N89-15945

RELATIVE MOTION USING ANALYTICAL DIFFERENTIAL GRAVITY

by

Robert G. Gottlieb

MCDONNELL DOUGLAS ASTRONAUTICS CO.
16055 SPACE CENTER BLVD.
HOUSTON, TEXAS 77062

Presented at the

FLIGHT MECHANICS/ESTIMATION THEORY SYMPOSIUM
GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND 20771

MAY 10-11, 1988

This work was performed for NASA
under Contract NAS 9-17885

Relative Motion Using Analytical Differential Gravity

1.0 INTRODUCTION

This paper presents a new approach to the computation of the motion of one satellite relative to another. The trajectory of the reference satellite is computed accurately subject to geopotential perturbations. This precise trajectory is used as a reference in computing the position of a nearby body, or bodies.

The problem that arises in this approach is differencing nearly equal terms in the geopotential model, especially as the separation of the reference and nearby bodies approaches zero. By developing closed form expressions for differences in higher order and degree geopotential terms, the numerical problem inherent in the differencing approach is eliminated.

2.0 ANALYSIS

The equations of motion for a satellite moving under the influence of gravity are written

$$\ddot{\underline{r}} = - \frac{\partial V}{\partial \underline{r}} \tag{1}$$

where $\underline{r}^T = (r_1 \ r_2 \ r_3)$

where V is the potential function

$$V = - \frac{\mu}{r} - \sum_{n=2}^{\infty} \sum_{m=0}^{n} \frac{\mu}{r} \left(\frac{a_e}{r}\right)^n P_{nm}(\varepsilon) \qquad (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) \tag{2}$$

and where $P_{nm}(\varepsilon)$ is the Legendre polynomial of degree n and order m, ε is r_3/r, C_{nm} and S_{nm} are the geopotential model coefficients, $\tan \lambda = r_2/r_1$, and a_e is earth equatorial radius.

Using the recursive formulation given in [2], and considering only terms through $n = 2, m = 2$, the equation for $\ddot{\underline{r}}$ may be written:

$$\ddot{\underline{r}} = - \frac{\mu}{r^2} \left(\frac{\underline{r}}{r} + \left(\frac{a_e}{r}\right)^2 \left\{ (\Gamma_2 + \varepsilon H_2) \frac{\underline{r}}{r} - \begin{pmatrix} J_2 \\ K_2 \\ H_2 \end{pmatrix} \right\} \right) \tag{3}$$

where Γ_2, J_2, K_2 and H_2 may be shown to be

$$\Gamma_2 = \frac{\underline{r}^T}{r} G \frac{\underline{r}}{r} - \frac{3}{2} C_{20}$$

where

$$G = \begin{bmatrix} 15C_{22} & (12S_{21} + 30S_{22}) & 12C_{21} \\ 0 & -15C_{22} & 0 \\ 0 & 0 & \frac{9}{2} C_{20} \end{bmatrix}$$

$$J_2 = \underline{j}^T \frac{\underline{r}}{r}$$

where

$$\underline{j}^T = (6C_{22} \;\; 6S_{22} \;\; 3C_{21})$$

$$K_2 = \underline{k}^T \frac{\underline{r}}{r}$$

where

$$\underline{k}^T = (6S_{22} \;\; 6C_{22} \;\; 3C_{21})$$

and

$$H_2 = \underline{h}^T \frac{\underline{r}}{r}$$

where

$$\underline{h}^T = (3C_{21} \;\; 3S_{21} \;\; 3C_{20})$$

Using these, we may write equation (3) as,

$$\ddot{\underline{r}} = -\frac{\mu}{r^2}\left(\frac{\underline{r}}{r} + \left(\frac{a_e}{r}\right)^2 \left\{ \left(\frac{\underline{r}^T}{r} G \frac{\underline{r}}{r} - \frac{3C_{20}}{2} + \frac{r_3}{r} \underline{h}^T \frac{\underline{r}}{r} \right) \frac{\underline{r}}{r} - \begin{bmatrix} \underline{j}^T \\ \underline{k}^T \\ \underline{h}^T \end{bmatrix} \frac{\underline{r}}{r} \right\}\right) \quad (4)$$

by noting that r_3 may be written

$$r_3 = \underline{r}^T \underline{a}$$

where

$$\underline{a} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and then defining

$$\Theta = G + \underline{a}\,\underline{h}$$

$$\Phi = I - \begin{bmatrix} \underline{j}^T \\ \underline{k}^T \\ \underline{h}^T \end{bmatrix}$$

we may write equation (4) as

$$\ddot{\underline{r}} = - \mu \, \Phi \, \frac{\underline{r}}{r^3} + \frac{3}{2} \mu \, C_{20} \, a_e^{\,2} \, \frac{\underline{r}}{r^5} - \mu \, a_e^{\,2} \, (\underline{r}^T \Theta \, \underline{r}) \, \frac{\underline{r}}{r^7} \tag{5}$$

In fact, it can be shown that in general for the geopotential

$$\ddot{\underline{r}} = M \, \underline{r}$$

where M is a matrix whose elements may depend on $\underline{r}$.

