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A Mathematical Formulation
of the SCOLE Control Problem,
Part II: Optimal Compensator Design

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A MATHEMATICAL FORMULATION

OF THE SCOLE CONTROL PROBLEM, PART II:

OPTIMAL COMPENSATOR DESIGN

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Abstract

In this report we conclude the study initiated in Part I and go on to consider optimal feedback control (compensator) design for stability augmentation, following the mathematical formulation developed in Part I. We assume co-located (rate) sensors and (force and moment) actuators, and allowing for both sensor and actuator noise, formulate stabilization as a stochastic regulator problem. Specializing the general theory developed by the author, a complete, "closed form" solution, (believed to be new with this report) is obtained, taking advantage of the fact that the inherent structural damping is light. In particular we are able to solve in closed form the associated infinite-dimensional steady-state Riccati equations. The SCOLE model involves associated partial differential equations in a single space variable, but the compensator design theory developed is far more general since it is given in the abstract wave equation formulation. Our results thus hold for any multibody system so long as the basic model is linear.

A MATHEMATICAL FORMULATION

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OPTIMAL COMPENSATOR DESIGN

1. Introduction

In this report we conclude the study initiated in Part I [1] and go on to consider optimal feedback control (compensator) design for stability augmentation, following the mathematical formulation developed in Part I. We assume co-located (rate) sensors and (force and moment) actuators, and allowing for both sensor and actuator noise, formulate stabilization as a stochastic regulator problem. Specializing the general theory developed in [2] we are able to obtain a complete, "closed form" solution, (believed to be new with this report) taking advantage of the fact that the inherent structural damping is light. In particular we are able to solve in closed form the associated infinite-dimensional steady-state Riccati equations. The SCOLE model involves associated partial differential equations in a single space variable, but the compensator design theory developed is far more general since it is given in the abstract wave equation formulation. Our results thus hold for any multibody system so long as the basic model is linear.

The organization is as follows. To make the report self-contained, Section 2 reviews the flexible-structure dynamic equations for a variation of the SCOLE model illustrating in particular the generating of the abstract Hilbert space wave-equation formulation. The formulation of the stabilization as a stochastic regulator problem in the abstract wave-equation setting is given in Section 3. The optimal compensator transfer function is derived in Section 4 where the steady-state Riccati equtions are also solved explicitly. The compensator is shown to have a simple, easily realized structure as a series of band-pass filters centered at the closed-loop modal frequencies.

2. Review: Structure Equations

We shall derive our results using the abstract formulation developed in [1]. To illustrate the generality of this formulation and incidentally also make this report self contained as much as possible, we shall re-derive the abstract wave equation formulation for a close variation of SCOLE where the controls are concentrated at the antenna end, and for simplicity, exclude proof mass controllers. In addition, since we are concerned only with mast stabilization rather than the antenna slewing problem we shall neglect the kinematic nonlinearity term as well.

We have then the basic equations for the mast displacements $u_0(\cdot)$, $u_\theta(\cdot)$ and $u_\psi(\cdot)$, (see [1]):

$$
\left.
\begin{aligned}
\rho A \ddot{u}_0(t, s) + EI_0 u_0''''(t, s) &= 0 \\
\rho A \ddot{u}_\theta(t, s) + EI_\theta u_\theta''''(t, s) &= 0 \\
\rho I_\psi \ddot{u}_\psi(t, s) - GI_\psi u_\psi''(t, s) &= 0
\end{aligned}
\right\} \quad 0 < t; \ 0 < s < L \qquad (2.1)
$$

where the dots represent derivatives with respect to time t and the primes with respect to the space variable s, as in [1]. With the ("clamped") boundary conditions at the shuttle end ($s = 0$):

$$
\left.
\begin{aligned}
u_0(t, 0) = u_\theta(t, 0) = u_\psi(t, 0) &= 0 \\
u_0'(t, 0) = u_\theta'(t, 0) &= 0
\end{aligned}
\right\}. \qquad (2.2)
$$

For the controls at the antenna end ($s = L$) we have:

$$
\left.
\begin{aligned}
EI_0 u_0'''(t, L) &= m_a \ddot{u}_0(t, L) + F_y(t) \\
EI_\theta u_\theta'''(t, L) &= m_a \ddot{u}_\theta(t, L) + F_x(t)
\end{aligned}
\right\} \qquad (2.3)
$$

where $F_y(\cdot)$ and $F_x(\cdot)$ are applied forces. With $M_4(t)$ denoting the applied control moment (3×1 vector):

$$
\left|
\begin{aligned}
EI_0 \, u_0''(t, L) \\
EI_\theta \, u_\theta''(t, L) \\
GI_\psi \, u_\psi'(t, L)
\end{aligned}
\right| + I_4 \dot{\omega}_4 + M_4(t) = 0 \qquad (2.4)
$$

where ω_4 is the angular velocity defined by:

$$\omega_4 \;=\; \begin{vmatrix} \dot{u}_0'(t,\,L) \\ \dot{u}_\theta'(t,\,L) \\ \dot{u}_\psi(t,\,L) \end{vmatrix} \tag{2.5}$$

and I_4 denote the moment of inertia at the antenna end.

