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SUMMARY

A single elastic fiber embedded in an infinite elastic matrix is considered. An incident plane SH wave is assumed in the infinite matrix, and an expression is derived for the total energy in the fiber due to the incident SH wave. A nondimensional form of the fiber energy is plotted as a function of the nondimensional wavenumber of the SH wave. It is shown that the fiber energy attains maximum values at specific values of the wavenumber of the incident wave. The results obtained here are interpreted in the context of phenomena observed in acousto-ultrasonic experiments on fiber reinforced composite materials.

1

INTRODUCTION

Quantitative ultrasonic nondestructive evaluation methods require analyses of wave propagation in the material or structure which is to be evaluated. In fiber reinforced composite materials, wave propagation characteristics depend on properties and arrangement of the constituent materials, as well as on the overall geometry and both the microscopic and macroscopic defect state of the composite.

Analyses of wave propagation in fiber reinforced composites are reviewed in [1] and [2]. According to these reviews, wave propagation analyses may be divided into two broad categories. In the first category, the composite is treated as an equivalent homogeneous medium, and mechanical responses such as stress and strain are described in terms of averages over the constitutive elements. It has been stated that the homogeneous or equivalent continuum approach is valid "when the scale of the changes in stress level (rise distance, wavelength, etc.) is much larger than the sizes of the constituents of the composite (fiber or particle diameter, fiber spacing, ply spacing, etc.)" [1]. In the second category of wave propagation analyses, the composite is modeled as a heterogeneous mixture of constituents, and the dynamic interaction between the constituent materials is considered. The heterogeneous analyses are complex, but are necessary if the scale of changes in stress level is smaller than or of the same order as the size of the composite constituents, or if small-scale phenomena such as interface bond failure are of interest.

Recent acousto-ultrasonic experiments by Kautz [3] show some interesting wave propagation behavior in graphite fiber reinforced epoxy composite panels. Kautz introduces a broadband signal into a composite panel, and observes that the response of the composite appears to be divided into a high frequency component traveling at a relatively fast wave speed and a low frequency component traveling at a relatively low wave speed. When the same broadband signal is introduced into a geometrically similar panel consisting only of the epoxy matrix, the response is observed to consist only of the low frequency component. Kautz hypothesizes that the composite panel acts as a heterogeneous medium, with wave propagation along the fibers at the higher wave speed and wave propagation through the matrix at the lower wave speed.

In this report, an infinite elastic matrix containing a single cylindrical elastic fiber is considered. An incident plane SH wave is assumed to exist in the elastic matrix, and

an expression is derived for the total energy of the fiber due to the incident SH wave. It is shown that the total fiber energy attains maximum values at specific values of the wavelength of the incident wave. The purpose of this work is to investigate, in a simple setting, the issue of energy transfer between the matrix and the fibers in a composite material. Recommendations are given for future research which may give further insight into the experimentally observed phenomena in [3].

ANALYSIS

A cylindrical fiber having circular cross section is shown in Fig. 1. Rectangular coordinates (x, y, z) and cylindrical coordinates (r, θ, z) are defined as shown in Fig. 1. Displacements u_x, u_y and u_z are defined in the x, y and z directions, respectively, and displacements u_r, u_θ and u_z are defined in the r, θ and z directions, respectively. The fiber is assumed to have radius a and to be of infinite extent in the z direction. The fiber is assumed to be elastic, with mass density ρ_f and Lamé constants μ_f and λ_f. It is assumed that the fiber is embedded in an infinite elastic matrix with mass density ρ and Lamé constants μ and λ. Continuity of relevant stresses and displacements is assumed at the materials interface $r = a$.

The infinite elastic matrix is assumed to contain an incident harmonic plane wave of the form

$$u_x^{(i)}(x, y, z, t) = 0 \tag{1}$$

$$u_y^{(i)}(x, y, z, t) = 0 \tag{2}$$

$$u_z^{(i)}(x, y, z, t) = Re\left\{ w_0 e^{j(kz - \omega t)} \right\} \tag{3}$$

where the superscript (i) denotes the incident wave, w_0 is the amplitude of the incident wave, k is the wavenumber of the incident wave, ω is radian frequency, t is time and $j = \sqrt{-1}$. The wavenumber k and frequency ω are related by

$$\omega = c_s k \tag{4}$$

where

$$c_s = \left(\frac{\mu}{\rho} \right)^{1/2} . \tag{5}$$

3

Eqns. (1), (2) and (3) define a wave which propagates in the positive x direction with particle motion parallel to the fiber in the z direction. Such a wave, in this context, is denoted as an SH wave.

