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WAVES AND LONGITUDINAL VORTICES IN
BOUNDARY-LAYER TRANSITION

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NONLINEAR INTERACTION OF NEAR-PLANAR TS WAVES AND LONGITUDINAL VORTICES IN BOUNDARY-LAYER TRANSITION

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Abstract

The nonlinear interactions that evolve between a planar or nearly planar Tollmien-Schlichting (TS) wave and the associated longitudinal vortices are considered theoretically, for a boundary layer at high Reynolds numbers. The vortex flow is either induced by the TS nonlinear forcing or is input upstream, and similarly for the nonlinear wave development. Three major kinds of nonlinear spatial evolution, Types I-III, are found. Each can start from secondary instability and then becomes nonlinear, Type I proving to be relatively benign but able to act as a pre-cursor to the Types II, III which turn out to be very powerful nonlinear interactions. Type II involves faster streamwise dependence and leads to a finite-distance blow-up in the amplitudes, which then triggers the full nonlinear three-dimensional triple-deck response, thus entirely altering the mean-flow profile locally. In contrast, Type III involves slower streamwise dependence but a faster spanwise response, with a small TS amplitude thereby causing an enhanced vortex effect which, again, is substantial enough to entirely alter the mean-flow profile, on a more global scale. Streak-like formations in which there is localized concentration of streamwise vorticity and/or wave amplitude can appear, and certain of the nonlinear features also suggest by-pass processes for transition and significant changes in the flow structure downstream. The powerful nonlinear 3D interactions II, III are potentially very relevant to experimental findings in transition.

1. INTRODUCTION

Experiments in boundary-layer transition tend to show that, depending on the input amplitude and spectrum of the disturbance upstream, a two-dimensional Tollmien-Schlichting (2D TS) incident wave readily succumbs to 3D secondary instabilities downstream which then induce fully 3D nonlinear responses in the subsequent transitional boundary-layer flow: see, e.g., Arnal's (1984) review. The resonant-triad nonlinear mechanism (Craik 1971, Smith and Stewart 1987) provides one theoretical explanation for this nonlinear 3D interaction for low input amplitudes. Also on the theoretical front, for boundary-layer transition, is the related work by Hall and Smith (1984, 1988a) on oblique TS input or incident vortex flow with nonlinear interaction, while Benney and Lin (1960) make other interesting suggestions and Herbert (1984) presents interesting accounts of the first, but only linear, secondary 3D phase: see also the criticism in Hall and Smith (1988a) and later comments. Again, there are recent analyses tackling the corresponding channel-flow nonlinear interactions by Srivastava and Dallmann (1987), Hall and Smith (1987, 1988b,c). Our aim in the present theoretical study is to derive the scales, structure, governing equations and solution properties for nonlinear 3D interactions set up between incident 2D, or nearly 2D, TS waves and their induced vortex motion, in incompressible boundary-layer transition.

As TS waves are involved it is possible, and in some ways natural, to start with 3D triple-deck theory (we extend this subsequently) since that is known to identify in a rational way the framework of 2D and 3D linear and nonlinear TS waves (Smith 1979, Hall and Smith 1984) near the first, lower-branch, neutral station, say for the Blasius boundary layer on a flat plate. This theory however also incorporates directly the induced longitudinal-vortex motion provided the latter's streamwise length scale is not excessive, as described subsequently. The three regions of the triple-deck structure are the lower, main, and upper decks. The lower deck lies closest to the plate surface, inside the boundary layer, and the flow there responds to a 3D nonlinear unsteady-inertial-pressure gradient-viscous force balance. This deck coincides with the critical layer for very small, linearized, disturbances, although our emphasis here is necessarily on the nonlinear range. Above the lower deck the motion in the main deck spanning the boundary layer is displaced simply in a quasi-steady planar fashion, due to the lower deck, and the displacement effect is thereby transmitted to the upper deck lying outside the boundary layer. There the inviscidly provoked pressure response has to be consistent with the wall pressure driving the lower-deck flow, thus producing viscous-inviscid interaction. The whole nonlinear evolution is governed mainly by the flow features in the lower deck, where the nondimensionalized velocity components

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and pressure have the form

$$[\varepsilon\lambda^{\frac{1}{4}}u, \varepsilon^3\lambda^{\frac{3}{4}}v, \varepsilon\lambda^{\frac{1}{4}}w, \varepsilon^2\lambda^{\frac{1}{2}}p] \qquad (1.1a)$$

in terms of the nondimensionalized local coordinates and time

$$[x_0 + \varepsilon^3\lambda^{-\frac{5}{4}}X, \varepsilon^5\lambda^{-\frac{3}{4}}Y, z_0 + \varepsilon^3\lambda^{-\frac{5}{4}}Z, \varepsilon^2\lambda^{-\frac{3}{2}}T]. \qquad (1.1b)$$

Here $Re \equiv \varepsilon^{-8}$ is assumed large, in tune with the large experimental values of the global Reynolds number Re of interest, and our attention is focussed initially around a typical $0(1)$ station $x = x_0 > 0, z = z_0$ on the plate $Y = 0, x > 0$, the undisturbed flow being predominantly in the x direction and giving an $0(1)$ skin-friction factor λ locally at x_0, z_0. With (1.1a,b) holding, the Navier-Stokes equations reduce to the 3D unsteady interactive boundary-layer equations for u, v, w, p as functions of X, Y, Z, T in the lower deck, subject among other conditions to matching with the displaced main-deck solution. The unknown displacement $\propto -A$ involved there is related to the unknown pressure p in (1.1a) by the upper-deck properties applying outside the original boundary layer.

This nonlinear interactive system is addressed in Section 2 with regard to vortex/TS interaction at reduced amplitudes; three principal types (I-III) of interaction are identified, in fact, and we choose to concentrate first on "Type I." If the input amplitudes are sufficiently low, on the other hand, nonparallel-flow effects due to the slow variation of λ with x matter considerably and structural scales different from those in (1.1a,b) come into play, as described in Section 3. These scales are derived from an adaptation of those above; alternative derivations come from the related Hall and Smith studies (1984, 1988a, and work in progress). Another point of note here is that our concern is with spatial evolution. This seems to tie in better with the experimental and real-life behavior than does a temporal-instability analogy of the sort used by Herbert (1984) for example, who also makes the irrational shape-factor assumption for the 2D TS contribution as opposed to the present rational approach based on nonlinear evolution equations for both the TS wave and the induced vortex pattern.

The flow properties resulting from the vortex/TS nonlinear interaction equations derived in Sections 2,3 are considered in Section 4. The flow-structural analysis and results in Sections 2-4 are specifically for "Type I" interactions in which the typical streamwise (X) variation is relatively slow, as is the spanwise (Z) variation associated with an almost 2D input upstream, producing a "warped" effect. Faster streamwise responses lead to the "Type II" nonlinear interaction discussed in Section 5, while faster spanwise dependence leads to the "Type III" interaction which is described in Section 6. Further discussion of these and other aspects is presented in Section 7. The Type I vortex/TS interaction,

for instance, which we study first, is found to provoke either a finite-distance breakdown or a long-distance sustained effect, both of these producing a marked alteration in scales and hence structural change downstream and possibly leading on subsequently to Type II and Type III interactions. Further, the Type II, III interactions turn out to be much more powerful than the Type I interaction and suggest (e.g.) streak-like formations downstream, associated with local spanwise concentration of the streamwise vorticity and the TS amplitudes. Both Types II, III are potentially very relevant to the experimental findings as (unlike other theories) they lead to complete alteration of the mean-flow profiles.

2. THE TRIPLE-DECK VERSION FOR WARPED TS-VORTEX INTERACTION (TYPE I)

We start by posing the triple-deck problem which governs the linear or nonlinear evolution of 2D and 3D TS waves and their mean-flow effects, initially at least, and which stems from substitution of (1.1a,b) into the Navier-Stokes equations. This requires us to tackle the unsteady 3D nonlinear interactive boundary-layer equations

$$u_X + v_Y + w_Z = 0, \tag{2.1a}$$

$$u_T + uu_X + vu_Y + wu_Z = -p_X(X, Z, T) + u_{YY}, \tag{2.1b}$$

$$w_T + uw_X + vw_Y + ww_Z = -p_Z(X, Z, T) + w_{YY}, \tag{2.1c}$$

describing the lower-deck response, in scaled form, for the unknowns $(u, v, w)(X, Y, Z, T)$, $(p, A)(X, Z, T)$ subject to the boundary conditions

$$u = v = w = 0 \text{ at } Y = 0, \tag{2.1d}$$

$$u \sim Y + A(X, Z, T), w \to 0, \text{ as } Y \to \infty, \tag{2.1e}$$

$$p(X, Z, T) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial^2 A(\xi, \eta, T)/d\xi^2 d\xi d\eta}{[(x - \xi)^2 + (Z - \eta)^2]^{\frac{1}{2}}}. \tag{2.1f}$$

Here the normal momentum balance reduces to the usual requirement that $\partial p/\partial Y = 0$ as assumed in the above, and (2.1d-f) represent in turn the no-slip condition at the wall, the displacement effect ($\propto -A$, unknown) on the main-deck motion further from the wall, and the pressure-displacement interaction with the flow outside the boundary layer, produced via the upper deck where linearized potential-flow theory holds, yielding the double Cauchy-Hilbert integral relation shown. This relation can be replaced by the complete upper-deck formulation of solving Laplace's equation for the pressure, $\hat{p}$ say, with $\hat{p}$ zero in the farfield ($X^2 + \hat{y}^2 \to \infty$) and equal to $p(X, Z, T)$ at the wall ($\hat{y} \to 0+$),

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where also the gradient $\partial \hat{p}/\partial \hat{y}$ is to equal $\partial^2 A/\partial X^2(X, Z, T)$, in terms of the upper-deck's scaled normal coordinate $\hat{y}$. The nature of the triple-deck flow in (2.1a-f) appropriate to nonlinear interaction between a warped 2D TS input wave and its induced vortex flow is discussed here, with the next section then considering the required extension of the theory to encompass global nonparallelism effects.

The starting point is that the interactive triple-deck system (2.1) captures traditional linear TS waves (Smith 1979) as small unsteady perturbations of the basic boundary-layer shear flow $u \equiv Y, v = w = p = A \equiv 0$. The waves are stable or unstable for a prescribed scaled input frequency Ω according as $\Omega \lessgtr \Omega_C(\approx 2.30)$, where the critical frequency $\Omega = \Omega_c$ defines the lower branch of the neutral curve for the boundary layer. Near the critical frequency, or equivalently, near the critical station $x_0 = x_c$ for an imposed dimensional frequency, interplay usually occurs between nonlinear amplitude-dependent effects and the relatively weak initial growth/decay present, for a small 2D or 3D disturbance. Supposing the specified input disturbance amplitude to be h, relative to the normalized form (2.1), and the input wave to be almost 2D, with a small "warping" factor β say, so that the characteristic Z-variation has $\partial_Z \sim \beta$, we argue by orders of magnitude as follows (see also Fig. 1). The vortex flow produced by the small warped-TS wave arises from inertial forces as an amplitude-squared response but with the w-component of the vortex multiplied also by β, since the typical TS w-velocity is reduced to $0(h\beta)$, thus reducing the the inertial secondary forcing to $0(h^2\beta)$ via contributions such as $u\partial w/\partial X$. That $0(h^2\beta)$ w-component in the vortex flow grows logarithmically with large Y, however, due to the decreased inertial reaction of the vortex. So an outer buffer zone is brought into play. This has Y-extent of order $k^{-\frac{1}{3}}$ to bring about an inertial-viscous balance for the vortex, i.e. balancing Yk against ∂_Y^2 because of the basic shear ($u \approx Y$) and the unknown slower X-variation $\partial_X \sim k$ (say) in the vortex flow due to the slight warping present. In the buffer zone the vortex velocity w is still $0(h^2\beta)$, apart from a logarithmic factor, and so continuity suggests a vortex u-component of size $\beta^2 h^2/k$ (giving $u_X \sim w_Z \sim \beta^2 h^2$) and hence a shear-correction effect $\partial u/\partial Y$ of order $\beta^2 h^2/k^{\frac{2}{3}}$. Our reasoning then is that a sensitive interaction is likely to take place when the three relative corrections, $0(\beta^2)$ from the warping, $0(h^2)$ in the traditional nonlinear amplitude-cubed feedback, and $0(\beta^2 h^2/k^{\frac{2}{3}})$, due to the nonlinearly induced vortex shear, are comparable. So the regime

$$\beta \sim h, k \sim h^3 \tag{2.2}$$

is indicated as a central one, which we take as defining "Type I" interactions. Other regimes of the warping factor β and the streamwise-variation factor k can be examined but they may be regarded either as extreme limits of that in (2.2) or as distinct ones. In

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fact a matter of significance at this stage concerns the small order h^3 in (2.2) corresponding to the Type-I streamwise variation. This is formally less than the traditional influence of relative size h^2 usually associated with amplitude-cubed nonlinear spatial evolution in a purely 2D input wave. That aspect leads on to the study of the alternative Type II and Type III interactions as described subsequently in Sections 5,6.

