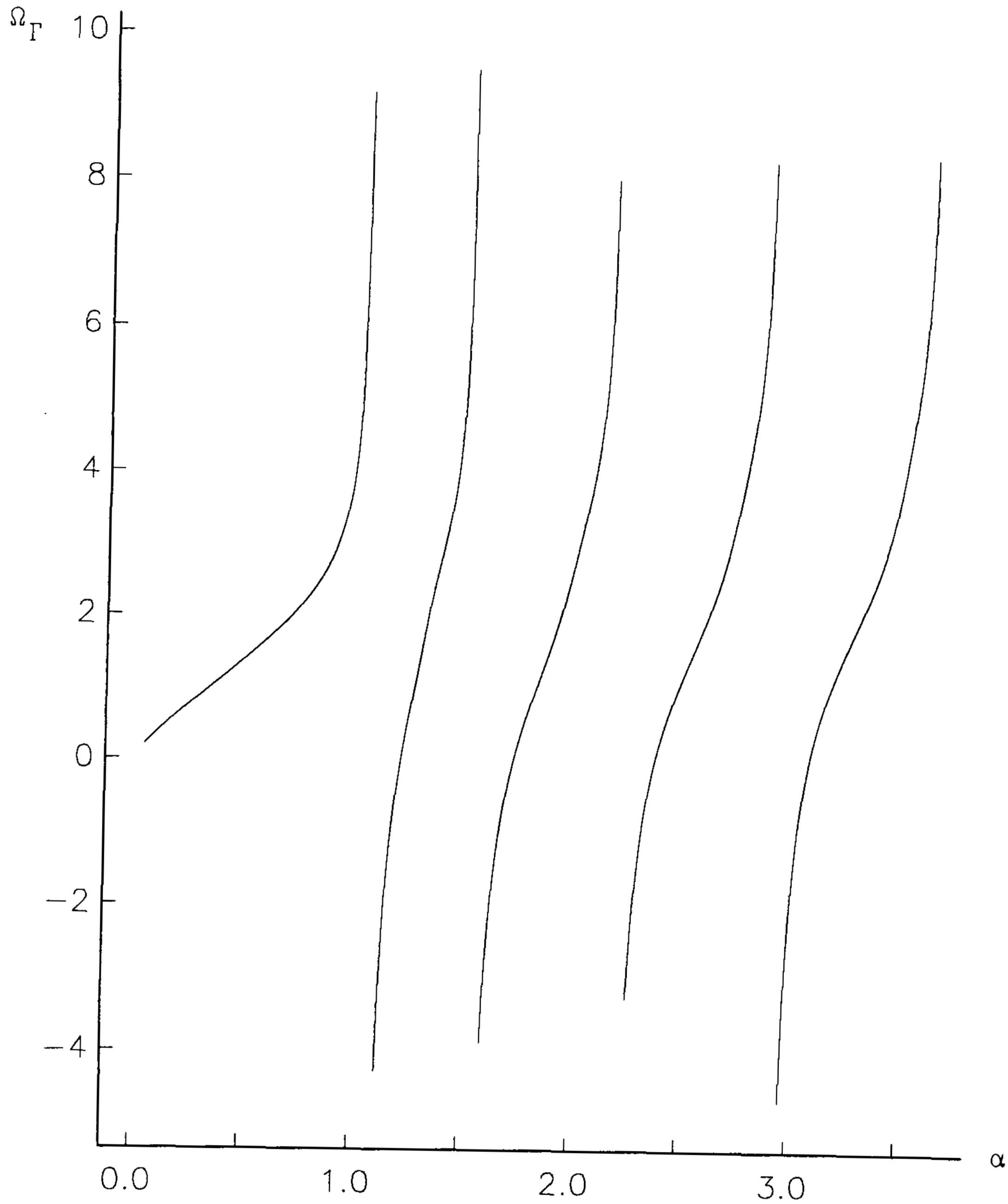

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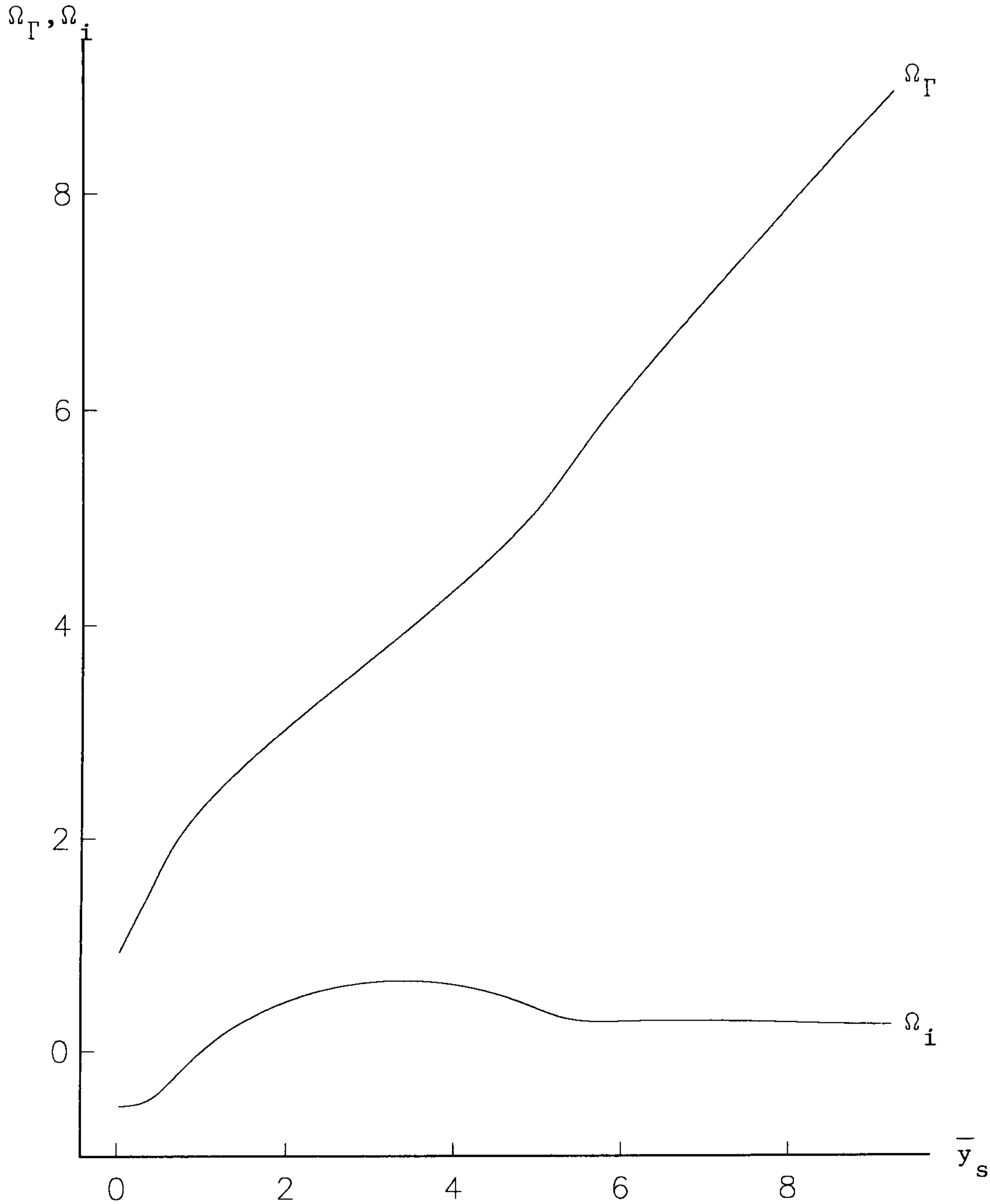
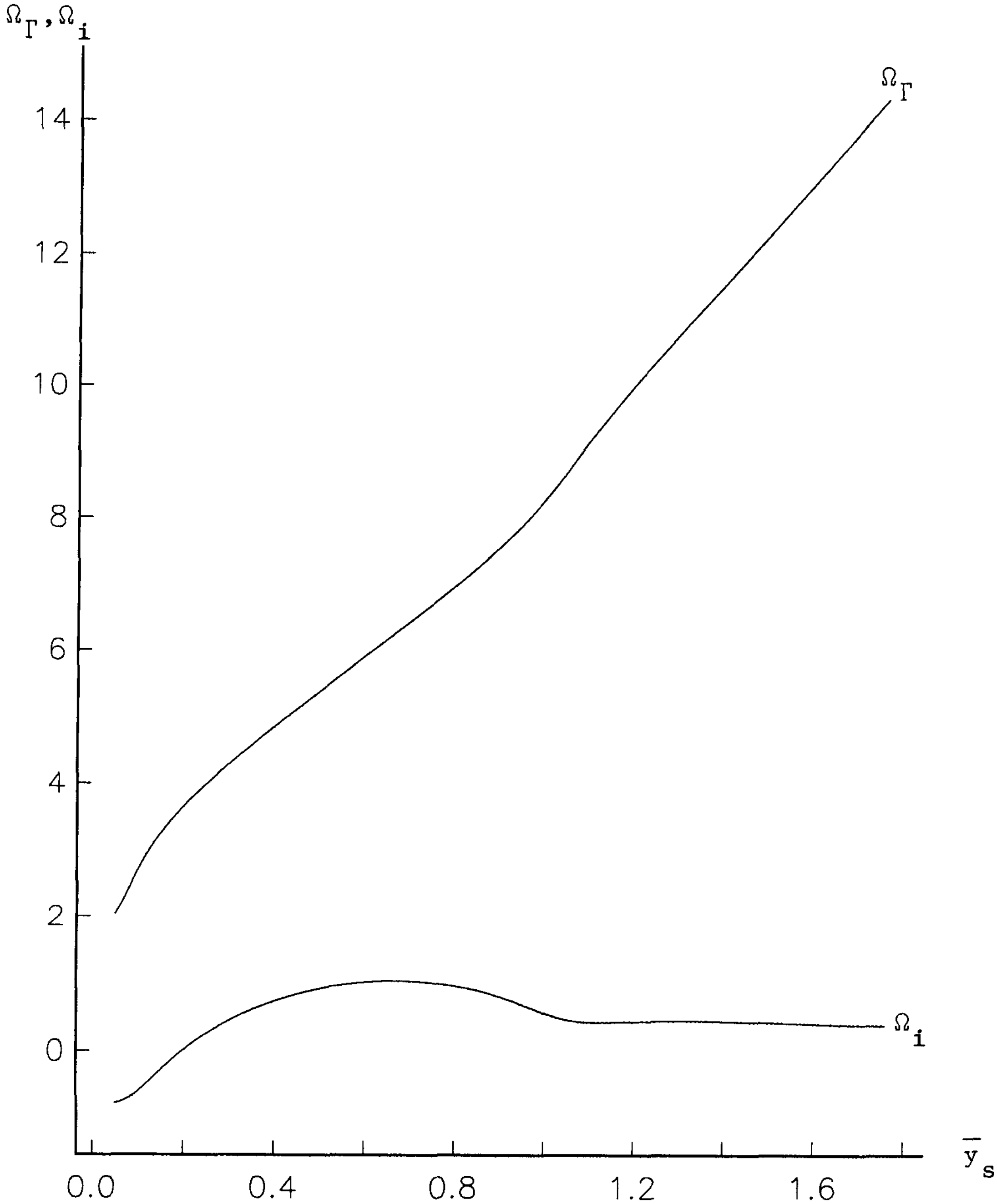


Ω_Γ, Ω_i
8
6
4
2
0
Ω_Γ
Ω_i
0
2
4
6
8
$\overline{y}_s$

Figure 4a.



Ω_Γ, Ω_i
Ω_Γ
Ω_i
14
12
10
8
6
4
2
0
0.0
0.4
0.8
1.2
1.6
$\bar{y}_s$

Figure 4b.

