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COSMOLOGICAL COSMIC STRINGS

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ABSTRACT

We investigate the effect of an infinite cosmic string on a cosmological background. We find that the metric is approximately a scaled version of the empty space string metric, i.e. conical in nature. We use our results to place bounds on the amount of cylindrical gravitational radiation currently emitted by such a string. We then analyse explicitly the gravitational radiation equations and show that even initially large disturbances are rapidly damped as the expansion proceeds. The implications for the gravitational radiation background and the limitations of the quadrupole formula are discussed.

1 Introduction.

Cosmic strings[1] have attracted a lot of attention recently as they give a compelling and plausible description of structure in the universe. They are topological defects which can be formed when the universe undergoes a suitable phase transition to a spontaneously broken symmetry state. The condition for string formation is that the vacuum manifold (the set of vacuum states of the theory) is not simply connected[2]. The strings responsible for galaxy formation are GUT strings, i.e. those formed during phase transitions at the GUT scale, 10^{15}GeV. Such strings are very thin ($r_s \sim 10^{-29}$cm) and extremely massive ($\mu \sim 10^{22}$ g cm^{-1}), but knowledge of their gravitational properties is not yet complete. The spacetime structure surrounding a local string in an otherwise empty spacetime has been analysed[3−5], and the spacetime was found to be asymptotically conical. Whilst this is a useful test calculation, otherwise empty universes hardly correspond with our own, which currently on a large scale behaves like a matter dominated Friedmann-Robertson-Walker (FRW) cosmology. Since we do on average expect at least one string per horizon volume[6], we must check that these properties are not modified in any essential way when we consider a more realistic background for the string. In particular, we expect the radius of the string to be fixed by the local microphysics, which will mean in general that the string is contracting with respect to the surrounding spacetime. This could give rise to interesting dynamical effects which could potentially be observable, such as cylindrical gravitational radiation, or particle radiation. As the most stringent constraints on the cosmic string scenario currently come from measurements of gravitational radiation[7], this last feature could be of great relevance.

In this paper we address the problem of a string in a cylindrically symmetric universe. There are two approaches one could take: one can either choose a particular model for the energy momentum tensor of the string and calculate the metric, or one can try the more complete treatment of regarding the string as a vortex solution to the equations of motion of some suitable spontaneously broken local gauge theory. The former has the advantage of simplicity, but neglects the more detailed structure of the string accounted for by the latter.

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We begin our investigation of the problem by using the most common model for a string: the wire or delta function source. This gives an idea of the essential structure of the spacetime (which is again conical), however there are complications in using distributional sources in general relativity[8] and the validity of this approximation should be justified. Therefore we proceed to analyse the full coupled equations for a Nielsen-Olesen vortex in matter and radiation dominated Friedmann universes which confirms the conical structure and also allows us to examine the question of cylindrical gravitational radiation from strings.

2 The Wire Approximation.

We start by summarising the formalism for dealing with cylindrically symmetric systems. We use the work of Thorne[9] as a basis. The metric of our universe may be written as

$$ds^2 = e^{2(\Gamma-\Psi)}(dt^2 - dr^2) - e^{2\Psi}dz^2 - \mathcal{A}^2 e^{-2\Psi}d\theta^2 \,, \tag{2.1}$$

where $\mathcal{A}$, Ψ and Γ are functions of r and t only. We will assume that the energy momentum tensor splits into the form $T_{ab} + S_{ab}$, where T_{ab} is the energy momentum tensor for the background matter fields (which are homogeneous and isotropic) and S_{ab} the string energy momentum tensor. With this splitting, the field equations take the form:

$$\ddot{\mathcal{A}} - \mathcal{A}'' = 8\pi G\sqrt{-g}(T^0_0 + T^r_r + S^0_0 + S^r_r) \tag{2.2a}$$

$$\frac{d}{dt}(\mathcal{A}\dot{\Psi}) - \frac{d}{dr}(\mathcal{A}\Psi') = 4\pi G\sqrt{-g}(T^0_0 + T^r_r + T^\theta_\theta - T^z_z + S^0_0 + S^r_r + S^\theta_\theta - S^z_z) \tag{2.2b}$$

$$\begin{aligned}(\mathcal{A}'^2 - \dot{\mathcal{A}}^2)\Gamma' = {} & 8\pi G\sqrt{-g}[\mathcal{A}'(T^0_0 + S^0_0) + \dot{\mathcal{A}}(T^r_0 + S^r_0)] \\ & + \mathcal{A}\mathcal{A}'(\dot{\Psi}^2 + \Psi'^2) - 2\mathcal{A}\dot{\mathcal{A}}\Psi'\dot{\Psi} + \mathcal{A}'\mathcal{A}'' - \dot{\mathcal{A}}\dot{\mathcal{A}}'\end{aligned} \tag{2.2c}$$

$$\begin{aligned}(\dot{\mathcal{A}}^2 - \mathcal{A}'^2)\dot{\Gamma} = {} & 8\pi G\sqrt{-g}[\dot{\mathcal{A}}(T^0_0 + S^0_0) + \mathcal{A}'(T^r_0 + S^r_0)] \\ & + \mathcal{A}\dot{\mathcal{A}}(\dot{\Psi}^2 + \Psi'^2) - 2\mathcal{A}\mathcal{A}'\Psi'\dot{\Psi} + \dot{\mathcal{A}}\mathcal{A}'' - \mathcal{A}'\dot{\mathcal{A}}'\end{aligned} \tag{2.2d}$$

$$\ddot{\Gamma} - \Gamma'' = \Psi'^2 - \dot{\Psi}^2 + 8\pi G e^{2(\Gamma-\Psi)}(T^\theta_\theta + S^\theta_\theta). \tag{2.2e}$$

where $\sqrt{-g} = \mathcal{A}e^{2(\Gamma-\Psi)}$ is the square root of the determinant of the metric, a dot denotes time derivative and a prime radial derivative.