It is convenient at this point to make the transition to a more precise the problem formulation by employing the vocabulary of "abstract" (or "function space") analysis. Thus let $\mathcal{I}$ denote the class of 3×1 functions

$$\begin{vmatrix} u_0(s) \\ u_\theta(s) \\ u_\psi(s) \end{vmatrix} \;, \qquad 0 < s < L$$

such that $u_0(\cdot)$, $u_0'(\cdot)$, $u_0''(\cdot)$, $u_0'''(\cdot)$; $u_\theta(\cdot)$, $u_\theta'(\cdot)$, $u_\theta''(\cdot)$, $u_\theta'''(\cdot)$; $u_\psi(\cdot)$, $u_\psi'(\cdot)$, $u_\psi''(\cdot)$, all belong to $L_2(0,\,L)$, and satisfy the boundary conditions:

$$u_0(0) \;=\; u_0'(0) \;=\; 0$$

$$u_\theta(0) \;=\; u_\theta'(0) \;=\; 0$$

$$u_\psi(0) \;=\; 0 \;.$$

Introduce the inner product in $\mathcal{I}$ defined by: for any two elements f, g in $\mathcal{I}$, where

$$f(s) \;=\; \begin{vmatrix} f_0(s) \\ f_\theta(s) \\ f_\psi(s) \end{vmatrix} \;, \qquad g(s) \;=\; \begin{vmatrix} g_0(s) \\ g_\theta(s) \\ g_\psi(s) \end{vmatrix}$$

$$[f,\,g] \;=\; \int_0^L (f_0(s)g_0(s) + f_\theta(s)g_\theta(s) + f_\psi(s)g_\psi(s))\,ds \;+\; f_0(L)g_0(L)$$

$$+\; f_\theta(L)g_\theta(L) \;+\; f_\psi(L)g_\psi(L) \;+\; f_0'(L)g_0'(L) \;+\; f_0''(L)g_0''(L) \;. \tag{2.6}$$

The space completed using this inner product will be designated $\mathcal{H}$. It is not difficult to see that

$$\mathcal{H} \;=\; L_2[0,\,L]^3 \;\times\; R^5 \;.$$

It is convenient to use the notation for any element x in $\mathcal{H}$:

$$x = \begin{vmatrix} u_0(\cdot) \\ u_\theta(\cdot) \\ u_\psi(\cdot) \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{vmatrix} \qquad (2.7)$$

such that for x in $\mathcal{I}$

$$b_1 = u_0(L)$$

$$b_2 = u_\theta(L)$$

$$b_3 = u_0'(L) \qquad (2.8)$$

$$b_4 = u_\theta'(L)$$

$$b_5 = u_\psi(L) .$$

Note that the boundary (at L where the controls are) is part of the state. Define now the linear operator A mapping $\mathcal{I}$ into $\mathcal{H}$ by:

$$x = \begin{vmatrix} u_0(\cdot) \\ u_\theta(\cdot) \\ u_\psi(\cdot) \\ u_0(L) \\ u_\theta(L) \\ u_0'(L) \\ u_\theta'(L) \\ u_\psi(L) \end{vmatrix} \quad ; \qquad Ax = \begin{vmatrix} EI_0\, u_0''''(\cdot) \\ EI_\theta\, u_\theta''''(\cdot) \\ -GI_\psi\, u_\psi''(\cdot) \\ -EI_0 u_0'''(L) \\ -EI_\theta u_\theta'''(L) \\ EI_0\, u_0''(L) \\ EI_\theta\, u_\theta''(L) \\ GI_\psi\, u_\psi'(L) \end{vmatrix} . \qquad (2.9)$$

Thus defined, A is self-adjoint and nonnegative definite with domain dense in $\mathcal{H}$:

$$[Ax, y] = [x, Ay] \qquad \text{for } x, y \text{ in } \mathcal{I} .$$

$$[Ax, x] \geq 0 \qquad \text{for } x \text{ in } \mathcal{I} .$$

Then as in [1] we can combine (2.1), (2.2), (2.3), (2.4) to yield the abstract "wave equation" in $\mathcal{H}$:

$$M\ddot{x}(t) + Ax(t) + Bu(t) = 0 \qquad (2.10)$$

where $u(t)$ represents the control:

$$u(t) = \begin{vmatrix} F_y(t) \\ F_x(t) \\ M_4(t) \end{vmatrix} \tag{2.11}$$

and B is the linear (finite-dimensional) operator mapping R^5 into $\mathcal{H}$ by:

$$Bu = x ; \qquad x = \begin{vmatrix} 0 \\ 0 \\ 0 \\ u \end{vmatrix} , \tag{2.12}$$

and M is the "inertia" operator:

$$Mx = y ; \qquad y = \begin{vmatrix} \rho A u_o \\ \rho A u_\theta \\ \rho I_\psi u_\psi \\ m_a b_1 \\ m_a b_2 \\ I_4 \begin{vmatrix} b_3 \\ b_4 \\ b_5 \end{vmatrix} \end{vmatrix} . \tag{2.13}$$

Note that M is positive definite and has a bounded inverse. It is convenient to use the notation:

$$y = \begin{vmatrix} \rho A u_o \\ \rho A u_\theta \\ \rho I_\psi u_\psi \\ M_b b \end{vmatrix} \tag{2.14}$$

where M_b is thus a 5×5 matrix, nonsingular and positive definite. In particular, we have for the adjoint B^* mapping $\mathcal{H}$ into R^5:

$$B^*x = b$$

$$B^*Mx = M_b b .$$

We shall call b the "boundary" vector or boundary "trace."