Because of the cylindrical geometry of the fiber, it is convenient to write the incident SH wave in the cylindrical coordinates (r, θ, z) as [4]

$$u_r^{(i)}(r, \theta, z, t) = 0 \tag{6}$$

$$u_\theta^{(i)}(r, \theta, z, t) = 0 \tag{7}$$

$$u_z^{(i)}(r, \theta, z, t) = Re\left\{ w_0 \sum_{n=0}^{\infty} \epsilon_n j^n J_n(kr) \cos n\theta e^{-j\omega t} \right\} \tag{8}$$

where J_n is a Bessel function of the first kind and of order n, and where

$$\epsilon_n = \begin{cases} 1, & n=0 \\ 2, & n=1,2,\ldots \end{cases} \tag{9}$$

It is shown in [4] that eqns. (1) through (3) and eqns. (6) through (8) are mathematically identical representations of the same incident wave.

The interface conditions at $r = a$ and the three-dimensional equations of elasticity which govern the infinite medium and the fiber are satisfied by the total displacement field [4]

$$u_r(r, \theta, z, t) = 0 \tag{10}$$

$$u_\theta(r, \theta, z, t) = 0 \tag{11}$$

$$u_z(r, \theta, z, t) = u_z^{(i)}(r, \theta, z, t) + u_z^{(s)}(r, \theta, z, t) \tag{12}$$

$\left. \right\} r \geq a$

$$u_r(r, \theta, z, t) = 0 \tag{13}$$

$$u_\theta(r, \theta, z, t) = 0 \tag{14}$$

$$u_z(r, \theta, z, t) = u_z^{(f)}(r, \theta, z, t) \tag{15}$$

$\left. \right\} r \leq a$

where

$$u_z^{(s)}(r, \theta, z, t) = Re\left\{ w_0 \sum_{n=0}^{\infty} A_n H_n^{(1)}(kr) \cos n\theta \, e^{-j\omega t} \right\} \tag{16}$$

$$u_z^{(f)}(r, \theta, z, t) = Re\left\{ -w_0 \sum_{n=0}^{\infty} C_n J_n(k_f r) \cos n\theta e^{-j\omega t} \right\} \tag{17}$$

4

$$A_n = (-j^n \epsilon_n / \Delta) \left[\mu_f k_f J_n(ka) J_n'(k_f a) - \mu k J_n'(ka) J_n(k_f a) \right] \tag{18}$$

$$C_n = (-j^n \epsilon_n / \Delta) \, \mu k \left[J_n'(ka) H_n^{(1)}(ka) - H_n^{(1)'}(ka) J_n(ka) \right] \tag{19}$$

$$\Delta = \mu_f k_f H_n^{(1)}(ka) J_n'(k_f a) - \mu k H_n^{(1)'}(ka) J_n(k_f a) \tag{20}$$

$$k_f = \frac{c_s}{c_{sf}} k \tag{21}$$

$$c_{sf} = \left(\frac{\mu_f}{\rho_f} \right)^{1/2} \tag{22}$$

The quantities A_n and C_n are dimensionless complex coefficients. $H_n^{(1)}$ is a Hankel function of the first kind and of order n, and the prime denotes the derivative of a function with respect to its argument.

Eqn. (16) represents a scattered wave which propagates outward from the fiber. As $r \rightarrow \infty$, the amplitude of the scattered wave approaches zero, so that far from the fiber the total displacement field approaches the displacement field of the incident SH wave.

Eqn. (17) represents a refracted or transmitted wave in the fiber. Eqn. (17) is a standing wave with time-harmonic particle motion in the z direction. Over any fiber cross section $z =$ constant, the fiber displacement in the z direction varies as a function of r and θ according to eqn. (17).