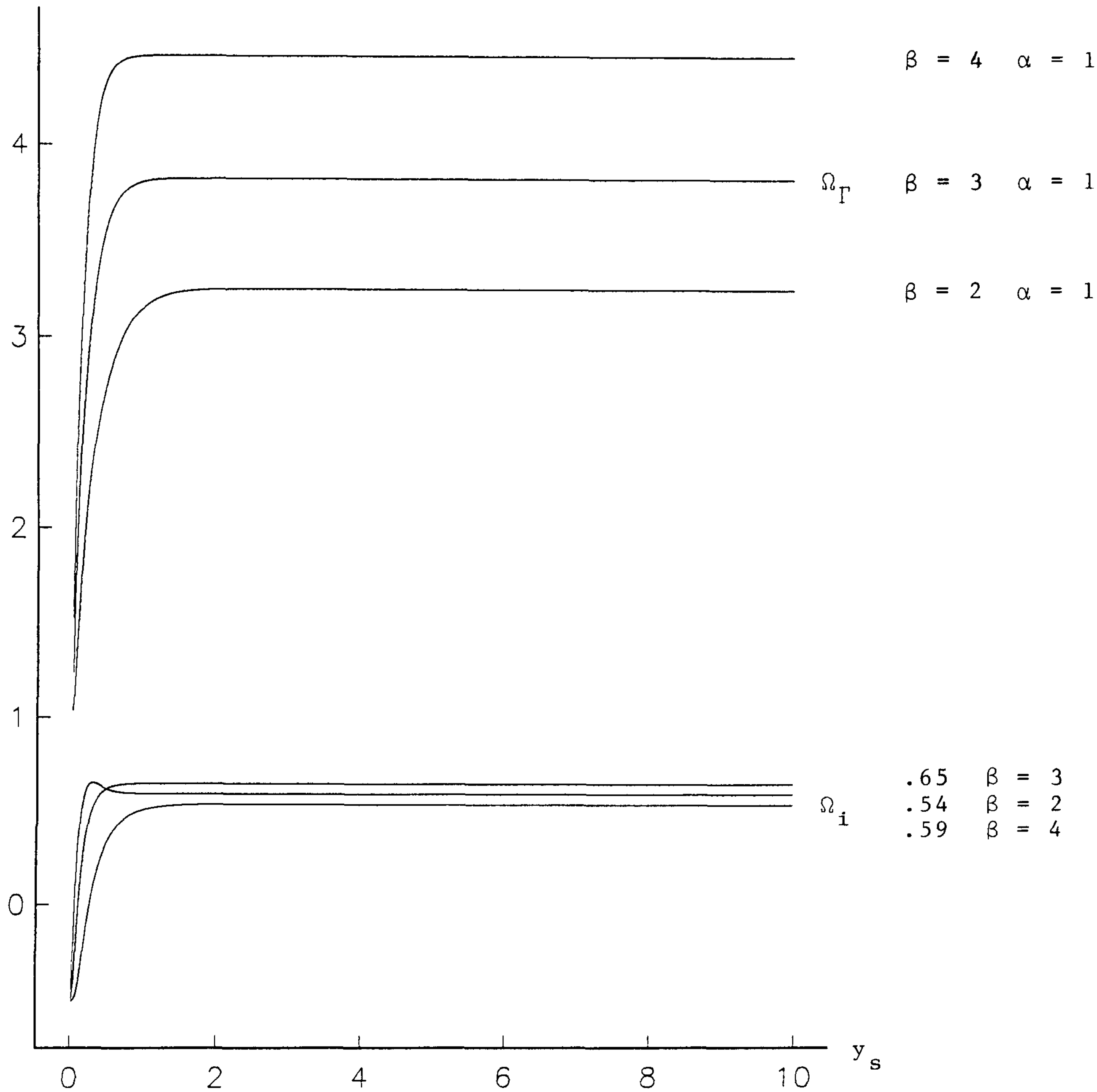
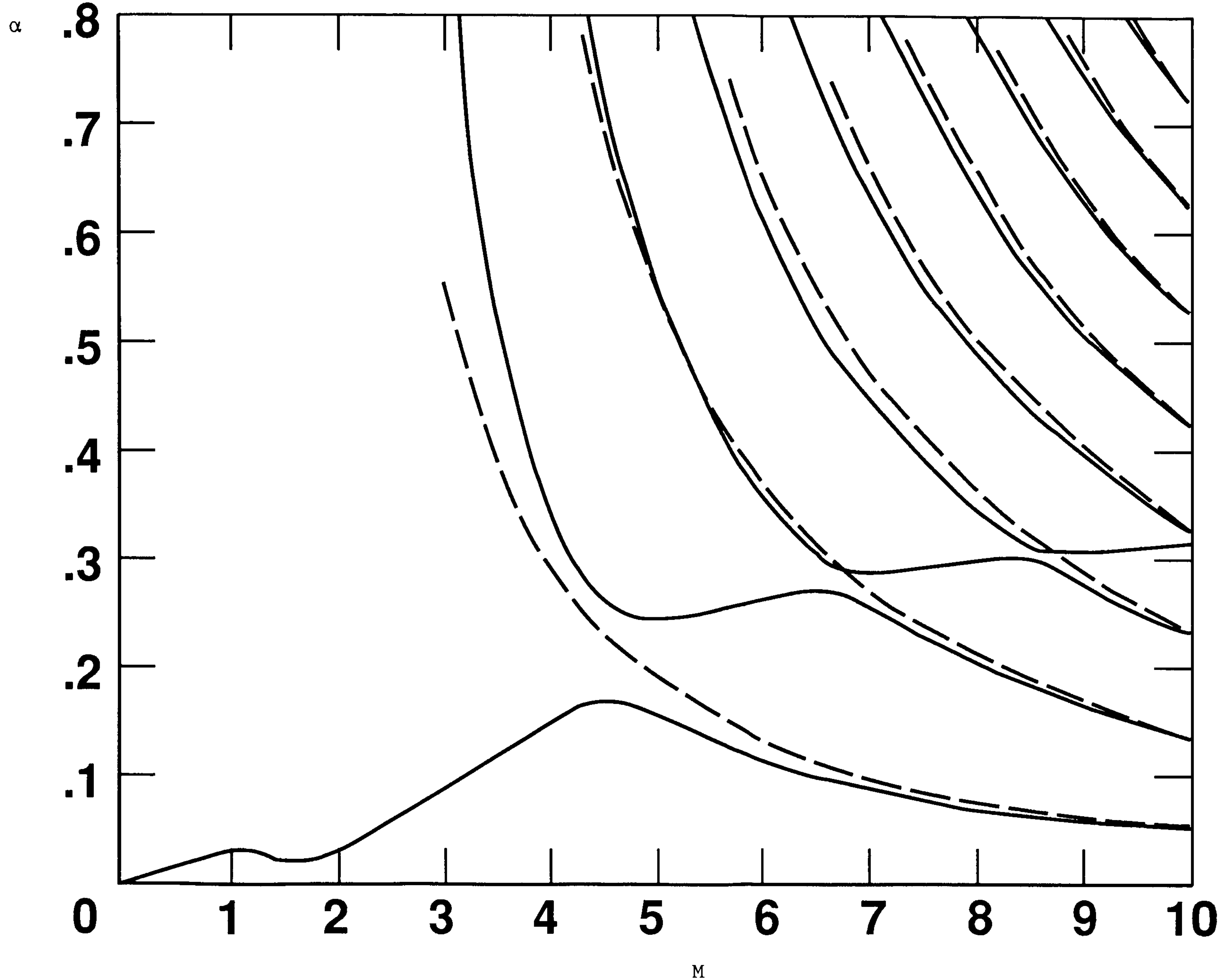

β = 4 α = 1
Ω_Γ β = 3 α = 1
β = 2 α = 1
.65 β = 3
Ω_i .54 β = 2
.59 β = 4
0
1
2
3
4
0
2
4
6
8
10
y_s

Figure 5.