Next we need to take into account the random noise input associated with the control. Let $N_s(\cdot)$ denote white Gaussian noise with (diagonal) spectral density matrix

$$d_s I$$

where I denotes the 5×5 Identity matrix. Then we modify (2.10) to read

$$M\ddot{x}(t) + Ax(t) + Bu(t) + BN_s(t) = 0 . \tag{2.15}$$

We are assuming co-located rate sensors. Let $v(t)$ denote the sensor ouptut. Then we can model it as:

$$v(t) = B^*\dot{x}(t) + N_0(t) \tag{2.16}$$

where $N_0(\cdot)$ is white Gaussian with spectral density

$$d_0 I .$$

The equations (2.15), (2.16) together yield the state space or abstract formulation that we shall employ from now on.

In concluding this section, let us note the essential properties of A: (i) the resolvent of A is compact so that A has a pure point spectrum (ii) zero is in the resolvent set of A — we consider only the "flexible body" modes.

3. Formulation of Stochastic Regulator Problem

The stabilization (or stability augmentation) problem is to make (if possible) the "boundary" displacements and displacement rates zero using controls which must be based on avilable sensor data. Here the specific version of the problem we shall consider is to minimize the steady state "boundary" rates: minimize in other words the time average

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \|\dot{b}(t)\|^2 \, dt \tag{3.1}$$

where

$$b(t) = B^*x(t) ,$$

$$\dot{b}(t) = B^*\dot{x}(t) .$$

We shall also impose a soft constraint on the controls and thus consider the problem of minimizing

$$\lim_{T \to \infty} \left[\frac{1}{T} \int_0^T \|\dot{b}(t)\|^2 \, dt \; + \; \frac{\lambda}{T} \int_0^T \|u(t)\|^2 \, dt \right] \qquad (3.2)$$

for fixed $\lambda > 0$. This can be seen to be a special case of the stochastic regulator problem in a Hilbert space setting treated in [2]. For this purpose we first rewrite (2.15) as a first order (in time) equation. Define the state

$$Y(t) \; = \; \left| \begin{array}{c} x(t) \\ \dot{x}(t) \end{array} \right| . \qquad (3.3)$$

$Y(\cdot)$ is thus an element of the cross product space $\mathcal{H} \times \mathcal{H}$. On this space we define, following (1), a new norm — the "energy" norm — defined on the cross product space:

$$\mathcal{D}(\sqrt{A}) \times \mathcal{H}, \qquad \mathcal{D}(\sqrt{A}) \text{ denoting domain of } \sqrt{A}, \qquad (3.4)$$

$$\|Y\|_E^2 \; = \; (\sqrt{A} \, y_1, \sqrt{A} \, y_1) \; + \; [My_2, y_2] \qquad (3.5)$$

where

$$Y \; = \; \left| \begin{array}{c} y_1 \\ y_2 \end{array} \right|$$

and

$$y_1 \; \in \; \text{domain of } \sqrt{A} \; .$$

Here $\sqrt{A}$ is the unique positive (self-adjoint and nonnegative definite) square root of A, and as is well known the domain of $\sqrt{A}$ includes that of A. The cross product space is actually complete under this norm, because zero belongs to the reolvent set of A and hence also to the resolvent set of $\sqrt{A}$. We shall denote the space by $\mathcal{H}_E$. The requirement that the first component should belong to the domain of $\sqrt{A}$ will cause no problem for us in what follows. Note that calculation of $\sqrt{A}$ is possible using the Balakrishnan formula [6] but will involve the boundary "trace" and can be tedious.

We can now proceed to the reformulation of (2.15) as a Cauchy problem in "state space" form in $\mathcal{H}_E$ (see [1]). Thus let

$$\mathcal{A} \; = \; \left[\begin{array}{cc} 0 & I \\ -M^{-1}A & 0 \end{array} \right] \qquad (3.6)$$

$$\mathcal{B}u \;=\; \begin{vmatrix} 0 \\ -M^{-1}Bu \end{vmatrix} \tag{3.7}$$

Then with

$$Y(t) \;=\; \begin{vmatrix} x(t) \\ \dot{x}(t) \end{vmatrix} . \tag{3.8}$$

(2.15) can be written:

$$\dot{Y}(t) \;=\; \mathcal{A}Y(t) \;+\; \mathcal{B}u(t) \;+\; \mathcal{B}N_s(t) \tag{3.9}$$

where $N_s(t)$ is white Gaussian noise with spectral density

$$d_s I .$$

It is easily verified that

$$\mathcal{B}^*Y \;=\; -B^*y_2 \tag{3.10}$$

where

$$Y \;=\; \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} .$$

Also:

$$\mathcal{A}^* \;=\; \begin{bmatrix} 0 & -I \\ M^{-1}A & 0 \end{bmatrix} . \tag{3.11}$$

Let

$$C \;=\; -\mathcal{B}^* . \tag{3.12}$$

Then we can write (2.16) in the form

$$v(t) \;=\; CY(t) \;+\; N_0(t) . \tag{3.13}$$

Finally, the criterion (3.2) now becomes

$$\lim_{T \to \infty} \left[\frac{1}{T} \int_0^T \|\mathcal{B}^*Y(t)\|^2 \, dt \;+\; \frac{\lambda}{T} \int_0^T \|u(t)\|^2 \, dt \right] . \tag{3.14}$$

In this formulation, we may now invoke Theorem 6.9.1 in [2], where we need to verify the stabilizability requirement imposed therein. For this purpose we prove first:

Theorem 3.1. $(\mathcal{A} \sim \mathcal{B})$ is Controllable.

Proof. We need the following Lemma.