(a) <u>The near-wall zone.</u>

Guided by the ordering arguments above, the pattern of the flow solution in the near-wall region where Y is $0(1)$ has

$$u = Y + hU_1 + h^2(U_2 + \lambda_3 Y) + h^3 U_3 + \cdots + h^4 U_4 + h^5 U_5 + \cdots, \qquad (2.3a)$$

$$v = hV_1 + h^2 V_2 + h^3 V_3 + h^4 V_4 + h^5 V_5 + \cdots, \qquad (2.3b)$$

$$w = h^2 W_1 + h^3 W_2 + h^4 W_3 + h^5 W_4 + h^6 W_5 + \cdots, \qquad (2.3c)$$

$$p = hP_1 + h^2 P_2 + h^3 P_3 + h^4 P_4 + \cdots, \qquad (2.3d)$$

$$A = hA_1 + h^2 A_2 + \cdots, \qquad (2.3e)$$

along with the extra scaling

$$\partial_X \to \partial_X + h^3 \partial_{\overline{X}} + \cdots, \partial_z \to h \partial_{\overline{Z}}. \qquad (2.4)$$

Here (U_1, V_1, W_1, P_1) represent the slightly warped 2D TS input wave, (U_2, V_2, P_2) are mean-flow correction and second-harmonic effects that are quasi-planar, similarly (U_3, V_3, P_3) include the amplitude-cubed forced TS response, while the induced-vortex motion is represented in the terms $\lambda_3 Y$ and U_5, V_5, W_2 and is independent of the faster scales X, T. The induced shear $\lambda_3(\overline{X}, \overline{Z})$ is an unknown feedback effect from the outer buffer zone which is studied subsequently in (b). In addition, a streamwise scale $X_1 \equiv h^2 X$ for the modulation of the TS amplitudes due to essentially 2D nonlinear forcing is absent here (except in the phase, see below), leaving the slower scale $\overline{X}(\equiv h^3 X)$ to operate on the TS modulation in which the nonlinear TS/induced-vortex interplay is set up. An assumption of near-criticality, e.g. $\Omega = \Omega_c + 0(h^2)$ or $x_0 = x_c + 0(h^2)$ in unscaled terms, is also implicit here in view of the relatively small amplitude growth present initially. The disparity in certain of the velocity scales in (2.3), e.g. in the main TS contributions U_1, V_1, W_1 and the induced vortex contributions U_5, V_5, W_2, is caused by the input warping defined in (2.2), (2.4).

The dominant governing equations that we need to address are obtained formally from substitution of (2.3), (2.4) into (2.1) and are the following: first, for continuity

$$U_{1X} + V_{1Y} = 0, \qquad (2.5a)$$

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$$U_{2X} + V_{2Y} = 0, \tag{2.5b}$$

$$U_{3X} + V_{3Y} + W_{1\overline{Z}} = 0; \tag{2.5c}$$

second, from the X-momentum balance,

$$U_{1T} + YU_{1X} + V_1 = -P_{1X} + U_{1YY}, \tag{2.6a}$$

$$U_{2T} + YU_{2X} + [U_1U_{1X} + V_1U_{1Y}] + V_2 = -P_{2X} + U_{2YY}, \tag{2.6b}$$

$$U_{3T} + YU_{3X} + \lambda_3 YU_{1X} + [U_1U_{2X} + U_2U_{1X} + V_1U_{2Y} + V_2U_{1Y}] + V_3 + V_1\lambda_3 + W_1U_{1\overline{Z}} = -P_{3X} + U_{3YY}; \tag{2.6c}$$

and, third, in the Z-momentum balance,

$$W_{1T} + YW_{1X} = -P_{1\overline{Z}} + W_{1YY}, \tag{2.7a}$$

$$[U_1W_{1X} + V_1W_{1Y}] = W_{2YY}. \tag{2.7b}$$

In (2.7b), only the induced-vortex terms, which are independent of the fastest scales X, T, are retained. Further, the successive governing equations in (2.5) - (2.7) show the gradual intrusion of three of the four major influences present, the warping through the $\overline{Z}$ variations, the amplitude-squared, -cubed, -etc., nonlinear responses and the feedback induced-vortex-shear contributions through the $\lambda_3(\overline{X}, \overline{Z})$ factors, but not yet the streamwise modulation in $\overline{X}$ which acts more significantly in (b) below.

The velocity and pressure fields above can be written now in component form,

$$\left. \begin{array}{l} U_1 = U_{11}E + c.c, \quad U_2 = (U_{22}E^2 + U_{21}E) + c.c. + U_{20}, \\[2mm] U_3 = U_{33}E^3 + \cdots, \quad U_4 = U_{44}E^4 + \cdots, \end{array} \right\} \tag{2.8}$$

and so on, and the solutions for the various components are obtained in sequence. Here the powers of

$$E = \exp[i(\alpha X - \Omega T)] \tag{2.9}$$

present contain all the fast dependence, with the main wavenumber α and frequency Ω being assumed real (see (2.12) below), so that the unsteady flow solution is near neutral. Also c.c. or, later, an asterisk denotes the complex conjugate function. The dominant, almost planar, TS wave is then controlled by the equations and constraints

$$i\alpha U_{11} + V_{11Y} = 0, \tag{2.10a}$$

$$-i\Omega U_{11} + i\alpha Y U_{11} + V_{11} = -i\alpha P_{11} + U_{11YY}, \tag{2.10b}$$

$$U_{11} = V_{11} = 0 \text{ at } Y = 0, U_{11} \to A_{11} \text{ as } Y \to \infty, \tag{2.10c,d}$$

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$$P_{11} = \alpha A_{11}, \tag{2.10e}$$

from (2.5a), (2.6a), with (2.1d-f), and (2.10a-d) yield the shear-stress solution

$$\left. \begin{array}{l} U_{11Y} = B_{11} Ai(\xi), \xi = (i\alpha)^{\frac{1}{3}}(Y - \Omega/\alpha), \\[2mm] B_{11} = (i\alpha)^{\frac{2}{3}} P_{11}/Ai_0', (i\alpha)^{\frac{1}{3}} P_{11}\kappa = A_{11} Ai_0' \end{array} \right\} \tag{2.11}$$

where the subscript zero refers to evaluation at $\xi = \xi_0 = -(i\alpha)^{\frac{1}{3}}\Omega/\alpha$, i^n denotes $\exp(in\pi/2)$, Ai is the Airy function, and $\kappa = \int_{\xi_0}^{\infty} Ai(s)ds$. Hence the pressure-displacement law (2.10e) leads to the eigenrelation

$$i^{\frac{1}{3}}\alpha^{\frac{4}{3}} = Ai_0'/\kappa \tag{2.12a}$$

between α, Ω. Both of these are real as required only for the values $\xi_0 = -d_1 i^{\frac{1}{3}}, Ai_0'/\kappa = d_2 i^{\frac{1}{3}}[d_1 \approx 2.30, d_2 \approx 1.00]$, i.e. the values

$$\alpha = d_2^{\frac{3}{4}}, \Omega = d_1 d_2^{\frac{1}{2}} \tag{2.12b}$$

are fixed. The fundamental pressure amplitude $P_{11}(\overline{X}, \overline{Z}, \cdots)$ remains undetermined at this level: see also (2.14)ff below. In addition, the spanwise velocity component W_{11} associated with the incident TS warping is only a passive response so far, given by

$$W_{11} = P_{11\overline{Z}}\mathcal{L}(\xi)/(i\alpha)^{\frac{2}{3}} \tag{2.13}$$

from (2.7a), where $\mathcal{L}$ is the solution of $\mathcal{L}'' - \xi\mathcal{L} = 1, \mathcal{L}_0 = \mathcal{L}(\infty) = 0$.

Moving on to the second-order responses, we may proceed similarly to determine $U_{20,21,22}$ from the component equations inferred from (2.5b), (2.6b), which again are quasi-2D. The working for the forced second-harmonic TS term U_{22} and the forced mean-flow correction U_{20} has been done before, however, by Smith (1979), giving in particular V_{20}, P_{20} identically zero but U_{20}, A_{20} nonzero. The un-forced extra fundamental which may be produced here for compatibility at higher order simply has (U_{21}, V_{21}, A_{21}) equal to P_{21}/P_{11} times (U_{11}, V_{11}, A_{11}) with P_{21} still unknown.

The third-order working for U_3, V_3 components is then as in Smith (1979) except that they feel the additional influences of the input warping, in the two $\partial/\partial\overline{Z}$ terms in (2.5c), (2.6c), and of the induced-vortex flow, in the two λ_3 terms in (2.6c). With the warping and vortex influences incorporated, then, the E-components contained in those two equations lead, through a compatibility requirement, to an amplitude-modulation equation part of which can be picked out from the above paper or from Hall and Smith (1984). We find therefore the modulation equation

$$i\sigma P_{11} + \hat{c}\lambda_3 P_{11} + \hat{d}\frac{\partial^2 P_{11}}{\partial\overline{Z}^2} = \hat{e}P_{11} + \hat{f}P_{11}|P_{11}|^2 \tag{2.14}$$

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for the unknown TS pressure amplitude P_{11}, on use of (2.13). The coefficients involved here,

$$\hat{c} = 2Di\xi_0\alpha^2 Ai_0/3\Delta^{\frac{2}{3}}Ai_0' - 5/3, \quad \hat{d} = 1/8\alpha^2, \qquad (2.15a,b)$$

$$\hat{e} = i\tilde{\alpha}_1\hat{a}, \quad \hat{f} = a_1\hat{a}, \hat{a} = -i(\hat{c}+3)/\alpha, \qquad (2.15c,d,e)$$

are in keeping with those in the previous studies mentioned earlier, for small β, with $\tilde{\alpha}_1$ (complex) representing the effective wavenumber shift and a_1 the Stuart-Landau constant for the pressure. Also $D \equiv 1 + \kappa\xi_0/Ai_0'$, $\Delta \equiv i\alpha$. Clearly, the effects of the induced-vortex motion and the warping on the TS wave are represented by the terms in λ_3 and $\partial^2/\partial\overline{Z}^2$ in (2.14). The other term on the left in (2.14) is the relatively fast phase effect noted earlier, corresponding to an extra $\exp(i\sigma X_1)$ factor necessary in P_{11} where the phase σ is real and constant, or associated with a contribution $h^2 i\sigma$ added to the streamwise variation in (2.4). This does not affect directly the vortex properties below and indeed σ could be absorbed into the term $\hat{e}$ in (2.14).

Lastly here, the driven vortex motion in (2.7b) needs to be addressed. The relevant E°-components give the equation

$$W_{20YY} = i\alpha(-U_{11}W_{11}^* + U_{11}^*W_{11}) + V_{11}W_{11Y}^* + V_{11}^*W_{11Y} \qquad (2.16a)$$

for W_{20}, which is subject to

$$W_{20} = 0 \text{ at } Y = 0, W_{20Y} \to 0 \text{ as } Y \to \infty. \qquad (2.16b,c)$$

Due to the forcing terms in (2.16a) decaying algebraically like Y^{-2} at large Y (from (2.13)), the integration for W_{20} is found to produce the logarithmic-growth result

$$W_{20} \sim \partial_{\overline{Z}}(|P_{11}^2|)[-\ell nY + \phi](\text{as } Y \to \infty) \qquad (2.16d)$$

where the constant ϕ is to be determined numerically from (2.16a-c). The behavior (2.16d) and the corresponding growth $\propto Y\ell nY, Y^3\ell nY$ in V_{50}, U_{50} are responsible for the strength of the induced-vortex flow set up in the outer zone discussed next.

(b) The outer-buffer zone.

In the buffer zone Y is larger, $Y = h^{-1}\overline{Y}$ with $\overline{Y}$ of order unity, and the flow solution is of the form

$$u = h^{-1}\overline{Y} + h(A_1 + u_3) + \cdots + h^2 A_2 + \cdots + h^5 R_1\overline{Y}^{-1} + \cdots \qquad (2.17a)$$

$$v = -A_{1X}\overline{Y} + h(Q_1 - A_{2X}\overline{Y}) + \cdots + h^3 v_3 + \cdots, \qquad (2.17b)$$

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$$w = h^3(\ell n h)w_{3L} + h^3(C_1\overline{Y}^{-1} + w_3) + \cdots, \tag{2.17c}$$

with the pressure p and displacement $-A$ remaining as in (2.3) of course. Here the unknown induced-vortex contribution of interest is represented by (u_3, v_3, w_3), dependent mainly on $\overline{X}, \overline{Y}, \overline{Z}$; although logarithmically larger vortex velocities [w_{3L} and similarly in u, v] are also provoked, we note, these react only passively to the forcing present in the matching constraints for $\overline{Y} \to 0+$ due to (2.16d), analogous to those in the related studies mentioned earlier. Further, the momentum balances in (a) above show that $Q_1 \equiv -P_{1X} - A_{1T}$ and C_1, R_1 satisfy $C_{1X} = -P_{1\overline{Z}}, R_{1X} + C_{1\overline{Z}} = 0$, i.e.,

$$Q_{11} = -i\alpha P_{11} + i\Omega A_{11}, i\alpha C_{11} = -P_{11\overline{Z}}, i\alpha R_{11} + C_{11\overline{Z}} = 0, \tag{2.18}$$

in component form.