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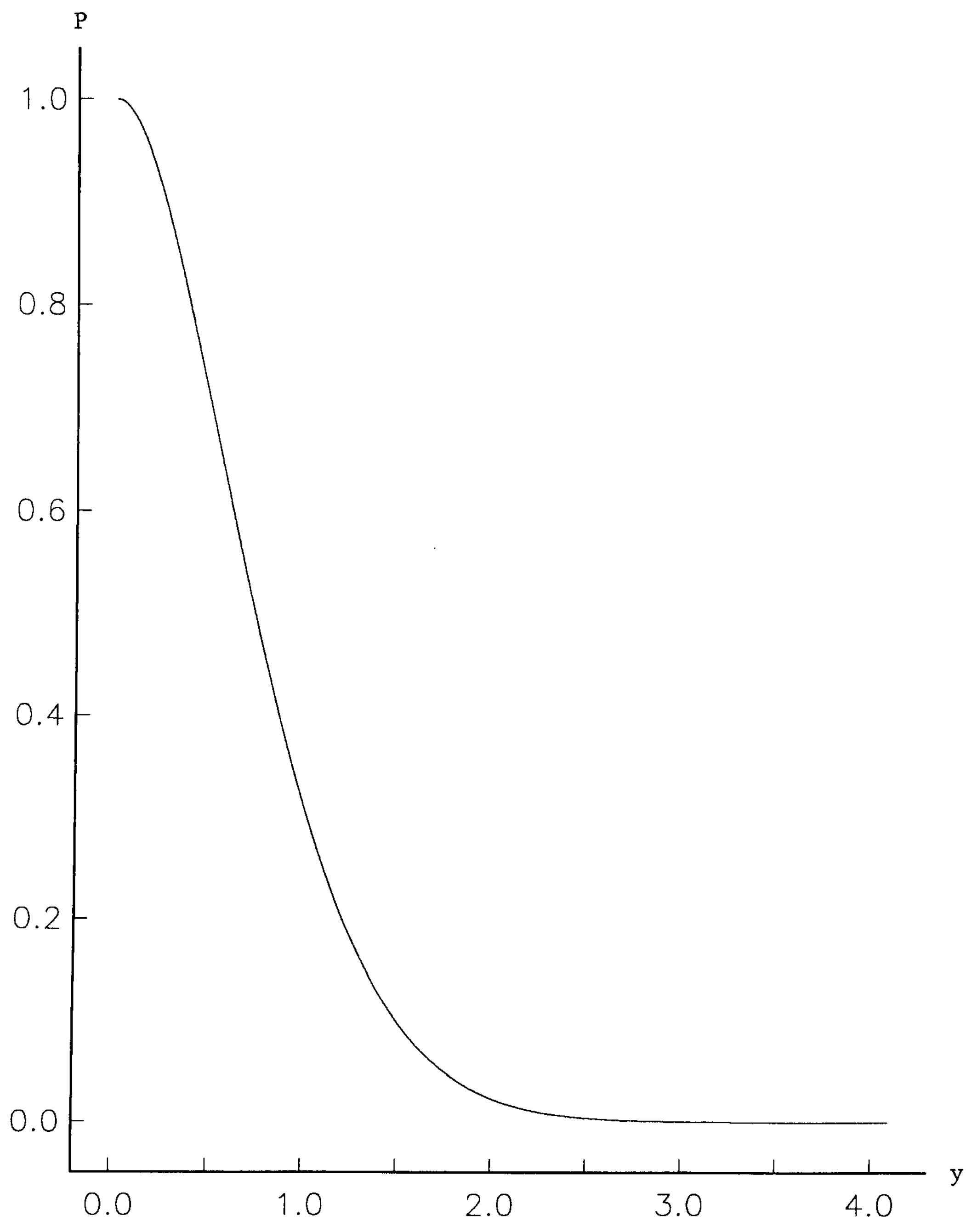

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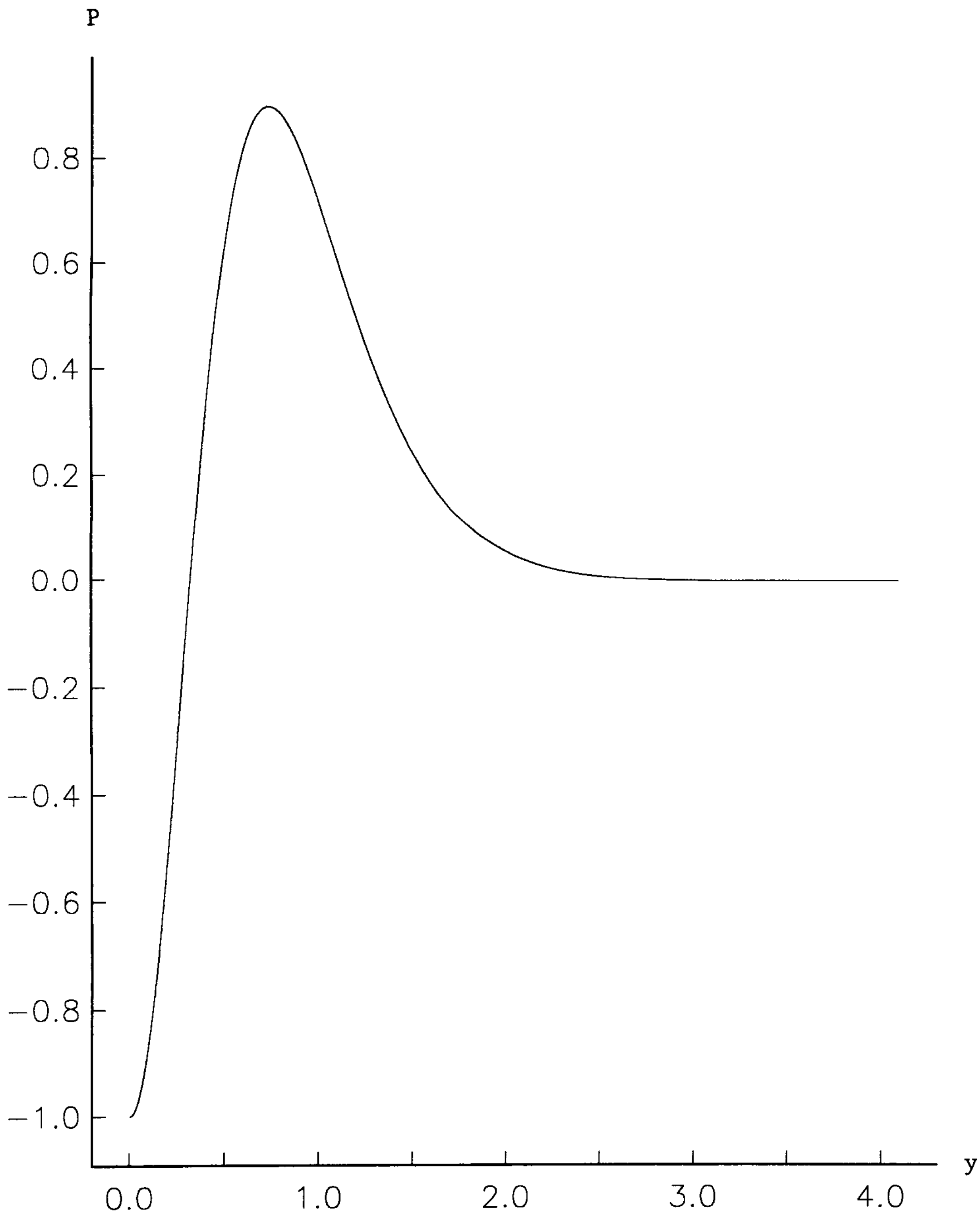