Lemma. Let $\{\phi_k\}$ denote the eigenvectors of A corresponding to the eigenvalues $\{\lambda_k\}$;

$$A\phi_k = \lambda_k M\phi_k ,\tag{3.15}$$

$$[M\phi_k, \phi_k] = 1 .$$

Then the corresponding boundary trace

$$B^*\phi_k \neq 0 \qquad \text{for any } k .\tag{3.16}$$

Proof. Suppose

$$B^*\phi_k = 0$$

for some k. Then

$$B^*(M\phi_k) = 0 .$$

Since

$$A\phi_k = \lambda_k M\phi_k$$

we have:

$$B^*(A\phi_k) = 0 .$$

Let

$$\phi_k = \begin{vmatrix} u_0 \\ u_\theta \\ u_\psi \\ B^*\phi_k \end{vmatrix} .$$

Then

$$u_0(L) = u_0'(L) = u_0''(L) = u_0'''(L) = 0$$

$$u_\theta(L) = u_\theta'(L) = u_\theta''(L) = u_\theta'''(L) = 0$$

$$u_\psi(L) = u_\psi'(L) = 0$$

and of course

$$u_\phi(0) = u_\phi'(0) = 0$$

$$u_\theta(0) = u_\theta'(0) = 0 .$$

In addition:

$$EI_\phi u_\phi''''(s) = \lambda_k u_\phi(s) , \qquad 0 < s < L$$

$$EI_\theta u_\theta''''(s) = \lambda_k u_\theta(s) , \qquad 0 < s < L$$

$$GI_\psi u_\psi''(s) = -\lambda_k u_\psi(s) , \qquad 0 < s < L .$$

Since the conditions include those of a beam clamped at both ends, it follows that $u_\phi(\cdot)$, $u_\theta(\cdot)$ must be identically zero. Similarly the boundary conditions

$$u_\psi'(L) = 0$$

along with

$$u_\psi(0) = u_\psi(L) = 0$$

are sufficient to make $u_\psi(\cdot)$ identically zero. Hence $B^*\phi_k$ cannot be zero, for any k.

Getting back now to the Theorem, let $\mathcal{L}(t)$, $t \geq 0$, denote the semigroup generated by $\mathcal{A}$. Then if $(\mathcal{A} \sim \mathcal{B})$ is not controllable, there must be a nonzero element, say Z, in $\mathcal{H}_E$ such that

$$[Z, \mathcal{L}(t)\mathcal{B}u]_E = 0 , \qquad t \geq 0 \qquad (3.17)$$

for every u in R^s. Because A has a compact resolvent, so does $\mathcal{A}$ and in fact (see also [4]):

$$\psi_k = \left| \begin{array}{c} \phi_k \\ i\omega_k\phi_k \end{array} \right| \left(\frac{1}{\sqrt{2\omega_k^2}} \right) \qquad (3.18)$$

$$\overline{\psi}_k = \left| \begin{array}{c} \phi_k \\ -i\omega_k\phi_k \end{array} \right| \left(\frac{1}{\sqrt{2\omega_k^2}} \right) \qquad (3.19)$$

where

$$\omega_k = \sqrt{\lambda_k}$$

are orthonormalized eigenvectors of $\mathcal{A}$. Let P_k denote the projection operator corresponding to the space spanned by ψ_k, $\overline{\psi}_k$. Then

$$\mathcal{L}(t)\psi_k = e^{i\omega_k t}\psi_k$$

$$\mathcal{L}(t)\overline{\psi}_k = e^{-i\omega_k t}\psi_k$$

$$\mathcal{L}(t)P_k = P_k\mathcal{L}(t)P_k .$$

Also, the $\{\psi_k, \overline{\psi}_k\}$ provide an orthonormal basis in $\mathcal{H}_E$ and

$$\mathcal{L}(t)\mathcal{B}u = \sum_k \mathcal{L}(t)P_k\mathcal{B}u . \tag{3.20}$$

We note that $\{\psi_k, \overline{\psi}_k\}$ is an orthonormal basis for $P_k\mathcal{H}_E$. Since

$$\mathcal{B}u = \left|\begin{array}{c} 0 \\ -M^{-1}Bu \end{array}\right|$$

it follows that

$$P_k\mathcal{B}u = \frac{i}{\sqrt{2}}[u, b_k](\psi_k - \overline{\psi}_k) \tag{3.21}$$

where

$$b_k = B^*\phi_k \tag{3.22}$$

Hence

$$\mathcal{L}(t)P_k\mathcal{B}u = \frac{i}{\sqrt{2}}[u, b_k][e^{i\omega_k t}\psi_k - e^{-i\omega_k t}\overline{\psi}_k] = [u, b_k]\left|\begin{array}{c} -\sqrt{2}(\sin \omega_k t)\phi_k \\ -\sqrt{2}\omega_k(\cos \omega_k t)\phi_k \end{array}\right| \tag{3.23}$$

Hence

$$[Z, \mathcal{L}(t)P_k\mathcal{B}u]_E = -\sqrt{2}[u, b_k]([M\phi_k, z_1]\omega_k^2 \sin \omega_k t + [M\phi_k, z_2]\omega_k \cos \omega_k t) \tag{3.24}$$

where

$$Z = \left|\begin{array}{c} z_1 \\ z_2 \end{array}\right| .$$

It follows from (3.17) and (3.24) that

$$0 = [u, b_k][z_1, M\phi_k] = [u, b_k][z_2, M\phi_k] , \qquad \text{for every } k .$$

Since $b_k \neq 0$ for any k, and u is arbitrary, it follows that

$$0 = [z_1, M\phi_k] = [z_2, M\phi_k]$$

or that

$$Z = 0$$

which is a contradiction.