Most of the leading-order balances that result from substitution of (2.17) into (2.1) are trivially satisfied but the E^0-components require the vortex motion to satisfy

$$u_{3\overline{X}} + v_{3\overline{Y}} + w_{3\overline{Z}} = 0, \tag{2.19a}$$

$$\overline{Y}u_{3\overline{X}} + v_3 = u_{3\overline{YY}}, \tag{2.19b}$$

$$\overline{Y}w_{3\overline{X}} + \partial_{\overline{Z}}(|P_{11}|^2)/\overline{Y}^2 = w_{3\overline{YY}}, \tag{2.19c}$$

after some manipulation of the forcing terms involving (2.18). The boundary conditions on (2.19a-c) are

$$u_3 = v_3 = 0 \text{ at } \overline{Y} = 0, \tag{2.19d}$$

$$w_3 \sim \partial_{\overline{Z}}(|P_{11}|^2)[-\ell n\overline{Y} + \phi] \text{ as } \overline{Y} \to 0+, \tag{2.19e}$$

$$u_{3\overline{Y}} \to 0, w_3 \to 0(\overline{Y}^{-3}) \text{ as } \overline{Y} \to \infty, \tag{2.19f}$$

where (2.19d,e) merge the solution with the near-wall vortex form in (a), e.g. in (2.16d), while (2.19f) achieves the outermost displacement behavior of (2.1e) as required. In contrast with the mostly viscous response of the vortex motion in zone (a), i.e. in (2.16a), the vortex here in zone (b) reacts in a viscous-inviscid fashion through (2.19a-c), in response to the warped-TS-forcing ($\propto |P_{11}|^2$) acting in (2.19c,e). The forced vortex flow in (2.19a-f) produces the (unknown) wall-shear contribution referred to earlier,

$$\lambda_3 \equiv \partial_{\overline{Y}}u_3 \text{ at } \overline{Y} = 0+, \tag{2.20}$$

which then helps to drive the TS amplitude P_{11} (via (2.14)) which in turn drives the vortex flow (2.19), and so on, thus producing nonlinear interaction between the warped TS disturbance and the induced vortex.

The nonlinear TS/vortex interaction found, as represented by (2.14), (2.19a-f) [with ϕ determined from (2.16a-d)], is studied further in Section 4, after the extension of the theory which is described next.

3. EXTENSION OF THE THEORY TO INCORPORATE NONPARALLEL-FLOW EFFECTS (TYPE I)

The influence of the nonparallelism of the basic boundary layer is secondary within the scope of the triple-deck scales leading to (2.1) because the latter's streamwise length scale [see (1.1)] is typically $0(\varepsilon^3)$, much less than the $0(1)$ length usually associated with the basic nonparallelism. The difference diminishes however for the reduced $0(h)$-sized disturbances considered in Section 2, since their main modulation length becomes extended to $0(h^{-3})$ times the ε^3 scale, from (2.4), while the characteristic nonparallel-flow length scale contracts, to $0(h^2)$, in view of the shift away from the neutral position. Hence, formally, the two length scales can become comparable if the disturbance amplitude h is reduced to

$$h \sim \varepsilon^m, m = \frac{3}{5}, \tag{3.1}$$

with significant nonparallelism then entering the fray. Strictly, the setting of h as an inverse power of the Reynolds number here requires us to start again from scratch, replacing (1.1a,b) by a new expansion as implied by (2.3) with (3.1) inserted and similarly in the main and upper decks, as well as in the extra buffer zone indicated in Section 2. But the only new substantial effect to come into play then is that corresponding to the global nonparallelism (involving $\lambda_1 \equiv d\lambda/dx$ at x_0). So we may cut matters short simply by adding in that effect to the previous working, along with (3.1), in retrospect. Here again the additional term involved may be picked out from the Hall and Smith (1984) study, so that the new equation for the warped-TS amplitude is

$$i\sigma P_{11} + (\hat{b}\hat{X} + \hat{c}\lambda_3)P_{11} + \hat{d}\frac{\partial^2 P_{11}}{\partial \overline{Z}^2} = \hat{e}P_{11} + \hat{f}P_{11}|P_{11}|^2 \tag{3.2}$$

when (3.1) holds, with (2.19) remaining intact. In (3.2) $\hat{b} = \lambda_1\hat{c}$ captures the nonparallel-flow effect. The TS/vortex interaction is thus controlled by (2.19), (3.2) in the regime defined by (3.1), as distinct from (2.19), (2.14) which apply for the higher-amplitude regime where $1 >> h >> \varepsilon^m$ in effect.

4. NONLINEAR INTERACTION PROPERTIES (TYPE I).

The warped-TS/induced-vortex interaction of Type I is governed by the nonlinear linked p.d.e. system implied by (2.19), (3.2):

$$W_{\overline{YY}} - \overline{Y}W_{\hat{X}} = \partial_{\overline{Z}}(|P|^2)/\overline{Y}^2, \tag{4.1a}$$

$$\tau_{\overline{YY}} - \overline{Y}\tau_{\hat{X}} = -W_{\overline{Z}} \tag{4.1b}$$

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subject to

$$W \sim \partial_{\overline{Z}}(|P|^2)\{-\ell n \overline{Y} + \phi\}, \tau_{\overline{Y}} \to 0, \text{ as } \overline{Y} \to 0, \qquad (4.1c)$$

$$\tau \to 0, W \to 0, \text{ as } \overline{Y} \to \infty, \qquad (4.1d)$$

for the vortex, coupled with the TS amplitude P which is controlled by

$$0 = (c_1 + c_2 \lambda_1 \hat{X} + c_3 \lambda_3)P + c_4 \frac{\partial^2 P}{\partial \overline{Z}^2} + c_5 P |P|^2 \qquad (4.1e)$$

where the vortex skin-friction factor λ_3 is

$$\lambda_3 = \tau(\hat{X}, 0, \overline{Z}), \qquad (4.1f)$$

and with ϕ determined from integration of (2.16). The coefficients involved are given by $c_1 \propto (\hat{e} - i\sigma)$ and

$$\left.\begin{array}{l} c_2 = c_3 = -0.375 + 0.117i, c_4 = -0.0156 + 0.0465i \\[2mm] c_5 = -0.719 - 0.766i, \phi = 4.15, \end{array}\right\} \qquad (4.2)$$

from a computation of (2.15), (2.16) with (2.21b). Also we have introduced $\tau \equiv \partial U / \partial \overline{Y}$ to obtain the $W - \tau$ formulation above instead of the $U - V - W$ version earlier. Here U, V, W are related to be velocities u_3, v_3, w_3 of Section 2, while P is related to P_{11}, and in the context of Section 2 the factor λ_1 is replaced by zero, thus suppressing the nonparallel-flow effect, as opposed to positive or (more likely) negative values of λ_1 which incorporate nonparallelism as in Section 3.

Computational studies of (4.1a-f) with (4.2) for various starting conditions upstream were made by use of a spectral method analogous to the treatment in Hall and Smith (1988a) except that for the vortex part (4.1a-d) we applied an implicit scheme like that in Hall and Smith (1988c) and the updating of P through (4.1e) was done differently, via a Newton iterative procedure. An alternative formulation in terms of an integral equation for λ_3 (see Appendix A) was treated in the same iterative way, yielding similar trends. The results are summarized later on.

We consider analytically now possible ultimate forms of the nonlinear interaction as $\hat{X}$ increases. It is useful to express P in polar form first, $P = r \exp(i\theta)$ with r, θ real, so that (4.1e) is replaced by the two real equations

$$0 = (c_{1r} + c_{2r}\lambda_1 \hat{X} + c_{3r}\lambda_3)r + c_{4r}(r_{\overline{ZZ}} - r\theta_{\overline{Z}}^2) - c_{4i}(2r_{\overline{Z}}\theta_{\overline{Z}} + r\theta_{\overline{ZZ}}) + c_{5r}r^3, \qquad (4.3a)$$

$$0 = (c_{1i} + c_{2i}\lambda_1 \hat{X} + c_{3i}\lambda_3)r + c_{4i}(r_{\overline{ZZ}} - r\theta_{\overline{Z}}^2) + c_{4r}(2r_{\overline{Z}}\theta_{\overline{Z}} + r\theta_{\overline{ZZ}}) + c_{5i}r^3 \qquad (4.3b)$$

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linked with (4.1a-d,f) by $|P|^2 = r^2$. Then one possibility (option 1) suggested by related studies is that of a nonlinear exponential interaction continuing to downstream infinity, giving the response

$$(r, \theta) \sim e^{s\hat{X}}(r_\infty, \theta_\infty), (W, \tau, \lambda_3) \sim e^{2s\hat{X}}(W_\infty, \tau_\infty, \lambda_{3\infty}) \qquad (4.4)$$

as $\hat{X} \to \infty$, with s an unknown positive constant. From (4.3) r_∞, θ_∞ must satisfy

$$0 = c_{3r}\lambda_{3\infty} - c_{4r}\theta^2_{\infty\overline{Z}} + c_{5r}r^2_\infty, \qquad (4.5a)$$

$$0 = c_{3i}\lambda_{3\infty} - c_{4i}\theta^2_{\infty\overline{Z}} + c_{5i}r^2_\infty, \qquad (4.5b)$$

where $\lambda_{3\infty} \propto (r^2_\infty)_{\overline{ZZ}}$ is τ_∞ at $\overline{Y} = 0+$ and τ_∞, W_∞ are controlled by forced Airy equations, the solutions of which can be written down explicitly. The coefficients involved in (4.5a,b) [see (4.2)] however are such that (4.5a,b) admit no positive solutions for $r^2_\infty, \theta^2_{\infty\overline{Z}}$, and hence the nonlinear exponential response seems to be ruled out. This conclusion might be avoided if the $\overline{Z}$ scale also decreases appropriately, we note.

A second option to consider is the 2D equilibrium solution where, in the shorter-length case of zero λ_1 for example, $|P| = r$ is a constant, say $r \to r_0$ where from (4.3) $r^2_0 = -c_{1r}/c_{5r}$ is positive and consistent with the value $-c_{1i}/c_{5i}$ by virtue of the absorbed phase shift σ. This 2D equilibrium is nominally an exact solution for all $\hat{X}$, with zero induced-vortex flow, but it is unstable to 3D perturbations. For, if $r = r_0 + \delta r_1, \theta = \delta \theta_1$ with δ small, then (4.1), (4.3) yield the linearized system

$$W_{1\overline{Y}\overline{Y}} - \overline{Y}W_{1\hat{X}} = 2r_0 r_{1\overline{Z}}/\overline{Y}^2, \qquad (4.6a)$$

$$\tau_{1\overline{Y}\overline{Y}} - \overline{Y}\tau_{1\hat{X}} = -W_{1\overline{Z}}, \qquad (4.6b)$$

$$\tau_{1\overline{Y}} \to 0, W_1 \sim 2r_0 r_{1\overline{Z}}\{-\ell n\overline{Y} + \phi\} \text{ as } \overline{Y} \to \infty, \qquad (4.6c)$$

$$0 = c_{1r}r_1 + c_{3r}\lambda_{31}r_0 + c_{4r}r_{1\overline{ZZ}} - c_{4i}r_0\theta_{1\overline{ZZ}} + 3c_{5r}r^2_0 r_1, \qquad (4.6d)$$

$$0 = c_{1i}r_1 + c_{3i}\lambda_{31}r_0 + c_{4i}r_{1\overline{ZZ}} - c_{4r}r_0\theta_{1\overline{ZZ}} + 3c_{5i}r^2_0 r_1, \qquad (4.6e)$$

with $(W, \tau) \approx \delta(W_1, \tau_1), \lambda_{31} = \tau_1(\hat{X}, 0, \overline{Z})$, and W_1, τ_1 to vanish as $\overline{Y} \to \infty$. The solution of (4.6a-c) may be decomposed in the form

$$r_1 = \tilde{r}_1 \exp(i\overline{\beta}\overline{Z} + \overline{\gamma}\hat{X}) \qquad (4.7)$$

and similarly for $W_1, \tau_1, \lambda_{31}, \theta_1$ and then the vortex solution is given by

$$\tilde{W}_1 = 2i\overline{\beta}r_0\tilde{r}_1 M(\tilde{y}), \tilde{y} = \overline{\gamma}^{\frac{1}{3}}\overline{Y},$$