The nonzero fiber strains associated with the displacements given by eqns. (13) through (15) are

$$\gamma_{\theta z} = \frac{1}{r} \frac{\partial u_z^{(f)}}{\partial \theta} \tag{23}$$

$$\gamma_{rz} = \frac{\partial u_z^{(f)}}{\partial r} \tag{24}$$

and the nonzero stresses in the fiber are

$$\sigma_{\theta z} = \mu_f \frac{1}{r} \frac{\partial u_z^{(f)}}{\partial \theta} \tag{25}$$

$$\sigma_{rz} = \mu_f \frac{\partial u_z^{(f)}}{\partial r} \tag{26}$$

The elastic strain energy U_f per unit length of fiber in the z direction is

$$U_f = \int_0^a \int_0^{2\pi} \frac{1}{2} \left(\sigma_{rz} \gamma_{rz} + \sigma_{\theta z} \gamma_{\theta z} \right) r \, d\theta dr \tag{27}$$

5

The kinetic energy K_f per unit length of fiber in the z direction is

$$K_f = \int_0^a \int_0^{2\pi} \frac{1}{2} \rho_f \left(\dot{u}_z^{(f)} \right)^2 r \, d\theta dr \tag{28}$$

The total energy E_f per unit length of fiber is, therefore,

$$E_f = U_f + K_f \tag{29}$$

The energy E_f given by eqn. (29) is a function of time. In order to obtain a single numerical measure of energy, it is customary to average the energy E_f over one time period of the harmonic oscillation. The time-averaged energy $< E >_f$ is defined by

$$< E >_f = \frac{1}{T} \int_0^T E_f \, dt \tag{30}$$

where

$$T = \frac{2\pi}{\omega} \tag{31}$$

When eqn. (17) and eqns. (23) through (29) are substituted into eqn. (30), a long algebraic calculation gives the following result for the time-averaged fiber energy:

$$< E >_f = \frac{\pi}{4} \mu_f w_0^2 \sum_{n=0}^{\infty} \left\{ C_n^* C_n \frac{1}{\beta_n} \left[\frac{(k_f a)^2}{2} \left\{ J_{n+1}^2(k_f a) - J_n(k_f a) J_{n+2}(k_f a) \right. \right. \right.$$
$$\left. + J_n^2(k_f a) - J_{n-1}(k_f a) J_{n+1}(k_f a) \right\} \tag{32}$$
$$\left. \left. + n J_n^2(k_f a) \right] \right\}$$

where

$$\beta_n = \begin{cases} 1/2, & n = 0 \\ 1, & n = 1, 2, \ldots \end{cases} \tag{33}$$

and where the asterisk denotes the complex conjugate. The derivation of eqn. (32) requires the complex variable identity

$$Re\{z\} = \frac{1}{2}(z + z^*) \,, \tag{34}$$

the trigonometric identies [5]

$$\int_0^{2\pi} \cos n\theta \cos m\theta \, d\theta = \frac{\pi}{\beta_n} \delta_{mn} \tag{35}$$

6

$$\int_0^{2\pi} \sin n\theta \sin m\theta = \begin{cases} 0 & \text{for n=0 or m=0} \\ \pi \delta_{mn} & \text{otherwise} \end{cases} \tag{36}$$

and the Bessel function identities [6]

$$\frac{\partial}{\partial r} J_n(k_f r) = -k_f J_{n+1}(k_f r) + \frac{n}{r} J_n(k_f r) \tag{37}$$

$$\int_0^a r J_n^2(k_f r) dr = \frac{a^2}{2} \left\{ J_n^2(k_f a) - J_{n-1}(k_f a) J_{n+1}(k_f a) \right\} \tag{38}$$

$$\int_0^a J_n(k_f r) J_{n+1}(k_f r) dr = \frac{1}{k_f} \left\{ \frac{1}{2} \left[1 - J_0^2(k_f a) \right] - \sum_{\ell=1}^n J_\ell^2(k_f a) \right\} \tag{39}$$

$$\int_0^a \frac{1}{r} J_n^2(k_f r) dr = \frac{1}{2n} \left\{ 1 + J_0^2(k_f a) + J_n^2(k_f a) - 2 \sum_{k=0}^n J_k^2(k_f a) \right\}, \quad n > 0 \tag{40}$$

The quantity δ_{mn} in eqns. (35) and (36) is the Kronecker delta.

