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ON A CLASS OF NONSTATIONARY STOCHASTIC PROCESSES

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ABSTRACT. A new class of nonstationary stochastic processes is introduced and some of the essential properties of its members are investigated. This class is richer than the class of stationary processes and has the potential of modeling some nonstationary time series.

Let X_n be a second order stochastic process with mean zero and covariance

$$R(m,n) = EX_m\bar{X}_n.$$

A stochastic process X_n is called stationary if $R(m,n)$ depends only on m-n, i.e., if $R(m,n) = R(m+1,n+1)$, for all $m,n \in Z$. The process X_n is called periodically correlated with period T if

$$R(m,n) = R(m+T, n+T) \text{ for all } m,n \in Z.$$

As a natural extension of these well-known stochastic processes, a linearly correlated processes is defined to be one for which there exist scalars a_j such that

$$R(m,n) = \sum_j a_j R(m+j, n+j), \text{ for all } m,n \in Z.$$

The relation between these newly defined processes with other important classes of nonstationary processes is investigated. Several examples of linearly correlated processes which are not stationary, periodically correlated, or harmonizable are given.

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1. *INTRODUCTION*. The purpose of this paper is to introduce a new class of stochastic processes which contains all stationary and periodically correlated processes as well as some other interesting processes such as the Wiener process. As will be seen, this new class is a natural extension of the well studied class of stationary processes.

As we all know, a zero mean second order stochastic process X_n, $n \in Z$, is stationary if

$$R(m,n) = R(m+1, n+1); \text{ for all } m,n \in Z,$$

where $R(m,n) = EX_mX_n$ is the correlation function of the process X_n. In other words, for X_n to be stationary, $R(m+\tau,n+\tau)$ must be the same for all integers τ. Since for many time series in practice, $R(m+\tau,n+\tau)$ does not stay the same for all τ, it is then not appropriate to model them as a stationary process. However, for our new class, $R(m,n)$ does not have to be equal to $R(m+1,n+1)$ or $R(m+2,n+2)$, etc., but equal to a linear combination of these correlations. So some time series can be modeled more accurately as a member of this new class.

Recently, there has been a considerable interest in studying classes of nonstationary processes [2],[3],[4]. One of these classes which again extends another aspect of stationary processes in a very natural way is the class of harmonizable processes. For the definition and properties of harmonizable processes and some other classes of nonstationary processes, the reader is referred to the interesting exposition [1] and the references therein.

In section 2, we introduce our new class of nonstationary processes and give a few examples and some preliminary discussion regarding them. In section 3, we study some basic properties of this new class. In particular, we will look for their relations with the class of harmonizable processes and the support of their power spectra.

2. *LINEARLY CORRELATED AND CONVEXLY CORRELATED PROCESSES*.

In this section we introduce our new class of nonstationary stochastic processes which is a natural extension of the well-known classes of stationary and periodically correlated processes. We also recall some definitions and results which are needed throughout this paper.

The classes of linearly correlated and convexly correlated processes which we introduce shortly are much richer than that of stationary processes and hence can serve us better for the purpose of time series modeling. As an example, an important nonstationary stochastic process is the Wiener process W_n, $n \in Z$, whose correlation function is given by

$$R\,(m,n) = \begin{cases} \min(m,n); & m,n \geq 0 \\ 0 & ;\ \text{otherwise} \end{cases}$$

One can see that this important process is not even harmonizable (see Definition 2.7). Nevertheless, we will see that W_n is a linearly correlated process. In fact, although $R(m,n)$ is neither equal to $R(m-1,\ n-1)$ or $R(m+1,\ n+1)$, it is equal to their average, namely

$$R(m,n) = \tfrac{1}{2}\, R(m-1,n-1) + \tfrac{1}{2}\, R(m+1,n+1),$$

for all positive integers m and n. This then shows that the Wiener process W_n is in the new class (see Definition 2.1 and Remark 2.3).

Let (Ω,β,P) be a probability space and X_n, $n \in Z$, be a sequence of random variables on this probability space. Throughout, we assume our stochastic process X_n to be of second order, namely $E|X_n|^2 < \infty$ for all n, and to have zero mean, namely $EX_n = 0$ for all n.

