Generation of Three-Dimensional Body-Fitted Grids by Solving Hyberbolic Partial Differential Equations

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SUMMARY

Hyperbolic grid generation procedures are described which have been used in external flow simulations about complex configurations. For many practical applications a single well-ordered (i.e., structured) grid can be used to mesh an entire configuration, in other problems, composite or unstructured grid procedures are needed. Although the hyperbolic partial differential equation grid generation procedure has mainly been utilized to generate structured grids, an extension of the procedure to semiunstructured grids is briefly described. Extensions of the methodology are also described using two-dimensional equations.

INTRODUCTION

In finite difference simulations it is often advantageous to generate a body-fitted grid about an arbitrary closed body. Such a grid can simplify the accurate application of boundary conditions, enhance computer vectorization, allow clustering to action areas of the field, and may allow the use of certain simplifications to be made to the partial differential governing equations. For a large class of problems (e.g., external aerodynamics), the grid must conform only to the body surface and does not necessarily have to coincide with a specified outer boundary. For such cases grids can often be efficiently generated by solving hyperbolic or parabolic partial differential grid generation equations.

THREE-DIMENSIONAL HYPERBOLIC GRID GENERATION EQUATIONS

An extension of the cell-volume hyperbolic grid generation scheme [1] to three dimensions has been described in [2,3]. In this approach, as in most other partial differential grid generation procedures, the equations are transformed to a uniform computational space, ξ, η, ζ. Here, the body surface was chosen to coincide with $\zeta(x, y, z) = 0$ and the surface coordinates were taken as ξ and η. The outer boundary $\zeta(x, y, z) = \zeta_{max}$ cannot be specified for a hyperbolic set of equations.

Two orthogonality relations with respect to the ζ-outward ray and a user-specified cell-volume condition are used as the governing equations:

$$x_\xi x_\zeta + y_\xi y_\zeta + z_\xi z_\zeta = 0 \qquad (1a)$$

$$x_\eta x_\zeta + y_\eta y_\zeta + z_\eta z_\zeta = 0 \qquad (1b)$$

$$x_\xi y_\eta z_\zeta + x_\zeta y_\xi z_\eta + x_\eta y_\zeta z_\xi - x_\xi y_\zeta z_\eta - x_\eta y_\xi z_\zeta - x_\zeta y_\eta z_\xi = \Delta V \qquad (1c)$$

These equations comprise a system of nonlinear partial differential equations in which x, y, and z are specified as initial data at $\zeta = 0$. Local linearization of the equations about a nearby state, 0, results in the system of grid generation equations which can be written as

$$C_0^{-1} A_0 \vec{r}_\xi + C_0^{-1} B_0 \vec{r}_\eta + \vec{r}_\zeta = C_0^{-1} \vec{f} \qquad (2)$$

with $\vec{r} = (x, y, z)^t$, coefficient matrices A_0, B_0, C_0, and $\vec{f} = (0, 0, \Delta V + 2\Delta V_0)^t$. Although the algebraic verification is not trivial, $C_0^{-1} A_0$ and $C_0^{-1} B_0$ are symmetric matrices so the linearized system Equation (2) is hyperbolic and can be marched with ζ serving as the "time–like" direction.

The nonlinear system of grid generation equations given by equation (2) has been solved with an approximate factorization noniterative implicit finite difference scheme using central differencing in ζ and η, with the nearby known state 0 taken from the previous ζ-step. Numerical dissipation terms are also appended, and if carefully chosen [2,4,5], the hyperbolic grid generation scheme can be quite robust. An unconditionally stable implicit scheme has the advantage that the marching step size in ζ can be arbitrarily selected based only on considerations of generating the grid. Applications are given in [3,6] while figure 1 illustrates a grid for the space shuttle orbiter taken from [6].

Composite Grids

If only a single, body-fitted, structured grid is used to mesh a complex configuration, the resulting grid will be too skewed or poorly clustered. For this reason, body-fitted composite grid or unstructured grid strategies are generally used to discretize complex configurations.

