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Modeling and Control of Flexible Structures

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ABSTRACT

This monograph presents integrated modeling and controller design methods for flexible structures. The controllers, or compensators, developed are optimal in the linear-quadratic-Gaussian sense. The performance objectives, sensor and actuator locations and external disturbances influence both the construction of the model and the design of the finite dimensional compensator. The modeling and controller design procedures are carried out in parallel to ensure compatibility of these two aspects of the design problem. Model reduction techniques are introduced to keep both the model order and the controller order as small as possible.

A linear distributed, or infinite dimensional, model is the theoretical basis for most of the text, but finite dimensional models arising from both lumped-mass and finite element approximations also play an important role. A central purpose of the approach here is to approximate an optimal infinite dimensional controller with an implementable finite dimensional compensator. Both convergence theory and numerical approximation methods are given. Simple examples are used to illustrate the theory.

ACKNOWLEDGEMENTS

CONTENTS

Tables

Figures

1. Introduction

Recent years have seen increasing research in active control of flexible structures. The primary motivation for this research is control of large flexible aerospace structures, which are becoming larger and more flexible at the same time that their performance requirements are becoming more stringent. For example, in tracking and other applications, satellites with large antennae, solar collectors and other flexible components must perform fast slew maneuvers while maintaining tight control over the vibrations of their flexible elements. Both of these conflicting objectives can be achieved only with a sophisticated controller. There are applications also to control of robotic manipulators with flexible links, and possibly to stabilization of large civil engineering structures such as long bridges and tall buildings.

A fundamental question that often arises with regard to designing a controller for a flexible structure is whether a finite dimensional model is sufficient as a basis for a controller that will produce the required performance, or is a distributed model necessary? The philosophy of this text is that, for almost any structure, a near optimal controller can be based on some finite dimensional approximate model but that, for large flexible space structures, the necessary order of the approximate model can be determined only in the controller design process, with some reference to the distributed model of the structure.

The main topic of this monograph is the design of LQG compensators based on finite dimensional approximations of distributed models of flexible structures. A primary objective of the approximation methods in subsequent chapters is convergence criteria that indicate the appropriate order of a finite dimensional

1

compensator for an infinite dimensional flexible structure. The basic idea in determination of the compensator order and gains is to approximate an ideal infinite dimensional compensator with an implementable finite dimensional compensator.

This monograph deals with linear-quadratic-Gaussian (LQG) compensators. The LQG optimal control problem for distributed, or infinite dimensional, systems is a generalization to Hilbert space of the now classical LQG problem for finite dimensional systems. The solution to the infinite dimensional problem yields an infinite dimensional state-estimator-based compensator, which is optimal in the context of this monograph. By a separation principle [Ba1, CP2], the problem reduces to a deterministic linear-quadratic optimal control problem and an optimal estimation, or filtering, problem with gaussian white noise. In infinite dimensions, the control system dynamics are represented by a semigroup of bounded linear operators instead of the matrix exponential operators in finite dimensions, and the plant noise process may be an infinite dimensional random process. The solutions to both the control and filtering problems involve Riccati operator equations, which are generalizations of the Riccati matrix equations in the finite dimensional case. Current results on the infinite dimensional LQG problem are most complete for problems where the input and measurement operators are bounded, as this monograph requires throughout. This boundedness also permits the strongest approximation results. For related control problems with unbounded input and measurement, see [Cu1, CS1, LT1, LT2, LT3].

Our primary objective is to approximate the optimal infinite dimensional LQG compensator for a distributed model of a flexible structure with finite dimensional compensators based on approximations to the structure, and to have these

2

finite dimensional compensators produce near optimal performance of the closed-loop system. We discuss how the gains that determine the finite dimensional compensators converge to the gains that determine the infinite dimensional compensator, and we examine the sense in which the finite dimensional compensators converge to the infinite dimensional compensator. With this analysis, we can predict the performance of the closed-loop system consisting of the distributed plant and a finite dimensional compensator that approximates the infinite dimensional compensator.

Our design philosophy is to let the convergence of the finite dimensional compensators indicate the order of the compensator that is required to produce the desired performance of the structure. The two main factors that govern rate of convergence are the desired performance (e.g., fast response) and the structural damping. We should note that any one of our compensators whose order is not sufficient to approximate the infinite dimensional compensator closely will not in general be the optimal compensator of that fixed order; i.e., the optimal fixed-order compensator that would be constructed with the design philosophy in [BH1, BH2]. But as we increase the order of approximation to obtain convergence, our finite dimensional compensators become essentially identical to the compensator that is optimal over compensators of all orders.

An important question, of course, is how large a finite dimensional compensator we must use to approximate the infinite dimensional compensator. In [GM1, GM2, GM3, MG1], we have found that our complete design strategy yields compensators of reasonable size for distributed models of complex space structures. This strategy in general requires two steps to obtain an implementable compensator that is essentially identical to the optimal infinite dimensional compensator: the first step determines the optimal compensator by letting the finite

dimensional compensators converge to it; the second step reduces, if possible, the order of a large (converged) approximation to the optimal compensator. The first step, which is the one involving control theory and approximation theory for distributed systems, is the subject of Chapters 7-9. For the second step, a simple modal truncation of the large compensator sometimes is sufficient, but there are more sophisticated methods in finite dimensional control theory for order reduction. For example [GM2, MG1], we have found balanced realizations, discussed in Chapter 5, to work well for reducing large compensators.

The approximation theory in this monograph follows from the application of approximation results in [BK1, Gi3, Gi4] to a sequence of finite dimensional optimal LQG problems based on a Ritz-Galerkin approximation of the flexible structure. For the optimal linear-quadratic control problem, the approximation theory here is a substantial improvement over that in [Gi1] because here we allow rigid-body modes, more general structural damping (including damping in the boundary), and much more general finite element approximations. These generalizations are necessary to accommodate common features of complex space structures and the most useful finite element schemes. For example, we write the equations for constructing the approximating control and estimator gains and finite dimensional compensators in terms of matrices that are built directly from typical mass, stiffness and damping matrices for flexible structures, along with actuator influence matrices and measurement matrices. This means that the numerical methods in this text for compensator design can be used for almost any finite element model of a flexible structure, without reference to the infinite dimensional theory that establishes the validity of the numerical methods for a distributed model of the structure.

For the estimator problem, this monograph presents rigorous approximation theory that has evolved from less complete results in previous research [GM1,

4

GM2, GM3, MG1]. As in the finite dimensional case, the infinite dimensional optimal estimation problem is the dual of the infinite dimensional optimal control problem, and the solutions to both problems have the same structure. Because we exploit this duality to obtain the approximation theory for the estimation problem from the approximation theory for the optimal control problem, the analysis in this monograph is almost entirely deterministic. We discuss the stochastic interpretation of the estimation problem and the approximating state estimators briefly, but we are concerned mainly with deterministic questions about the structure and convergence of approximations to an infinite dimensional compensator and the performance -- especially stability -- of the closed-loop systems produced by the approximating compensators.