Now, let's introduce our new class of nonstationary processes.

2.1 DEFINITION. Let X_n, $n \in Z$, be a zero mean second order stochastic process and let S be a subset of Z. The process X_n is called linearly correlated on S if there exists a positive integer p and some complex numbers a_j, $j=0, \pm 1, \pm 2, \ldots, \pm p$ such that

$$(2.2) \qquad R(m,n) = \sum_{|j|=1}^{p} a_j \, R(m+j, n+j), \qquad \text{for all } m, n \in S.$$

2.3 EXAMPLES. As we know, a process X_n is called stationary if its correlation satisfies the relation.

$$R(m,n) = R(m+1, n+1), \qquad \text{for all } m,n \in Z,$$

and hence (2.2) with $p=1$, $a_{-1} = 0$ and $a_1 = 1$. So any stationary process is linearly correlated on Z.

(b) A process X_n is called periodically correlated with period q if its correlation function satisfies

$$R(m,n) = R(m+q, n+q), \qquad \text{for all } m,n \in Z,$$

and hence (2.2) with $p=q$, $a_q = 1$ and all the other a_j equal to 0. So all periodically correlated processes are linearly correlated on Z.

(c) The Wiener process W_n mentioned before is linearly correlated on the set of positive integers. This is because one can easily check that

$$(2.4) \qquad R(m,n) = \tfrac{1}{2} R(m+1,n+1) + \tfrac{1}{2} R(m-1,n-1), \text{ for all } m,n > 0.$$

In fact, if m,n are two positive integers with say $m \geq n$, then (2.4) reduces to

$$n = \tfrac{1}{2} (n+1) + \tfrac{1}{2} (n-1)$$

which is clearly true.

(d) Let X_n be a stationary process and a be any complex number. Consider the stochastic process

$$Y_n = a^n X_n.$$

Then one can show that Y_n is periodically correlated on the set of all integers. In fact, one can see that the correlation function of Y_n satisfies

$$R(m,n) = \frac{1}{|a|^2+1} \quad R\ (m+1,\ n+1) + \frac{|a|^2}{|a|^2+1}\ R\ (m-1,\ n-1)$$

for all $m,n \in Z$.

(e) Let X_n be a stationary stochastic process and consider the process Y_n defined by

$$Y_n = nX_n.$$

Then one can easily check that

$$R(m,n) = 3R(m+1,n+1) - 3R(m+2,n+2) + R(m+3,n+3), \text{ for all } m,n \in Z.$$

Hence Y_n is linearly (even convexly) correlated.

Notes. (a) The linearly correlated process of example (d) is not periodically correlated unless $|a| = 1$. This is because in this example, $E|X_n|^2$ goes to ∞ with n. While for a periodically correlated process, these quantities stay bounded.

(b) One can find other examples by replacing the stationary process of examples (d) and (e) by a periodically correlated process.

An important class of non-stationary processes is that of harmonizable processes. It is thus essential to study the relations (if any) that exist between our classes of linearly correlated and convexly correlated processes introduced here and the class of harmonizable processes. We postpone this study until the next section. However, here we would like to recall the definition of harmonizable processes as well as some of their properties that we will need in the next section. For more detail on these, see [1], [5].

2.7 DEFINITION. A process X_n is called strongly harmonizable if its correlation function can be represented as

(2.8) $$R(m,n) = \int_{I^2} e^{-i(m\lambda - n\theta)}\ dF(\lambda,\theta),$$

where F is a complex-valued measure on the unit interval $I^2 = [0,2\pi] \times [0,2\pi]$

which is of finite Vitali variation. It is called weakly harmonizable if one has the relation (2.8) with F being only a bimeasure with finite Frechet variation on I^2. In either of those cases, F is called the spectral measure of the process X_n.

2.9 DEFINITION. Let X_n be a second order process, so that the X_n's belong to $H=L^2(\Omega,\beta,P)$. X_n is said to have stationary dilation if one can find a larger Hilbert space $K \supseteq H$ and a stationary process $Y_n \in K$ such that

$$X_n = \underline{P}Y_n; \text{ for all } n\in Z,$$

where $\underline{P}$ is the orthogonal projection of K onto H.