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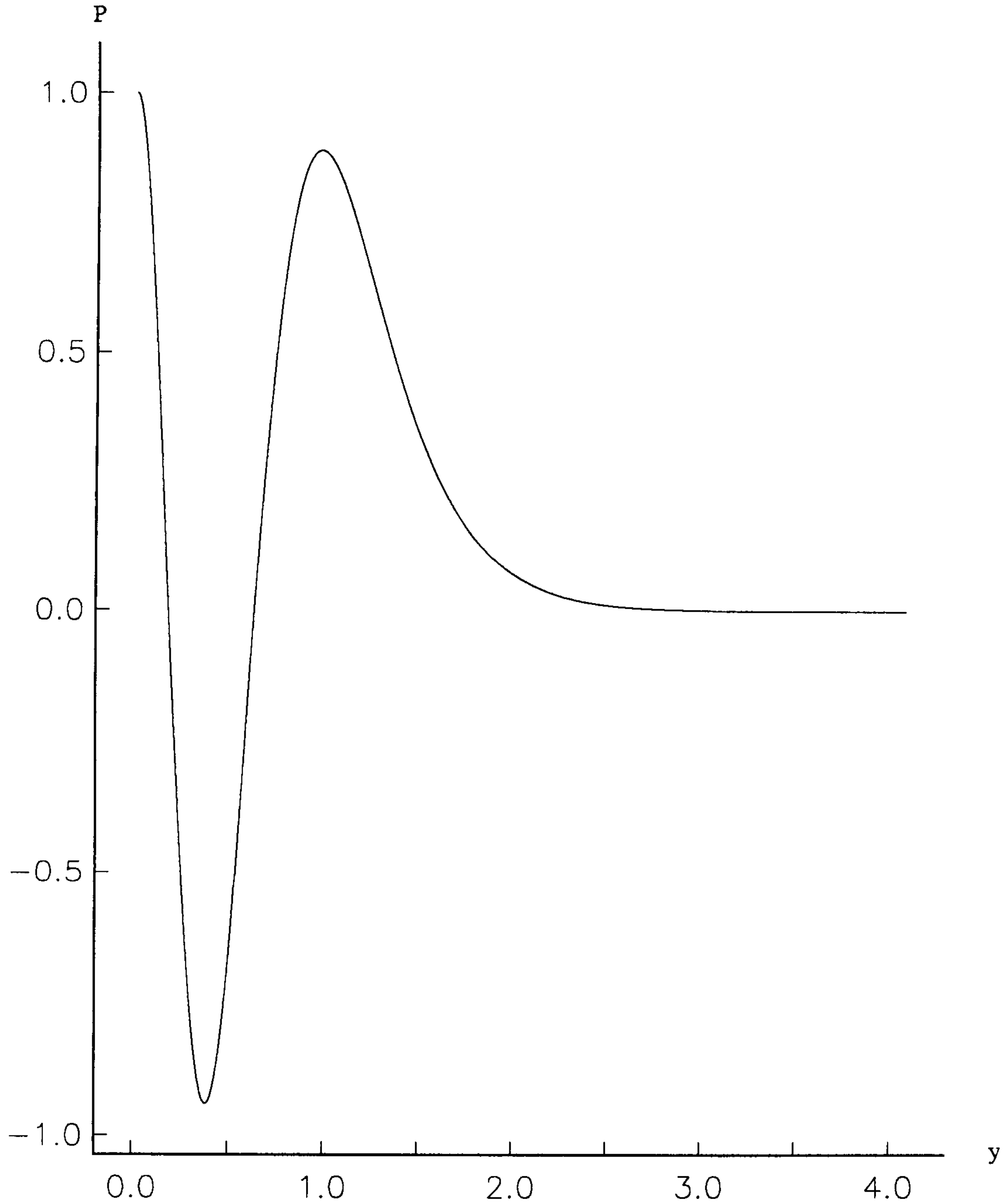