Now we will outline the technical contents of the monograph. Chapter 2 discusses a finite dimensional model for the forced linear vibrations of a flexible structure. Although the main goal is to design compensators for infinite dimensional flexible structures, the finite dimensional approximations upon which implementable compensators are based amount to finite dimensional models of flexible structures. These approximate models have the form of the finite dimensional model in Chapter 2. Important features of flexible structures, like natural modes and damping, are discussed in Chapter 2. Extensions of these notions for flexible structures with gyroscopic forces are also discussed. Gyroscopic terms appear whenever the structure rotates or contains rotating elements.

Chapter 3 defines the abstract distributed model of an infinite dimensional flexible structure and the energy spaces to be used in most of the subsequent chapters. We assume a finite number of actuators, since this is the case in all applications, and we assume that the actuator influence operator is bounded.

Chapter 3 also establishes certain mathematical properties of the open-loop system that are useful in control and approximation.

We develop the approximation of the distributed model of the structure in Chapter 4 and prove convergence of the approximating open-loop systems. The approximation scheme is essentially a Ritz-Galerkin method that includes modal, including component modal, approximations and most finite element approximations of flexible structures. For convergence, we use the Trotter-Kato semigroup approximation theorem, which was used in optimal open-loop control of hereditary systems in [BB1] and has been used in optimal feedback control of hereditary, hyperbolic and parabolic systems in [BK1, Gi1, Gi3] and other papers. The usual way to invoke Trotter-Kato is to prove that the resolvents of the approximating semigroup generators converge strongly. To prove this, we introduce an inner product that involves both the strain-energy inner product and the damping functional, and show that the resolvent of each finite dimensional semigroup generator is the projection, with respect to this special inner product, of the resolvent of the original semigroup generator onto the approximation subspace. The idea works as well for the adjoints of the resolvents, and when the open-loop semigroup generator has compact resolvent, it follows from our projection that the approximating resolvent operators converge in norm.

The speed of convergence to the optimal feedback law and optimal estimator is affected by the manner in which the model order is increased. It is useful to have a method of identifying important modes of the structure so they can be added to the model first. This issue is considered in Chapter 5. Moore's Balanced Realization Theory is introduced as a method for ordering the importance of states of a structural model. The approach takes into account input and output coupling as well as frequency and damping. An efficient approxima-

6

tion scheme is developed for structures which are lightly damped. Balanced Realization Theory may also be used to reduce the order of a general matrix transfer function, and in this context it provides a tool for reducing the order of the compensator.

Chapter 6 discusses the LQG optimal control problem for the distributed model of the structure and establishes some estimates involving bounds on solutions to infinite dimensional Riccati equations and open-loop and closed-loop decay rates. We need these estimates for the subsequent approximation theory. To get the approximation theory for the estimation problem, we have to give certain results on the control problem in a more general form than would be necessary were we interested only in the control problem for flexible structures. Therefore, in Chapter 6, we first give some generic results applicable to the LQG problem for a variety of distributed systems and then apply the generic results to the control of flexible structures.

For closing the loop on the control system, we assume a finite number of bounded linear measurements and construct the optimal state estimator, which is infinite dimensional in general. The gains for this estimator are obtained from the solution to an infinite dimensional Riccati equation that has the same form as the infinite dimensional Riccati equation in the control problem.

Since the approximation issues that this book treats are fundamentally deterministic, we make the book self-contained by defining the infinite dimensional estimator as an observer, although the only justification for calling this estimator and the corresponding compensator optimal is their interpretation in the context of stochastic estimation and control. We discuss the stochastic interpretation but do not use it. We say estimator and observer interchangeably to emphasize the deterministic definition of the estimator here.

Because we assume a finite number of actuators and a bounded input operator, the optimal feedback control law consists of a finite number of bounded linear functionals on the state space, which is a Hilbert space. This means that the feedback law can be represented in terms of a finite number of vectors, which we call functional control gains, whose inner products with the generalized displacement and velocity vectors define the control law. For any finite-rank, bounded linear feedback law for a control system on a Hilbert space, the existence of such gains is obvious and well known. A functional control gain for a flexible structure will have one or more distributed components, or kernels, corresponding to each distributed component of the structure and scalar components corresponding to each rigid component of the structure.

Analogous to the functional control gains are functional estimator gains corresponding to the finite number of sensor measurements. The functional gains play a prominent role in our analysis. They give a concrete representation of the infinite dimensional compensator and provide a criterion for convergence of the approximating finite dimensional compensators.

The optimal LQG compensator is infinite dimensional in general. The transfer function of this compensator is irrational, but it is still an m(number of actuators) x p(number of sensors) matrix function of a complex variable, as in finite dimensional control theory. The optimal closed-loop system consists of the distributed model of the structure controlled by the optimal compensator.

We develop the approximation scheme for the LQG control problem in Chapter 7. We define a sequence of finite dimensional LQG problems, whose solutions approximate the solution to the infinite dimensional problem of Chapter 6. The solution to each finite dimensional problem is based on the solutions to two

8

Riccati matrix equations, and we give formulas for using these solutions to compute approximations to the functional control and estimator gains as linear combinations of the basis vectors. The approximating functional gains indicate how closely the finite dimensional compensators approximate the infinite dimensional compensator. The brief discussion in Section 7.6 is the only place in the monograph where stochastic estimation theory is necessary, and none of the analysis in the rest of the monograph depends on this discussion.

Chapter 7 presents the finite dimensional equations to be used in designing the finite dimensional compensators. This chapter contains no convergence analysis. The equations in Chapter 7 can be used for numerical design of the finite dimensional compensators, without worrying about the meaning of these compensators with respect to the infinite dimensional control problem.

Chapter 8 contains convergence theory that gives conditions under which the approximating compensators in Chapter 7 converge to the infinite dimensional compensator in Chapter 6. This theory also describes the sense in which the finite dimensional compensators converge. The compensator convergence is discussed in terms of both the convergence of the approximating functional gains and the convergence of the transfer functions of the finite dimensional compensators.

In Chapter 9, we compute approximating finite dimensional compensators for a compound structure that consists of an Euler-Bernoulli beam attached on one end to a rotating rigid hub and on the other end to a lumped mass. We emphasize the fact that we do not solve, or even write down, the coupled partial and ordinary differential equations of motion. For both the definition and numerical solution of the problem, only the kinetic and strain energy functionals and a dissi-

pation functional for the damping are required. We show the approximating functional control gains obtained by using a standard finite element approximation of the beam, and we discuss the effect on convergence of structural damping and of the ratio of state weighting to control weighting in the performance index. As suggested by a theorem in Chapter 8, the functional gains do not converge when no structural damping is modeled.

We study the convergence of the finite dimensional compensators by examining convergence of the approximating functional gains and the frequency responses of the finite dimensional compensators. Also, we compute the eigenvalues of the closed-loop system consisting of a finite dimensional compensator and larger-order model of the structure. These eigenvalues indicate how many modes are controlled how much by the infinite dimensional compensator and by the finite dimensional compensators that are essentially identical to the infinite dimensional compensator as far as the input/output map. The functional gains and the compensator frequency response indicate the order of a finite dimensional compensator necessary to approximate the infinite dimensional compensator closely, and the closed-loop eigenvalues confirm this order.