The following useful result is now well-known. (cf. [1], [5], [6])

2.10 THEOREM. A second order process X_n is weakly harmonizable if and only if it has a stationary dilation.

Finally, it is well-known that any stationary process is strongly harmonizable and its spectral measure is concentrated only on the diagonal $D=\{(\lambda,\theta): \lambda=\theta\}$ of I^2. It is also known (see [4]) that every periodically correlated process with period q is also strongly harmonizable. However, in this case, the spectral measure is supported on 2q-1 equidistant straight lines parallel to the main diagonal of I^2, namely on the lines

$$D_k = \{(\lambda,\theta); \lambda=\theta + \frac{2\pi k}{q}\}, \qquad k = 0,\pm 1,\dots,\pm(q-1)$$

3. *LINEARLY AND CONVEXLY CORRELATED PROCESSES AND THEIR HARMONIZABILITY*. In this section, we will study the relation between our newly introduced linearly and convexly correlated processes and the harmonizable processes. As we have shown, stationary and periodically correlated processes are both linearly correlated and harmonizable. Thus, the following questions are natural to ask and reasonable answers to them seem essential for any further study of the linearly correlated processes.

Question 1. Are all linearly correlated processes strongly harmonizable?

Question 2. For those linearly correlated processes which are strongly harmonizable, what is the support of their spectral measure?

The answer to Question 1 is negative: To see this, consider the linearly correlated process $Y_n = 2^n X_n$ of Example 2.6. If this process were strongly harmonizable, then

$$R(m,n) = \int_{I^2} e^{-i(m\lambda - n\theta)}\, dF(\lambda,\theta),$$

for some spectral measure F which is of finite Vitali variation. Thus

$$|R_y(m,n)| \le \int_{I^2} |e^{-i(m\lambda-n\theta)}|\, d\|F\|(\lambda,\theta)$$

and hence

$$|R_y(m,n)| \le \|F\|(I,I) < \infty$$

for all $m,n \in Z$, i.e. $R_y(m,n)$ must be bounded which clearly contradicts the fact that $R_y(n,n) = 2^{2n} R_x(0,0)$ goes to ∞ with n.

This then closes Question 1, but instead raises the following:

Question 3. Is there any linearly correlated process, other than the stationary and periodically correlated processes, which is weakly harmonizable?

Question 4. How about if we replace weakly with strongly in Question 3?

In this section, we first address Question 2 furnishing a complete answer to that (see Theorem 3.1) and then prove some other related results. At the end of this section, we give a partial answer to Question 3 (see Theorem 3.11) by proving a dilation result (see Theorem 3.7) as well as giving an example showing the answer to Question 4 is on the positive side (see Example 3.12).

The following theorem gives a complete description of the support of the spectral measure of linearly correlated stochastic processes.

3.1 THEOREM. A strongly harmonizable stochastic process X_n is linearly correlated if and only if its spectral measure is concentrated on a finite number of straight lines parallel to the main diagonal of I^2 (not necessarily equidistance).

PROOF. Suppose that the spectral measure F of a stochastic process X_n is concentrated on say k straight lines parallel to $D = \{(\lambda,\theta): \lambda=\theta\}$, so one can find real numbers d_j, $j = 1, 2, \ldots, k$ such that the support of F is on the set $\bigcup_{j=1}^{k} \{(\lambda,\theta): \theta = \lambda + d_j\}$. Since the function $\prod_{j=1}^{k} (e^{-i(\lambda-\theta)} - e^{id_j})$ vanishes on the support of F, we can write

$$(3.2) \qquad \int_{I^2} e^{-i(\lambda m-\theta n)} \prod_{j=1}^{k} (e^{-i(\lambda-\theta)} - e^{id_j})\, dF(\lambda,\theta)=0 \text{ for all } m,n\in Z.$$

Multiplying the product in (3.2) out, we get

$$\int_{I^2} e^{-i(\lambda m-n\theta)} (\sum_{j=0}^{k} c_j(e^{-i(\lambda-\theta)j})\, dF(\lambda,\theta)=0, \text{ for all } m,n\in Z.$$