Remark. We may note that while

$$B^*\phi_k \neq 0 \qquad \text{for any } k,$$

we do have that

$$\|b_k\| = \|B^*\phi_k\| \to 0 \qquad \text{as } k \to \infty. \tag{3.25}$$

In fact, from

$$\|\mathcal{B}u\|_E^2 = \sum_1^\infty \left(|[\mathcal{B}u, \psi_k]_E|^2 + |[\mathcal{B}u, \overline{\psi}_k]_E|^2 \right)$$

where the left side

$$= [M_b^{-1}u, u]$$

and the right side

$$= \sum_1^\infty \|P_k \mathcal{B}u\|_E^2 = \sum_1^\infty |[u, b_k]|^2 .$$

It follows in particular that

$$\sum_1^\infty \|b_k\|^2 < \infty. \tag{3.26}$$

The result (3.25) has been explored numerically by Taylor and Naidu in [3].

Theorem 3.2. Let $\mathcal{L}_b(t)$, $t \geq 0$, denote the semigroup generated by

$$(\mathcal{A} - \mathcal{B}\mathcal{B}^*) .$$

Then

$$\|\mathcal{L}_b(t)Y\|_E \to 0 \qquad \text{as } t \to \infty \tag{3.27}$$

for every Y. In other words the semigroup $\mathcal{L}_b(t)$ is strongly stable.

Proof. Follows from the general result of this kind due to Benchimol (see [2]) as a consequence of the fact that $\mathcal{A}$ has a compact reolvent, $(\mathcal{A} \sim \mathcal{B})$ is controllable and $\mathcal{A}$ is dissipative.

Theorem 3.2 is adequate to satisfy the stabilizability conditions requirement of Theorem 6.9.1 in [2]. Applying that Theorem, we note that the optimal control that minimizes (3.14) is given by

$$u_0(t) = -\frac{\mathcal{B}^*P_c\hat{Y}(t)}{\lambda} \tag{3.28}$$

where

$$\dot{\hat{Y}}(t) = \left[\mathcal{A} - P_f\mathcal{B}\mathcal{B}^* - \frac{\mathcal{B}\mathcal{B}^*P_c}{\lambda}\right]\hat{Y}(t) - P_fd_0^{-1}\mathcal{B}v(t) \tag{3.29}$$

and

$$\hat{Y}(0) \text{ can be chosen arbitrarily.}$$

Here P_c is the unique self-adjoint solution of algebraic Riccati equation

$$0 = [P_cY, \mathcal{A}Y]_E + [\mathcal{A}Y, P_cY]_E + [\mathcal{B}^*Y, \mathcal{B}^*Y] - [\mathcal{B}^*P_cY, \mathcal{B}^*P_cY], \tag{3.30}$$

$$Y \in \mathcal{D}(\mathcal{A})$$

and similarly, P_f is the unique self-adjoint solution of

$$0 = [P_fY, \mathcal{A}^*Y]_E + [\mathcal{A}^*Y, P_fY]_E + [d_s\mathcal{B}^*Y, \mathcal{B}^*Y] - [\mathcal{B}^*P_fY, d_0^{-1}\mathcal{B}^*P_fY] \tag{3.31}$$

$$Y \in \mathcal{D}(\mathcal{A}^*).$$

The corresponding (minimal) value of (3.14) is

$$\text{Tr. } \mathcal{B}P_f\mathcal{B} + \frac{\text{Tr. } \mathcal{B}^*P_fP_cP_f\mathcal{B}}{d_0}. \tag{3.32}$$

We are fortunate that we can solve (3.30) and (3.31) exactly. In fact it is readily verified that

$$P_c = \sqrt{\lambda}\, I$$

and that

$$P_f = \sqrt{d_s d_0}\, I \ .$$

Hence

$$u_0(t) = -\frac{\mathcal{B}^* \hat{Y}(t)}{\sqrt{\lambda}} \tag{3.33}$$

and

$$\dot{\hat{Y}}(t) = (\mathcal{A} - \gamma \mathcal{B}\mathcal{B}^*)\hat{Y}(t) - \sqrt{\frac{d_s}{d_0}}\, \mathcal{B}v(t) \tag{3.34}$$

where

$$\gamma = \sqrt{d_s d_0} + \frac{1}{\sqrt{\lambda}}\ . \tag{3.35}$$

Correspondingly, (3.32) becomes:

$$\sqrt{d_s d_0} + \sqrt{\lambda}\, d_s\ . \tag{3.36}$$

We note that the semigroup $\mathcal{L}_\gamma(t)$, $t \geq 0$, generated by $(\mathcal{A} - \gamma\mathcal{B}\mathcal{B}^*)$ is strongly stable. We have thus obtained a complete closed-form solution of the stochastic regulator problem considered herein.

4. Optimal Compensator Transfer Function

In this section we shall study the problem of determining the compensator transfer function corresponding to the optimal control law (3.33). Strictly speaking (3.34) holds only in the weak sense (see [2]), since $v(t)$ contains white noise and is not differentiable. Hence we must replace (3.34) by

$$\hat{Y}(t) = \int_0^t \mathcal{L}_\gamma(t-\sigma)\left(-\sqrt{\frac{d_s}{d_0}}\right)\mathcal{B}v(\sigma)\, d\sigma + \mathcal{L}_\gamma(t)\hat{Y}(0)\ . \tag{4.1}$$