One composite mesh scheme is the chimera overset grid method [7,8] in which each body component of a complex configuration is meshed independently and the composite grid is made up of the superposition of the individual grids. Figure 2 shows body symmetry planes of a set of grids generated for the integrated space shuttle vehicle in which individual grids are formed for the shuttle orbiter, external tank, and the solid rocket boosters. The individual grids are then connected by cutting out nonrealistic grid points (e.g., points of one grid that fall within another body) and setting up interpolation links between cut-out-hole and outer boundaries. An efficient structured grid flow solver can then be used to solve the flow equations on each grid and only a small segment of the code must deal with the unstructured boundary data supplied from other grids. Because the grids are superimposed in this approach, the outer boundary location of each individual grid is arbitrary. Thus, the use of a hyperbolic grid generation procedure is generally quite adequate, and all of the shuttle grids shown have been generated in this way. Computational results obtained with these grids can be found in [6].

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Semiunstructured Grids

A practical grid for high Reynolds number viscous flow applications could be formed as follows. From a body boundary surface grid $\zeta = 0$, lines can be radiated outward by marching hyperbolic grid generation equations from one $\Delta\zeta$-plane to the next so as to form a three-dimensional grid. All of the interface logic, link lists, etc. set up and specified on the body surface – a two dimensional (2-D) problem – would be maintained for each ζ=constant plane. Because a ray-like structure along which ordered grid points in the outward direction exists, it is easy to tightly cluster grid points to the surface to resolve thin viscous flow layers and it is also easy to use an implicit operator in this direction. Away from the body a fully unstructured grid could be used if the rays coalesce.

The hyperbolic grid generation procedure has been extended to generate this kind of mesh by treating the semi-unstructured data in a ζ=constant plane as follows. The ADI implicit algorithm used to solve the governing equations at each ζ=constant plane was discarded and an explicit marching scheme was substituted. However, an explicit method places limitations on the marching step size that are not always compatible with how the grid should be varied in the outward direction. Because the governing equations have been formulated with respect to a curvilinear coordinate system which no longer exists on the unstructured grid, a local coordinate system was defined for each point. Finally, differencing schemes and link-lists for the semiunstructured grid and pointwise local coordinates were coded.

A choice was made to difference the equations at node points (not cell centers), and the points were linked together as sketched in figure 3. On a ζ=constant surface, a local ξ, η coordinate for each grid point was defined and derivatives in ξ and η were formed using the four linked points. First-order-accurate, centered, difference approximations were used, and for a derivative with respect to ξ formed as (see notation, fig. 3)

$$f_\xi|_0 = \frac{f_{+1} - f_{-1}}{\Delta\xi_{+1} - \Delta\xi_{-1}}$$

with $\Delta\xi_{\pm1} = (\vec{r}_{\pm1} - \vec{r}_0) \cdot \hat{e}_\xi$ where $\hat{e}_\xi$ is the unit vector in ξ. Because central differencing is used in ξ and η, numerical dissipation is added by using some averaging of the four linked points.

An explicit method composed of a forward Euler predictor and followed by two applications of either backward Euler or trapezoidal rule correctors was used to difference and march the equations in ζ. This scheme is conditionally stable for central differencing of ξ and η terms for ordered grids, but the impact on stability due to the use of local coordinates has not been analyzed or severely tested.

Although all of the logic for treating unstructured grids was coded, ordered patched grids were used to test the grid generation scheme because general procedures to set up node connectivities and link lists were not available. Test calculations were carried out using either an ordered grid or, to stress the unstructured logic, patched grids were set up on a sphere by rotating segments of the grid and/or locally halving the

grid spacing. The two axes and the disjoint patched boundaries required the general link logic. Control volumes were chosen to simply radiate the spherical surface shell outward so an affine grid was generated.

TWO-DIMENSIONAL ELLIPTIC-HYPERBOLIC GRID GENERATION EQUATIONS

The hyperbolic grid generation equations represent conditions to control mesh size and orthogonality (skewness), while the conventional elliptic grid generation equations [9,10] represent conditions of averaging (or smoothing) an interior distribution of grid points to specified boundary-point distributions. A combination or blend of the two equation sets should give good control of the natural constraints usually imposed on a grid.