In the otherwise empty space case the approximation of the string by a delta function or wire source readily gave the essential asymptotic structure of the spacetime[3]. We adopt this approach here, using the wire model to investigate asymptotic structure. Therefore for the string source we set

$$S_0^0 = S_z^z = \frac{\mu}{2\pi\mathcal{A}}\,\delta(r)\ ;\ S_r^r = S_\theta^\theta = 0. \tag{2.3}$$

The boundary conditions are

$$\mathcal{A}'(0,t) = e^{\Gamma(0,t)},\ \Gamma(0,0) = 2\Psi(0,0)\ .$$

These now become formal, since $r = 0$ is a (distributional) curvature singularity. We do not expect $\lim_{r\to 0}\mathcal{A}'(r,t) = \mathcal{A}'(0,t)$ as prescribed by the boundary conditions, indeed it is this discontinuity which gives rise to the curvature singularity necessitated by the distributional string source.

Examination of the field equations shows that the equation for $\mathcal{A}$, (2.2a), is modified from the pure cosmology case

$$\ddot{\mathcal{A}} - \mathcal{A}'' = 8\pi G\mathcal{A}e^{2(\Gamma-\Psi)}(T_0^0 + T_r^r) + 4G\mu e^{2(\Gamma-\Psi)}\delta(r)\ . \tag{2.4}$$

The equation for Ψ, (2.2b), explicitly remains the same, as does the final equation, (2.2e). Although at first sight the first order equations for Γ, (2.2c,d), appear modified, if we substitute for $\mathcal{A}''$ we see that in fact the distributional term does not contribute. Notice also that all the equations are invariant under scaling of $\mathcal{A}$ by a constant.

Integrating (2.4) over a disc of radius δ gives

$$\begin{aligned}\mathcal{A}'(0,t) - \mathcal{A}'(\delta,t) &= 8\pi G\frac{\mu}{2\pi}e^{2(\Gamma(0,t)-\Psi(0,t))}\\ &= 4G\mu\mathcal{A}'(0,t)\\ \Rightarrow\qquad \mathcal{A}'(\delta,t) &= (1-4G\mu)\mathcal{A}'(0,t)\ .\end{aligned} \tag{2.5}$$

Thus we see that the effect of the string is to multiply the $\mathcal{A}$ in the original cosmology by a factor $(1 - 4G\mu)$ to give the required discontinuity in $\mathcal{A}'$ at the origin. The modified string-plus-cosmology metric is now

$$ds^2 = e^{2(\Gamma-\Psi)}(dt^2 - dr^2) - e^{2\Psi}dz^2 - (1-4G\mu)^2\mathcal{A}^2e^{-2\Psi}d\theta^2\ \ , \tag{2.6}$$

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which is indeed a conical spacetime, with a deficit angle of $8\pi G\mu$.

This model is however static, in the sense that the effect of the string is a time independent one. The string, being infinitesimally thin, does not participate in the expansion of the universe. In a more realistic situation where the string has width and internal structure, we might expect the dynamics of the surrounding spacetime to produce dynamical effects in the string. This is the problem we shall consider next.

3 Vortex solutions in Friedmann universes.

We now try to take account of the fact that strings are in reality thick objects that will in general be affected by the dynamics of the surrounding spacetime. The string radius r_s is generally seen as being fixed by the microphysics but a string of fixed proper radius need not have fixed coordinate radius. In order to maintain a constant proper radius fixed by the microphysics, the string will have to resist the expansion (or otherwise) of the surrounding spacetime in which it sits. It is possible that this "variation" of the string radius with respect to the surrounding spacetime could give rise to interesting dynamical effects, for example cylindrical gravitational radiation or particle radiation, which may be observable in our universe. We will consider only local strings, since global strings have badly behaved asymptotic structures. In particular, we consider the case of a U(1) local string: the Nielsen-Olesen vortex.

The Nielsen-Olesen string is a vortex solution to the lagrangian

$$\mathcal{L}[\phi, A_\mu] = D_\mu\phi^\dagger D^\mu\phi - \frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{\lambda}{4}(\phi^\dagger\phi - \eta^2)^2, \tag{3.1}$$

where $D_\mu = \nabla_\mu + ieA_\mu$ is the usual gauge covariant derivative, and $\tilde{F}_{\mu\nu}$ the field strength associated with A_μ. However, we will express the field content in a slightly different manner, in which the physical degrees of freedom are more manifest. We define the (real) fields X, χ and P_μ by

$$\phi(x^\alpha) = \eta X(x^\alpha)e^{i\chi(x^\alpha)} \tag{3.2a}$$

$$A_\mu(x^\alpha) = \frac{1}{e}\left[P_\mu(x^\alpha) - \nabla_\mu\chi(x^\alpha)\right] . \tag{3.2b}$$

In terms of these new variables, the lagrangian becomes

$$\mathcal{L} = \eta^2 \nabla_\mu X \nabla^\mu X + \eta^2 X^2 P_\mu P^\mu - \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda \eta^4}{4} (X^2 - 1)^2, \tag{3.3}$$

where $F_{\mu\nu}$ is the field strength associated with P_μ.

We can see that a vacuum state is characterised by $X = 1$. However, this is not the only stable ground state which solves the equations of motion. Nielsen and Olesen[10] showed that there exists a non-trivial stable ground state solution to the above equations of motion which has a vortex-like structure.

The Nielsen-Olesen vortex solution corresponds to an infinite, straight static string aligned with the z-axis. In this case, we can choose a gauge in which

$$\phi = \eta X(\rho) e^{i\theta} \quad ; \quad P^\mu = P(\rho) \nabla^\mu \theta \tag{3.4}$$

in cylindrical polar coordinates. This string has winding number one.