Eqn. (32) is an expression for the total fiber energy per fiber length in the z direction. In order to nondimensionalize the fiber energy $< E >_f$, consider an infinite uniform elastic matrix containing only the incident SH wave given by eqns. (6) through (8), and consider the time-averaged energy $< E >_m$ contained, per unit length in the z direction, in the region $0 < r < a$. By a calculation similar to the calculation leading to eqn. (32), the time-averaged energy $< E >_m$ is

$$< E >_m = \frac{\pi}{4} \mu w_0^2 \sum_{n=0}^\infty \left\{ D_n^* D_n \frac{1}{\beta_n} \left[\frac{(ka)^2}{2} \left\{ J_{n+1}^2(ka) - J_n(ka) J_{n+2}(ka) \right. \right. \right.$$
$$\left. \left. \left. + J_n^2(ka) - J_{n-1}(ka) J_{n+1}(ka) \right\} + n J_n^2(ka) \right] \right\}$$
$$\tag{41}$$

where

$$D_n = \epsilon_n j^n \tag{42}$$

A nondimensional fiber energy $< e >_f$ is now defined by

$$< e >_f = \frac{< E >_f}{< E >_m} \tag{43}$$

The dimensionless quantity $< e >_f$ gives a measure of the fiber energy relative to the energy the matrix would contain if there were no fiber present.

By introducing the nondimensional parameters

$$r_\rho = \frac{\rho_f}{\rho} \tag{44}$$

$$r_\mu = \frac{\mu_f}{\mu} \tag{45}$$

eqns. (43) and (19) can be written as

$$
<e>_f = \frac{<E>_f}{<E>_m}
$$

$$
= r_\mu \frac{\sum_{n=0}^{\infty} C_n^* C_n \frac{1}{\beta_n} \left[\frac{(\gamma ka)^2}{2} \left\{ \begin{array}{c} J_{n+1}^2(\gamma ka) - J_n(\gamma ka) J_{n+2}(\gamma ka) \\ + J_n^2(\gamma ka) - J_{n-1}(\gamma ka) J_{n+1}(\gamma ka) \end{array} \right\} + n J_n^2(\gamma ka) \right]}{\sum_{n=0}^{\infty} D_n^* D_n \frac{1}{\beta_n} \left[\frac{(ka)^2}{2} \left\{ \begin{array}{c} J_{n+1}^2(ka) - J_n(ka) J_{n+2}(ka) \\ + J_n^2(ka) - J_{n-1}(ka) J_{n+1}(ka) \end{array} \right\} + n J_n^2(ka) \right]} \tag{46}
$$

$$
C_n = \frac{(-j^n \epsilon_n) \left[J_n'(ka) H_n^{(1)}(ka) - H_n^{(1)'}(ka) J_n(ka) \right]}{r_\mu \gamma H_n^{(1)}(ka) J_n'(\gamma ka) - H_n^{(1)'}(ka) J_n(\gamma ka)} \tag{47}
$$

respectively, where

$$\gamma = \left(\frac{r_p}{r_\mu} \right)^{1/2} \tag{48}$$

From eqns. (46), (47) and (48) it can be seen that the nondimensional fiber energy $<e>_f$ depends only on the nondimensional parameters ka, r_p and r_μ. The parameter ka is a nondimensional wavenumber of the incident wave, and can be written in terms of the incident wavelength λ_i as

$$ka = \frac{2\pi a}{\lambda_i}$$

or in terms of frequency as

$$ka = \frac{\omega a}{c}$$

The nondimensional fiber energy $<e>_f$ is plotted as a function of the nondimensional wavenumber ka in Fig. 2 for the numerical values $r_\rho = 2.0$ and $r_\mu = 25.0$. These numerical values correspond to an E-glass fiber in a PMR (polyimide monomeric reactant) matrix [7]. In Fig. 3, the nondimensional fiber energy $<e>_f$ is plotted as a function of ka for the numerical values $r_p = 1.5$ and $r_\mu = 7.5$. These numerical values correspond to a Celion 6000 fiber in a PMR matrix [7]. The numerical values used to generate Fig. 3 are estimates of the material properties of the graphite/epoxy composite used in [3]. The infinite series in eqn. (46) are computed by truncating when the absolute value of the individual terms becomes less than 10^{-12} times the absolute value of the accumulated sum.