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$$M(\tilde{y}) \equiv B_1 Ai(\tilde{y}) - \ell n\tilde{y} - Ai(\tilde{y})\int_0^{\tilde{y}} Ai^{-2}(y_1)\{\int_\infty^{y_1} y_2 Ai(y_2)\ell n y_2 dy_2\}dy_1 \qquad (4.8a)$$

$$\tilde{r}_1 = 2\overline{\gamma}^{-\frac{2}{3}}\overline{\beta}^2 r_0 \tilde{r}_1 N(\tilde{y}),$$

$$N(\tilde{y}) = B_2 Ai(\tilde{y}) + M'(\tilde{y}) + \tilde{y}^{-1} + Ai(\tilde{y})\int_0^{\tilde{y}} Ai^{-2}(y_1)\{\int_\infty^{y_1} Ai(y_2)dy_2\}dy_1 \qquad (4.8b)$$

where $B_1 Ai(0) = (\ell n\overline{\gamma})/3 + \phi$ and $B_2 Ai'(0) = 1/(3Ai(0))$. Hence $\tilde{\lambda}_{31} = 2\overline{\gamma}^{-\frac{2}{3}}\overline{\beta}^2 r_0 \tilde{r}_1 N(0)$ with $N(0) = -e_1 - e_2 \ell n\overline{\gamma}$ and $e_1 = 1.21 + 0.73\phi, e_2 = 0.243$, after some working. Then (4.6d,e) yield two coupled equations for $\tilde{r}_1, \tilde{\theta}_1$, and for these to have a nontrivial solution the eigenrelation

$$\overline{\beta}^2[(c_{4r}^2 + c_{ri}^2) + 2r_0^2(c_{3r}c_{4r} + c_{3i}c_{4i})\overline{\gamma}^{-\frac{2}{3}}(e_1 + e_2 \ell n\overline{\gamma})] = 2r_0^2(c_{5r}c_{4r} + c_{5i}c_{4i}) \qquad (4.9a)$$

must hold, determining the 3D spatial growth rate $\overline{\gamma}$ in terms of the input 2D amplitude r_0 and the scaled spanwise wavenumber $\overline{\beta}$. With the numerical values in (4.2) inserted into (4.9a) we then have the relation

$$\overline{\gamma}^{-\frac{2}{3}}(e_1 + e_2 \ell n\overline{\gamma}) = -d_1/r_0^2 - d_2/\overline{\beta}^2 \qquad (4.9b)$$

for $\overline{\gamma}(r_0, \overline{\beta})$ where the constants e_1, e_2, d_1, d_2 are all real and positive. See Fig. 2. Positive growth factors $\overline{\gamma}$ exist for all nonzero $r_0, \overline{\beta}$, the maximum growth factor being for large $r_0, \overline{\beta}$ where $\overline{\gamma}$ approaches the value $\exp(-e_1/e_2)$ from below. Thus secondary 3D instability occurs for the 2D equilibrium state.

The secondary 3D modes given by (4.9b) are completely absent in the undisturbed state of steady Blasius flow, we observe, since then (4.9a) is simply replaced by $\overline{\beta}^2(c_{4r}^2 + c_{4i}^2) = (c_{1r}c_{4r} + c_{1i}c_{4i})$ or $\tilde{r}_1 \equiv 0$. The above secondary instability and those to be found in Sections 5,6 form some connection with Squire-mode destabilization (e.g., Herbert 1984, and see below), and the present one applies similarly to the nonparallel case $\lambda_1 \neq 0$ and rules out the 2D possibility as a terminal response of the full interaction. In addition the dependence of $\overline{\gamma}$ on $\overline{\beta}$ tends to suggest that some spanwise focussing may be significant in the full nonlinear system (4.1).

Third, there is the possibility of the nonlinear interaction continuing to downstream infinity in an algebraic form. The orders of magnitude present then suggest that

$$W \sim \hat{X}^{-\frac{1}{6}}\tilde{W} + \cdots, (\tau, \lambda_3) \sim \hat{X}(\tilde{\tau}, \tilde{\lambda}_3) + \cdots, \qquad (4.10a,b)$$

$$r = |P| \sim \hat{X}^{-\frac{1}{3}}(\ell n\hat{X})^{-\frac{1}{2}}\tilde{r}(\tilde{z}), \theta \sim \tilde{\theta}(\tilde{z}) + \cdots \qquad (4.10c,d)$$

as $\hat{X} \to \infty$, with the $\overline{Z}$ scale contracting such that $\overline{Z} = \hat{X}^{-\frac{1}{2}}\tilde{z}$ with $\tilde{z}$ of 0(1). Hence the vortex equations become, with $\eta = \overline{Y}/\hat{X}^{\frac{1}{3}}$ of order unity,

$$\partial^2\tilde{W}/\partial\eta^2 + (\eta^2/3)\partial\tilde{W}/\partial\eta - (\eta\tilde{z}/2)\partial\tilde{W}/\partial\tilde{z} + (\eta/6)\tilde{W} = 0, \qquad (4.11a)$$

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$$\partial^2 \tilde{\tau}/\partial \eta^2 + (\eta^2/3)\partial \tilde{\tau}/\partial \eta - (\eta \tilde{z}/2)\partial \tilde{\tau}/\partial \tilde{z} - \eta \tilde{\tau} = -\partial \tilde{W}/\partial \tilde{z}, \qquad (4.11b)$$

subject to

$$\tilde{W} = [d(\tilde{\tau}^2)/d\tilde{z}]/3, \partial \tilde{\tau}/\partial \eta = 0, \tilde{\tau} = \tilde{\lambda}_3 \text{ at } \eta = 0, \qquad (4.11c)$$

$$\tilde{W} \to 0, \tilde{\tau} \to 0 \text{ as } \eta \to \infty, \qquad (4.11d)$$

and coupled with the TS equation

$$0 = (c_2 \lambda_1 + c_3 \tilde{\lambda}_3)\tilde{P} + c_4 d^2 \tilde{P}/d\tilde{z}^2 \qquad (4.11e)$$

where $\tilde{P} \equiv \tilde{r} \exp(i\tilde{\theta})$. The TS-forcing of the vortex motion now appears only through the inner restraint on $\tilde{W}$ in (4.11c), while the vortex forcing on the TS wave is due to the $\tilde{\lambda}_3$ term in (4.11e), and it is seen also that the amplitude-cubed TS effects become secondary in this type of spatial evolution. The solution of the vortex part (4.11a-d) leads to an induced skin-friction factor $\tilde{\lambda}_3$ which enables (4.11e) to be written as an integro-differential equation for $\tilde{P}(\tilde{z})$,

$$0 = \left[c_2 \lambda_1 - c_3 \tilde{\gamma} \tilde{z}^{-2} \int_0^{\tilde{z}} \frac{\tilde{\rho}''(\xi)\xi^{\frac{5}{3}}}{(\tilde{z}^2 - \xi^2)^{\frac{1}{3}}} d\xi \right] \tilde{P} + c_4 \tilde{P}'' \qquad (4.12a)$$

$$\tilde{\rho} = |\tilde{P}|^2, \qquad (4.12b)$$

where $\tilde{\gamma} = 2/[3^{\frac{2}{3}}\Gamma(1/3)]$ is a positive constant and symmetry about $\tilde{z} = 0$ is assumed for convenience. The next question is whether an acceptable solution exists or not. Finite-difference computations have been performed for (4.12a,b) and these all suggest that there is no solution smooth for all $\tilde{z}$. Analytically, the large $-|\tilde{z}|$ behavior poses difficulties since the main suggested asymptote there, $\tilde{\rho} \sim k_3 \tilde{z}^2$ with k_3 a positive constant, makes the integral in (4.12a) positive, so that the square-bracketed term cannot tend to zero then. Instead the solution appears to hit a singularity at finite $\tilde{z} = \tilde{z}_0-$ in which $\tilde{\rho}$ tends to infinity like $(\tilde{z}_0 - \tilde{z})^{-\frac{7}{3}}$, and this seems in line with the computational findings. The singularity cannot be smoothed out locally, it seems, and so option 3 fails in that sense.

The fourth option is that the nonlinear interaction (4.1) terminates in an algebraic breakdown at a finite distance $\hat{X}$, say as $\hat{X} \to \hat{X}_s-$. The vortex part suggests that, in terms of order of magnitude, $\overline{Y} \sim (\hat{X}_s - \hat{X})^{\frac{1}{3}}, W \sim r^2 \overline{Z}^{-1}, \tau \sim \lambda_3 \sim r^2 \overline{Y}^2 \overline{Z}^{-2}$ and hence the dominant effects in the TS part are due to the c_3, c_4, c_5 contributions, giving terms $\sim r^3(\hat{X}_s - \hat{X})^{\frac{2}{3}}\overline{Z}^{-2}, r\overline{Z}^{-2}, r^3$ respectively. For these to be comparable $\overline{Z} \sim (\hat{X}_s - \hat{X})^{\frac{1}{3}} \sim r^{-1}$. This indicates that the breakdown as $\hat{X} \to \hat{X}_s-$ takes the form

$$W \sim (\hat{X}_s - \hat{X})^{-1}L^{-\frac{1}{2}}\hat{W}, (\tau, \lambda_3) \sim (\hat{X}_S - \hat{X})^{-\frac{2}{3}}L^{-1}(\hat{\tau}, \hat{\lambda}_3), \qquad (4.13a, b)$$

$$r = |P| \sim (\hat{X}_s - \hat{X})^{-\frac{1}{3}}L^{-\frac{1}{2}}\hat{r}(\hat{z}), \theta \sim \hat{\theta}(\hat{z}), \qquad (4.13c, d)$$

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to leading order, focussed near $\overline{Z} = \overline{Z}_*$ say with

$$\overline{Z} - \overline{Z}_* = (\hat{X}_* - \hat{X})^{\frac{1}{3}} L^{\frac{1}{2}} \hat{z}, \qquad (4.13e)$$

and $Y = (\hat{X}_* - \hat{X})^{\frac{1}{6}} \hat{\eta}$, $L \equiv -\ell n(\hat{X}_* - \hat{X})$, the extra logarithmic factors here and in (4.10) being required by the inner condition in (4.1c). The coupled local governing equations and constraints are, from (4.1),

$$\partial^2 \hat{W}/\partial \hat{\eta}^2 - (\hat{\eta}^2/3)\partial \hat{W}/\partial \hat{\eta} - (\hat{\eta}\hat{z}/3)\partial \hat{W}/\partial \hat{z} - \hat{\eta}\hat{W} = 0, \qquad (4.14a)$$

$$\partial^2 \hat{\tau}/\partial \hat{\eta}^2 - (\hat{\eta}^2/3)\partial \hat{\tau}/\partial \hat{\eta} - (\hat{\eta}\hat{z}/3)\partial \hat{\tau}/\partial \hat{z} - (2\hat{\eta}/3)\hat{\tau} = -\partial \hat{W}/\partial \hat{z}, \qquad (4.14b)$$

$$\hat{W}(\infty, \hat{z}) = \hat{\tau}(\infty, \hat{z}) = (\partial \hat{\tau}/\partial \hat{\eta})(0, \hat{z}) = \hat{W}(0, \hat{z}) - (\hat{\tau}^2)'/3 = \hat{\tau}(0, \hat{z}) - \hat{\lambda}_3 = 0 \qquad (4.14c)$$

for the vortex motion, and

$$0 = c_{3r}\hat{\lambda}_3\hat{\tau} + c_{4r}(\hat{\tau}'' - \hat{\tau}\hat{\theta}'^2) - c_{4i}(2\hat{\tau}'\hat{\theta}' + \hat{\tau}\hat{\theta}'') + c_{5r}\hat{\tau}^3, \qquad (4.14d)$$

$$0 = c_{3i}\hat{\lambda}_3\hat{\tau} + c_{4i}(\hat{\tau}'' - \hat{\tau}\hat{\theta}'^2) + c_{4r}(2\hat{\tau}'\hat{\theta}' + \hat{\tau}\hat{\theta}'') + c_{5i}\hat{\tau}^3 \qquad (4.14e)$$

for the TS part. Here once again the TS-forcing on the vortex motion appears only through the inner condition, in (4.14c), with the vortex forcing on the TS wave appearing via $\hat{\lambda}_3$ in (4.14d,e). In contrast with the previous possibility however the amplitude-cubed TS contribution here is a primary effect. The solution of (4.14a-c) can now be used to express $\hat{\lambda}_3$ as an integral involving $(\hat{\tau}^2)''$ and from this and (4.14d,e) we obtain the integro-differential equation

$$0 = c_3\hat{\gamma}\left\{\hat{z}^{-1}\int_0^{\hat{z}} \frac{s\hat{p}''(s)ds}{(\hat{z}^3 - s^3)^{\frac{1}{3}}}\right\}\hat{p} + c_4\hat{p}'' + c_5\hat{p}|\hat{p}|^2 \qquad (4.15)$$

with $\hat{p} = |\hat{p}|^2$ and $\hat{p}(\hat{z}) \equiv \hat{\tau}\exp(i\hat{\theta})$, while $\hat{\gamma} = 3^{\frac{1}{3}}/\Gamma(1/3)$ is a positive constant and again symmetry about $\hat{z} = 0$ is assumed. The issue therefore depends on whether (4.15) admits an acceptable solution or not. We note that as $|\hat{z}| \to \infty$ matching could be achieved with the flow solution outside, since then (4.15) suggests the behavior $\hat{p} \propto |\hat{z}|^{-1}$ and hence P, W, etc., become $0(1)$ when $\overline{Z} - \overline{Z}_*$ is $0(1)$. Finite-difference computations performed for (4.15) suggest the same conclusion as for option 3, however, namely the occurrence of a singularity at finite distance, this time with $\hat{p}$ tending to infinity like $(\hat{z}_0 - \hat{z})^{-\frac{5}{3}}$ as $\hat{z} \to \hat{z}_0-$, say. Again the singularity seems non-removable, thus ruling out option 4.