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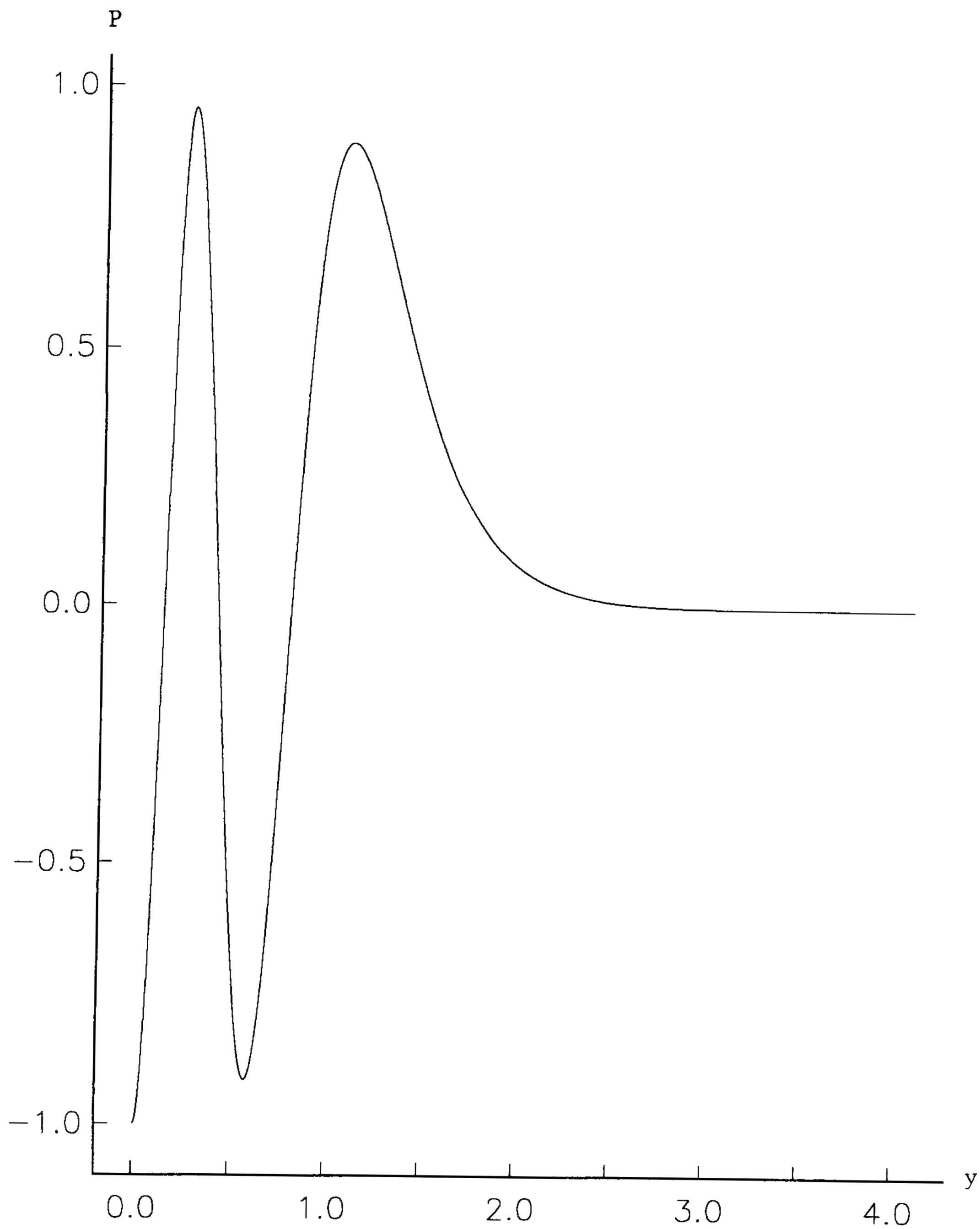

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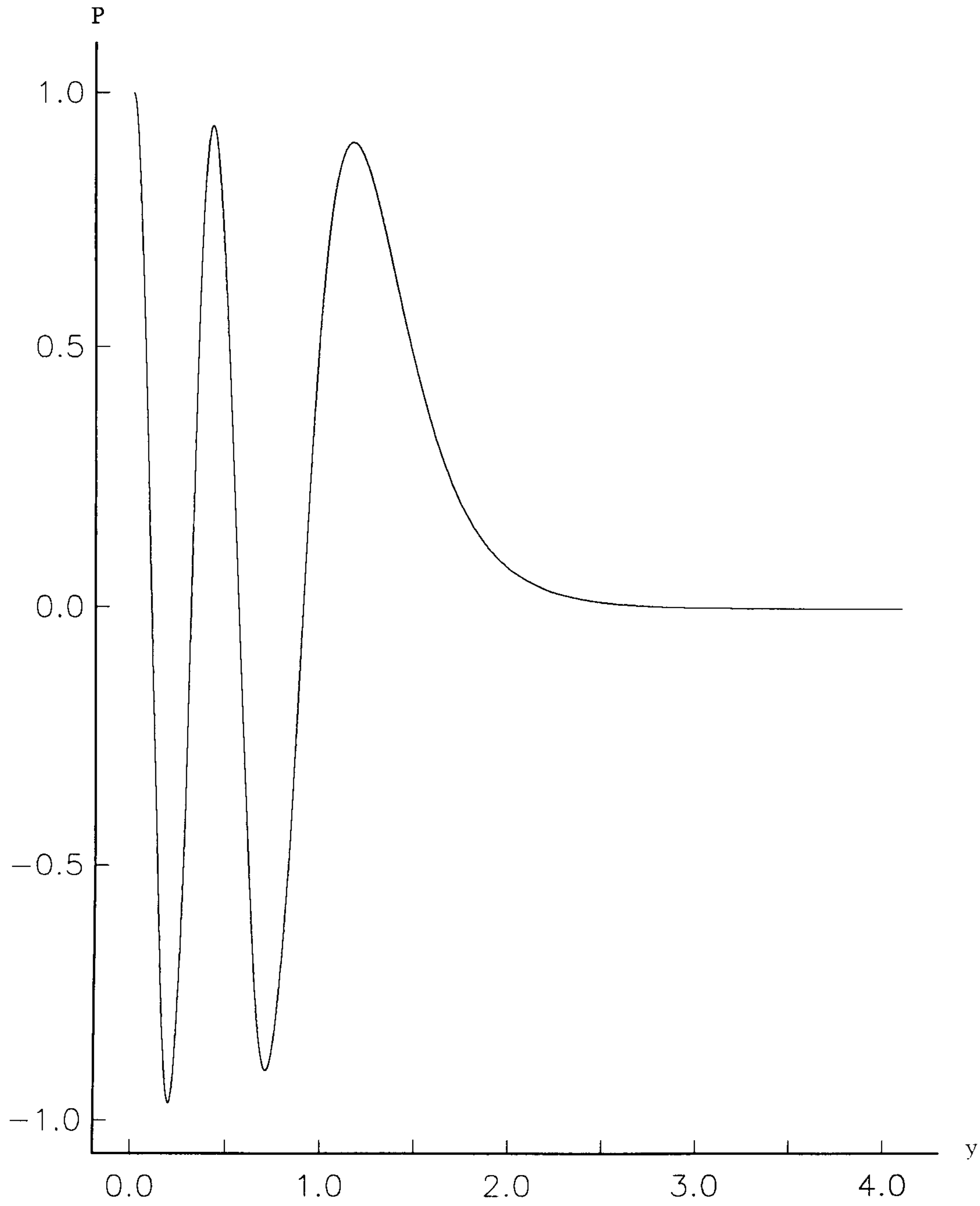