Chapter 10 discusses several further issues important for implementation of compensators designed by the methods in this book, and suggests further research to make the ideas presented here useful for application to other control problems for large flexible structures.

2. A Finite Dimensional Model of a Flexible Structure

2.1 Forced Linear Vibrations

The forced vibrations of a large class of flexible structures may be described approximately by the second-order finite dimensional differential equation

$$(2.1.1) \qquad M \ddot{x} + G \dot{x} + D \dot{x} + K x + F x = f_0$$

where the generalized displacement $x(t)$ and the generalized forcing function $f_0(t)$ are n-vectors, the mass matrix M, damping matrix D and stiffness matrix K are real symmetric n x n matrices, and the gyroscopic matrix G and circulatory matrix F are real skew-symmetric n x n matrices. The mass matrix is positive definite, and the damping matrix is nonnegative. In most cases the stiffness matrix is also nonnegative, but some gyroscopic systems can have stiffness matrices which are indefinite. Any zero eigenvalue of K corresponds to a rigid-body mode. The gyroscopic matrix G results from rotating components, and the circulatory force matrix F may result from either follower forces or damping described in rotating reference frames. The terms $M \ddot{x}$, $G \dot{x}$ and $K x$ are conservative terms in (2.1.1) while $D \dot{x}$ and $F x$ are nonconservative. The forcing function f_0 is often expressed as

$$(2.1.2) \qquad f_0 = B_0 \, u(t)$$

In this expression, B_0 is a real n x m input distribution matrix and $u(t)$ is an m x 1 input vector which could represent feedback control forces, external forces or both. In the present development, u will represent control forces or moments. The input distribution matrix depends on where the control or external forces act on the system, and it determines the relative effect of the inputs on the various coordinates.

The first-order form of (2.1.1) is

(2.1.3) $\dot{z} = Az + f$

where the 2n x 2n state vector z is $z(t) = \begin{bmatrix} x^T & \dot{x}^T \end{bmatrix}^T$ and

(2.1.4) $A = \begin{bmatrix} 0 & I \\ -M^{-1}(K+F) & -M^{-1}(G+D) \end{bmatrix}$, $f = \begin{bmatrix} 0 \\ f_0 \end{bmatrix}$.

Linear measurements of the state vector have the form

(2.1.5) $y = C z + C_0 u = [C_p \quad C_v] \begin{pmatrix} x \\ \dot{x} \end{pmatrix} + C_0 u$

where C, C_0, C_p and C_v are real matrices of appropriate dimensions.

2.2 Natural Modes for Nongyroscopic Systems

Natural modes and natural frequencies may be defined by considering free vibrations of the conservative part of (2.1.1), i.e.,

(2.2.1) $M \ddot{x} + G \dot{x} + K x = 0$

We first examine the case where G is zero and K is nonnegative. In Laplace transform notation, (2.2.1) reduces to:

(2.2.2) $[s^2 M + K] x = 0$

Equation (2.2.2) defines an eigenvalue problem which may be solved for s and x. Since M and K are real symmetric matrices with M positive definite and K nonnegative, the solutions for s consist of 2n purely imaginary complex conjugates, $s_r = \pm j\omega_r$, r=1,2,...,n where the ω_r's are called the *natural frequencies*. The corresponding eigenvectors x_r are real and are called the *natural mode shapes*.

12

The mode shapes are complete in R^n and mode shapes corresponding to distinct frequencies are orthogonal with respect to both M and K.

If U is the matrix whose columns are the natural mode shapes normalized with respect to the mass matrix M, then substituting

(2.2.3) $x(s) = U a(s)$

in (2.2.2) and multiplying the equation on the left by U^T yields

(2.2.4) $s^2 a + \Omega^2 a = U^T f_0$

where

(2.2.5) $\Omega = U^T K U = \begin{bmatrix} \omega_1^2 & & & & 0 \\ & \omega_2^2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & \omega_n^2 \end{bmatrix}$

The components of the n-vector $a(t)$ are called the modal coordinates of the structure, and the n scalar equations in (2.2.4) are called the modal equations of motion.

2.3 Modal Damping

If a damping matrix D is added to (2.2.2), then (2.2.4) becomes

(2.3.1) $s^2 a + s U^T D U a + \Omega^2 a = U^T f_0$

The damping represented by the matrix D is called *modal damping* if $U^T D U$ is diagonal. In this case, the damping does not couple the modes of undamped free

vibration and the eigenvectors of the damped structure are the same as the eigenvectors of the undamped structure. The damping matrix $U^T D U$ is diagonal if and only if $M^{-1}D$ commutes with $M^{-1}K$, or equivalently, if and only if $M^{-1}D$ and $M^{-1}K$ have the same eigenvectors. In particular, $U^T D U$ is diagonal if $M^{-1}D$ is equal to a convergent power series in $M^{-1}K$ or if $M^{-1}D$ is equal to a linear combination of fractional powers of $M^{-1}K$. Although an assumption of modal damping greatly simplifies the analysis of vibrating structures, it is not always easy to reconcile this assumption with physical reality. (See Section 2.5.) In the present discussion, we are thinking primarily of the damping matrix D as resulting from some material damping model for the flexible components of the structure and/or models of joint damping or passive dampers attached to the structure. In practice it is common to assume that the matrix $U^T D U$ is diagonal and the values of modal damping ratios are estimated from experimental data. The only justification for this questionable assumption is that for sufficiently light damping, the eigenvectors of the finite dimensional model approach the natural mode shapes from (2.2.2). Hence, whatever the damping matrix D, as long as it is sufficiently small, a damping matrix $\tilde{D}$ for which $U^T \tilde{D} U$ is diagonal will produce both eigenvalues and eigenvectors for the structure model that are close to those produced by the correct D.

2.4 Gyroscopic Systems

If G is not zero in (2.2.1), then the procedure for finding natural frequencies and natural modes is conceptually and computationally more complicated. Meirovitch [Me1] has suggested a procedure which casts the equations in a form analogous to (2.2.2) where the coefficient matrices are symmetric. Once in this form, it is possible to determine the natural frequencies and natural modes

as above (with some differences in interpretation). The procedure depends on K being positive definite. If K is only positive semi-definite the procedure can still be applied, but first the rigid body modes must be removed from (2.2.1). The system which remains is then treated as follows. First, rewrite (2.2.1) in an augmented form which has a symmetric and a skew symmetric coefficient matrix:

$$(2.4.1) \qquad \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix} \begin{pmatrix} \ddot{x} \\ \dot{x} \end{pmatrix} + \begin{bmatrix} G & K \\ -K & 0 \end{bmatrix} \begin{pmatrix} \dot{x} \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Take the Laplace transform of (2.4.1) with $(\dot{x}^T \ x^T)^T$ replaced by q.