We may set $\hat{Y}(0) = 0$. Then we have:

$$u_0(t) = \frac{1}{\sqrt{\lambda}}\sqrt{\frac{d_s}{d_0}}\int_0^t \mathcal{B}^*\mathcal{L}_\gamma(t-\sigma)\mathcal{B}v(\sigma)\, d\sigma\ , \tag{4.2}$$

defining the compensator input-output relation. Let

$$W(t) = \mathcal{B}^* \mathcal{L}_\gamma(t)\mathcal{B}, \qquad t \geq 0 \tag{4.3}$$

denote the 5×5 matrix weighting function. Then we need to calculate the Laplace transform:

$$\int_0^\infty e^{-st} \mathcal{B}^* \mathcal{L}_\gamma(t)\mathcal{B} \, dt, \qquad \text{Re. } s > 0. \tag{4.4}$$

The integral is absolutely convergent for Re. $s > 0$, since

$$\|\mathcal{B}^* \mathcal{L}_\gamma(t)\mathcal{B}\| \rightarrow 0 \qquad \text{as } t \rightarrow \infty \tag{4.5}$$

because the semigroup $\mathcal{L}_\gamma(\cdot)$ is strongly stable. The Laplace transform

$$\int_0^\infty e^{-\mu t} \mathcal{L}_\gamma(t) Y \, dt = (\mu I - (\mathcal{A} - \gamma \mathcal{B}\mathcal{B}^*))^{-1}, \qquad \mu > 0 \tag{4.6}$$

is usually denoted

$$\mathcal{R}(\mu, \mathcal{A} - \gamma \mathcal{B}\mathcal{B}^*)$$

and called the "resolvent" of the semigroup (see [2]). $(\mathcal{A} - \gamma \mathcal{B}\mathcal{B}^*)$ has a pure point spectrum — or eigenvalue spectrum. To calculate the eigenvalues we need to solve:

$$(\mathcal{A} - \gamma \mathcal{B}\mathcal{B}^*)\Psi = \lambda \Psi; \qquad \Psi = \begin{vmatrix} \Phi_1 \\ \Phi_2 \end{vmatrix}$$

yielding:

$$\left. \begin{array}{c} \lambda^2 M\Phi_1 + A\Phi_1 + \lambda\gamma BB^*\Phi_1 = 0 \\ \\ \Phi_2 = \lambda\Phi_1 \end{array} \right\} \tag{4.7}$$

Let $\{\lambda_k\}$ denote the eigenvalues, $k = 1, ...,$ in order of increasing $|\lambda_k|$. Let, correspondingly,

$$\lambda_k^2 M\Phi_k + A\Phi_k + \lambda\gamma BB^*\Phi_k = 0 \tag{4.8}$$

and let us normalize so that

$$[M\Phi_k, \Phi_k] = 1. \tag{4.9}$$

These Φ_k should of course be distinguished from those in (3.15)). Let us note that the $\{\Phi_k\}$

are not orthogonal. We shall call $\{\lambda_k\}$ the closed-loop mode frequencies and $\{\Phi_k\}$ the closed-loop mode shapes. The closed-loop mode frequencies and mode shapes are complex valued. Let

$$\sigma_k = \text{Re. } (\lambda_k) ; \qquad \tilde{\omega}_k = \text{Im. } (\lambda_k) .$$

Then, taking inner products with respect to Φ_k in (4.8) we have:

$$\lambda_k^2 + [A\Phi_k, \Phi_k] + \lambda_k \gamma \|B^*\Phi_k\|^2 = 0 . \tag{4.10}$$

Let

$$\tilde{b}_k = \text{Tr. } \Phi_k = B^*\Phi_k .$$

We of course assume that γ is "small" in the sense that

$$\gamma \|\tilde{b}_k\|^2 \ll [A\Phi_k, \Phi_k] . \tag{4.11}$$

Then

$$\sigma_k = -\frac{\gamma \|\tilde{b}_k\|^2}{2}$$

$$\tilde{\omega}_k = \sqrt{[A\Phi_k, \Phi_k] - \frac{\gamma^2 \|\tilde{b}_k\|^4}{4}} \approx \sqrt{[A\Phi_k, \Phi_k]} .$$

Consistent with the small γ assumptions, we may approximate

$$[A\Phi_k, \Phi_k] \qquad \text{by} \qquad [A\phi_k, \phi_k] = \omega_k^2$$

$$\tilde{b}_k = B^*\Phi_k \qquad \text{by} \qquad b_k = B^*\phi_k$$

using the open-loop (undamped) modes. In this way we get a first approximation to the eigenvalues λ_k as

$$\lambda_k \approx (\text{appr.}) - \frac{\gamma \|b_k\|^2}{2} + i\omega_k . \tag{4.12}$$

We may also get a corresponding approximation to Φ_k as a perturbation of ϕ_k. We omit the details. The main point is that Φ_k will be complex with an approximation of the form

$$\Phi_k = \phi_k + i\sigma_k\Delta_k . \tag{4.13}$$

We can now state:

Lemma 4.1. (i) $\tilde{b}_k \neq 0$ for any k

(ii) $\|\tilde{b}_k\| \to 0$ as $k \to \infty$.