The third and fourth possibilities above both represent the formation of concentrated tongues of high vorticity and/or TS amplitude.

Option 5 for the present interaction is that the zero state is approached far downstream, with

$$(W, \tau, P) \to 0 \text{ as } \hat{X} \to \infty. \qquad (4.16)$$

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The stability of this state can be analyzed as for option 2, thus leading to (4.6) in the zero - λ_1 case but with r_0 replaced by zero. Hence at large $\hat{X}$ say λ_{31} is zero as well and no growth of the perturbation is possible as anticipated in earlier comments. The option (4.16) means that the Type I interaction peters out but that may in turn allow the Type III interaction addressed in Section 6 below to be set off subsequently further downstream, since the Type III involves smaller TS amplitudes.

The sixth option to consider has a finite-distance singularity in which the TS pressure remains finite but exhibits a singular behaviour in its gradient, as distinct from the strong singularity proposed in option 4. This option follows from a simple exact solution which is a generalization of the 2D one mentioned earlier to include a single spanwise-mode dependence, $P = P_0(\hat{X}) \exp(i\hat{\beta}\overline{Z})$ or $r = R_0(\hat{X}), \theta = \Theta_0(\hat{X}) + \hat{\beta}\overline{Z}$, say. Here there is no induced vortex $W - \tau$ flow, only the free $W - \tau$ flow due to the input upstream, giving the skin-friction effect $\lambda_3(\hat{X}, \overline{Z})$ in general. The solution fits together if λ_3 is independent of $\overline{Z}$ since then R_0 satisfies

$$0 = \left[c_{1r} - c_{4r}\hat{\beta}^2 + c_{2r}\lambda_1\hat{X} + c_{3r}\lambda_3(\hat{X})\right] R_0 + c_{5r}R_0^3. \tag{4.17}$$

Hence as $\hat{X}$ increases the amplitude R_0 can hit zero in a square-root fashion if $\lambda_3(\hat{X})$ goes sufficiently positive that the square-bracketed term above approaches zero. An extension to the general case of (4.1) may be made next (see also Fig. 3). The proposal then is that as $\hat{X} \to \hat{X}_s-$ for some finite station $\hat{X}_s$ the irregular response

$$P = P_0(\overline{Z}) + (\hat{X}_s - \hat{X})^{\frac{1}{2}} P_1(\overline{Z}) + (\hat{X}_s - \hat{X}) P_2(\overline{Z}) + \cdots, \tag{4.18a}$$

$$\lambda_3 = \lambda_{30}(\overline{Z}) + (\hat{X}_s - \hat{X})\lambda_{31}(\overline{Z}) + 0(\hat{X}_s - \hat{X})^{\frac{7}{6}} \tag{4.18b}$$

is encountered. This is consistent with (4.1) provided that the successive equations

$$0 = [c_1 + c_2\lambda_1\hat{X}_s + c_3\lambda_{30}(\overline{Z})]P_0 + c_4P_0'' + c_5P_0|P_0|^2 \tag{4.19a}$$

$$0 = [c_1 + c_2\lambda_1\hat{X}_s + c_3\lambda_{30}(\overline{Z})]P_1 + c_4P_1'' + c_5[2|P_0|^2P_1 + P_0^2P_1^*] \tag{4.19b}$$

$$0 = [c_1 + c_2\lambda_1\hat{X}_s + c_3\lambda_{30}(\overline{Z})]P_2 + [-c_2\lambda_1 + c_3\lambda_{31}(\overline{Z})]P_0$$

$$+ c_4P_2'' + c_5[2|P_0|^2P_2 + P_0^2P_2^* + P_1^2P_0^* + 2P_0|P_1|^2] \tag{4.19c}$$

hold for P_0, P_1, P_2 in turn. Here the vortex skin-friction factors $\lambda_{30}, \lambda_{31}$ remain arbitrary in the sense that they depend on the history of the flow upstream, and this arbitrariness makes the solution of (4.19a-c) difficult in general. As a model however suppose that λ_{30} is constant and the coefficients $c_n(n = 1 \to 5)$ are real. Then a solution of (4.19a) has P_0 equal to a real constant, while (4.19b) allows $P_1 \propto \sin(\hat{\beta}\bar{z})$ with $\hat{\beta}^2c_4 = 2c_5P_0^2$ and a

solution with the same periodicity requirement in $\overline{Z}$ can be found for P_2 in (4.19c). Hence the terminal form then is self-consistent. This model can also be enlarged to allow the c_n values to be complex as in (4.2), and other models support the proposal above. Another enlargement possible has spanwise focussing where the $\overline{Z}$-scale shrinks like a power of $\hat{X}_s - \hat{X}$ at the singular station and this appears to fit together as well.

The overall interpretation for this option 6 is that the induced-vortex skin friction can act to produce an irregular response in the TS amplitude, without a corresponding irregularity in the vortex motion. Further, the dependence on the history of the induced-vortex motion emphasizes the role of the upstream starting conditions in determining whether option 6 arises or not.

The computations referred to earlier seemed to confirm the availability of option 5, but they were not conclusive with regard to option 6. In particular it proved difficult to distinguish between numerical divergence of the iterative procedure used and appearance of a genuine square-root irregularity in the solutions. Failing other options (1-4 above and certain other ones tested), we tend to the view however that options 5,6 are the only ones attainable via the nonlinear interaction (4.1). The other options are associated with by-pass transitions: see Section 7.

The spanwise dependence in option 6 again allows streak-like formations to appear, although less firm than in the options 3,4 in Hall and Smith's (1988a) interactions and in the Type II interaction below. Option 6, with its shortening length scale streamwise, can lead into the type II interaction, which is studied next.

5. "TYPE II" INTERACTIONS

The earlier arguments for Type I interactions may be modified to account for an alternative, Type II, interaction as follows (again see Fig. 1). The relative errors referred to at the start of Section 2 have the orders

$$\beta^2 \text{ [from the warping]}, \tag{5.1a}$$

$$h^2 \text{ [from amplitude-cubed feedback]}, \tag{5.1b}$$

$$\beta^2 h^2 k^{-\frac{2}{3}} \text{ [from the induced vortex shear]}, \tag{5.1c}$$

$$h^2 \text{ [possible from streamwise TS modulation]}, \tag{5.1d}$$

$$k \text{ [from the streamwise vortex modulation]}, \tag{5.1e}$$

and in the Type I case above the balance of (5.1a-c) holds, yielding (2.2). The present alternative interaction (Type II) occurs for $\beta \sim h$ again, balancing the effects (5.1a,b),

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but $k \sim h^2$ now, so that the typical streamwise spatial evolution is faster here. Then the vortex-shear effect (5.1c) is of order $h^{\frac{8}{3}}$ and is relatively small, giving a reduced vortex influence on the nonlinear TS evolution.

The Type II interaction therefore has

$$\beta \sim h, k \sim h^2, h << 1, \qquad (5.2)$$

and it can be seen from modification of the working in sections 2,3 that the new sizes (5.2) lead to a TS amplitude equation in which $\partial P_{11}/\partial X_1$ replaces the interaction term $\propto \lambda_3$ on the left-hand side in (2.14) and absorbs the σ-term there also. Hence we are left with solving the nonlinear equation

$$\frac{\partial P}{\partial \hat{X}} = (c_1 + c_2\lambda_1\hat{X})P + c_4\frac{\partial^2 P}{\partial \overline{Z}^2} + c_5 P|P|^2 \qquad (5.3)$$

instead of (4.1e), the vortex effect (4.1a-d,f) then being of secondary significance. The constants c_1, c_2, c_4, c_5 are as in Section 4.

The governing equation (5.3) for the TS pressure $P(\hat{X}, \overline{Z})$ is a generalized cubic Schrodinger or Ginzburg-Landau equation, and its main properties seem to be clear. First, the pure 2D version where the c_4 term is suppressed produces a supercritical bifurcation, namely that in Smith (1979), Hall and Smith (1984), leading to a saturation amplitude downstream as $\hat{X}$ increases: see below. Second, however, the 3D version with $\overline{Z}$-dependence present shows that there is unbounded secondary instability of the 2D state. This is due mainly to the coefficient c_{4r} begin negative. Thus in polar form $P = r\exp(i\theta)$, which replaces (5.3) by the two real equations

$$r_{\hat{X}} = (c_{1r} + c_{2r}\lambda_1\hat{X})r + c_{4r}(r_{\overline{Z}\overline{Z}} - r\theta_{\overline{Z}}^2) - c_{4i}(2r_{\overline{Z}}\theta_{\overline{Z}} + r\theta_{\overline{Z}\overline{Z}}) + c_{5r}r^3, \qquad (5.4a)$$

$$r\theta_{\hat{X}} = (c_{1i} + c_{2i}\lambda_1\hat{X})r + c_{4i}(r_{\overline{Z}\overline{Z}} - r\theta_{\overline{Z}}^2) + c_{4r}(2r_{\overline{Z}}\theta_{\overline{Z}} + r\theta_{\overline{Z}\overline{Z}}) + c_{5i}r^3, \qquad (5.4b)$$

the pure 2D flow solution has $(r, \theta) = (r_0, \theta_0)(\hat{X})$ where

$$r_0' = (c_{1r} + c_{2r}\lambda_1\hat{X})r_0 + c_{5r}r_0^3, \text{ and } \theta_0' = (c_{1i} + c_{2i}\lambda_1\hat{X}) + c_{5i}r_0^2, \qquad (5.5a,b)$$

so that with $\overline{\kappa} \equiv c_{2r}\lambda_1$ positive (the nonparallel-flow case)

$$r_0 = \left[\exp(\overline{\kappa}\overline{X}^2)/\{-2c_{2r}\int_{\overline{X}_0}^{\overline{X}} \exp(\overline{\kappa}q^2)dq\}\right]^{\frac{1}{2}}, \hat{X} \equiv \overline{X} + c_{1r}\overline{\kappa}^{-1}, \qquad (5.5c)$$

for $\overline{X}$ beyond an initial station downstream of $\overline{X}_0$, while if λ_1 is zero (the parallel-flow case)

$$r = [c_{1r}a_1\exp(2c_{1r}\hat{X})/\{1 - a_1c_{5r}\exp(2c_{1r}\hat{X})\}]^{\frac{1}{2}} \qquad (5.5d)$$

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where the constants a_1, c_{1r} are both positive. The saturated 2D solution downstream exhibits parabolic growth $r_0 \sim (\overline{\kappa}\hat{X}/(-c_{5r}))^{\frac{1}{2}}$ or tends to the constant value $r_0 = r_\infty \equiv (c_{1r}/(-c_{5r}))^{\frac{1}{2}}$ as $\hat{X} \to \infty$. A small 3D perturbation of the 2D solution,

$$[r, \theta] = [r_0, \theta_0](\hat{X}) + \tilde{\delta}[r_1, \theta_1](\hat{X}) \exp(i\overline{\beta}\overline{Z}) + 0(\tilde{\delta}^2), \tag{5.6}$$

with spanwise wavenumber $\overline{\beta}$, is then controlled by the linearized system

$$r_1' = (c_{1r} + c_{2r}\lambda_1\hat{X} - \overline{\beta}^2 c_{4r})r_1 + \overline{\beta}^2 c_{4i}r_0\theta_1 + 3c_{5r}r_0^2 r_1, \tag{5.7a}$$

$$r_0\theta_1' = -\overline{\beta}^2 c_{4i}r_1 - \overline{\beta}^2 c_{4r}r_0\theta_1 + 2c_{5i}r_0^2 r_1, \tag{5.7b}$$

from (5.4a,b). As posed, the secondary instability problem (5.7a,b) represents a nonparallel marching problem in $\hat{X}$ for the spatial evolution of the 3D perturbation, from given starting conditions at some finite $\hat{X}$ and with $r_0(\hat{X})$ specified in (5.5c or d). Downstream at large positive $\hat{X}$ however the 2D equilibrium $r_0 \to r_\infty$ (constant) holds if we focus on the case of zero λ_1 and there the solution of (5.7) is sought in the form

$$(r_1, \theta_1) = (\hat{r}_1, \hat{\theta}_1) \exp(\hat{Q}\hat{X}) \tag{5.8}$$

with the spatial-growth factor $\hat{Q}$ to be found. In effect, each of the primes in (5.7a,d) is then replaced by a factor $\hat{Q}, r_\infty$ replaces $r_0(\hat{X})$, and so substituting for $\hat{\theta}_1$ from the equivalent of (5.7b) into (5.7a) we obtain, for nontrivial solutions, an eigenrelation determining $\hat{Q}$ in terms of $\overline{\beta}, r_\infty$ which gives

$$\hat{Q} = c_{5r}r_\infty^2 - c_{4r}\overline{\beta}^2 \pm [c_{5r}^2 r_\infty^4 + 2c_{4i}c_{5i}r_\infty^2\overline{\beta}^2 - c_{4i}^2\overline{\beta}^4]^{\frac{1}{2}}. \tag{5.9}$$

The maximum spatial growth rate $\hat{Q}_r$ therefore arises at large spanwise wavenumbers $\overline{\beta}$, for which $\hat{Q}_r \sim -c_{4r}\overline{\beta}^2$ is large and positive. Indeed, the same large unbounded growth also occurs at finite $\hat{X}$, within the framework of (5.7) (with λ_1 zero or nonzero), as a pronounced short-wave instability. This again raises the possibility of spanwise focussing taking place in the nonlinear system (5.3).