The elliptic and hyperbolic 2-D grid generation equations can be combined as

$$\nu(B^{-1}A\vec{r}_\xi + \vec{r}_\eta) = \nu B^{-1}\vec{f} + \mu(\alpha\vec{r}_{\xi\xi} - 2\beta\vec{r}_{\xi\eta} + \gamma\vec{r}_{\eta\eta}) \qquad (3)$$

where $\alpha = x_\eta^2 + y_\eta^2$, $\beta = x_\xi x_\eta + y_\xi y_\eta$, $\gamma = x_\xi^2 + y_\xi^2$, and

$$A = \begin{pmatrix} x_\eta & y_\eta \\ y_\eta & -x_\eta \end{pmatrix}, \quad B = \begin{pmatrix} x_\xi & y_\xi \\ -y_\xi & x_\xi \end{pmatrix}, \quad \vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \vec{f} = \begin{pmatrix} 0 \\ \Delta V + \Delta V \end{pmatrix}$$

If the parameter $\mu = 0$ the hyperbolic equations alone are recovered for marching in η; otherwise, the equations are elliptic.

Iterative solution algorithms such as Gauss Seidel and certain implicit approximate factorization schemes can be built to solve equation (3) such that when $\mu = 0$ a single-sweep hyperbolic scheme is recovered. For example a Gauss Seidel scheme (for marching outwards, $\nu > 0$) is given by

$$[\nu I + \nu B^{-1}A\delta_\xi - (\mu + \epsilon)\alpha\nabla_\xi\Delta_\xi + 2\mu\gamma I]\vec{r}_k^{n+1} =$$
$$+ \nu B^{-1}\vec{f} + (\nu + \mu\gamma)\vec{r}_{k-1}^{n+1} + \mu(-2\beta\delta_\xi\delta_\eta\vec{r}_k^n + \gamma E_\eta\vec{r}_{k+1}^n)$$

where α, β, γ, B and A are evaluated using $\vec{r}_{k-1}^{n+1}$ (if $\mu = 0$) and/or $\vec{r}_k^n$. The small positive parameter ϵ is used to maintain a smoothing term if $\mu = 0$, and $\Delta\eta = 1$ so $\eta = k$.

The hyperbolic terms (and perhaps the fortified approach discussed below) can give an efficient alternative to the usual control-function approach used with elliptic equations [9,10]. Figure 4 shows an example (developed by William R. Gutierrez) of using this kind of scheme where μ was set to zero near the body and rapidly increased so that away from the body the usual elliptic equation solely dominates.

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FORTIFIED HYPERBOLIC GRID GENERATION IN TWO DIMENSIONS

As a marching scheme, hyperbolic grid generation methods can break down if unrealistic conditions are imposed. For example, consider using a hyperbolic grid generation scheme to generate the interior of a cylindrical grid. As the method marches inward, it will simply break down as the singularity is approached. The hyperbolic grid generation scheme has sometimes broken down at airfoil sharp trailing edges because the constraints of controlled volume and orthogonality can be meaningless there. In each case, though, the singular grid solution is known and could be specified, and this information can be imposed on the solution algorithm by a simple fortified [11] approach.

For example, to fortify the 2-D hyperbolic difference equations, they are modified as

$$(1 + \chi)(\vec{r}_{k+1} - \vec{r}_k) + B_k^{-1} A_k \delta_\xi (\vec{r}_{k+1} - \vec{r}_k) = B_k^{-1} \begin{pmatrix} 0 \\ \Delta V \end{pmatrix} + \chi(\vec{r}_f - \vec{r}_k) \tag{4}$$

where the forcing term $\chi(\vec{r}_f - \vec{r}_{k+1})$ is appended. Here $\vec{r}_f$ is the specified solution near the singularity or breakdown, and $\chi \geq 0$ is a user-specified function of ξ and η. For $\chi >> 1$, $\vec{r}_{k+1} \to \vec{r}_f$, while if χ is set to 0 the algorithm is unaltered.

By specifying the grid-line points emanating out from the trailing edge and letting χ be large nonzero just along this line, the regular grid generation solution is neatly overwritten at this line with the specified solution. As the solution is marched out, the singularity disappears and χ can be allowed to approach zero. A grid generated in this way is shown in figure 5.