$$(2.4.2) \qquad s \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix} q + \begin{bmatrix} G & K \\ -K & 0 \end{bmatrix} q = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The eigenvalue problem defined by (2.4.2) may be solved for s and q. The solutions for s consist of 2n purely imaginary complex conjugates, $s_r = \pm j\omega_r$, r=1,2,...,n. As before, the ω_r's are called the *natural frequencies*. The corresponding solutions for q also occur in complex conjugate pairs and have the form $q_r = v_r + j \ w_r$, r=1,2,...,n. Setting $s_r = j\omega_r$ and $q_r = v_r + j \ w_r$ in (2.4.2) and equating the real and imaginary parts to zero separately yields two equations for v_r and w_r.

$$(2.4.3) \qquad \omega_r \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix} v_r + \begin{bmatrix} G & K \\ -K & 0 \end{bmatrix} w_r = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$(2.4.4) \qquad -\omega_r \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix} w_r + \begin{bmatrix} G & K \\ -K & 0 \end{bmatrix} v_r = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

These equations may be rearranged in the form:

$$
(2.4.5) \qquad \omega_r^2 \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix} v_r - \begin{bmatrix} (K-GM^{-1}G) & -GM^{-1}K \\ KM^{-1} & KM^{-1}K \end{bmatrix} v_r = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

$$
(2.4.6) \qquad \omega_r^2 \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix} w_r - \begin{bmatrix} (K-GM^{-1}G) & -GM^{-1}K \\ KM^{-1} & KM^{-1}K \end{bmatrix} w_r = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The appeal of (2.4.5) and (2.4.6) is that in each equation, the left hand matrix is symmetric and positive definite and the right hand matrix is symmetric as in the case where G is zero (see Eq. (2.2.2)). These features permit the use of simpler numerical algorithms, and the eigenvalue analysis proceeds as in the nongyroscopic case. There are, however, some differences in the interpretation of the eigenvectors since these vectors now include velocities as well as position coordinates. These eigenvectors are not in general referred to as natural modes. Refs. [Me1, Me2, Me3] discuss the solution and interpretation of (2.4.5) and (2.4.6) at some length.

It will also be noted that Eqs. (2.4.5) and (2.4.6) are identical. This does not mean that v_r and w_r are identical. From (2.4.3) and (2.4.4) one can show that v_r and w_r must satisfy the equation

$$
(2.4.7) \qquad w_r = \begin{bmatrix} 0 & \omega_r I \\ -\omega_r^{-1} I & 0 \end{bmatrix} v_r
$$

Thus only one eigenvalue problem must be solved to obtain both w_r and v_r. If (2.4.5) is used to obtain v_r, then (2.4.7) can be used to calculate w_r. The w_r obtained in this way will satisfy (2.4.6).

2.5 A Lumped Mass Example

As noted in section 2.3, there may be some difficulty in justifying the assumption of modal damping. This will now be illustrated by means of the simple example shown in Figure 2.1. This structure consists of three masses, two springs and three dampers. The three degrees of freedom include one rigid-body mode and two elastic modes. It is instructive to consider the implications of assuming modal damping with the same damping ratio in each mode. Because we want the damping in each mode to be proportional to the undamped frequency of that mode, there should be no damping in the rigid-body mode, so there are no dampers attached to the ground.
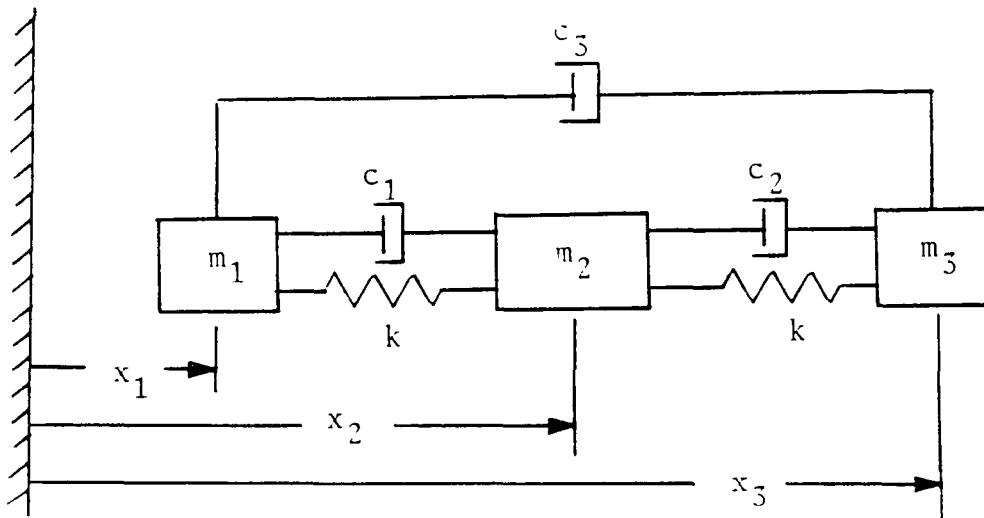
Figure 2.1. Lumped Mass Example

For the coordinates x_1, x_2 and x_3 shown in Figure 2.1, the mass stiffness and damping matrices are

$$(2.5.1) \qquad M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \qquad K = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$$

and

17

$$(2.5.2) \qquad D = \begin{bmatrix} (c_1+c_2) & -c_1 & -c_3 \\ -c_1 & (c_1+c_2) & -c_2 \\ -c_3 & -c_2 & (c_2+c_3) \end{bmatrix}$$

The damping coefficients c_1, c_2 and c_3 are nonnegative. Suppose that we have the (dimensionless) mass and spring constants

$$m_1 = 0.1 \qquad m_2 = 1.0 \qquad m_3 = 0.1 \qquad k = 1.0.$$

For the matrix $U^T D U$ in the modal equation (2.3.1) to be equal to, say, .02 times the square root of the matrix on the right hand side of (2.2.5) with n = 3 (i.e., for one percent critical damping in each mode) the damping matrix D must be

$$(2.5.3) \qquad D = \begin{bmatrix} 0.010379 & -0.014434 & 0.004055 \\ -0.014434 & 0.028868 & -0.014434 \\ 0.004055 & -0.014434 & 0.010379 \end{bmatrix} .$$

No combination of the damping coefficients in Eq. (2.5.2) produces this damping matrix.

With more elaborate combinations of pulleys and dampers, infinitely many damping configurations are possible, so it may be possible to realize the damping matrix in (2.5.3) by some arrangement of dampers. But even if a realization of the required damping exists, it may be a very unlikely physical configuration. The difficulty of physically realizing constant-damping-ratio modal damping in this example suggests that assuming a form, particularly a diagonal form, for the damping matrix in the modal equations of motion, without considering the physical sources of the damping, can produce improbable damping models.