DISCUSSION

In both Fig. 2 and Fig. 3, the nondimensional fiber energy $< e >_f$ is approximately equal to one when ka is close to zero, or, equivalently, when the wavelength of the incident wave is long compared to the fiber diameter. This means that for long wavelengths, the energy in the fiber is approximately the same as the energy which would exist in a uniform matrix. In other words, the presence of the fiber does not significantly affect the energy distribution for long wavelengths. As ka becomes larger (or, equivalently, as the wavelength becomes smaller) the energy in the scattered wave becomes larger, and the energy in the fiber becomes smaller than the energy which would exist in the region $0 < r < a$ in the uniform matrix.

The most interesting features of Fig. 2 and Fig. 3 are the local maxima in the values of $< e >_f$. For an incident wave containing a broad band of wavenumber, the fiber energy will be greatest at frequencies corresponding to the values of ka where the local maxima occur. If the fiber response could be measured, the local maxima would appear in a frequency spectrum of the measured response. Thus it is sensible, at least in this simple case, to suppose that certain frequency components in a measured response may contain information about the fiber.

The incident SH wave assumed here is independent of the spatial coordinate z and so, therefore, is the displacement field in the fiber. Therefore, it is impossible, in this analysis, to discuss the issue of wave propagation along the fiber in the z direction. The analysis of an incident wave which has a z component may allow consideration of wave propagation along the fiber. Equations which describe the scattering and refraction of such an obliquely incident wave are given in [4].

CONCLUSIONS AND RECOMMENDATIONS

The purpose of this report is to investigate the transfer of energy to a single fiber in an infinite elastic matrix containing an incident SH wave. An expression is derived for the energy in the fiber, and it is shown that the fiber energy attains local maxima at specific values of the wavenumber of the incident wave.

Future research should consider the problems of incident P and SV waves. As mentioned above, an obliquely incident wave may allow analysis of wave propagation along

the fiber. Also, properties of the scattered wave given by eqn. (16) should be investigated, since the scattered wave may be available for experimental measurement. It would be interesting to see if the energy flux of the scattered wave has wavelength dependence similar to the wavelength dependence of the energy of the fiber.