We consider the case of this cylindrically symmetric string embedded in a spatially flat FRW universe. We will assume that the string is a perturbation in the sense that it is confined to a cylinder of radius r_s, which is small compared to the Hubble radius R_H. If this were not the case, i.e. if the string width were comparable to the Hubble radius, then a significant proportion of the Hubble volume would lie within the core of the string. The dominant term in the energy-momentum tensor would be the potential term $\frac{1}{4}(X^2 - 1)^2 g_{\mu\nu}$, which would drive a rapid expansion. In short, such a régime would be extremely gravitationally active. Furthermore, such a régime would not be of particular physical relevance, since if strings form in a post-inflationary era, the string radius will be considerably smaller than the Hubble radius, in fact, by a factor of about 10^{-20}.

Assuming $r_s \ll R_H$, the string does not affect the overall average cosmological expansion. We therefore rewrite the metric in a form more suited to the problem:

$$ds^2 = \Omega^2(t) \left[e^{2(\gamma - \psi)} (dt^2 - dr^2) - e^{2\psi} dz^2 - \alpha^2 e^{-2\psi} d\theta^2 \right] , \tag{3.5}$$

which factors out the immediate time dependence due to the expansion of the universe. In terms of the previous variables

$$\Gamma = \gamma + 2 \log \Omega \ , \quad \Psi = \psi + \log \Omega \ , \quad \mathcal{A} = \alpha \Omega^2 \ . \tag{3.6}$$

The pure FRW universe corresponds to $\gamma = \psi = 0$, $\alpha = r$

$$ds^2 = \Omega^2(t)[dt^2 - dr^2 - dz^2 - r^2 d\theta^2] \ , \tag{3.7}$$

and the Thorne equations give

$$\begin{aligned}
\left(\frac{\dot\Omega}{\Omega}\right)^2 + \frac{\ddot\Omega}{\Omega} &= 4\pi G(T_{00} - T_{rr}) \\
3\left(\frac{\dot\Omega}{\Omega}\right)^2 &= 8\pi G T_{00} \\
2\frac{\ddot\Omega}{\Omega} - \left(\frac{\dot\Omega}{\Omega}\right)^2 &= -8\pi G T_{\theta\theta}/r^2 = -8\pi G T_{rr} \ ,
\end{aligned} \tag{3.7}$$

which will be useful in the simplification of the field equations later. For a matter dominated cosmology only the energy density is non-zero, which leads to a conformal factor $\Omega(t) = t^2$. For a radiation dominated cosmology $T^r_r = T^\theta_\theta = T^z_z = -\frac{1}{3}T^0_0$ giving $\Omega(t) = t$. Note that t here is conformal time rather than cosmological time.

We consider the Nielsen-Olesen vortex string embedded in one of these cosmologies, and consider only the generalisation of a simple string in vacuo to a time dependent string in FRW by setting

$$\phi = \eta X(r,t)e^{i\theta} \tag{3.9a}$$

$$A_\mu = \frac{1}{e}(P(r,t) - 1)\nabla_\mu \theta \tag{3.9b}$$

In terms of these fields, the Lagrangian is given by

$$\mathcal{L} = \frac{\eta^2 e^{2\psi-2\gamma}}{\Omega^2}(\dot X^2 - X'^2) + \frac{e^{4\psi-2\gamma}}{2e^2\alpha^2\Omega^4}(\dot P^2 - P'^2) - \frac{\eta^2 X^2 P^2 e^{2\psi}}{\alpha^2\Omega^2} - \frac{\lambda\eta^4}{4}(X^2-1)^2 \ . \tag{3.10}$$

To simplify the discussion we will set $2e^2 = \lambda$ in order that the gauge and scalar parts of the string have the same width; this will not qualitatively affect the results.

To clarify the analysis of the equations, we will scale out the λ, η dependence by introducing the dimensionless variables:

$$\rho = \sqrt{\lambda}\eta r \qquad \tau = \sqrt{\lambda}\eta t \qquad \epsilon = 8\pi G\eta^2 \qquad \tilde\alpha = \sqrt{\lambda}\eta\alpha \tag{3.11a}$$

$$E = S^0_0/\lambda\eta^4 \quad -P_\rho = S^r_r/\lambda\eta^4 \quad -P_\theta = S^\theta_\theta/\lambda\eta^4 \quad -P_z = S^z_z/\lambda\eta^4 \ . \tag{3.11b}$$

Thus the characteristic energies and pressures of the string are 0(1) and the radius of the string likewise. Notice that the static equality $E + P_z = 0$ no longer need hold due to the potential time dependence of the string fields.

Using equations (3.7) for simplification, we may write the full set of equations for the system:

$$\ddot{\tilde{\alpha}} - \tilde{\alpha}'' + 4\dot{\tilde{\alpha}}\tfrac{\dot{\Omega}}{\Omega} = \epsilon\tilde{\alpha}e^{2(\gamma-\psi)}\Omega^2(E - P_\rho) \tag{3.12a}$$

$$\ddot{\psi} + \frac{\dot{\tilde{\alpha}}}{\tilde{\alpha}}\dot{\psi} - \psi'' - \frac{\tilde{\alpha}'}{\tilde{\alpha}}\psi' = \frac{1}{2}\epsilon e^{2(\gamma-\psi)}\Omega^2(E - P_\rho - P_\theta + P_z) + \tfrac{\dot{\Omega}}{\Omega}\left(2\dot{\psi} + \frac{\dot{\tilde{\alpha}}}{\tilde{\alpha}}\right) \tag{3.12b}$$