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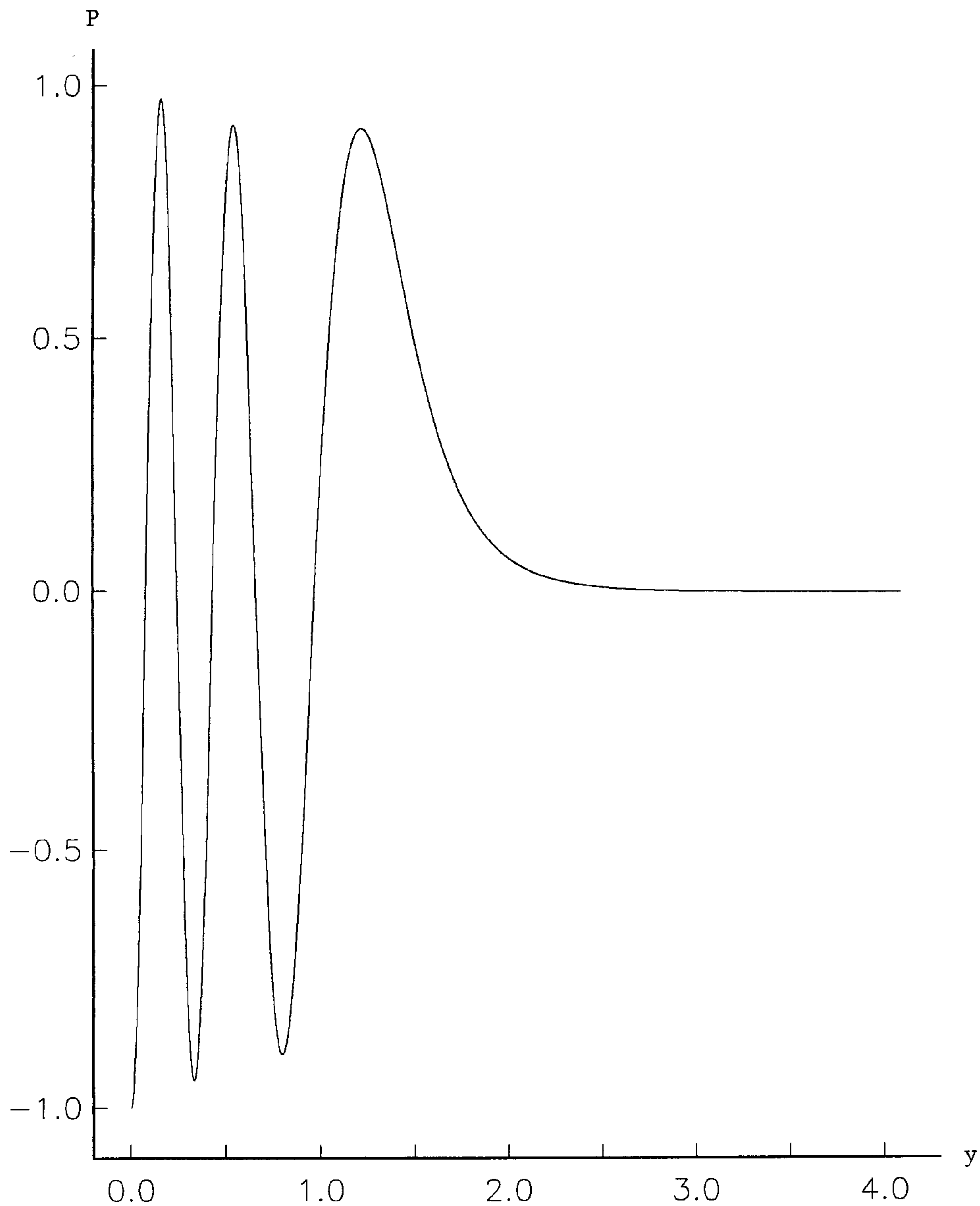

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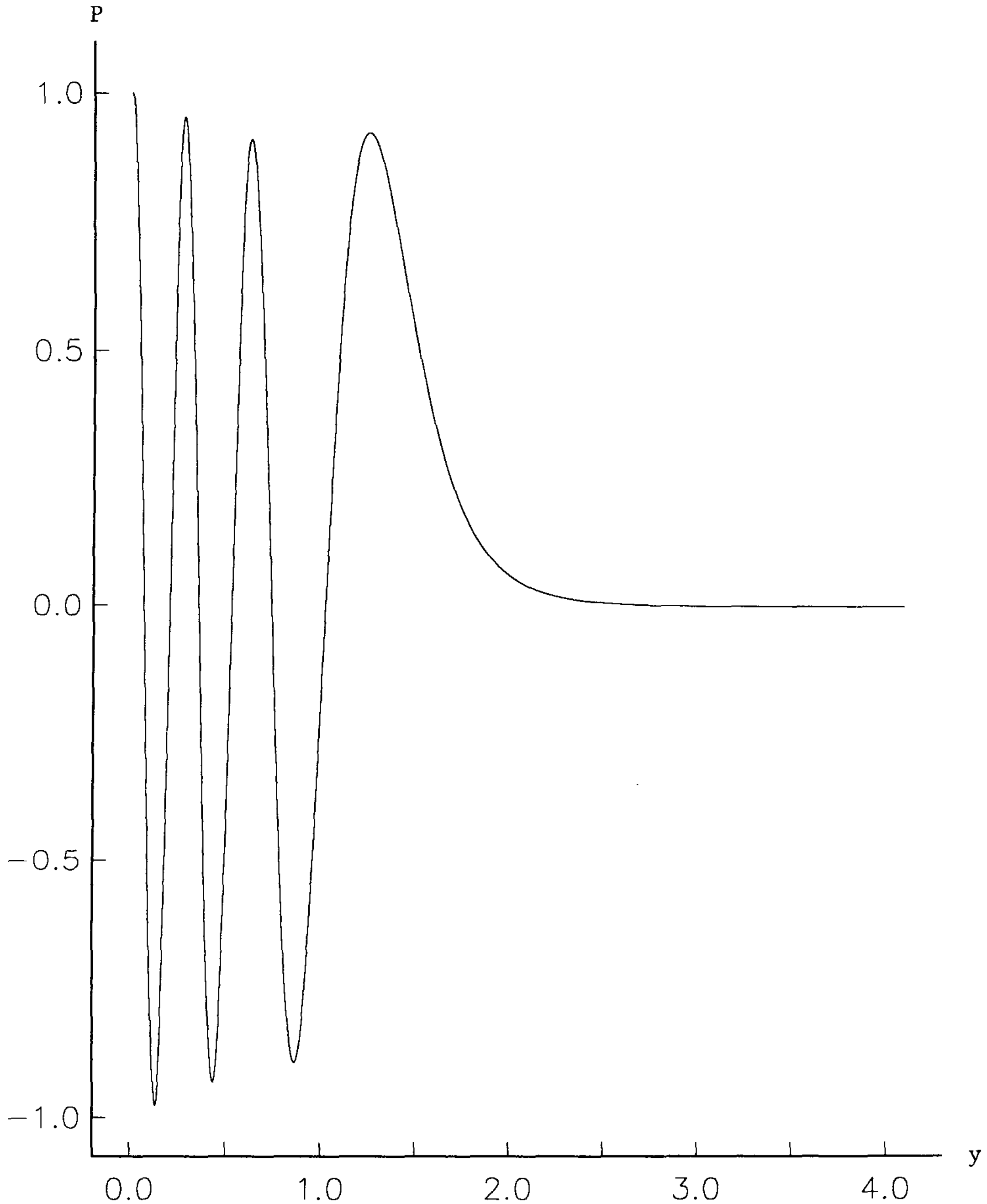