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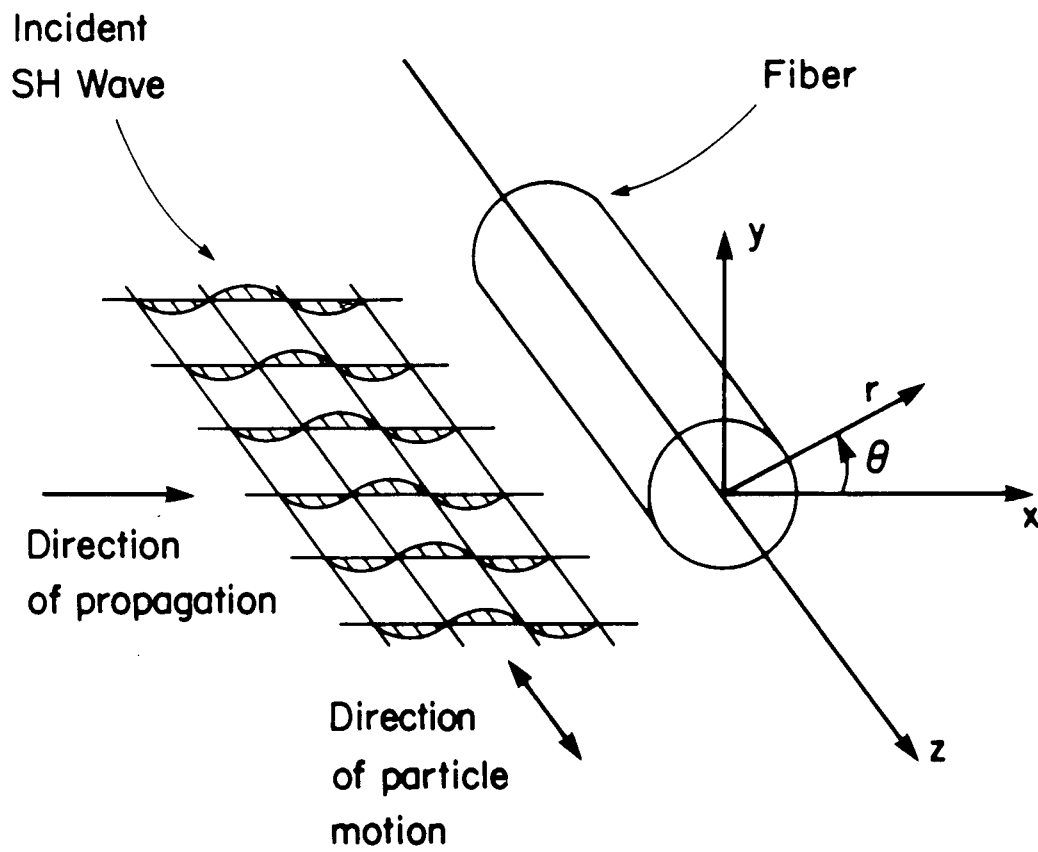
Fig. 1 Elastic fiber embedded in infinite elastic matrix containing incident SH wave.
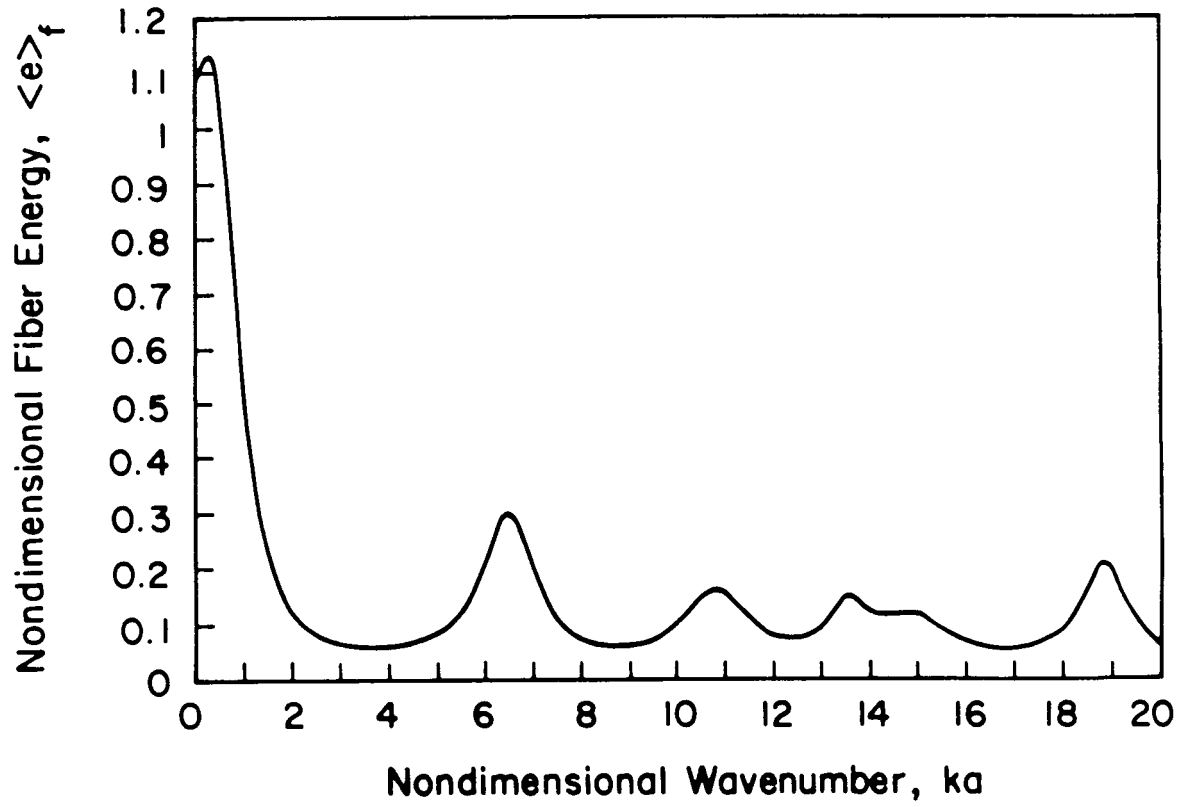
Fig. 2 Nondimensional fiber energy, $<e>_f$ as a function of nondimensional wavenumber ka for $r_p = 2.0$, $r_\mu = 25.0$.
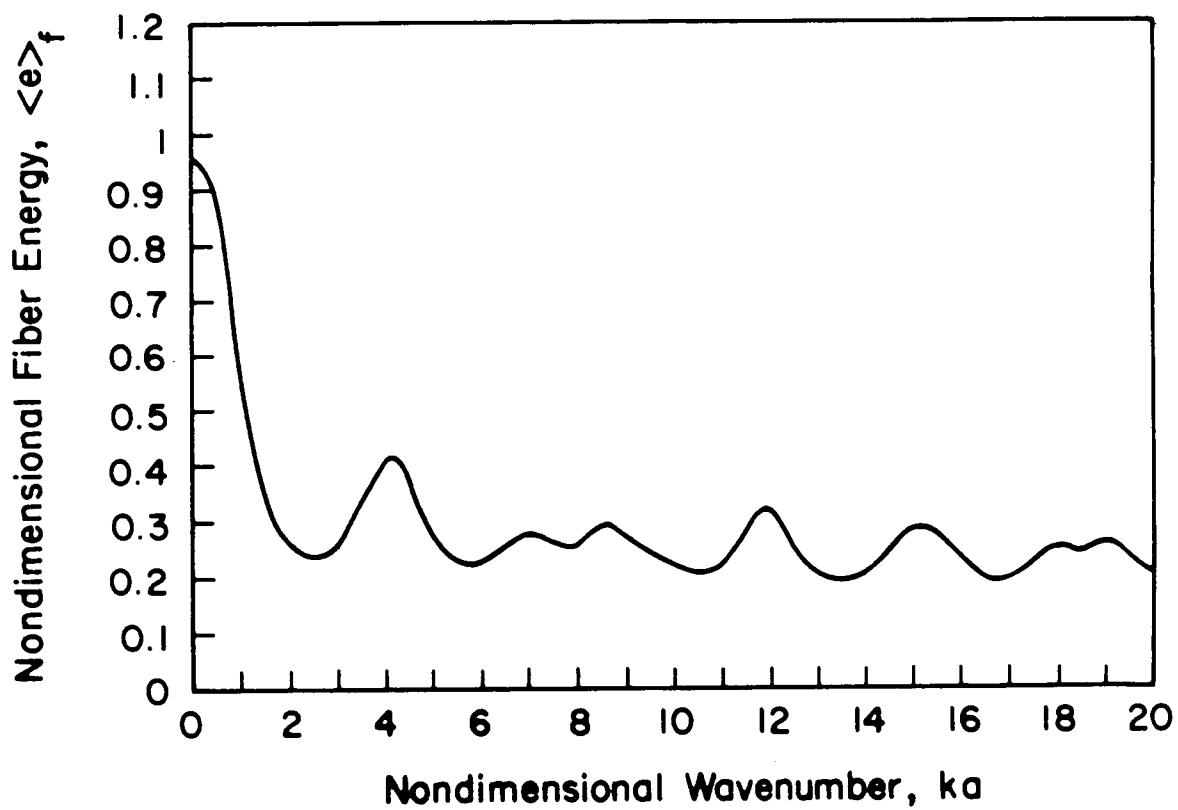
Fig. 3 Nondimensional fiber energy, $< e >_f$ as a function of nondimensional wavenumber ka for $r_p = 1.5$, and $r_\mu = 7.5$.

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16. Abstract

A single elastic fiber embedded in an infinite elastic matrix is considered. An incident plane SH wave is assumed in the infinite matrix, and an expression is derived for the total energy in the fiber due to the incident SH wave. A nondimensional form of the fiber energy is plotted as a function of the nondimensional wavenumber of the SH wave. It is shown that the fiber energy attains maximum values at specific values of the wavenumber of the incident wave. The results obtained here are interpreted in the context of phenomena observed in acousto-ultrasonic experiments on fiber reinforced composite materials.

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