$$\begin{aligned}\left[\tilde{\alpha}'^2 - \dot{\tilde{\alpha}}^2 - 4\dot{\tilde{\alpha}}\tfrac{\dot{\Omega}}{\Omega} - 4\tilde{\alpha}^2\left(\tfrac{\dot{\Omega}}{\Omega}\right)^2\right]\gamma' &= 8\pi G\Omega^2\tilde{\alpha}e^{2(\gamma-\psi)}\left[\tilde{\alpha}'S_0^0 + (\dot{\tilde{\alpha}} + 2\tilde{\alpha}\tfrac{\dot{\Omega}}{\Omega})S_0^r\right] - \dot{\tilde{\alpha}}'(\dot{\tilde{\alpha}} + 2\tilde{\alpha}\tfrac{\dot{\Omega}}{\Omega}) \\ &+ \tilde{\alpha}\tilde{\alpha}'(\dot{\psi}^2 + 2\dot{\psi}\tfrac{\dot{\Omega}}{\Omega} + \psi'^2) - 2\tilde{\alpha}\psi'(\dot{\tilde{\alpha}} + 2\tilde{\alpha}\tfrac{\dot{\Omega}}{\Omega})(\dot{\psi} + \tfrac{\dot{\Omega}}{\Omega}) + \tilde{\alpha}\tilde{\alpha}'' - 2\tilde{\alpha}'\dot{\tilde{\alpha}}\tfrac{\dot{\Omega}}{\Omega}\end{aligned} \tag{3.12c}$$

$$\begin{aligned}\left[\dot{\tilde{\alpha}}^2 + 4\dot{\tilde{\alpha}}\tfrac{\dot{\Omega}}{\Omega} + 4\tilde{\alpha}^2\left(\tfrac{\dot{\Omega}}{\Omega}\right)^2 - \tilde{\alpha}'^2\right]\dot{\gamma} &= 8\pi G\Omega^2\tilde{\alpha}e^{2(\gamma-\psi)}\left[(\dot{\tilde{\alpha}} + 2\tilde{\alpha}\tfrac{\dot{\Omega}}{\Omega})S_0^0 + \tilde{\alpha}'S_0^r\right] + \tilde{\alpha}''(\dot{\tilde{\alpha}} + 2\tilde{\alpha}\tfrac{\dot{\Omega}}{\Omega}) \\ &+ \tilde{\alpha}(\dot{\tilde{\alpha}} + 2\tilde{\alpha}\tfrac{\dot{\Omega}}{\Omega})(\dot{\psi}^2 + 2\dot{\psi}\tfrac{\dot{\Omega}}{\Omega} + \psi'^2) - 2\tilde{\alpha}\tilde{\alpha}'\psi'(\dot{\psi} + \tfrac{\dot{\Omega}}{\Omega}) - 2\dot{\tilde{\alpha}}^2\tfrac{\dot{\Omega}}{\Omega} - \tilde{\alpha}'\dot{\tilde{\alpha}}'\end{aligned} \tag{3.12d}$$

$$\ddot{\gamma} - \gamma'' = \psi'^2 - \dot{\psi}^2 - 2\dot{\psi}\tfrac{\dot{\Omega}}{\Omega} - \epsilon e^{2(\gamma-\psi)}\Omega^2 P_\theta \tag{3.12e}$$

for the gravitational fields, and

$$2\Omega^{-2}e^{2\psi-2\gamma}\left[\ddot{X} - X'' + \left(\frac{\dot{\tilde{\alpha}}}{\tilde{\alpha}} + \frac{2\dot{\Omega}}{\Omega}\right)\dot{X} - \frac{\tilde{\alpha}'}{\tilde{\alpha}}X'\right] + \frac{2XP^2e^{2\psi}}{\tilde{\alpha}^2\Omega^2} + X(X^2 - 1) = 0 \tag{3.12f}$$

$$\Omega^{-2}e^{2\psi-2\gamma}\left[\ddot{P} - P'' + \left(2\dot{\psi} - \frac{\dot{\tilde{\alpha}}}{\tilde{\alpha}}\right)\dot{P} - \left(2\psi' - \frac{\tilde{\alpha}'}{\tilde{\alpha}}\right)P'\right] + X^2P = 0 \tag{3.12g}$$

for the string fields, where prime and dot now denote derivative with respect to the scaled parameters σ and τ respectively.

So far the discussion has been general, however the equations (3.12) are so involved that we clearly need some physical simplifications in order to make progress. Recall that the string radius r_s was seen as being fixed by the microphysics. This statement implicitly assumes that $r_s \ll R_H$, the Hubble radius, otherwise the notion of the string as a separate entity would break down. Since $r_s \sim 10^{-29}$cm for GUT strings this statement is clearly valid in the post inflationary era, indeed $r_s/R_H \sim 10^{-57}$ at the present time! In the dimensionless variables $r_s/R_H = \dot{\Omega}/\Omega^2$ therefore a reasonable first approximation to the problem is certainly to discard all terms of order $\dot{\Omega}/\Omega^2$ or higher. But in fact we can go further than this. The value of r_s/R_H at the time of string formation is of the order of 10^{-20}. At such a time, the universe is radiation dominated, and $r_s/R_H \propto \tau^{-2}$, thus $\dot{\Omega}/\Omega$, which is proportional to τ^{-1}, is of the order of 10^{-10} at the time of string formation. We may therefore conclude that at subsequent times $\dot{\Omega}/\Omega$ will be less than 10^{-10} and can therefore also be ignored.