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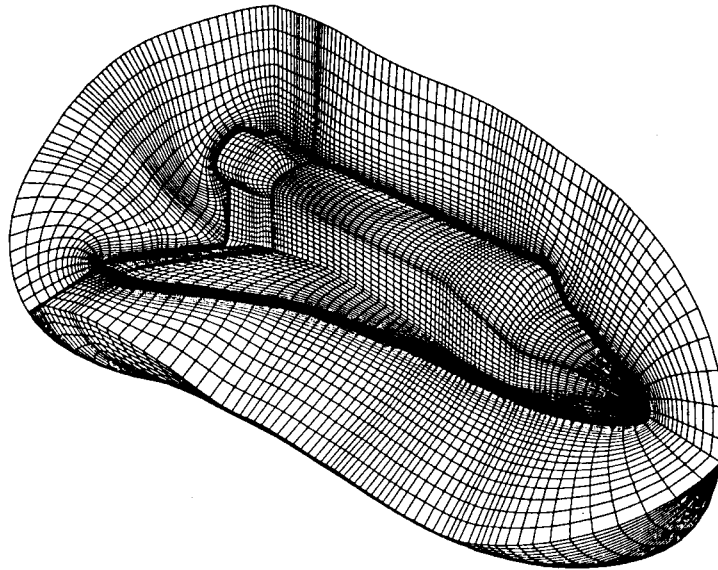
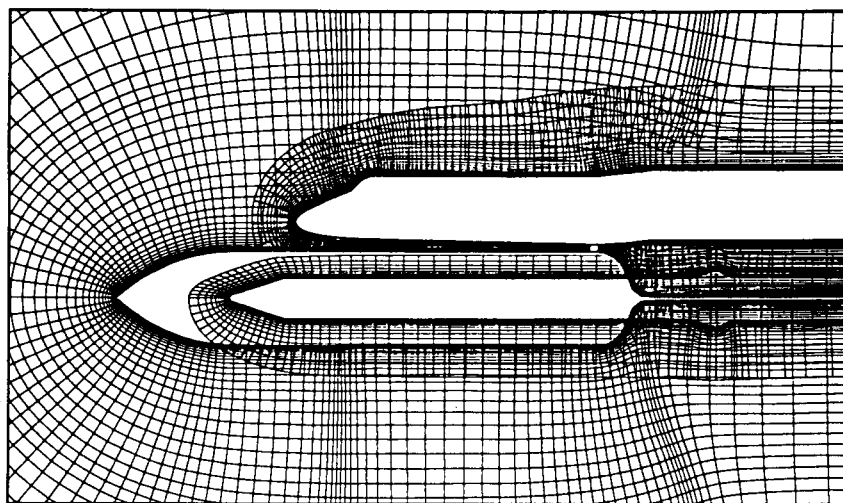Figure 1.– Various computational planes of the orbiter grid.

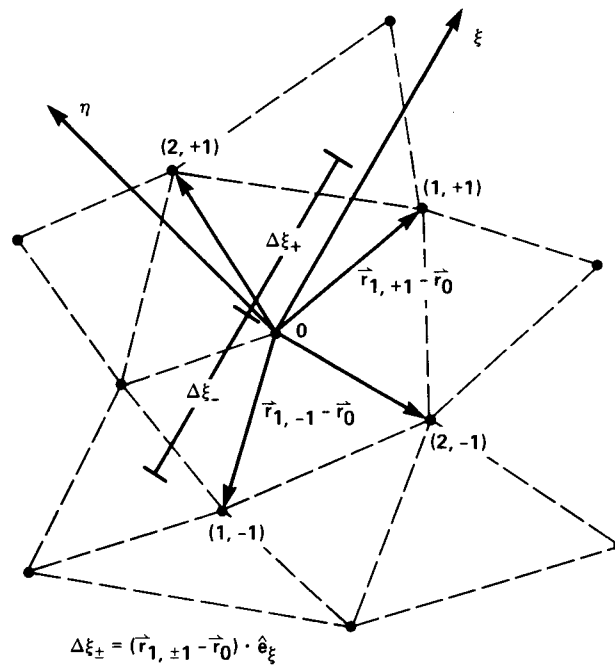Figure 2.– Symmetry planes of all grids.