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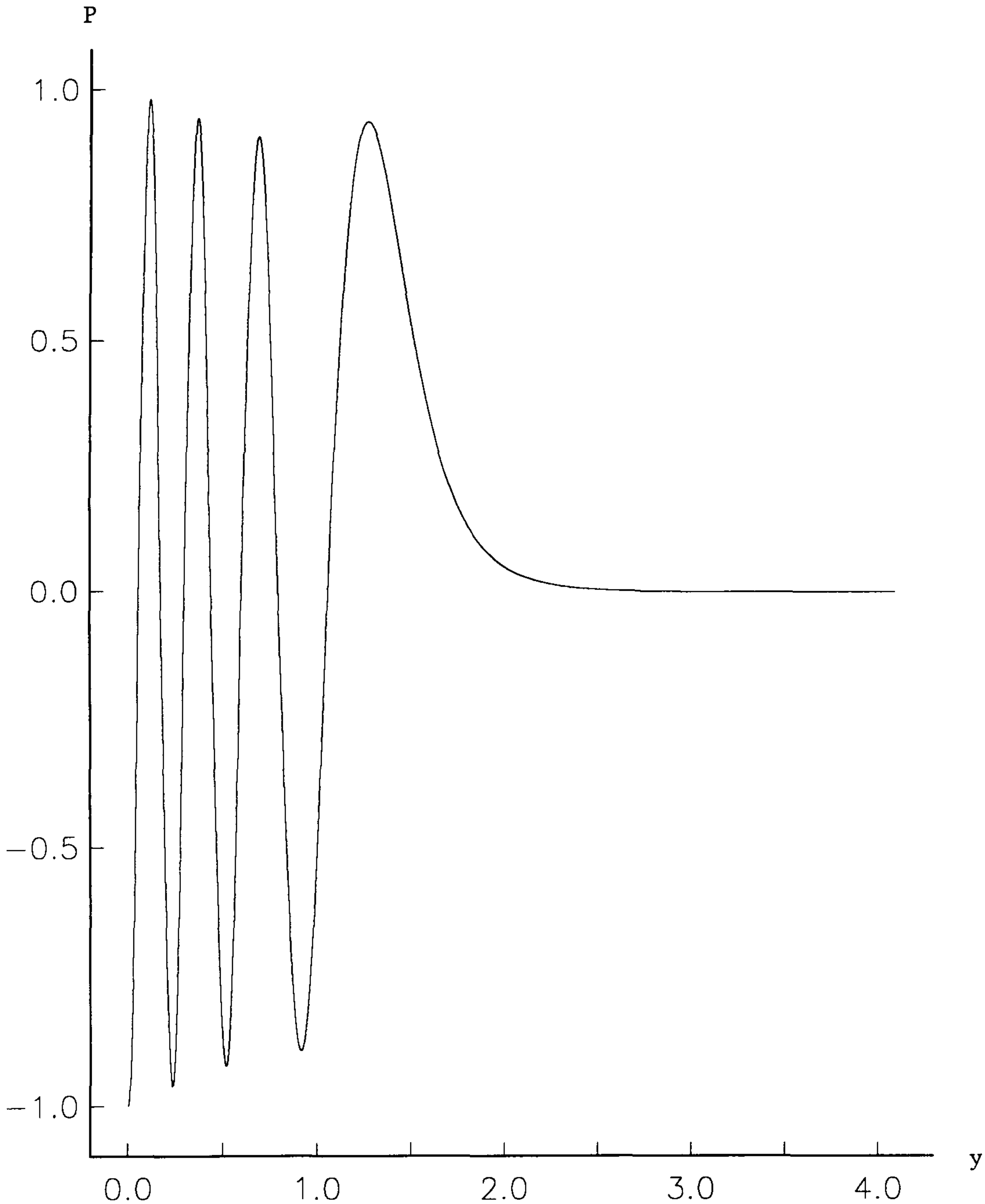

Figure 7h.

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Final Report

16. Abstract

The instability of a compressible flow past a wedge is investigated in the hypersonic limit. Particular attention is given to Tollmien-Schlichting waves governed by triple-deck theory though some discussion of inviscid modes is given. It is shown that the attached shock has a significant effect on the growth rates of Tollmien-Schlichting waves. Moreover, the presence of the shock allows for more than one unstable Tollmien-Schlichting wave. Indeed an infinite discrete spectrum of unstable waves is induced by the shock, but these modes are unstable over relatively small but high frequency ranges. The shock is shown to have little effect on the inviscid modes considered by previous authors and an asymptotic description of inviscid modes in the hypersonic limit is given.

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