This means

$$\sum_{j=0}^{k} c_j \; R(m+j,\, n+j) = 0, \quad \text{for all } m,n\in Z,$$

which clearly means that our process X_n is linearly correlated. Conversely, suppose X_n is a linearly correlated process which is strongly harmonizable. So we have

$$R(m,n) = \sum_{j=0}^{k} a_j \; R(m+j,\, n+j); \quad \text{for all } m,n\in Z.$$

Without loss of generality, we can take $a_j=0$ for all negative j's. So, for all $m,n\in Z$, we can write

$$\int_{I^2} e^{-i(m\lambda-n\theta)}\, dF(\lambda,\theta) = \sum_{j=0}^{k} a_j \int_{I^2} (e^{-i(m\lambda-n\theta)}\, e^{-ij(\lambda-\theta)}\, dF(\lambda,\theta),$$

or

$$\int_{I^2} e^{-i(m\lambda-n\theta)}\, [1- \sum_{j=0}^{k} a_j\, e^{-ij(\lambda-\theta)}]\, dF(\lambda,\theta)=0.$$

This means that $1- \sum_{j=0}^{k} a_j\, e^{-ij(\lambda-\theta)}=0$ for a.e. (λ,θ) with respect to the measure dF. Thus, the support of F must be included in the set $\{(\lambda,\theta)\in I^2\colon \sum_{j=1}^{k} a_j\, e^{-ij(\lambda-\theta)} =1\}$ for $\lambda=\theta$. Solving the equation $\Sigma\, a_j\, e^{-ij(\lambda-\theta)}=1$ for $\lambda-\theta$ and listing its distinct real solutions by d_1, d_2, $\ldots d_\ell$, we conclude that the support of F is included in the set

$$\bigcup_{j=1}^{\ell} \{(\lambda,\theta)\colon \lambda-\theta = d_j\}$$

consisting of ℓ straight lines parallel to the main diagonal of I^2.

The last theorem gives a complete description for the support of the spectral measure of a harmonizable linearly correlated process. Its support, as in the case of stationary and periodically correlated processes, is on a finite number of straight lines parallel to the main diagonal. However, now these lines do not have to be equidistant.

3.3 REMARK. One can easily check that for strongly harmonizable stochastic processes every linearly correlated process is convexly correlated. In fact, if we have a linearly correlated process which is strongly harmonizable, then as we saw in the second half of the proof, its spectral measure is concentrated on $\bigcup_{j=1}^{\ell} \{(\lambda,\theta)\colon \lambda-\theta = -d_j\}$. So

$$\prod_{j=0}^{\ell} (e^{-i(\lambda-\theta)} - e^{id_j}) = 0 \qquad \text{a.e. } dF$$

where $d_0 = 0$. This means that

$$\int_{I^2} -i(m\lambda - n\theta) \prod_{j=0}^{\ell} (e^{-i(\lambda-\theta)} - e^{id_j}) = 0, \qquad \text{for all } m,n \in Z.$$

Expanding the product and writing the resulting equation in terms of the correlation function, we get

$$\sum_{j=0}^{\ell+1} b_j \, R\,(m+j,n+j) = 0; \qquad \text{for all } m,n \in Z,$$

where $b_{\ell+1} = 1$ and the other b_j's come from

$$\sum_{j=0}^{\ell+1} b_j \, e^{-i(\lambda-\theta)j} = (e^{i(\lambda-\theta)} - 1) \prod_{j=1}^{\ell} (e^{i(\lambda-\theta)} - e^{id_j})$$

If we put 0 for $\lambda-\theta$ on both sides of (3.4), we get

$$1 + \sum_{j=0}^{1} b_j = 0 \qquad \text{or} \qquad \sum_{j=0}^{1} (-b_j) = 1$$

This fact together with (3.5) proves our claim.

Let's call a process positively convexly correlated if it is convexly correlated and if its correlation function satisfies

3.5 $\left\{ R(m,n) = \sum_{|j|=1}^{p} a_j \, R(m+j,n+j), \text{ with } \sum_{|j|=1}^{p} a_j = 1 \text{ and } a_j \geq 0 \text{ for all } j. \right.$

The following result is somewhat surprising.