Finally, the fully nonlinear 3D version, in (5.3), produces in general a break-up of the nonlinear TS solution within a finite distance, associated with the formation of "vorticity tongues." The break-up, at the station $\hat{X} \to \hat{X}_s-$ say, has

$$|P| = r \sim (\hat{X}_s - \hat{X})^{-\frac{1}{2}}\hat{r}(\hat{Z}) + \cdots, \tag{5.10a}$$

$$\arg P = \theta \sim k_1 \ell n(\hat{X}_s - \hat{X}) + \hat{\theta}(\hat{Z}) + \cdots, \tag{5.10b}$$

$$\overline{Z} - \overline{Z}_s = (\hat{X}_s - \hat{X})^{\frac{1}{2}}\hat{Z}, \tag{5.10c}$$

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focussed at a particular spanwise station $\overline{Z} = \overline{Z}_s$: see Fig. 4. Here k_1 is a constant. So the governing equations (5.3) or (5.4) reduce to the nonlinear ordinary-differential system

$$\frac{1}{2}(\hat{r} + \hat{Z}\hat{r}') = c_{4r}(\hat{r}'' - \hat{r}\hat{\theta}'^2) - c_{4i}(2\hat{r}'\hat{\theta}' + \hat{r}\hat{\theta}'') + c_{5r}\hat{r}^3, \qquad (5.11a)$$

$$\hat{r}(-k_1 + \frac{1}{2}\hat{Z}\hat{\theta}') = c_{4i}(\hat{r}'' - \hat{r}\hat{\theta}'^2) + c_{4r}(2\hat{r}'\hat{\theta}' + \hat{r}\hat{\theta}'') + c_{5i}\hat{r}^3 \qquad (5.11b)$$

for $\hat{r}, \hat{\theta}$. The existence of acceptable solutions of this terminal form (see also Hocking and Stewartson (1972) and references therein) is suggested simply by a linearized version for example in which the cubic terms in (5.11) are negligible. One solution then has

$$\hat{r} \propto \exp[-\hat{Z}^2(-c_{4r})/4|c_4|^2],$$

$$\hat{\theta} = \text{constant} - \hat{Z}^2 c_{4i}/4|c_4|^2, \qquad (5.12a)$$

and the other solution is expressible in terms of error functions, giving an algebraic $0(\hat{Z}^{-1})$ decay in $\hat{r}$ at large $|\hat{Z}|$ which matches with the flow solution outside where r is $0(1)$ at $0(1)$ values of $\overline{Z} - \overline{Z}_s$. Both of the above solutions are acceptable because the coefficient c_{4r} is negative.

The finite-distance break-up in (5.10) - (5.11) is clearly a strong one and it may be interpreted as producing strong tongues of concentrated streamwise vorticity and TS amplitudes in the singular form (5.10). New higher-amplitude shorter-scale phenomena must then come into operation, closer to the break-up point $\hat{X}_s, \overline{Z}_s$, to continue the development of the tongue. Specifically, the full 3D nonlinear system (2.1) is then reinstated. This is mainly because the pressure amplitude p due to (5.10) is becoming of order $h(\hat{X}_s - \hat{X})^{-\frac{1}{2}}$, which reduces $0(1)$ when the streamwise distance $|X - h^{-2}\hat{X}_s|$ reduces to $0(1)$, from (5.2). Simultaneously the Z-scale contracts to $0(1)$, from (5.1) with (5.10), the induced-vortex strength rises to $0(1)$ also, and of course an $0(1)$ time scale T is present because of the primary wave (2.9). Hence the further evolution of this tongue or streak is controlled by the fully nonlinear triple-deck system (2.1). Some computational solutions for such 3D unsteady triple-deck flows are given by Smith (1988a), while the likelihood of finite-time break-ups occurring even in the full triple-deck framework is shown by Smith (1988b).

6. "TYPE III" INTERACTIONS

Here once again the reasoning concerning the error sizes in (5.1a-e) may be modified/extended to describe a new kind of nonlinear interaction possible, Type III (see also Fig. 1). This last type has β being of order unity, i.e. faster spanwise dependence than

before, but with h still being small and k again being $O(h^3)$ as in the first type of interaction. So now the vortex-shear influence (5.1c) is also of order unity, balancing the warping effect (5.1a).

This Type-III nonlinear interaction, in which

$$\beta \sim 1, k \sim h^3, h << 1 \tag{6.1}$$

therefore provokes a nonlinear effect on the *whole* flow, in particular completely altering the mean-flow profile, despite the smallness [$h << 1$] of the input TS wave. In more detail, some allowance has to be made for the logarithmic interplay (see Section 2) between the buffer and the near-wall zones, and in fact the near-wall zone has the expansion

$$h' \equiv \frac{h}{(-\ell n h)^{\frac{1}{2}}} \begin{cases} u = \lambda_v Y + h'(u_1 E + c.c.) + h'^2 u_v + \cdots, & (6.2a) \\[2mm] v = h'(v_1 E + c.c.) + h'^2 v_v + \cdots, & (6.2b) \\[2mm] w = h'(w_1 E + c.c.) + h'^2 w_v + \cdots & (6.2c) \end{cases}$$

now, while in the buffer zone where $Y = h^{-1}\overline{Y}$ we have

$$u = h^{-1}U_v(\overline{X}, \overline{Y}, Z) + \cdots, \tag{6.3a}$$

$$v = hV_v(\overline{X}, \overline{Y}, Z) + \cdots, \tag{6.3b}$$

$$w = h^2 W_v(\overline{X}, \overline{Y}, Z) + \cdots. \tag{6.3c}$$

In (6.2a), $\lambda_v(\overline{X}, Z)$ is the unknown vortex-induced shear, and a short- and a long-scale streamwise dependence hold, in that

$$\frac{\partial}{\partial X} \rightarrow \frac{\partial}{\partial X} + h^3 \frac{\partial}{\partial \overline{X}}. \tag{6.4}$$

The subscript v refers to induced-vortex quantities. Also the unknown pressure and displacement take the forms

$$p = h'(PE + c.c.) + \cdots, \tag{6.5a}$$

$$A = h^{-1}A_v + h'(\tilde{A}E + c.c.) + \cdots \tag{6.5b}$$

where, as in (6.2), the smallness of the input TS disturbance is evident.

The principal feature of the near-wall-zone form in (6.2) is that it involves a small perturbation of the unknown shear flow $u = \lambda_v Y$ and as such it leads at first order to the linear equation

$$\frac{\partial^2 P}{\partial Z^2} - \frac{1}{\lambda_v} \frac{\partial \lambda_v}{\partial Z} \frac{\partial P}{\partial Z} \mathcal{F} - \alpha^2 P = \tilde{A}\mathcal{G} \tag{6.6a}$$

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for the TS pressure P, after some working as in Smith (1980). In (6.6a), $\tilde{A}$ may be replaced by a linear integral of P, stemming from the interaction law (2.1f), or the equivalent upper-deck-flow setting may be invoked, i.e. solve

$$(\partial_{\hat{y}}^2 + \partial_Z^2 - \alpha^2)\hat{p} = 0, \tag{6.6b}$$

$$\hat{p} = P, \partial_{\hat{y}}\hat{p} = -\alpha^2\tilde{A} \text{ at } \hat{y} = 0+, \tag{6.6c}$$

with $\hat{p}$ bounded at infinity, to obtain the $P - \tilde{A}$ law. Again, with some notation of Section 2,

$$\mathcal{F} = \frac{3}{2} + \frac{\xi_0 Ai_0'}{2Ai_0}\left(1 + \frac{\xi_0\kappa}{Ai_0'}\right), \mathcal{G} = (i\alpha\lambda_v)^{\frac{5}{3}}\frac{Ai_0'}{\kappa} \tag{6.6d}$$

and α, Ω must be real. We observe that (6.6a) looks linear but strong nonlinear dependence is still present, nevertheless, through the unknown shear λ_v. At second order a vortex-flow contribution (among others) is induced due to amplitude-squared forcing and that yields the logarithmic growth

$$w_v \sim -\lambda_v^{-2}\partial_Z[|P|^2 + \alpha^{-2}P_Z P_Z^*]\ell n Y \text{ as } Y \to \infty \tag{6.7}$$

similar to the growth described in Section 2, with the subscript v again referring to the vortex component.

The growth (6.7) and the corresponding growths proportional to $Y^3\ell n Y, Y\ell n Y$ in u_v, v_v, respectively, are responsible for the considerable vortex effect set up in the buffer zone as described by (6.3). The controlling equations in this buffer are the full nonlinear 3D boundary-layer equations,

$$\frac{\partial U_v}{\partial \overline{X}} + \frac{\partial V_v}{\partial \overline{Y}} + \frac{\partial W_v}{\partial Z} = 0, \tag{6.8a}$$

$$\frac{U_v\partial U_v}{\partial \overline{X}} + \frac{V_v\partial U_v}{\partial \overline{Y}} + \frac{W_v\partial U_v}{\partial Z} = \frac{\partial^2 U_v}{\partial \overline{Y}^2}, \tag{6.8b}$$

$$\frac{U_v\partial W_v}{\partial \overline{X}} + \frac{V_v\partial W_v}{\partial \overline{Y}} + \frac{W_v\partial W_v}{\partial Z} = \frac{\partial^2 W_v}{\partial \overline{Y}^2} \tag{6.8c}$$

from substitution of (6.3) into (2.1a-c). It is noteworthy that the induced pressure gradient for the vortex motion is negligible here. This is because the main displacement (in (6.5b)) associated with (6.3), although of order h^{-1}, depends mostly on the slow variable $\overline{X}$, not the fast one X, and hence the corresponding induced pressure p is only $0(h^5)$, from (2.1f) or from analysis of the upper-deck behavior of the vortex flow, which then provokes a negligible feedback in the momentum balances (6.8b,c). Thus the outer boundary conditions here are

$$U_v \sim \overline{Y} + A_v(\overline{X}, Z), W_v \to 0 \text{ as } \overline{Y} \to \infty \tag{6.8d}$$

where the displacement function $A_v(\overline{X}, Z)$ is to be found from the solution of (6.8a-c). The inner boundary conditions account for the match with (6.2) combined with (6.7), so that

$$U_v = V_v = 0, W_v = -\lambda_v^{-2} \partial_Z [|P|^2 + \alpha^{-2} P_Z P_Z^*] \text{ at } \overline{Y} = 0. \tag{6.8e}$$

Finally here, the unknown vortex shear

$$\lambda_v = \frac{\partial U_v}{\partial \overline{Y}}(\overline{X}, 0, Z) \tag{6.9}$$

links the buffer and near-wall solutions together, providing the overall nonlinear interaction between (6.6), (6.8). Another noteworthy point, to repeat, is the smallness of the TS amplitude compared with the considerable vortex motion that it provokes; and the TS-squared forcing of the vortex motion in the buffer zone is dominated by the effective spanwise-slip-velocity condition in (6.8e); other forcings such as those through the momentum equations are of less importance, at least at the current stage.

The full Type-III interaction is controlled then by (6.8a-e) subject to (6.6), (6.9), for the driven vortex flow and the driven TS pressure, respectively.