3. An Infinite Dimensional Model

3.1 Infinite Dimensional Equation of Motion

Now we will describe the abstract model that we will use throughout the book to represent an infinite dimensional flexible structure. The generalized displacement vector $x(t)$ is in a real Hilbert space H, the control vector $u(t)$ is in R^m for some finite m and $x(t)$ is a mild solution to

$$(3.1.1) \quad \ddot{x}(t) + D_0\dot{x}(t) + A_0x(t) = B_0u(t), \qquad\qquad t > 0.$$

In this equation of motion, the linear stiffness operator A_0 is densely defined and selfadjoint with compact resolvent and at most a finite number of negative eigenvalues. The stiffness operators for structures involving beams, plates and membranes will satisfy the hypotheses here on A_0. Any nonpositive eigenvalues of A_0 will represent rigid-body modes. For now, we will assume that the damping operator D_0 is symmetric, nonnegative and bounded relative to A_0; this includes linear material damping in continuous structural components and linear damping from a viscous fluid surrounding the structure (not likely in space applications). In Section 3.4, we will discuss a more general type of damping that we will allow from then on, except where otherwise noted. The input operator B_0 is a linear operator from R^m to H, and hence bounded (so that this model does not include boundary control).

For the distributed model of a flexible structure, we will not discuss a gyroscopic operator corresponding to the matrix G in Chapter 2. It is straightforward to add a bounded skew selfadjoint operator to the damping operator D_0 and generalize our infinite dimensional system theory and approximation theory appropriately, but we do not want to complicate the exposition of this and subsequent chapters by carrying this extra detail along.

Remark 3.1. Our analysis includes the system

$$(3.1.1') \quad M_0\ddot{x}(t) + D_0\dot{x}(t) + A_0x(t) = B_0u(t), \qquad t > 0,$$

where the mass operator M_0 is a selfadjoint, bounded and coercive linear operator on a real Hilbert space H_0. The operators A_0, B_0 and D_0 in $(3.1.1')$ have the same properties with respect to H_0 that the corresponding operators in (3.1.1) have with respect to H. To include $(3.1.1')$ in our analysis, we need only take H to be H_0 with the norm-equivalent inner product $\langle\cdot,\cdot\rangle_H = \langle M_0\cdot,\cdot\rangle_{H_0}$, and multiply $(3.1.1')$ on the left by M_0^{-1}. In H, the operator $M_0^{-1}A_0$ is self-adjoint with compact resolvent, and $M_0^{-1}D_0$ is symmetric and nonnegative. With no loss of generality, then, we will refer henceforth only to (3.1.1) and assume that the H-inner product accounts for the mass distribution. ▽▽▽

By natural modes of a structure represented by (3.1.1), we will mean the eigenvectors ϕ_j of the eigenvalue problem

$$(3.1.2) \quad \lambda_j\phi_j = A_0 \phi_j.$$

From our hypotheses on A_0, it follows that these eigenvalues form an infinitely increasing sequence of real numbers, of which all but a finite number are positive. Also, the corresponding eigenvectors are complete in H and satisfy

$$(3.1.3) \quad \langle\phi_i, \phi_j\rangle_H = \langle A_0 \phi_i, \phi_j\rangle_H = 0, \qquad\qquad i \neq j.$$

(For these standard properties of the eigenvalue problem, see [Ba1, Ka2] and other texts.) For $\lambda_j > 0$, $\omega_j = \sqrt{\lambda_j} > 0$ is a *natural frequency.*

3.2 The Energy Spaces

To discuss the damping in (3.1.1) more precisely, to derive the first-order form of (3.1.1) and to specify the class of measurements that we will consider,

we need to define two additional Hilbert spaces, the *strain energy space V* and *the total energy space E*, in terms of the basic space H and the stiffness operator A_0. We choose a bounded selfadjoint linear operator A_1 on H such that A_1 is positive definite on the eigenspace of A_0 corresponding to nonpositive eigenvalues of A_0 and the null space of A_1 is the closed span of the eigenvectors of A_0 corresponding to positive eigenvalues. Thus $\tilde{A}_0 = A_0 + A_1$ is coercive; i.e., there exists $\rho > 0$ such that

$$(3.2.1) \quad \langle \tilde{A}_0 x, x \rangle \geq \rho \, |x|^2, \qquad\qquad x \in D(\tilde{A}_0) = D(A_0).$$

We define the Hilbert space V to be the completion of $D(A_0)$ with respect to the inner product

$$(3.2.2) \quad \langle v_1, v_2 \rangle_V = \langle \tilde{A}_0 \, v_1, v_2 \rangle_H, \qquad\qquad v_1, v_2 \in D(A_0).$$

Equivalently, $V = D(\tilde{A}_0^{\frac{1}{2}})$ and $\langle v_1, v_2 \rangle_V = \langle \tilde{A}_0^{\frac{1}{2}} v_1, \tilde{A}_0^{\frac{1}{2}} v_2 \rangle_H$.

In the usual way, we will use the imbedding

$$(3.2.3) \quad V \subset H = H' \subset V',$$

where the injection from V into H and from H' into V' are continuous with dense ranges. We denote by Λ_V the Riesz map from V onto its dual V'; i.e.,

$$(3.2.4) \quad \langle v, v_1 \rangle_V = (\Lambda_V v_1) v, \qquad\qquad v_1, v \in V.$$

Then $\tilde{A}_0$ is the restriction of Λ_V to $D(A_0)$ in the sense that

$$(3.2.5) \quad (\Lambda_V v_1) v = \langle v, \tilde{A}_0 v_1 \rangle_H, \qquad\qquad v_1 \in D(A_0), v \in V.$$

Remark 3.2. We began our description of the control system model with (3.1.1) because its form is familiar in the context of flexible structures. The stiffness operator A_0, for example, is the infinite dimensional analogue of the stiffness matrix in Chapter 2. In applications, though, it is often easier to begin with a strain-energy functional from which the correct strain-energy inner

product is obvious. There is a one-to-one correspondence between the stiffness operator $\tilde{A}_0$ and the strain-energy space V. We have seen how V is defined in terms of $\tilde{A}_0$. If V is specified first, then $\tilde{A}_0$ is defined in terms of the Riesz map for V by (3.2.5) with $D(\tilde{A}_0) = \Lambda_V^{-1} H$ (see [Sh1] also for this approach). Either way, the relationship between $\tilde{A}_0$ and V is the same.

This means that if the space H and its inner product (determined by kinetic energy; recall Remark 3.1) and the space V and its inner product (determined by strain energy and geometric boundary conditions) are written down, then the operators $\tilde{A}_0$ and Λ_V are determined implicitly. For both our theoretical analysis and our numerical computation, only the H and V inner products are needed; neither $\tilde{A}_0$, Λ_V nor Λ_V^{-1} need be written down (although it is instructive to write the partial differential operator A_0 for simple examples, as in Section 3.3). $\triangledown\triangledown\triangledown$

Now we define the total energy space $E = V \times H$, noting that when A_0 is coercive and x(t) is the solution to (3.1.1), then $|(x(t),\dot{x}(t))|_E^2$ is twice the total energy (kinetic plus potential) in the system. The total energy space is the natural space for the infinite dimensional state-space model of the flexible structure; i.e., the first order form of (3.1.1), which we will derive in Section 3.5.