$$\Delta\xi_\pm = (\vec{r}_{1,\,\pm1} - \vec{r}_0) \cdot \hat{e}_\xi$$
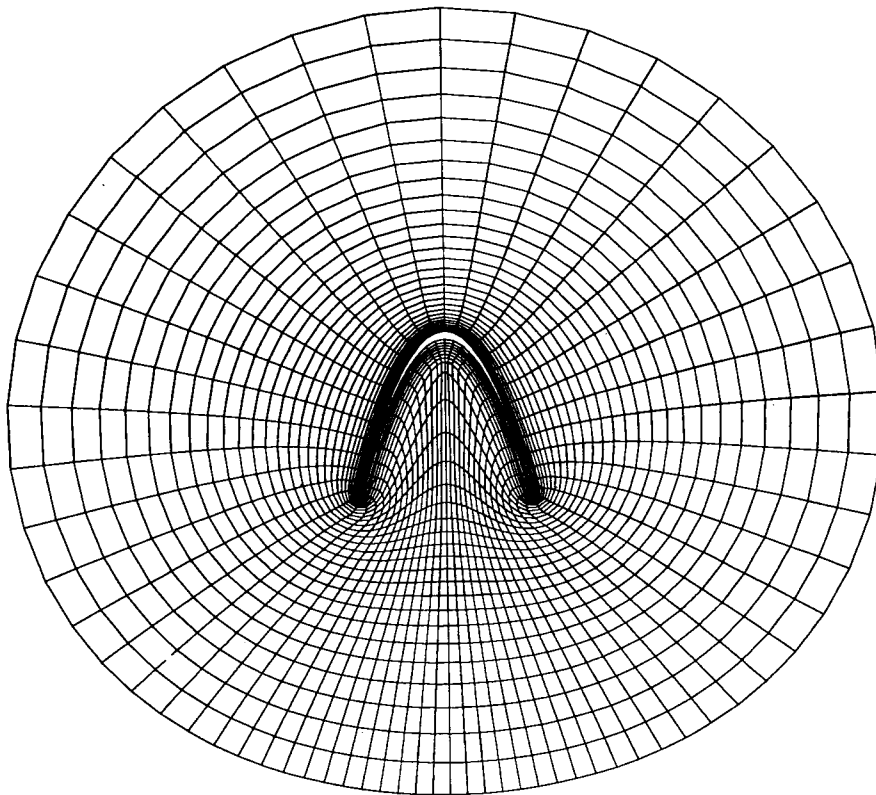
Figure 3.– Local differencing notation.

Figure 4.– Elliptic hyperbolic grid.
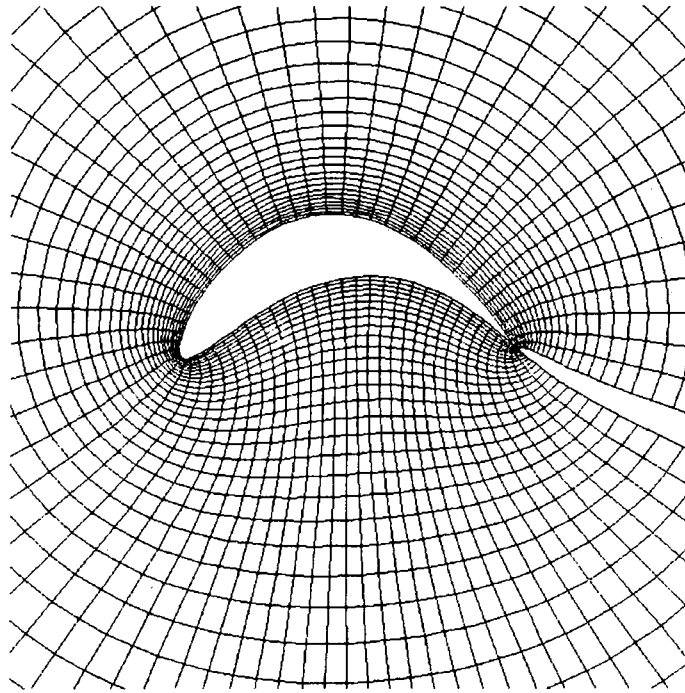
Figure 5.– Fortified hyperbolic grid (periodic points not connected).

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16. Abstract

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