Proof. Let

$$\Psi_k = \frac{1}{\alpha_k} \begin{vmatrix} \Phi_k \\ \lambda_k \Phi_k \end{vmatrix}$$

(these Ψ_k should be distinguished from ψ_k in (3.18)) where

$$\alpha_k = \sqrt{[A\Phi_k,\ \Phi_k]\ +\ |\lambda_k|^2}\ .$$

Then Ψ_k is the eigenvector corresponding to λ_k. Since

$$\sigma_k = -\frac{\gamma \|\tilde{b}_k\|^2}{2}$$

we shall use the strong stability property of the semigroup $\mathcal{S}_\gamma(\cdot)$ to prove that

$$\sigma_k \neq 0 \qquad \text{for any } k\ .$$

This is immediate from:

$$\|\mathcal{S}_\gamma(t)\Psi_k\| = e^{\sigma_k t} \to 0 \qquad \text{as } t \to \infty\ .$$

Hence

$$\sigma_k < 0\ ,$$

proving (i). To prove (ii) we may show that the $\{\Phi_k\}$ are linearly independent in $\mathcal{H}$ by examining the function space part of (4.8). Here rather we shall use what we already have. Thus since the $\{\Psi_k\}$ corresponding to distinct eigenvalues are linearly independent, we know that

$$\|\mathcal{B}^*\Psi_k\| \to 0 \qquad \text{as } k \to \infty\ .$$

Now

$$\|\mathcal{B}^* \Psi_k\| = \frac{|\lambda_k| \, \|\tilde{b}_k\|}{\alpha_k} = \frac{\|\tilde{b}_k\|}{\sqrt{1 + \frac{|A\Phi_k + \Phi_k|}{|\lambda_k|^2}}}$$

and

$$[A\Phi_k, \Phi_k] = \tilde{\omega}_k^2 + \sigma_k^2 = |\lambda_k|^2 \, .$$

Hence

$$\|\tilde{b}_k\| \to 0 \, .$$

Note in particular that the closed-loop modes Φ_k approximate the clamped-clamped modes as $k \to \infty$, just as the open-loop mode shapes ϕ_k do.

Next let us note that $\bar{\lambda}_k$ is an eigenvalue if λ_k is, and has eigenvector $\overline{\Psi}_k$. Since

$$\mathcal{R}(\mu; \, \mathcal{A} - \gamma \mathcal{B}\mathcal{B}^*) \Psi_k = \frac{\Psi_k}{\mu - \lambda_k} \, , \tag{4.14}$$

for any Y in $\mathcal{H}_E$, with the representation:

$$Y = \sum_1^\infty \left(a_k \Psi_k + b_k \overline{\Psi}_k \right) \tag{4.15}$$

we have:

$$\mathcal{R}(\mu; \, \mathcal{A} - \gamma \mathcal{B}\mathcal{B}^*) Y = \sum_1^\infty \left[\frac{a_k}{\mu - \lambda_k} \Psi_k + \frac{b_k}{\mu - \bar{\lambda}_k} \overline{\Psi}_k \right] \, . \tag{4.16}$$

We want to specialize to

$$Y = \mathcal{B}u \, .$$

Also we assume that $\mathcal{B}u$ is in the (closed) subspace spanned by the $\{\Psi_k\}$, so that (4.15) holds. In physical terms, this is equivalent to saying that we need not consider responses in which no modes are excited. We can actually prove this by assuming that there is some (negligibly small) strictly proportional damping so that the semigroup $\mathcal{S}(\cdot)$ is analytic, implying then the same for $\mathcal{S}_\gamma(\cdot)$ — see [4].

Since complex-valued functions and vectors are involved, we now fix our notation for the inner product in C^5: with asterisk denoting conjugate transpose:

$$[u, v] = u^*v .$$

Since u is real:

$$\mathcal{B}u = \overline{\mathcal{B}u}$$

bar denoting conjugate complex. Hence we can write

$$-\mathcal{B}u = \frac{1}{2}\left(\sum_1^\infty [u, c_k]\Psi_k + \sum_1^\infty [u, \overline{c}_k]\overline{\Psi}_k\right) . \tag{4.17}$$

Taking inner products with respect to Ψ_j and $\overline{\Psi}_j$ respectively we have:

$$+\frac{1}{\alpha_j}[u, \lambda_j\tilde{b}_j] = \frac{1}{2}\left(\sum_1^\infty [u, c_k][\Psi_k, \Psi_j]_E + \sum_1^\infty [u, \overline{c}_k][\overline{\Psi}_k, \Psi_j]_E\right)$$

$$+\frac{1}{\alpha_j}[u, \overline{\lambda}_j\overline{\tilde{b}}_j] = \frac{1}{2}\left(\sum_1^\infty [u, c_k][\Psi_k, \overline{\Psi}_j]_E + \sum_1^\infty [u, \overline{c}_k][\overline{\Psi}_k, \overline{\Psi}_j]_E\right) .$$

These equations can be submerged into the form

$$\sum_1^\infty \mathcal{Z}_k M_{kj} = \frac{1}{\alpha_j}|\lambda_j\tilde{b}_j \quad \overline{\lambda}_j\overline{\tilde{b}}_j| \tag{4.18}$$

where $\mathcal{Z}_j$ are 2×5 matrices:

$$\mathcal{Z}_j = |c_j \quad \overline{c}_j|$$

and M_{kj} is 2×2 defined by

$$M_{kj} = \frac{1}{2}\begin{bmatrix} [\Psi_k, \Psi_j]_E & [\Psi_k, \overline{\Psi}_j]_E \\ [\overline{\Psi}_k, \Psi_j]_E & [\overline{\Psi}_k, \overline{\Psi}_j]_E \end{bmatrix} .$$