3.3 Clamped-Free Beam Example

For a simple example, we consider the clamped-free Euler-Bernoulli beam in Figure 3.1. The length of the beam is 1 and the product of the modulus of elasticity and the second moment of the cross section is EI. The single control force u is spread uniformly over the last five percent of the beam at the right end.
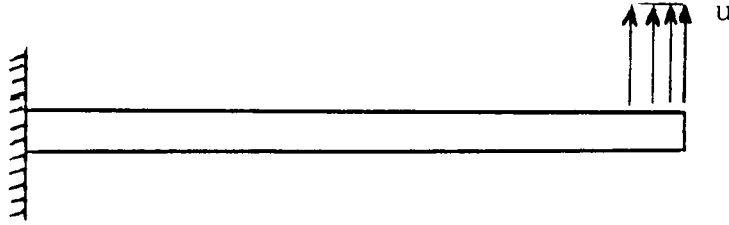
Figure 3.1: Clamped-free Euler-Bernoulli Beam

For this example, the most natural Hilbert space H is $L_2(0,1)$, and the stiffness operator is

$$(3.3.1) \quad A_0 = EI \, d^4/ds^4 \, , \quad D(A_0) = \{g \in H^4(0,1): g(0) = g'(0)$$

$$= g''(1) = g'''(1) = 0\}.^{[1]}$$

We model Kelvin-Voigt viscoelastic damping in the beam [CP1], which means that the damping operator is

$$(3.3.2) \quad D_0 = c_0 A_0$$

where c_0 is a positive constant. The actuator influence operator is given by

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[1] We denote by $H^k(0,1)$ the kth-order Sobolev space of functions on $(0,1)$. A function f is in this Hilbert space if f, along with its derivatives of orders up through k-1, are absolutely continuous on $[0,1]$ and $f^{(k)} \in L_2(0,1)$. The square of the $H^k(0,1)$ norm of f is

$$|f|_k^2 = \sum_{j=1}^{k} \int_0^1 [d^j f(t)/dt^j]^2 \, dt.$$

$$0, \quad 0 < s < .95,$$

(3.3.3) $B_0 u = u b_0, \quad b_0(s) =$

$$1, \quad .95 < s < 1.$$

Since A_0 in this example is coercive, $A_1 = 0$ and $\tilde{A}_0 = A_0$. The strain-energy space V in this example is $\{v \in H^2(0,1): v(0) = v'(0) = 0\}$, which follows from defining the inner product

$$(3.3.4) \quad \langle v_1, v_2 \rangle_V = \langle A_0 v_1, v_2 \rangle_H = \int_0^1 v_1'' v_2'' \, ds, \qquad v_1, v_2 \quad D(A_0),$$

according to (3.2.2) and completing $D(A_0)$ with respect to the norm induced by this inner product. Although the geometric boundary conditions $v(0) = v'(0) = 0$ are the only boundary conditions retained explicitly in V, the boundary conditions $v''(0) = v'''(1) = 0$ are retained implicitly by the imbedding of V in H and the Riesz map for V because specifying H and V is equivalent to specifying H and A_0, as discussed in Remark 3.2. That H and V determine all of the boundary conditions for A_0 should remind readers experienced in the structural dynamics of applying Hamilton's principle to derive the equations of motion for a distributed model of a flexible structure: once the energy functionals and geometric boundary conditions are defined, the natural boundary conditions follow from integrating by parts in the spatial domain.

We should note that the coercive, selfadjoint operator $A_0^{1/2}$ exists for this example, but that it is not a differential operator. This square-root operator cannot be written down in closed form, but its infinite dimensional matrix representation with respect to its eigenvectors is diagonal with the natural frequencies of the clamped-free beam on the diagonal. The operators A_0 and $A_0^{1/2}$ have the same eigenvectors, which are complete and mutually orthogonal in

both H and V.

3.4 The Damping Functional and Operator

Before we discuss the first-order form of (3.1.1), we will state the damping hypothesis that we will use from here on and discuss the representation of the damping admitted under this hypothesis. To construct the first-order form of (3.1.1), we do not require an operator D_0 defined from some subset of H into H. Rather, we need only the weaker assumption that there exists a damping (or dissipation) functional

$$(3.4.1) \quad d_0(\cdot,\cdot) : V \times V \longrightarrow R$$

such that d_0 is bilinear, symmetric, continuous on $V \times V$ and nonnegative. Under this hypothesis on d_0, there is a unique nonnegative, selfadjoint operator $D_V \in L(V)$ such that

$$(3.4.2) \quad d_0(v_1, v_2) = \langle D_V v_1, v_2 \rangle_V = \langle v_1, D_V v_2 \rangle_V, \qquad v_1, v_2 \in V.$$

If we have a symmetric, nonnegative damping operator D_0 defined from $D(A_0)$ into H such that D_0 is bounded relative to A_0, then $\langle D_0 \cdot, \cdot \rangle_H$ defines a bilinear, symmetric, continuous, nonnegative functional on a dense subset of $V \times V$. In this case, d_0 is the unique extension of this functional to $V \times V$ and

$$(3.4.3) \quad D_V|_{D(A_0)} = A_0^{-1} D_0.$$

(That D_0 being A_0-bounded implies continuity of $\langle D_0 \cdot, \cdot \rangle_H$ with respect to the V norm follows from [Ka2, Theorem 4.12, page 292].)

In applications, either D_0 or d_0 should be straightforward to write from the physics of the structure, but D_V is difficult to determine except for simple structures or damping that is a simple function of stiffness.

Remark 3.3. For our theoretical analysis and our numerical work, the damping functional d_0 is sufficient; it is not necessary to write down either of the damping operators D_V and D_0. (As with the stiffness operator, it is instructive to write the damping operators for simple examples; see Section 3.7.)

3.5 *The Semigroup Generator and the First-Order Form of the Equation of Motion*

We want to write (3.1.1) as a well-posed first-order evolution equation on the total energy space $E = V \times H$. To do this, we will determine the appropriate semigroup generator for the open-loop system, so that C_0-semigroup theory will guarantee the existence and uniqueness of solutions to the state-space differential equation under appropriate conditions on $u(t)$. We will derive the semigroup generator by constructing its inverse explicitly, and then we will try to convince the reader that we do have the appropriate semigroup generator. This approach seems mathematically efficient, and we will need the inverse of the generator for approximation theory.

We define $\tilde{A}^{-1} \in L(E,E)$ by

$$(3.5.1) \qquad \tilde{A}^{-1} = \begin{bmatrix} -D_V & -\tilde{A}_0^{-1} \\ I & 0 \end{bmatrix}.$$

This operator is clearly one-to-one, and its range is dense, since V is dense in H and $D(A_0)$ is dense in V. Now, we take

$$(3.5.2) \quad \tilde{A} = (\tilde{A}^{-1})^{-1}.$$

Direct calculation of the inner product shows

$$(3.5.3) \quad \langle \tilde{A}^{-1} \begin{pmatrix} v \\ h \end{pmatrix} , \begin{pmatrix} v \\ h \end{pmatrix} \rangle_E = -d_0(h,h),$$

so that $\tilde{A}$ is dissipative with dense domain. Also, since $D(\tilde{A}^{-1}) = E$, $\tilde{A}$ is maximal dissipative by [Gi1, Theorem 2.1]. Therefore, $\tilde{A}$ generates a C_0-contraction semigroup on E.