3.6 THEOREM. The only positively convexly correlated processes which are strongly harmonizable are the periodically correlated ones.

PROOF. Let X_n be a positively convexly correlated process satisfying (3.5). Suppose also that X_n is strongly harmonizable with spectral measure F. As in the proof of Theorem 3.1, we see that the support of F is included in the set

$$\left\{ (\lambda,\theta): \quad \sum_{|j|=1}^{p} a_j \, e^{-ij(\lambda-\theta)} = 1 \right\}$$

which can be rewritten as

$$\left\{ (\lambda,\theta): \sum_{|j|=1}^{p} a_j \cos j(\lambda-\theta)=1 \text{ and } \sum_{|j|=1}^{p} \operatorname{sgn} j \; a_j \sin j\,(\lambda-\theta) = 0 \right\}.$$

Now, since all $a_j \geq 0$ and $\Sigma a_j = 1$, one can see that $\sum_{|j|=1}^{p} a_j \cos j(\lambda-\theta)=1$ is possible only if $\cos j(\lambda-\theta)=1$ whenever $a_j + a_{-j} \neq 0$. So, letting $S = \{j: a_j + a_{-j} \neq 0\}$, we conclude that the support of F is included in the set $\bigcap_{j\in S} \{(\lambda,\theta): \cos j(\lambda-\theta)= 1\}$. Let q be a common divisor for the integers in S. Then one can see

$$\bigcap_{j\in S} \{(\lambda,\theta): \quad \cos j(\lambda-\theta)= 1\} \subseteq \bigcup_{n=-q+1}^{q-1} \left\{(\lambda,\theta): \lambda=\theta + \frac{2\pi n}{q}\right\}$$

Thus, F is concentrated on 2q-1 lines

$$\left\{(\lambda,\theta): \lambda = \theta + \frac{2\pi n}{q}\right\}, \qquad n=0, \pm 1, \ldots, \pm (q-1).$$

This in turn means that

$$\int_{I^2} e^{-i(m\lambda-n\theta)} \left(e^{iq(\lambda-\theta)}-1\right) dF(\lambda,\theta)=0, \qquad \text{for all } m,n.$$

This, rewritten in terms of the correlation function gives

$$R(m,n) = R(m+q,n+q), \qquad \text{for all } m,n\in Z.$$

Thus, X_n must be periodically correlated with period q.

Now, we turn to Question 3. However, before that we prove the following dilation theorem which is crucial for our development. This theorem is of course of independent interest. This not only shows that a class of convexly correlated processes have a stationary dilation, but also gives the precise form of their stationary dilation processes.

We say a process is one-sided positively convexly correlated if its correlation function satisfies

$$R(m,n) = \sum_{j=-p}^{-1} a_j \, R(m+j, n+j), \tag{3.7}$$

with $a_j \geq 0$ and $\sum_{j=-p}^{-1} a_j = 1$.

3.7 THEOREM. Any one-sided positively convexly correlated process has a stationary dilation.

PROOF. Let $Y_n = [c_{-p+1}X_{n-p+1}, c_{-p+2}X_{n-p+2}, \ldots c_0 X_n]$ be a vector random process with values in the Hilbert space H^p, with $c_0 = 1$ and, for $-p+1 \leq j \leq -1$, c_j is any complex number satisfying

$$|c_j|^2 = \sum_{k=-p}^{j-1} a_k \tag{3.9}$$

It is quite clear that

$$X_n = \underline{P}\, Y_n; \qquad \text{for all } m,n \in Z,$$

where $\underline{P}: H^q \to H$ is the orthogonal projection sending each element of H^q to its first component. So we must show that Y_n is actually stationary. To show this, we need to check that

$$(Y_n, Y_m) = (Y_{n-1}, Y_{m-1}).$$

To see this, consider the expression

$$(Y_{n-1}, Y_{m-1}) - (Y_n, Y_m) =$$

$$\sum_{j=1}^{p} |c_{-p+j}|^2 \, R(n-p-1+j,\ m-p-1+j) \; - \sum_{j=1}^{p} |c_{-p+j}|^2 \, R(n-p+j,\ m-p+j)$$