Assume that we have two satellites, the first with position vector $\underline{r}$ and the second with position vector $\underline{\rho}$. Both must satisfy equation (5). Subtracting the two differential equations and defining the difference between the two solution vectors to be

$$\underline{\delta} \equiv \underline{\rho} - \underline{r}$$

we may write

$$\ddot{\underline{\delta}} = - \mu \, \Phi \left(\frac{\underline{\rho}}{\rho^3} - \frac{\underline{r}}{r^3} \right) + 3 \mu \, \frac{C_{20}}{2} \, a_e^{\,2} \left(\frac{\underline{\rho}}{\rho^5} - \frac{\underline{r}}{r^5} \right) - \mu \, a_e^{\,2} \left[(\underline{\rho}^T \Theta \underline{\rho}) \frac{\underline{\rho}}{\rho^7} - (\underline{r}^T \Theta \underline{r}) \frac{\underline{r}}{r^7} \right] \tag{6}$$

Now collect coefficients of $\frac{1}{r^3}$, $\frac{1}{r^5}$, $\frac{1}{r^7}$ to get

$$\ddot{\underline{\delta}} = \frac{-\mu}{r^3}\,\Phi\left(\underline{\delta} + (\frac{r^3}{\rho^3} - 1)\,\underline{\rho}\right) + 3\,\mu\,\frac{C_{20}}{2}\,\frac{a_e^2}{r^5}\left(\underline{\delta} + (\frac{r^5}{\rho^5} - 1)\,\underline{\rho}\right)$$

$$- \frac{\mu\, a_e^2}{r^7}\left\{\left[(\underline{\rho}^T\Theta\underline{\rho})\,(\frac{r^7}{\rho^7} - 1) + 2\,(\underline{\delta}^T\,\Theta\,\underline{\rho}) - (\underline{\delta}^T\Theta\,\underline{\delta})\right]\underline{\rho} + (\underline{r}^T\Theta\underline{r})\,\underline{\delta}\right\} \quad (7)$$

Note that the factors

$$\frac{r^3}{\rho^3} - 1\ , \quad \frac{r^5}{\rho^5} - 1\ , \quad \text{and} \quad \frac{r^7}{\rho^7} - 1$$

should each approach zero as $\underline{\delta}$ approaches zero. Numerically this presents a problem since r and ρ are large and nearly equal.

These can all be computed using Potter's [1] approach by noting that

$$r^2 = (\underline{\rho} - \underline{\delta}) \cdot (\underline{\rho} - \underline{\delta}) = \rho^2 - 2\,\underline{\rho} \cdot \underline{\delta} + \delta^2$$

and

$$r^n = (\rho^2 - 2\,\underline{\rho} \cdot \underline{\delta} + \delta^2)^{n/2}$$

and that

$$\rho^n = (\rho^2)^{n/2}$$

$$\therefore \quad \frac{r^n}{\rho^n} - 1 = \left(\frac{\rho^2 - 2\,\underline{\rho} \cdot \underline{\delta} + \delta^2}{\rho^2}\right)^{n/2} - 1 \quad (8)$$

and we can write

$$\frac{r^n}{\rho^n} - 1 = \frac{(1+q)^n - 1}{(1+q)^{n/2} + 1} \tag{9}$$

where

$$q \equiv \frac{\delta^2 - 2\,\underline{\rho} \cdot \underline{\delta}}{\rho^2}$$

Note that $q \longrightarrow 0$ as $\delta \longrightarrow 0$

We can now write these factors as

$$\frac{r^3}{\rho^3} - 1 = \frac{f}{(1+q)^{3/2} + 1} \tag{10}$$

where

$$f \equiv 3q + 3q^2 + q^3$$

$$\frac{r^5}{\rho^5} - 1 = \frac{f + (f+1)(2q + q^2)}{(1+q)^{5/2} + 1} \tag{11}$$

and

$$\frac{r^7}{\rho^7} - 1 = \frac{q + (q+1)(2f + f^2)}{(1+q)^{7/2} + 1} \tag{12}$$

Note that from the definition of q and f, these factors approach zero directly as δ approaches zero.

3.0 DISCUSSION AND RECOMMENDATIONS

Equations (10), (11), (12) when substituted into equation (7) yield the companion set of differential equations for the second satellite relative to the first. The terms in the resulting equation all go to zero directly as δ approaches zero and do not contain differences of large nearly equal terms. These equations would be quite useful for both space station and tethered satellite analysis.

The technique presented here extends, albeit with effort, to higher order and degree terms in the geopotential. A recursive approach to the computation of the coefficient matrices would be a welcome development.

4.0 REFERENCES

1. Battin, R. H., Astronautical Guidance, McGraw-Hill Book Company, 1964.

2. Gottlieb, R. G., "A Fast Recursive Singularity Free Algorithm for Calculating the First and Second Derivatives of the Geopotential", MDAC Report No. AA:0028, 1988.