Before writing down the simplified field equations, we will make one other observation. When strings are formed during a phase transition we do not expect them to be in their equilibrium field configuration, however we do expect them to settle to this in a timescale of order $(\sqrt{\lambda}\eta)^{-1}$. This process of 'settling in' will generate some high frequency primordial gravitational radiation (which we will consider later), of a qualitatively different nature to that produced by the cosmological expansion. In searching for a solution representing a string in an expanding universe, we will consider the situation in which the string has already settled to its equilibrium configuration, so that the spacetime on the symmetry axis is locally Friedmannian, i.e. $\gamma, \psi \to 0$, $\alpha \sim r$ as $r \to 0$.

Making these approximations, we arrive at the considerably simpler looking set of equations:

$$\Omega^{-2}(\ddot{\tilde{\alpha}} - \tilde{\alpha}'') = \epsilon \tilde{\alpha} e^{2(\gamma-\psi)}(E - P_\rho) \tag{3.13a}$$

$$\Omega^{-2}(\ddot{\psi} + \frac{\dot{\tilde{\alpha}}}{\tilde{\alpha}}\dot{\psi} - \psi'' - \frac{\tilde{\alpha}'}{\tilde{\alpha}}\psi') = \frac{1}{2}\epsilon e^{2(\gamma-\psi)}(E - P_\rho - P_\theta + P_z) \tag{3.13b}$$

$$\Omega^{-2}\left[(\tilde{\alpha}'^2 - \dot{\tilde{\alpha}}^2)\gamma' + 2\tilde{\alpha}\dot{\tilde{\alpha}}\psi'\dot{\psi} - \tilde{\alpha}\tilde{\alpha}'' + \dot{\tilde{\alpha}}'\dot{\tilde{\alpha}} - \tilde{\alpha}\tilde{\alpha}'(\dot{\psi}^2 + \psi'^2)\right]$$
$$= 8\pi G \tilde{\alpha} e^{2(\gamma-\psi)}\left[\tilde{\alpha}' S^0_0 + \dot{\tilde{\alpha}} S^r_0\right] \tag{3.13c}$$

$$\Omega^{-2}\left[(\dot{\tilde{\alpha}}^2 - \tilde{\alpha}'^2)\dot{\gamma} + 2\tilde{\alpha}\tilde{\alpha}'\psi'\dot{\psi} - \tilde{\alpha}''\dot{\tilde{\alpha}} + \tilde{\alpha}'\dot{\tilde{\alpha}}' - \tilde{\alpha}\dot{\tilde{\alpha}}(\dot{\psi}^2 + \psi'^2)\right]$$
$$= 8\pi G \tilde{\alpha} e^{2(\gamma-\psi)}\left[\dot{\tilde{\alpha}} S^0_0 + \tilde{\alpha}' S^r_0\right] \tag{3.13d}$$

$$\Omega^{-2}(\ddot{\gamma} - \gamma'') = \Omega^{-2}(\psi'^2 - \dot{\psi}^2) - \epsilon e^{2(\gamma-\psi)} P_\theta \tag{3.13e}$$

for the spacetime fields, and

$$2\Omega^{-2} e^{2\psi - 2\gamma}\left[\ddot{X} - X'' + \frac{\dot{\tilde{\alpha}}}{\tilde{\alpha}}\dot{X} - \frac{\tilde{\alpha}'}{\tilde{\alpha}}X'\right] + \frac{2XP^2e^{2\psi}}{\tilde{\alpha}^2\Omega^2} + X(X^2 - 1) = 0 \tag{3.13f}$$

$$\Omega^{-2}e^{2\psi-2\gamma}\left[\ddot{P}-P''+\left(2\dot{\psi}-\frac{\dot{\tilde{\alpha}}}{\tilde{\alpha}}\right)\dot{P}-\left(2\psi'-\frac{\tilde{\alpha}'}{\tilde{\alpha}}\right)P'\right]+X^2P=0 \qquad (3.13g)$$

for the matter fields.

Apart from the factors of Ω^{-2}, these equations are reminiscent of the empty space equations for the string which have a minimum energy static solution[5]. Having made this observation, we will propose the following Ansatz for the fields

$$\begin{aligned} X(\rho,\tau) &= X_0(\rho\Omega) \\ P(\rho,\tau) &= P_0(\rho\Omega) \\ \gamma(\rho,\tau) &= \gamma_0(\rho\Omega) \\ \psi(\rho,\tau) &= \psi_0(\rho\Omega) \\ \tilde{\alpha}(\rho,\tau) &= \Omega^{-1}a(\rho\Omega) \end{aligned} \qquad (3.14)$$

X_0, P_0 etc. are the static empty space cylindrical solutions[5], the slightly unusual final scaling being to preserve boundary conditions on axis.

For $\rho\Omega > r_s$ these functions are given by the asymptotic forms of the empty space solutions in [5], viz.

$$\begin{aligned} &X \sim 1 \quad P \sim 0 \\ &\gamma \sim C \quad \psi \sim C/2 \\ &\quad \tilde{\alpha} \sim A\rho + B/\Omega\,, \end{aligned} \qquad (3.15)$$

which clearly solve (3.13) to $O(r_s/R_H)$. For $\rho\Omega < r_s$, we see that (e.g.) $\dot{X} = O(\dot{\Omega}/\Omega)X'$ and thus the equations are satisfied to $O(\dot{\Omega}/\Omega)^2$.

Thus the string-cosmology spacetime essentially looks like a scaled version of a string in a vacuum spacetime. Any corrections to the fields appear at order $(\dot{\Omega}/\Omega)^2$, which, as we have argued, is negligible (10^{-20}) in any physical régimes.

4 Cylindrical Gravitational Radiation.

In the previous section we derived the spacetime structure for a Nielsen-Olesen vortex string in a Friedmann universe. In this section we deal with the problem of gravitational radiation from such a string. The most stringent bounds so far on the mass per unit length of cosmic strings come from consideration of the gravitational radiation intensity of a network of cosmic string loops[7]. We expect about one infinite string per Hubble volume, therefore any source of gravitational radiation from such structures is potentially important. By considering the total C-energy of the string inside the Hubble volume we can place an upper bound on the amount of cylindrical gravitational (or other) radiation present without going into a detailed analysis of the first order corrections to the string fields.