$$= |c_{-p+1}|^2 R(n-p,\ m-p) + \sum_{j=2}^{p} |c_{-p+j}|^2 R(n-p-1+j,\ m-p-1+j)$$

$$- \sum_{j=1}^{p-1} |c_{-p+j}|^2 R(n-p+j,\ m-p+j) - |c_0|^2 R(n,m)$$

$$= |c_{-p+1}|^2 R(n-p,\ m-p) + \sum_{j=1}^{p-1} |c_{-p+j+1}|^2 R(n-p+j,\ m-p+j)$$

$$- \sum_{j=1}^{p-1} |c_{-p+j}|^2 R(n-p+j,\ m-p+j) - |C_0|^2 R(n,m)$$

$$(3.10) \quad = |c_{-p+1}|^2 R(n-p,\ m-p) + \sum_{j=1}^{p-1} \left[|c_{-p+j+1}|^2 - |c_{-p+j}|^2\right] R(n-p+j,\ m-p+j) - |c_0|^2 R(n,m).$$

Now, from (3.8) we see that

(i) $|c_o|^2 = 1$

(ii) $|c_{-p+1}|^2 = \sum_{k=-p}^{-p} a_k = a_{-p}$,

(iii) $|c_{-p+j+1}|^2 - |c_{-p+j}|^2 = \sum_{k=-p}^{-p+j} a_k - \sum_{k=-p}^{-p+j-1} = a_{-p+j}$, for $1 \le j \le p-2$

(iv) Adding the equations in (ii) to those in (iii), we get

$$|c_{-1}|^2 = \sum_{k=-p}^{-2} a_k = \sum_{k=-p}^{-1} a_k - a_{-1} = 1 - a_{-1},$$

which gives the coefficient $|c_{-p+j+1}|^2 - |c_{-p+j}|^2$ in (3.9), for $j = p-1$ to be

$$|c_0|^2 - |c_{-1}|^2 = 1 - (1-a_{-1}) = a_{-1}.$$

Substituting from (i)-(iv) in (3.10), we get

$$(Y_{n-1}, Y_{m-1}) - (Y_n, Y_m) = \sum_{j=-p}^{-1} a_j R(n+j), m+j) - R(m,n)$$

which is zero by (3.7). Q.E.D.

We can now prove the following which provides a partial positive answer to Question 3.

3.11 THEOREM. Every one-sided positively convexly correlated stochastic process X_n is weakly harmonizable.

PROOF. Since by Theorem 3.8 the process X_n has a stationary dilation. The proof can be completed by appealing to the main Theorem of [5] which says: A stochastic process is weakly harmonizable if and only if it has stationary dilation.

In connection with last result, one may ask whether linearly correlated processes are weakly harmonizable. The answer is no. For instance, the examples (d) and (e) mentioned in section 2 can not be weakly harmonizable. Because any weakly harmonizable process having a stationary dilation should have a bounded sequence of variances and this was not the case for these examples (as we noted before).

The following example shows the answer to Question 4 on the positive side.

3.12 EXAMPLE. Let $Z(\cdot)$ be an orthogonally scattered stochastic measure on $[0,2\pi]$ and define a new stochastic measure $Y(\cdot)$ on $[0,2\pi]$ by

$$Y(\Delta) = Z(\Delta\cap[0,2\pi]) + Z(\Delta\cap[0,2\pi]-\sqrt{2})$$

where $B-a = \{b-a:\in B\}$. Consider the strongly harmonizable processes

$$X_n = \int e^{-in\theta}dY(\theta)$$

One can easily see that the spectral measure of X_n is concentrated of the diagonal $\lambda=\theta$ and lines $\lambda-\theta=\pm\sqrt{2}$ of I^2. By theorem 3.1 X_n is linearly correlated and it is clearly not stationary or even periodically correlated.

Another important feature of stationary processes which turns out to be extremely useful is the existence of a unitary shift. However, when it comes to nonstationary processes, even the existence of a shift operator is not always guaranteed [3]. So, it is nice to know when a nonstationary process does have a well-defined shift and if it does have one whether it is bounded.