Finally, the open-loop semigroup generator is

$$(3.5.4) \qquad A = \tilde{A} + \begin{bmatrix} 0 & 0 \\ A_1 & 0 \end{bmatrix}, \qquad D(A) = D(\tilde{A}),$$

where A_1 is the bounded linear operator discussed in Section 3.2. With

$$(3.5.5) \qquad B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix} \epsilon L(R^m, E),$$

the first-order form of (3.1.1) is

$$(3.5.6) \qquad \dot{z}(t) = A z(t) + B u(t), \qquad\qquad t > 0,$$

where $z = (x, \dot{x}) \epsilon E$.

To see that A is indeed the appropriate open-loop semigroup generator, suppose that A_0 is coercive (so that $A_1 = 0$) and that we have a symmetric, nonnegative A_0-bounded damping operator D_0. Then the appropriate generator should be a maximal dissipative extension of the operator

$$(3.5.7) \qquad Å = \begin{bmatrix} 0 & I \\ -A_0 & -D_0 \end{bmatrix}, \qquad D(Å) = D(A_0) \times D(A_0).$$

It is shown in [Gi1, Section 2] that Å has a unique maximal dissipative extension, and after noting (3.4.3), it can be shown easily that the $A(= \tilde{A})$ defined

by (3.5.1) and (3.5.2) is an extension of A.

We should note that Showalter [Sh1, Chapter VI] elegantly derives a semigroup generator for a class of second-order systems that includes the flexible-structure model here. The presentation here is most useful for our approximation theory because of the explicit construction of the inverse of the semigroup generator. For the purposes of this paper, we do not need to characterize the operator A itself more explicitly, but we should make the following points.

First, from $\tilde{A}^{-1}$ we see

(3.5.8) $D(A) = \{(x,\dot{x}): \dot{x} \in V, x + D_V\dot{x} \in D(A_0)\}.$

In applications, the "natural boundary conditions" can be determined from (3.5.8) and the boundary conditions included in the definition of $D(A_0)$ (see Section 3.8), although we will not need the natural boundary conditions. In the case of a damping operator that is bounded relative to $A_0^{1/2}$, $D(A) = D(A_0) \times V$. If the damping operator is bounded relative to A_0^μ for $\mu < 1$, then A has compact resolvent.

In many structural applications, the open-loop semigroup is analytic, although this has been proved only for certain important cases. Showalter obtains an analytic semigroup when the damping functional is V-coercive; for example, when there exists a damping operator that is both A_0-bounded and as strong as A_0. Such a damping operator results from the Voigt-Kelvin viscoelastic material model. Also, it can be shown that the semigroup is analytic for a damping operator equal to $c_0A_0^\mu$ for $1/2 < \mu < 1$ and c_0 a positive scalar. The case $\mu = 1/2$, which produces the same damping ratio in all modes, is especially common in structural models, and Chen and Russell [CR1] have shown that the

semigroup is analytic for a more general class of damping operators involving $\tilde{A}_0^{\frac{1}{2}}$.

We can guarantee that the open-loop semigroup generator is a spectral operator (i.e., its eigenvectors are complete in E) only for a damping operator that is a linear combination of an H-bounded operator and a fractional power of A_0. However, nowhere do we use or assume anything about the eigenvectors of either the open-loop or the closed-loop semigroup generator. The natural modes -- of undamped free vibration -- in (3.1.2) are always complete in both H and V.

The Adjoint of the Semigroup Generator. Since D_V is selfadjoint on V, direct calculation shows that $\tilde{A}^{-*} = (\tilde{A}^{-1})^*$ -- the adjoint of $\tilde{A}^{-1}$ with respect to the E-inner product -- is

$$(3.5.9) \quad \tilde{A}^{-1} = \begin{bmatrix} -D_V & A_0^{-1} \\ -I & 0 \end{bmatrix}.$$

Then $\tilde{A}^* = (\tilde{A}^{-*})^{-1}$. Having $\tilde{A}^{-*}$ explicitly facilitates proving strong convergence for approximating adjoint semigroups.

Exponential Stability of the Open-Loop System. According to the following theorem, a sufficient condition for the open-loop system to be uniformly exponentially stable is that there be no rigid-body modes and the damping be coercive. Coercive damping means, basically, that all structural components have positive damping. That the decay rate given for the energy norm depends only on the lower bound for the stiffness operator and the upper and lower bounds for the damping functional is essential for convergence results for the approximating optimal control problems of subsequent chapters.

Theorem 3.4. Suppose that A_0 and d_0 are H-coercive. Let ρ be the positive constant in (3.2.1), and let δ_0 and δ_1 be positive constants such that

$$(3.5.10) \quad \delta_0 |v|_H^2 \le d_0(v,v) \le \delta_1 |v|_V^2, \qquad\qquad v \in V.$$

Then

$$(3.5.11) \quad |T(t)| \le \left(1 + \frac{\delta_0}{\sqrt{\rho}} + \delta_1 \delta_0/2 \right)^{\frac{1}{2}} \exp\left[\frac{-t}{2/\delta_0 + 2/\sqrt{\rho} + \delta_1} \right], \qquad t \ge 0.$$

For a proof of this theorem, see [GA1]. The proof uses an explicit Liapunov functional for the open-loop system.

3.6 Representation of the Open-Loop Semigroup in the Case of Uncoupled Modes

Since the eigenvectors ϕ_j of A_0 are complete in V, $x(t)$ can be written

$$(3.6.1) \quad x(t) = \sum_{j=1}^{\infty} a_j(t) \, \phi_j$$

where $a_j(t)$ is the modal amplitude for the jth mode. The energy space $E = V \times H$ is isomorphic to the space ℓ_{2E} of sequences

$$(3.6.2) \quad a = \begin{pmatrix} a_1 \\ a_1 \\ a_2 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

that are square summable in the sense of

$$(3.6.3) \quad |a|_E^2 = \sum_{j=1}^{\infty} (\omega_j^2 \, a_j^2 + a_j^2) < \infty$$

where ω_j is the jth natural frequency.

When a damping operator D_0 exists and has the same eigenvectors as A_0, the damping leaves the natural modes of free vibration uncoupled and (3.1.1) is equivalent to the infinite set of modal equations

(3.6.4) $\ddot{a}_j(t) + 2\,\zeta_j\,\omega_j\,\dot{a}_j(t) + \omega_j^2\,a_j(t) = B_{0j}\,u(t)$

where

(3.6.5) $\zeta_j = \langle D_0\,\phi_j,\,\phi_j\rangle_H\,/2\omega_j|\phi_j|_H^2$

is the damping ratio of the jth mode and B_{0j} is a 1 x m matrix. On ℓ_{2E}, the semigroup $T(t)$ is represented by an infinite dimensional matrix with all zeros except for 2 x 2 blocks $T_j(t)$ centered on the main diagonal and given by

(3.6.6) $T_j(t) = \exp\left(\begin{bmatrix} 0 & 1 \\ -\omega_j^2 & -2\zeta_j\omega_j \end{bmatrix} t\right)$.