We use the generalised C-energy density per unit length[9]:

$$\mathcal{E}(r) = \frac{1}{8G}\left[1 - (\mathcal{A}^2_{,r} - \mathcal{A}^2_{,t})e^{-2\Gamma}\right]|_r \,. \tag{4.1}$$

In order to clarify the interpretation of the C-energy of the composite system, we first calculate the C-energy of the pure cosmology and the pure string case.

For the pure cosmology, $\mathcal{A} = r\Omega^2$ and $\Gamma = 2\log\Omega$, which gives

$$\begin{aligned}\mathcal{E}_{\text{COSM}} &= \frac{1}{8G}\left[1 - (\Omega^4 - 4r^2\Omega^2\dot{\Omega}^2)\Omega^{-4}\right] \\ &= \frac{r^2}{2G}\left(\frac{\dot{\Omega}}{\Omega}\right)^2 \quad = \frac{4}{3}\pi r^2 T_{00}.\end{aligned} \tag{4.2}$$

Therefore if we evaluate the C-energy at a fixed proper radius $r_0 = r\Omega$,

$$\mathcal{E}_{\text{COSM}} = \frac{4}{3}\pi\frac{r_0^2}{\Omega^2}T_{00}.$$

For the pure string, $\mathcal{A} = a$, $\Gamma = \gamma_0$, giving

$$\begin{aligned}\mathcal{E}_{\text{STR}} &= \frac{1}{8G}\left[1 - a'^2 e^{-2\gamma_0}\right] \\ &= \mu \text{ for } r_0 > r_s.\end{aligned} \tag{4.3}$$

For the string plus cosmology case, computing the C-energy at a fixed proper radius $r_0 = r\Omega$ outside the source we obtain

$$\begin{aligned}\mathcal{E}|_{r_0} &= \frac{1}{8G}\left[1 - (\mathcal{A}^2_{,r} - \mathcal{A}^2_{,t})e^{-2\Gamma}\right]|_{r_0} \\ &= \frac{1}{8G}\left[1 - \left(\alpha'^2\Omega^4 - (\dot{\alpha}\Omega^2 + 2\alpha\Omega\dot{\Omega})^2\right)\Omega^{-4}e^{-2\gamma}\right]|_{r_0}\end{aligned}$$

Outside the source $\alpha \sim Ar + B/\sqrt{\lambda}\eta\Omega$, hence $\dot{\alpha} = O(r_s/R_H)$, and we obtain

$$\mathcal{E} = \frac{1}{8G}\left[1 - \alpha'^2 e^{-2\gamma} + 4\alpha^2\left(\frac{\dot{\Omega}}{\Omega}\right)^2 + O(r_s/R_H)\right]\Big|_{r_0}$$

$$= \frac{1}{8G}\left[1 - a'^2(r_0)e^{-2\gamma(r_0)} + 4e^{-2\gamma(r_0)}a^2(r_0)\tfrac{8\pi G}{3}T_{00} + O(r_s/R_H)\right]$$

$$= \mu + \frac{4\pi}{3}\frac{a^2(r_0)}{\Omega^2}e^{-2\gamma(r_0)}T_{00} + O(r_s/R_H) \tag{4.4}$$

μ is the energy per unit length of the string and is constant. The second term represents the cosmological energy density (adjusted for the angular deficit). Thus the C-energy is the sum of the cosmic string energy plus the energy due to the background cosmological matter distribution contained within the cylinder of radius r_0, plus a piece of order (r_s/R_H).

We may now place an order of magnitude bound on the energy contained in cylindrical gravitational radiation in a Hubble volume as follows. Since μ is constant, the energy per unit length of string being emitted is at most $O(r_s/R_H)$. The total length of string in a Hubble volume is approximately R_H, and the Hubble volume R_H^3, thus the average energy density per Hubble volume due to cylindrical gravitational radiation is at most $O(r_s/R_H^3)$. However, we see that the energy density of the universe is of order R_H^{-2}, thus the string radiation is suppressed by a factor of r_s/R_H. At the present time $r_s/R_H \sim 10^{-57}$ and even at the time of nucleosynthesis $r_s/R_H \sim 10^{-41}$, thus the cosmological radiation from strings is negligible and the geometry surrounding the string can be taken to be conical.

It only remains to show that the gravitational radiation produced by the strings during their formation and settling in does not affect this conclusion. As mentioned earlier such gravitational radiation will be composed mainly of high frequency components.

We will first comment on gravitational radiation in an otherwise empty background as this exhibits several interesting features we wish to illustrate. We will then show how these carry through and become modified in the case of a radiation dominated universe. Setting $\Omega = 1$, outside the string we obtain

$$\ddot{\tilde{\alpha}} - \tilde{\alpha}'' = 0 \tag{4.5a}$$

$$\ddot{\psi} + \frac{\dot{\tilde{\alpha}}}{\tilde{\alpha}}\dot{\psi} - \psi'' - \frac{\tilde{\alpha}'}{\tilde{\alpha}}\psi' = 0 \tag{4.5b}$$

$$\tilde{\alpha}\tilde{\alpha}'(\psi'^2 + \dot{\psi}^2) - 2\tilde{\alpha}\dot{\tilde{\alpha}}\dot{\psi}\psi' + \tilde{\alpha}'\tilde{\alpha}'' - \dot{\tilde{\alpha}}\dot{\tilde{\alpha}}' = (\tilde{\alpha}'^2 - \dot{\tilde{\alpha}}^2)\gamma' \tag{4.5c}$$