We conclude this work with the following result which examines the existence of shift operator and its boundedness for a positively linearly correlated process.

3.13 THEOREM. Let X_n be a positively convexly correlated process whose correlation function satisfies (3.5). If $a_1 > 0$, then X_n has a bounded shift. If $a_{-1} > 0$ too, then its shift is boundedly invertible as well.

PROOF. Take any finite linear combination $\Sigma\, b_k X_k$ of X_k's, then we can write

$$\Big\| \sum_k b_k X_k \Big\|^2 = \sum_{k,k'} b_k b_{k'} \,(X_k, X_{k'})$$

$$= \sum_{k,k'} b_k b_{k'} \left[\sum_{|j|=1} a_j \,(X_{k+j}, X_{k'+j}) \right]$$

$$= \sum_{|j|=1}^{p} a_j \left[\sum_{k,k'} b_k b_{k'} \,(X_k, X_{k'}) \right] = \sum_{|j|=1}^{p} a_j \,\Big\| \sum b_k X_{k+j} \Big\|^2$$

This implies that

$$\Big\| \sum_k b_k X_k \Big\| \geq \sqrt{a_1}\, \Big\| \sum_k b_k X_{k+1} \Big\| \tag{3.14}$$

Now if $a_1 > 0$, it is quite clear from (3.14) that the shift operator defined via

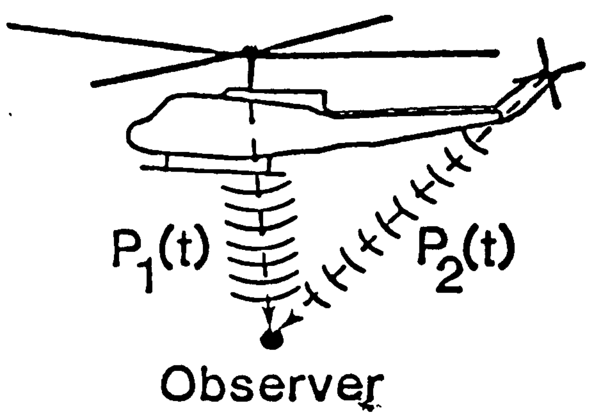
$$U\,(\Sigma b_k X_k) = \Sigma\, b_k X_{k+1}$$

is well defined and bounded. This completes the proof of the first part. The second part can be proven similarly using the inequality:

$$\| \Sigma b_k X_k \| \geq \sqrt{a_{-1}}\, \| \Sigma\, b_k X_{k-1} \|$$

3.15 REMARK. In this present work we have only considered discrete time processes. We are planning to study continuous time linearly correlated processes in future. It looks like they have nice applications in engineering. For example, in analyzing the helicopter noise where the sound reaching an observer consist of two periodically correlated random noise processes generated by the main and tail rotors. Ordinarily, the periods of these two signals are incommensurate. One can see that the helicopter noise is then not periodically correlated. However, this continuous time processes turns out to be linearly correlated and harmonizable. This application is illustrated in the following figure.

HELICOPTER NOISE FIELD



$$P(t) = P_1(t) + P_2(t)$$

REFERENCES

1. Chang, D.K. and Rao, M.M., Bimeasures and nonstationay processes in Real and Stochastic Analysis, (M.M. Rao, Ed.) Wiley, New York, 1986, 7-118.

2. Cramer, H. (1962). On some class of nonstationary stochastic processes, proc. IV Berkeley Sym. Math. Stat. Prob. 2, 57-78, Univ. Calif. Press.

3. Getoor, R.K. (1956). The shift operator for nonstationary stochastic processes. Duke Math. J. 23, 175-187.

4. Gladyshev, E.G. (1961). Periodically correlated random sequences, Soviet mathematics ,2, 385-388.

5. Miamee, A.G. and Salehi, H. (1978). Harmonizability, V-boundedness and stationary dilation of stochastic processes, Indiana Univ. Math. J. 27, 37-50.

6. Niemi, H. (1957). Stochastic process as Fourier transformation of stochastic measures. Ann. Acad. Sci. Fenn. AI Math. 591, 1-47.