Secondary instability of a 2D TS input can be established as for the previous Types I, II and as for the channel-flow interaction of Hall and Smith (1988b). The secondary instability of the pure 2D state, i.e. $P = P_0(\overline{X})$ with $U_v \equiv \overline{Y}, V_v = W_v = 0$ and $\alpha = \alpha_0 = d_2^{\frac{3}{4}}, \lambda_v = 1$ (which gives an exact solution of the system), arises as a small 3D perturbation in the form

$$[P, \tilde{A}, U_v, V_v, W_v] = [P_0, A_0, \overline{Y}, 0, 0]$$

$$+\delta[P^{(1)} \cos \beta Z, A^{(1)} \cos \beta Z, U^{(1)} \cos \beta Z, V^{(1)} \cos \beta Z, W^{(1)} \sin \beta Z] + \cdots \tag{6.10}$$

and $\lambda_v = 1 + \delta \lambda^{(1)} \cos \beta Z + \cdots$, with δ being small. Here we take the case of P_0, A_0 being constant, within the present length scales, and all the Z-dependence is as shown explicitly in (6.10). The 3D induced-vortex flow is therefore controlled by the linearized equations, from (6.8a-e),

$$U_{\overline{X}}^{(1)} + V_{\overline{Y}}^{(1)} + \beta W^{(1)} = 0, \tag{6.11a}$$

$$\overline{Y} u_{\overline{X}}^{(1)} + V^{(1)} = U_{\overline{Y}\overline{Y}}^{(1)}, \tag{6.11b}$$

$$\overline{Y} w_{\overline{X}}^{(1)} = W_{\overline{Y}\overline{Y}}^{(1)}, \tag{6.11c}$$

subject to the constraints

$$U_{\overline{Y}}^{(1)} \to 0, W^{(1)} \to 0 \text{ as } \overline{Y} \to \infty, \tag{6.11d}$$

$$U^{(1)} = V^{(1)} = 0, W^{(1)} = \beta[P_0 P^{(1)*} + P_0^* P^{(1)}], \tag{6.11e}$$

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with the last condition representing the TS-forcing effect. Along with this, the TS perturbation satisfies, from (6.6),

$$(\beta^2 + \alpha_0^2)P^{(1)} = \alpha_0^3 A^{(1)} + \mathcal{H}\lambda^{(1)}P_0\alpha_0^{-1}, \tag{6.11f}$$

$$(\alpha_0^2 + \beta^2)^{\frac{1}{2}}P^{(1)} = \alpha_0^2 A^{(1)}. \tag{6.11g}$$

Here (6.11g) follows from (6.6b,c) and $P_0 = \alpha_0 A_0$, while $\mathcal{H} = [\frac{5}{3} - \frac{2\xi_0 Ai_0}{3\kappa}(1 + \frac{\xi_0 \kappa}{Ai_0'})]\alpha_0^3 = -\alpha_0^3 \hat{c}$, $\xi_0, \hat{c}$ are defined just prior to (2.12b) and in (2.15), respectively, and the vortex-forcing term is $\lambda^{(1)} \equiv \partial U^{(1)}(\overline{X}, 0)/\partial \overline{Y}$. The whole 3D perturbation is governed by the interactive system (6.11a-g) or by its integro-differential counterpart analogous to that in Appendix A. So the spatial development of the perturbation with downstream distance $\overline{X}$ is in general a *nonparallel* one, starting from input conditions at some finite station upstream. We turn therefore to the far-downstream response where spatial instability is proposed,

$$P^{(1)} \sim P^{(e)} \exp(s\overline{X}) \text{ as } \overline{X} \to \infty, \tag{6.12}$$

say, and similarily for the other variables, with the constant growth factor s to be found. Then the vortex part (6.11a-e) yields

$$W^{(e)} = K^{(e)}\frac{Ai(\eta)}{Ai(0)}, \eta \equiv s^{\frac{1}{3}}\overline{Y}, \tag{6.13a}$$

$$\frac{\partial U^{(e)}}{\partial \overline{Y}} = \frac{-\beta K^{(e)}Ai'(\eta)}{s^{\frac{2}{3}}Ai(0)} \tag{6.13b}$$

with $K^{(e)} \equiv \beta[P_0 P^{(e)*} + c.c.]$, so that the vortex skin-friction factor is

$$\lambda^{(e)} = \frac{-\beta^2 Ai'(0)}{s^{\frac{2}{3}}Ai(0)}[P_0 P^{(e)*} + P_0^* P^{(e)}]. \tag{6.13c}$$

Hence the TS part (6.11f,g) reduces to the complex linear equation

$$(\beta^2 + \alpha_0^2)P^{(e)} = \alpha_0(\alpha_0^2 + \beta^2)^{\frac{1}{2}}P^{(e)} + \frac{\mathcal{H}\Gamma\beta^2}{\alpha_0 s^{\frac{2}{3}}}\left[P_0^2 P^{(e)*} + |P_0|^2 P^{(e)}\right] \tag{6.14}$$

where $\Gamma = -Ai'(0)/Ai(0)$ is a positive constant. The real and imaginary parts of (6.14) then provide an eigenrelation for the determination of s, giving the result

$$s^{\frac{2}{3}} = 2\Gamma\alpha_0^{-1}\mathcal{H}_r|P_0|^2\beta^2(\alpha_0^2 + \beta^2)^{-\frac{1}{2}}[(\alpha_0^2 + \beta_0^2)^{\frac{1}{2}} - \alpha_0]^{-1}. \tag{6.15}$$

So the main properties of the growth factor s are that, first, it is real and positive, since $\mathcal{H}_r > 0$; second, s decreases monotonically with increasing spanwise wavenumber β, from the $0(1)$ value s_0 at $\beta = 0+$ to $\frac{1}{2}s_0$ as $\beta \to \infty$, where $s_0^{\frac{2}{3}} \equiv 4\Gamma\mathcal{H}_r|P_0|^2$, and hence the

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fastest growth is for nearly planar secondary modes, cf. Sections 4,5; third, s increases monotonically with $|P_0|$, like $|P_0|^3$; and, fourth, this secondary growth is present if and only if the input 2D TS amplitude $|P_0|$ is present. Other aspects are similar to those in Section 4 including the spanwise focussing associated with the large-β instability for initial-value problems.

Secondary instability of a similar kind also occurs at relatively high frequencies.

In the nonlinear regime there are several options which could be put forward for the ultimate behavior of the full interaction (6.6), (6.8), (6.9) as the downstream distance $\overline{X}$ increases, some of the options resembling those in Section 4. Again, the Type-III interaction connects up with a global-scale nonlinear interaction currently being studied by the author and Professor P. Hall. The Type-III interaction involves short-scale/long-scale balancing (via (6.4)) and it is clearly a robust one in that it changes the mean-flow characteristics (see (6.3)), even though the TS amplitude is still small (see (6.2)). Its full nonlinear properties remain to be studied.

7. FURTHER COMMENTS

We finish by noting the following items (a)-(f) on the nonlinear TS/vortex interactions studied above.

(a) Each of the three Types I-III of interaction addressed in this work can be triggered by the input disturbance upstream, depending on the latter's amplitude spectrum as indicated by the sizes specified in (2.2), (5.2), (6.1) [see also (d) below]. Alternatively, each interaction can start as a form of secondary 3D stability of the 2D TS input upstream. Beyond that however nonlinearity takes control.

(b) There are some connections between the three Types I-II, as their governing equations in Sections 4-6 suggest, and a match can be established also with the Hall and Smith (1988a) study concerning oblique TS waves. Further, the main options 5,6 for the behavior of the Type I nonlinear interaction may lead on into the Type III, II interactions, respectively, downstream. Type II has a faster streamwise response than Type I, while Type III has a faster spanwise dependence: see the scales in Sections 2,5,6.

(c) The Types II, III appear to be much more powerful and dangerous nonlinear interactions than Type I. Type III involves only a small TS amplitude (see (6.2)) but despite that the mean-flow profile is completely altered (see (6.3)), due to the induced vortex motion. Similarly, Type II starts with small TS amplitudes (see (5.2)) but the ensuing

finite-distance break-up then causes the full triple-deck system (2.1) to be triggered [Smith 1988a,b], thus completely altering the mean-flow profile again.

(d) By-pass transition processes are also possible and of much interest. They are associated with other scales of input upstream, say, and in principle these can activate all the options 1-6 described for the Type-I interaction and the corresponding options available for Type III as well. These and other subsequent stages seem to merit further research.

(e) The extra effects of wall curvature, cross flow and compressibility on the nonlinear interactions remain to be studied. All three can be captured by modifications of the triple-deck starting point (2.1) for instance: see earlier studies of compressibility effects in Smith (1987), cross-flow effects in Stewart and Smith (1987) and wall-curvature effects in Hall and Smith (1987, 1988b), among others. In turn, any of the three effects could be incorporated initially as an extra contribution to the nonlinear interaction equations of Sections 4-6, and it would be interesting to see their influence on the interactive flow properties.

(f) No quantitative comparisons with transition experiments [see earlier references] have been attempted yet, but the possibility of a reasonably firm link exists there in view of the predicted formation of streak-like behavior (among other phenomena), with spanwise concentrations of the vortex flow and the TS amplitudes (Sections 4-6).

ACKNOWLEDGEMENT

APPENDIX A: AN INTEGRAL FORMULATION FOR THE TYPE I IN-TERACTION

An integral formula for the vortex skin-friction factor $\lambda_3(\hat{X}, \overline{Z})$ can be obtained from the vortex-flow equations (4.1a-d). A Fourier or Laplace transform in $\hat{X}$, for instance, enables the solution for the transforms of W, τ to be expressed in terms of the functions M, N introduced in (4.8a,b). Hence we find the result

$$\lambda_3(\hat{X}, \overline{Z}) = \frac{\partial^2}{\partial \overline{Z}^2} \int_{\hat{X}_0}^{\hat{X}} |P|^2(\xi, \overline{Z})(\hat{X} - \xi)^{-\frac{1}{3}}[g_1 + g_2 \ell n(\hat{X} - \xi)]d\xi \qquad (A1)$$

in the case of a vortex flow starting from rest, or zero shear, at a station $\hat{X} = \hat{X}_0$, say. Here the constants g_1, g_2 are given by

$$g_2 = -e_2/\Gamma\left(\frac{2}{3}\right), g_1 = \left\{e_1 - g_2\Gamma'\left(\frac{2}{3}\right)\right\}/\Gamma(\frac{2}{3}) \qquad (A2)$$

with e_1, e_2 specified just after (4.8b).

So the nonlinear Type-I interaction can be written concisely in an integro-differential form, by coupling (A1) with (4.1e). This reproduces the governing equations for all the options 1-6 presented in Section 4. A similar formulation can be constructed for the oblique-wave/vortex interaction in Hall and Smith (1988a) and is used in the wave/vortex interactions being studied by Mr. N. D. Blackaby and Mr. P. A. Stewart.

References

[1] Craik, A. D. D., 1971, "Nonlinear resonant instability in boundary layers," *J. Fluid Mech.* **50**, 393.

[2] Hocking, L. M. and Stewartson, K., 1972, "On the nonlinear response of a marginally unstable plane parallel flow to a two-dimensional disturbance," *Proc. Roy. Soc. A* **326**, 289.

[3] Hall, P. and Smith, F. T., 1984, "On the effects of non-parallelism, three dimensionality, and mode interaction in nonlinear boundary layer stability," *Studies in Appl. Math.* **70**, 91.

[4] ———— 1987, "The nonlinear interaction of Tollmien-Schlichting waves and Taylor-Görtler vortices in curved channel flows," ICASE Report 87-25; and *Proc. Roy. Soc. A.*, 1988, **417**, 255.

[5] Hall, P. and Smith, F. T., 1988a, "Nonlinear Tollmien-Schlichting/vortex interaction in boundary layers," ICASE Report 88-46; and submitted to *European J. Mech.*

[6] ———— 1988b, "The strong nonlinear interaction of Tollmien-Schlichting waves and Taylor-Görtler vortices in curved channel flow," ICASE Report 88-45; and submitted to *J. Fluid Mech.* (with J. Bennett).

[7] ———— 1988c, "Strong nonlinear interactions between Tollmien-Schlichting waves and longitudinal vortices in boundary layers," to be submitted as an ICASE Report.

[8] Herbert, T., 1984, "Analysis of the subharmonic route to transition in boundary layers," AIAA Paper 84-0009, presented in Reno, Nevada.

[9] Smith, F. T., 1979, "Nonlinear stability of boundary layers for disturbances of various sizes," *Proc. Roy. Soc. A* **366**, 91 and **368**, 573.

[10] ———— 1980, "Instability of flow through types of general cross-sections, Parts 1 and 2," *Mathematika* **26**, 187-210 and 211-223.

[11] ———— 1987, "On the first-mode instability in subsonic, supersonic, and hypersonic boundary layers," United Tech. Research Center Report UT87-52; and J. Fluid Mech., to appear.

[12] ———— 1988a, "Steady and unsteady 3-D interacting boundary layers," *Computers and Fluids*, to appear.

[13] —————— 1988b, "Finite-time break-up can occur in any unsteady interacting boundary layer," *Mathematika*, to appear December 1988.

[14] Smith, F. T. and Stewart, P. A., 1987, "The resonant-triad nonlinear interaction in boundary-layer transition," *J. Fluid Mech.* **179**, 227.

[15] Stewart, P. A. and Smith, F. T., 1987, "Three-dimensional instability in steady and unsteady non-parallel boundary layers, including effects of Tollmien-Schlichting disturbances and cross-flow," *Proc. Roy. Soc. A* **409**, 229.

[16] Srivastava, K. M. and Dallman, U., 1987, "Effect of streamwise vortices on Tollmien-Schlichting waves in growing boundary layers," *Phys. Fluids* **30**, 1005.

[17] Arnal, D., 1984, "Description and prediction of transition in 2-D incompressible flow," Special Course on Stability and Transition of Laminar Flow, AGARD-R-709, 2-1 to 2-71.

[18] Benney, D. J. and Lin, C. C., 1960, "On the secondary motion induced by oscillations in a shear flow," *Phys. Fluids* **3**, 656.

FIGURE CAPTIONS

Fig. 1. Nonlinear interaction Types I-III between long-scale vortices and short-scale waves, in normalized coordinates, for waves of low input amplitude $O(h)$. The values of the length scales β^{-1}, k^{-1} for I-III are given in the text (sections 2,5,6 in turn).

Fig. 2. The scaled 3D spatial growth rate $\bar{\gamma}$ versus spanwise wavenumber $\bar{\beta}$ for various input 2D amplitudes r_0, in secondary instability for Type-I interaction.