$$\tilde{\alpha}\dot{\tilde{\alpha}}(\psi'^2 + \dot{\psi}^2) - 2\tilde{\alpha}\tilde{\alpha}'\dot{\psi}\psi' + \dot{\tilde{\alpha}}\tilde{\alpha}'' - \tilde{\alpha}'\dot{\tilde{\alpha}}' = (\dot{\tilde{\alpha}}^2 - \tilde{\alpha}'^2)\dot{\gamma} \tag{4.5d}$$

$$\ddot{\gamma} - \gamma'' - \psi'^2 + \dot{\psi}^2 = 0 \tag{4.5e}$$

In the absence of radiation the asymptotic solutions are $\tilde{\alpha} \sim (1-4G\mu)\rho$ and $\gamma = 2\psi \sim -C$ as $\rho \to \infty$. The energy of the string fields is exponentially damped therefore 'asymptotically' essentially means 'outside the string'. We will therefore look for solutions of the form

$$\tilde{\alpha} = (1 - 4G\mu)\rho + \tilde{\alpha}_1(\rho, \tau) \tag{4.6a}$$

$$\gamma = -C + \gamma_1(\rho, \tau) \tag{4.6b}$$

$$\psi = -C/2 + \psi_1(\rho, \tau) \tag{4.6c}$$

with $\tilde{\alpha}_1 \ll \tilde{\alpha}$ etc. Linearising the above equations gives

$$\ddot{\tilde{\alpha}}_1 - \tilde{\alpha}_1'' = 0 \tag{4.7a}$$

$$\ddot{\psi}_1 - \psi_1'' - \psi_1'/\rho = 0 \tag{4.7b}$$

$$(1 - 4G\mu)^2\gamma_1' = (1 - 4G\mu)\tilde{\alpha}_1'' \tag{4.7c}$$

$$-(1 - 4G\mu)^2\dot{\gamma}_1 = -(1 - 4G\mu)\dot{\tilde{\alpha}}_1' \tag{4.7d}$$

$$\ddot{\gamma}_1 - \gamma_1'' = 0 \tag{4.7e}$$

Thus $\tilde{\alpha}$ and γ satisfy the two-dimensional wave equation, γ being determined by $\tilde{\alpha}_1$ up to a constant. ψ satisfies the three-dimensional† wave equation. Here we come to the first interesting feature: because ψ satisfies a wave equation in an odd number of dimensions, Huygen's principle does not hold. In other words, wave propagation in ψ is not clean, the waves have 'tails'. To see this explicitly, choose initial data $\psi_1 = p_1$, $\dot{\psi}_1 = p_2$ on $t = 0$, then the solution for ψ is[11]

$$\begin{aligned}\psi_1(\rho, \tau) =& \partial_\tau \int_0^\tau dr \int_0^{2\pi} d\theta \frac{r}{\sqrt{\tau^2 - r^2}} p_1(\sqrt{r^2 + \rho^2 - 2r\rho\cos\theta}) \\ &+ \int_0^\tau dr \int_0^{2\pi} d\theta \frac{r}{\sqrt{\tau^2 - r^2}} p_2(\sqrt{r^2 + \rho^2 - 2r\rho\cos\theta})\end{aligned} \tag{4.8}$$

† i.e. two space plus one time dimension

Thus the solution for ψ can be seen to depend not only upon the intersection of the past null cone with the surface $\tau = 0$, but also on the interior of this disc. Once a disturbance in ψ reaches a point, it does not only pass on, but persists, rather like the reverberations in a bell after the initial strike. This is in contrast with the propagation in $\tilde{\alpha}$ and γ, which is clean.

These features essentially pass over into the cosmological situation. We will consider the case in which the universe is radiation dominated, thus modelling the epoch in which strings were 'settling in'. For this background, $\Omega = \tau$ and therefore $r_s/R_H = 1/\tau^2$. In the absence of gravitational radiation $\tilde{\alpha} \sim (1 - 4G\mu)\rho$ and $\gamma = 2\psi \sim -C$ outside the string. Writing the same field decomposition as before (4.6), the linearised equations outside the string are

$$\ddot{\tilde{\alpha}}_1 - \tilde{\alpha}_1'' + \frac{4\dot{\tilde{\alpha}}_1}{\tau} = 0 \tag{4.9a}$$

$$\ddot{\psi}_1 - \psi_1'' - \frac{\psi_1'}{\rho} + \frac{2\dot{\psi}_1}{\tau} + \frac{\dot{\tilde{\alpha}}_1}{(1-4G\mu)\rho\tau} = 0 \tag{4.9b}$$

$$\frac{2\rho\dot{\psi}_1}{\tau} + \frac{\tilde{\alpha}_1''}{(1-4G\mu)} - \frac{2\rho\dot{\tilde{\alpha}}_1'}{(1-4G\mu)\tau} - \frac{2\dot{\tilde{\alpha}}_1}{(1-4G\mu)\tau} = \gamma_1' \tag{4.9c}$$

$$\frac{2\rho\psi_1'}{\tau} - \frac{2\rho\tilde{\alpha}_1''}{(1-4G\mu)\tau} + \frac{\dot{\tilde{\alpha}}_1'}{(1-4G\mu)} = \dot{\gamma}_1 \tag{4.9d}$$

$$\ddot{\gamma}_1 - \gamma_1'' + \frac{2\dot{\psi}_1}{\tau} = 0 \tag{4.9e}$$

having neglected terms of order τ^{-2}, but not $\tau^{-1}\partial_\tau$. (We are implicitly considering the behaviour of high frequency components of gravitational radiation.) Thus we see the expansion factor of the cosmology entering into the equations. Note that, as before, once $\tilde{\alpha}_1$ and ψ_1 are determined, γ_1 is fixed by the remaining equations,

$$\gamma_1 = \frac{\tilde{\alpha}_1'}{(1-4G\mu)} - \frac{2\rho\dot{\tilde{\alpha}}_1}{(1-4G\mu)\tau} + \int\int \frac{2\rho\ddot{\psi}_1}{\tau} d\rho d\tau \tag{4.11}$$