We can solve (4.18) in the form:

$$|c_j \quad \overline{c}_j| = \sum_1^\infty |\lambda_k\tilde{b}_k \quad \overline{\lambda}_k\overline{\tilde{b}}_k|\left(\frac{1}{\alpha_k}\right)r_{jk} \tag{4.20}$$

for appropriate 2×2 matrices, r_{jk}, and hence finally we get

$$\mathcal{R}(\mu, \mathcal{A} - \gamma\mathcal{B}\mathcal{B}^*)\mathcal{B}u = -\frac{1}{2}\left(\sum_1^\infty \frac{[u, c_k]\Psi_k}{\mu - \lambda_k} + \frac{[u, \overline{c}_k]\overline{\Psi}_k}{\mu - \overline{\lambda}_k}\right)$$

and hence:

$$\mathcal{B}^*\mathcal{R}(\mu, \ \mathcal{A} - \gamma\mathcal{B}\mathcal{B}^*)\mathcal{B} \ = \ \text{Re.} \sum_1^\infty \frac{\lambda_k \tilde{b}_k c_k^*}{\alpha_k(\mu - \lambda_k)}$$

$$= \ \frac{1}{2} \sum_1^\infty \left(\frac{\lambda_k \tilde{b}_k c_k^*}{\alpha_k(\mu - \lambda_k)} \ + \ \frac{\overline{\lambda}_k \overline{\tilde{b}}_k \overline{c}_k^*}{\alpha_k(\mu - \overline{\lambda}_k)} \right) \qquad (4.21)$$

where we note that from

$$\|\mathcal{B}u\|_E^2 \ = \ \left\| \frac{1}{2} \sum_1^\infty [u, c_k] \Psi_k \ + \ \sum_1^\infty [u, \overline{c}_k] \overline{\Psi}_k \right\|_E^2 \ = \ \frac{1}{2} \sum_1^\infty ([u, \lambda_k \tilde{b}_k c_k^* u] + [u, \overline{\lambda}_k \overline{\tilde{b}}_k \overline{c}_k^* u]) \left(\frac{1}{\alpha_k} \right)$$

we have that:

$$\frac{1}{2} \sum_1^\infty \left(\frac{\lambda_k}{\alpha_k} \tilde{b}_k c_k^* \ + \ \frac{\overline{\lambda}_k}{\alpha_k} \overline{\tilde{b}}_k \overline{c}_k^* \right) \ = \ I_{5 \times 5} \ . \qquad (4.22)$$

Hence finally we obtain that the transfer function (Laplace transform) of the compensator has the form

$$= \ \frac{1}{\sqrt{\lambda}} \sqrt{\frac{d_s}{d_0}} \frac{1}{2} \left(\sum_1^\infty \frac{\lambda_k \tilde{b}_k c_k^*}{(\mu - \lambda_k)} \ + \ \frac{\overline{\lambda}_k \overline{\tilde{b}}_k \overline{c}_k^*}{(\mu - \overline{\lambda}_k)} \right) \ . \qquad (4.23)$$

From (4.23) we can readily infer the structure of the compensator: we have a series of band pass filters centered at the closed-loop mode frequencies $\tilde{\omega}_k$, the Q-factor being determined by the damping $|\sigma_k|$. The filter amplitudes decrease to zero as the mode number increases, by virtue of (4.22). In the limiting case as $|\sigma_k| \to 0$, the compensator consists of line-filters at the open-loop mode frequences ($\omega_k/2\pi$).

As a first approximation to the solution of (4.18), we may take:

$$c_j \ = \ \frac{2\lambda_j \tilde{b}_j}{\alpha_j} \qquad (4.24)$$

yielding for the compensator transfer function:

$$\frac{1}{\sqrt{\lambda}} \sqrt{\frac{d_s}{d_0}} \frac{1}{2} \sum_1^\infty \left(\frac{\tilde{b}_k \tilde{b}_k^*}{(\mu - \lambda_k)} \ + \ \frac{\overline{\tilde{b}}_k \overline{\tilde{b}}_k^*}{(\mu - \overline{\lambda}_k)} \right) \qquad (4.25)$$

and a still further approximation by taking

$$\tilde{b}_k \ = \ b_k$$

yielding

$$\frac{1}{\sqrt{\lambda}} \sqrt{\frac{d_s}{d_0}} \sum_1^\infty \frac{(\mu - \frac{1}{2}\gamma \|b_k\|^2)b_k b_k^*}{(\mu - \lambda_k)(\mu - \overline{\lambda}_k)} \cdot \qquad (4.26)$$

Let us note that in (4.26) we have an approximation for the compensator transfer function (Laplace transform) which is quite general — independent of the particular flexible system configuration. Only the traces of the open-loop mode shapes on the control sensor boundary locations are required — as well of course as the mode frequencies. In particular we note that numerical calculations of $\{b_k\}$ for the SCOLE flight article have been reported by S. Joshi [5]. We can readily see from (4.23) that the feedback amplitude is inversely proportional to $\sqrt{\lambda}$: the smaller the λ the larger the control effort, as we expect. Also it increases with the actuator noise spectral density and decreases as the sensor noise increases, again as we should expect. But the point is that the precise dependence has been determined in (4.23).

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16. Abstract

In this report we conclude the study initiated in Part I and go on to consider optimal feedback control (compensator) design for stability augmentation, following the mathematical formulation developed in Part I. We assume co-located (rate) sensors and (force and moment) actuators, and allowing for both sensor and actuator noise, formulate stabilization as a stochastic regulator problem. Specializing the general theory developed by the author, a complete, "closed form" solution, (believed to be new with this report) is obtained, taking advantage of the fact that the inherent structural damping is light. In particular we are able to solve in closed form the associated infinite-dimensional steady-state Riccati equations. The SCOLE model involves associated partial differential equations in a single space variable, but the compensator design theory developed is far more general since it is given in the abstract wave equation formulation. Our results thus hold for any multibody system so long as the basic model is linear.

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