Fig. 3. The square-root spatial breakdown of the wave amplitude (option 6) for Type-I nonlinear interactions, due to the vortex-shear effect (λ_3), leading to shortening of the interaction length scale.

Fig. 4. Nonlinear breakdown of the Type-II interaction, via spanwise focussing and spatial blow-up of the wave amplitude.
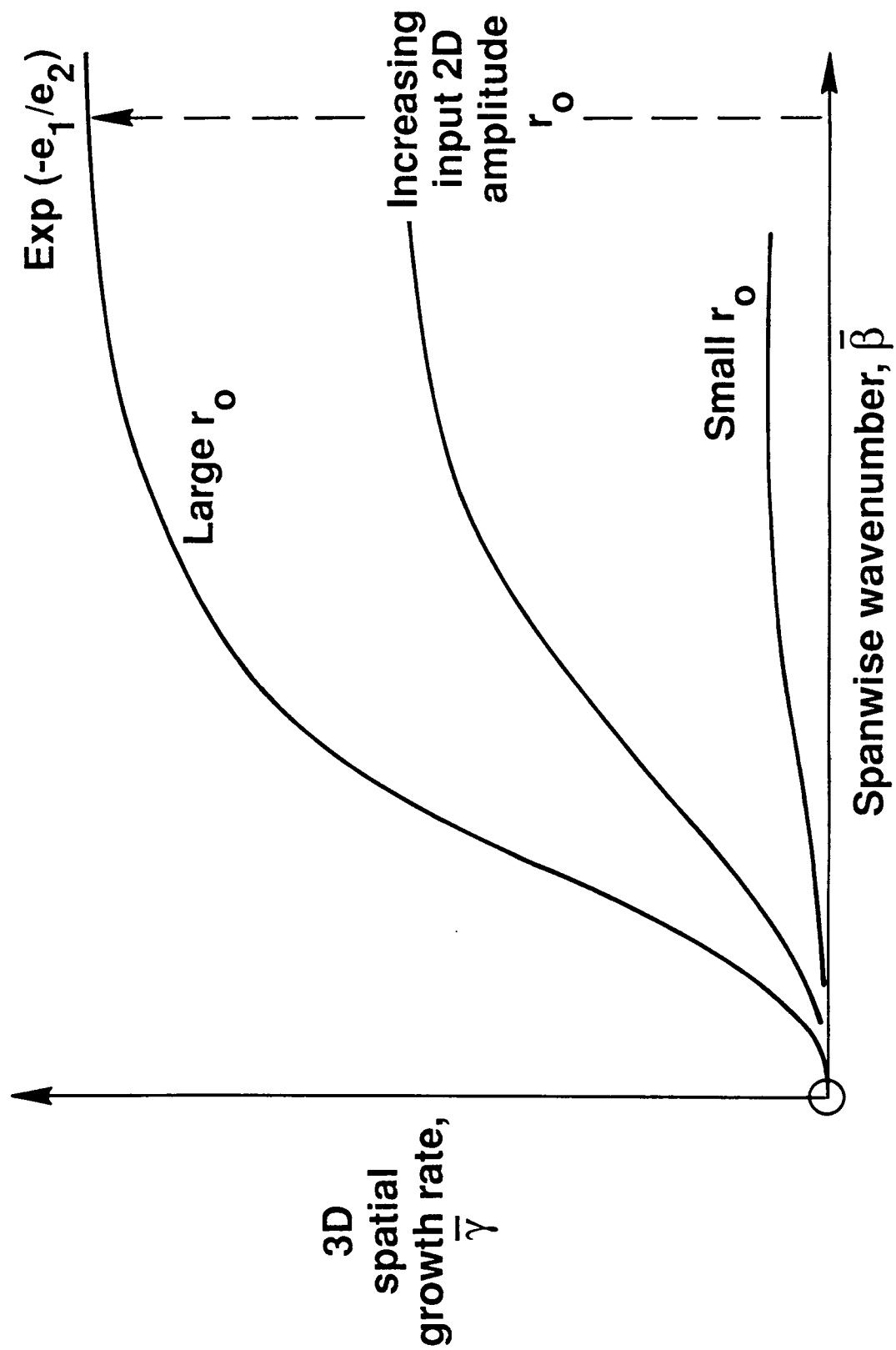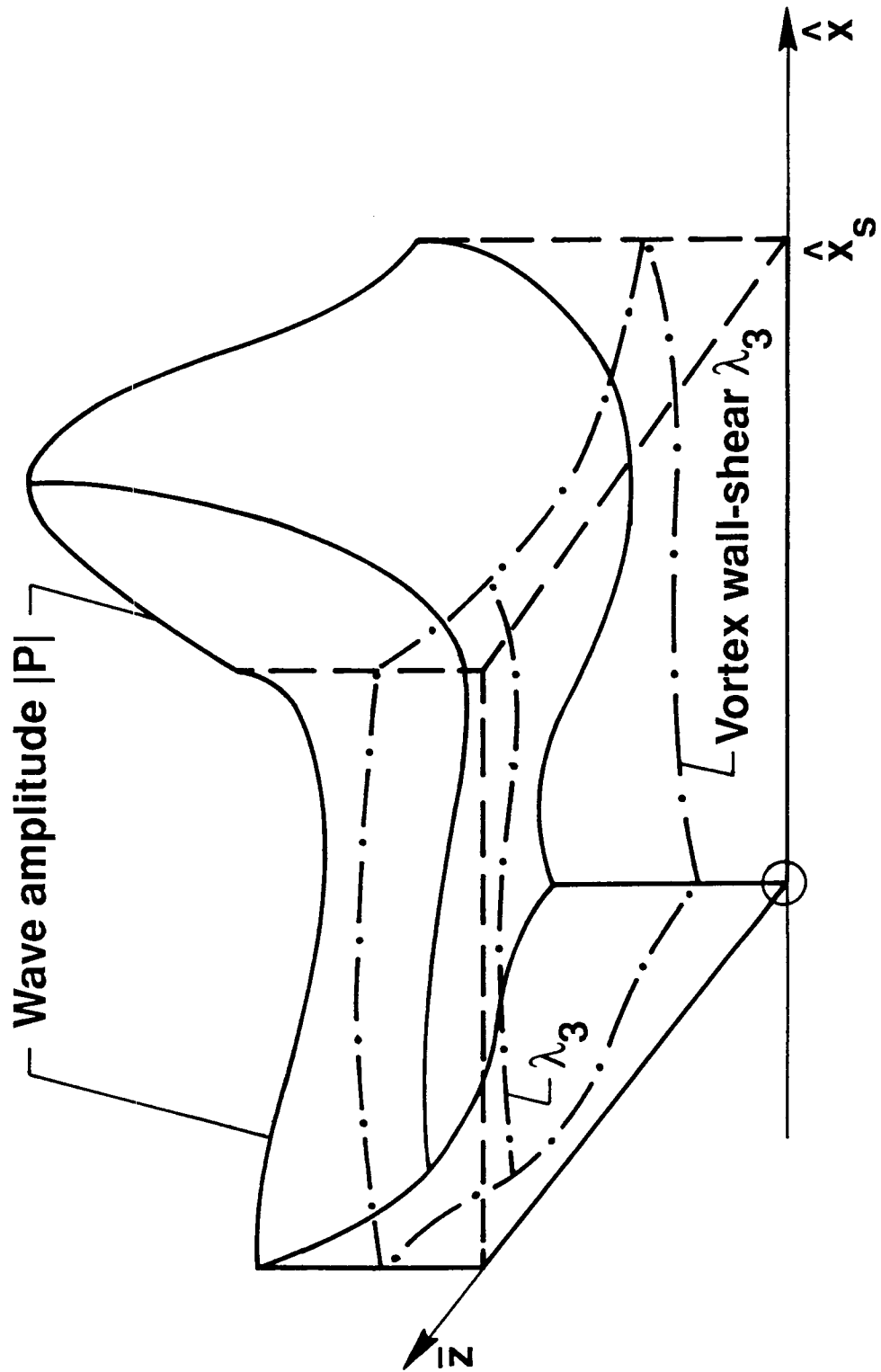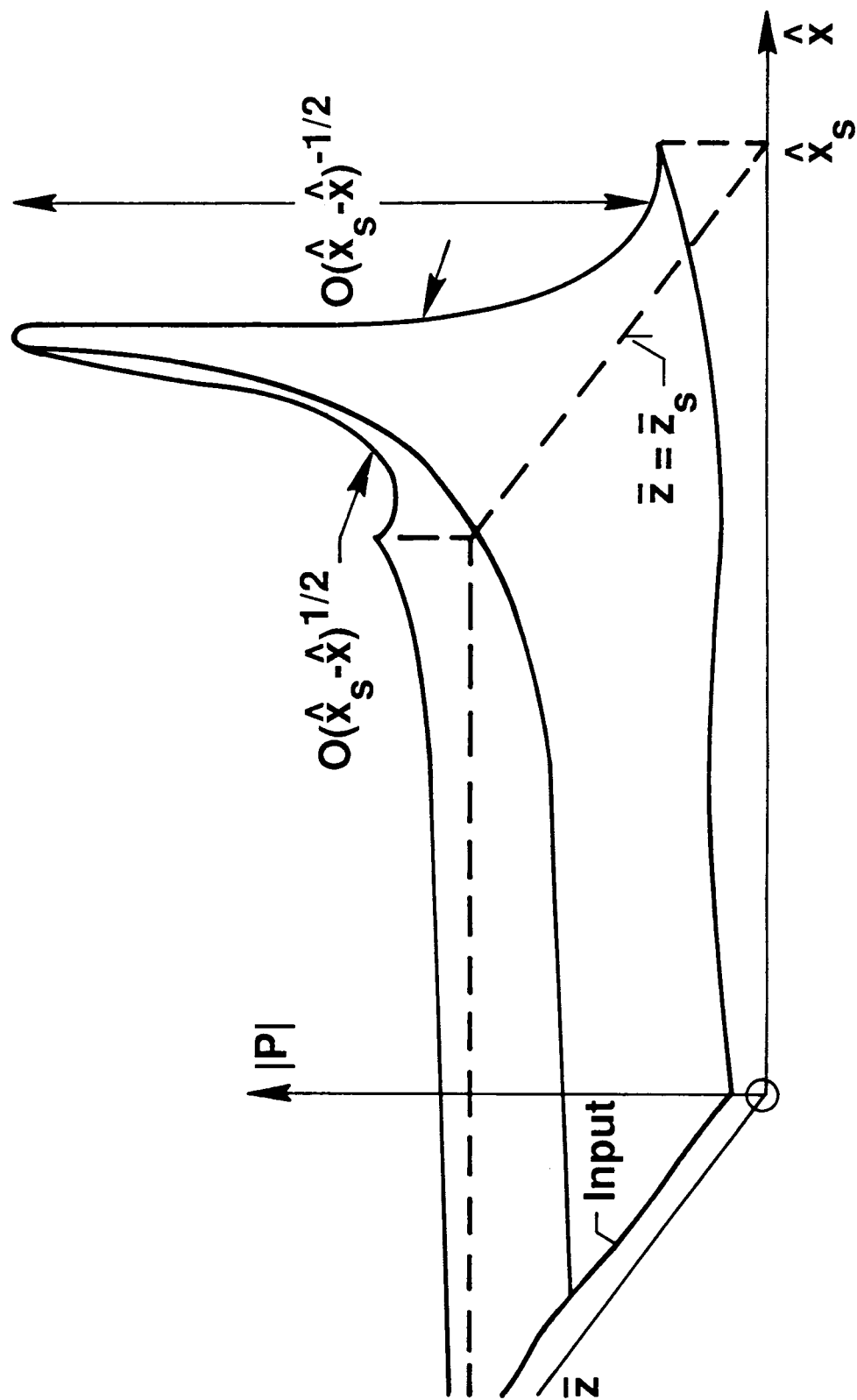
Figure 1.

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Figure 2.

Figure 3.

Figure 4.

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16. Abstract

The nonlinear interactions that evolve between a planar or nearly planar Tollmien-Schlichting (TS) wave and the associated longitudinal vortices are considered theoretically, for a boundary layer at high Reynolds numbers. The vortex flow is either induced by the TS nonlinear forcing or is input upstream, and similarly for the nonlinear wave development. Three major kinds of nonlinear spatial evolution, Types I-III, are found. Each can start from secondary instability and then becomes nonlinear, Type I proving to be relatively benign but able to act as a pre-cursor to the Types II, III which turn out to be very powerful nonlinear interactions. Type II involves faster streamwise dependence and leads to a finite-distance blow-up in the amplitudes, which then triggers the full nonlinear three-dimensional triple-deck response, thus entirely altering the mean-flow profile locally. In contrast, Type III involves slower streamwise dependence but a faster spanwise response, with a small TS amplitude thereby causing an enhanced vortex effect which, again, is substantial enough to entirely alter the mean-flow profile, on a more global scale. Streak-like formations in which there is localized concentration of streamwise vorticity and/or wave amplitude can appear, and certain of the nonlinear features also suggest by-pass processes for transition and significant changes in the flow structure downstream. The powerful nonlinear 3D interactions II, III are potentially very relevant to experimental findings in transition.

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