Consider (4.9a). Making the transformation

$$\tilde{\alpha}_1 = \hat{\alpha}/\tau^2$$

and neglecting terms of order τ^{-2}, we obtain

$$\ddot{\hat{\alpha}} - \hat{\alpha}'' = 0$$

for the high frequency alpha radiation. Hence the general solution for $\tilde{\alpha}_1$ is given by

$$\tilde{\alpha}_1 = \frac{f(\rho+\tau) + g(\rho-\tau)}{\tau_0^2} \frac{\tau_0^2}{\tau^2} \tag{4.11}$$

where τ_0 represents the time of string formation.

Thus even if we start with a very rapidly varying disturbance in $\tilde{\alpha}$, it becomes damped by a factor of $(\tau_0/\tau)^2 = (R_0/R_H)^2$ as the cosmological expansion proceeds.

In order to find the solution for ψ_1, note that a similar type of transformation

$$\psi_1 = \hat{\psi}/\tau$$

reduces the ψ_1 equation to a cylindrical wave equation with a driving term:

$$\ddot{\hat{\psi}}_1 - \hat{\psi}_1'' - \frac{\hat{\psi}_1'}{\rho} + \frac{\dot{\hat{\alpha}}_1}{(1-4G\mu)\rho} = 0.$$

The general solution for $\hat{\psi}$ consists of a homogeneous part, given by (4.8), with an additional inhomogeneous term arising from the $\dot{\hat{\alpha}}$ dependence[11]

$$\hat{\psi}_{IN}(\rho,\tau) = -\frac{1}{2\pi}\int_0^\tau dt \int_0^t dr \int_0^{2\pi} d\theta \frac{\dot{\hat{\alpha}}_1(r,\tau-t)}{(1-4G\mu)r\sqrt{(\tau-t)^2 - (r^2+\rho^2-2r\rho\cos\theta)}} \tag{4.12}$$

Again, even if we start off with very rapidly varying disturbances, we see that the above integrals (4.8,9) are damped roughly by a factor of τ^{-1}, and hence ψ_1 is damped by a factor of τ^{-2}.

We see therefore that cylindrical gravitational radiation in a cosmological background is damped by a factor of $(r_s/R_H)^2$ compared with empty-space gravitational radiation. The tails of the ψ-radiation persist in the cosmological case.

5 Conclusions.

We have examined the gravitational field of a cosmic string in an expanding universe, and shown that it is essentially a scaled version of the pure string spacetime. The corrections to this scaled solution appear at order (r_s/R_H), the ratio of the string radius to the Hubble radius. Clearly this ratio is negligible at the current time, but even at the time of string formation (assuming that strings form after inflation) it is extremely small (10^{-20}). Thus the expanding cosmology does not have significant ramifications for the asymptotic spacetime structure.

C-energy was then used to estimate the cosmological gravitational radiation from strings. Because the cosmic string is resisting the background cosmological expansion, we expect that it will produce cylindrical gravitational radiation. Our reason for supposing this lies in some earlier work by Cocke[12]. He considered contracting tubes of matter in a fixed exterior, and found gravitational radiation. Our set-up is that of a fixed tube (the string) in an expanding exterior, we therefore expect radiation. An estimate of the C-energy shows however that the amount of gravitational radiation is extremely small. Further evidence for the existence of such radiation is given by Stein-Schabes and Burd[13], who analysed the equations of motion of the string fields in a régime where the radius of the string was comparable to the Hubble radius. Although their analysis did not include back reaction of the string on the cosmology, the solutions obtained displayed an oscillatory behaviour, damping as the expansion proceeds. Such a result indicates that we might expect analogous behaviour of both string and gravity fields when full coupling is taken into account for régimes in which $r_s \ll R_H$. Incidentally, a cosmic string of fixed radius has fixed quadrupole moment, and yet can emit gravitational radiation. This is a rather nice illustration of the limits of the quadrupole formula[14] – which is derived in flat space. The terms which violate the approximation occur at the scale of the spacetime curvature.

Finally, we considered the subsequent behaviour of gravitational radiation generated by strings by explicitly analysing the gravitational field equations. Having shown that the string field configuration in its equilibrium state does not radiate significantly, we must not neglect the effect of primordial gravitational radiation, generated as the strings formed.

We show that this is damped by a factor of $(r_s/R_H)^2$, and therefore even if the initial disturbance is large, it rapidly becomes negligible as expansion proceeds. An interesting feature of this radiation is that it is not 'clean', the gravity waves develop tails. Thus the initial shock-wave formed as the deficit angle settles in cannot propagate cleanly away, however the cosmological expansion ensures that this is rapidly damped so as to become negligible.

Thus we have established that cosmic strings in an expanding universe do have the same spacetime signature as their empty-space cousins: asymptotic conicity. Using C-energy, we have shown that the cylindrical gravitational radiation emitted by such a string contributes negligibly to the background gravitational radiation intensity. Finally, by considering the gravitational radiation equations in a cosmological background, we have shown that any radiation generated while cosmic strings form (and we might expect some fairly spectacular effects as the deficit angle settles in) is rapidly damped as the expansion proceeds. We treated these problems in the context of a flat Friedmann model. It may be interesting to investigate the effect of a non-trivial spatial section on the cosmic string fields. Particularly in the case of a closed universe we might expect some interesting phenomena.

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