3.7 The Measurement Equation and Operators

Recall that the second-order equation (3.1.1) and the first-order equation (3.5.6) are equivalent. We assume that the measurement has the form

(3.7.1) $y(t) = C_0\,u(t) + C\,z(t)$

where $y(t)$ is a p-vector, C_0 is a real p x m matrix and C is a bounded linear operator from E to R^p. Since E = V x H, C must have the form

(3.7.2) $C = [C_1 \quad C_2]$

where C_1 and C_2 are bounded linear operators from V and H, respectively, to R^p. This means that (3.7.1) can be written

(3.7.3) $y(t) = C_0\, u(t) + C_1\, x(t) + C_2\, \dot{x}(t)$.

Also, according to Riesz, if $(C\, z(t))_i$ is the ith component of the p-vector $C\, z(t)$, then

(3.7.4) $(C\, z(t))_i = \langle c_{1i}, x \rangle_V + \langle c_{2i}, \dot{x} \rangle_H$, $i = 1, \ldots, p$,

where $c_{1i} \in V$ and $c_{2i} \in H$.

3.8 Further Comments on the Clamped-Free Beam

According to Section 3.5, once the basic space H was defined in Section 3.3 to be $L_2(0,1)$, the first-order form of the equation of motion for the beam in Section 3.3 was determined implicitly by the stiffness operator in (3.3.1), the damping operator in (3.3.2) and the actuator influence operator in (3.3.3). It follows from Theorem 3.4 that the open-loop beam (i.e., the free response of the beam) is uniformly exponentially stable for any positive damping coefficient c_0 in (3.3.2).

As discussed in Remark 3.2, specifying the strain-energy inner product in (3.3.4) and the boundary conditions on the left end of the beam would have been equivalent to specifying A_0. Also, according to Section 3.4, specifying the dissipation functional to be

(3.8.1) $d_0(v_1, v_2) = c_0 \langle v_1, v_2 \rangle_V$, $v_1, v_2 \in V$,

would have been equivalent to specifying the damping operator D_0.

To see the advantage of being able to use the damping functional instead of the damping operator, suppose that we attach a linear damper with positive coefficient c_1 between the right end of the beam and ground. Then there is no damping operator D_0, but it is easy to write the dissipation functional

(3.8.2) $d_0(v_1, v_2) = c_0 \langle v_1, v_2 \rangle_V + c_1 v_1(1)v_2(1)$, $v_1, v_2 \in V$,

since this means that the rate of energy dissipation for the open-loop beam is

(3.8.3) $\dot{E} = d_0(\dot{x}(t), \dot{x}(t)) = -c_0 \int_0^1 \dot{x}''(t,s)^2 \, ds - c_1 \dot{x}(t,1)^2$.

For none of our purposes in subsequent chapters do we need to go further in characterizing the damping. Nonetheless, it is instructive to look at the natural boundary conditions that the damping produces and how these boundary conditions can be determined with the results discussed so far. In this example (and probably only a few others), it is easy to write the operator D_V discussed in Section 3.4. The part of D_V corresponding to the Kelvin-Voigt damping is c_0 times the identity in V, which follows from (3.4.3). The part of D_V corresponding to the linear damper on the end of the beam is less obvious, but it can be determined with (3.4.2) and integration by parts. The result (for the total damping functional in (3.8.2)) is

(3.8.4) $D_V v = c_0 v + v(1) \, g_0$

where $g_0 \in V$ is

(3.8.5) $g_0(s) = (1 - s)^3/6 + s/2 - 1/6$, $0 \leq s \leq 1$.

The natural boundary conditions for the equation of motion for the beam follow from (3.5.8). With the $D(A_0)$ and V defined in Section 3.3 and D_V given by (3.8.4) and (3.8.5), (3.5.8) says that the domain of the semigroup generator contains those function pairs $(x, \dot{x})$ that satisfy $\dot{x} \in V$, $x + c_0 \dot{x} \in V \subset H^4(0,1)$,

(3.8.6) $(x + c_0 \dot{x})''(1) = 0$

and

(3.8.7) $(x + c_0 \dot{x})'''(1) + c_1 \dot{x}(1) = 0$.

These last two equations are the natural boundary conditions.

3.9 Hub-Beam-Tip-Mass Example

One end of the Euler-Bernoulli beam in Figure 3.2 is attached rigidly (cantilevered) to a rigid disc which is free to rotate about its center, point 0, which is fixed. Also, a point mass m_1 is attached to the other end of the beam. The control is a torque u applied to the disc, and all motion is in the plane.
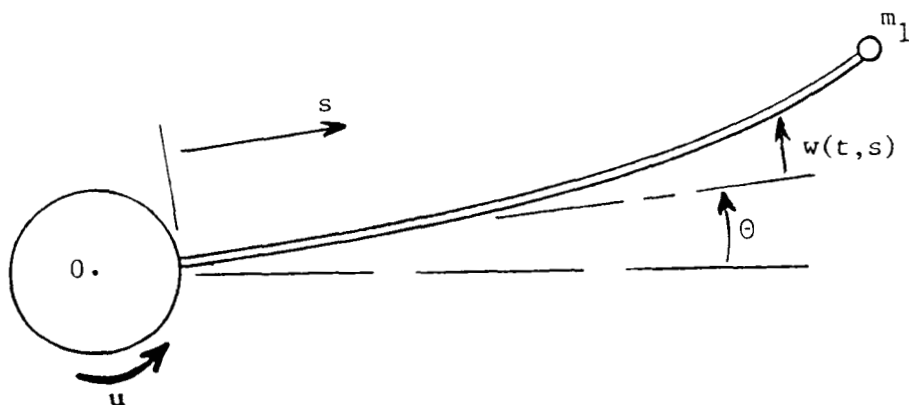
Figure 3.2. Flexible Structure

Table 3.1. Structural Data

r = hub radius	10 in
l = beam length	100 in
I_0 = hub moment of inertia about axis perpendicular to page through 0	100 slug in^2
m_b = beam mass per unit length	.01 slug/in
m_1 = tip mass	1 slug
EI = product of elastic modulus and second moment of cross section for beam	13,333 slg in^3/sec^2
Fundamental frequency of undamped structure	.9672 rad/sec

34

The angle θ represents the rotation of the disc (the rigid-body mode), $w(t,\cdot)$ is the elastic deflection of the beam from the rigid-body position, and $w_1(t)$ is the displacement of m_1 from the rigid-body position. For technical reasons, we do not yet impose the condition $w_1(t) = w(t,1)$; more on this later.

The control problem is to stabilize rigid-body motions and linear (small) transverse elastic vibrations about the state θ = 0 and w = 0. Our linear model assumes not only that the elastic deflection of the beam is linear but also that the axial inertial force produced by the rigid-body angular velocity has negligible effect on the bending stiffness of the beam. The rigid-body angle need not be small.

For this example, it is a straightforward exercise to derive the three coupled differential equations (one partial and two ordinary differential equations) of motion in θ, w and w_1, and they do have the form (3.1.1'). However, to emphasize the fact that we do not use the explicit differential equations, we will not write these equations here. Rather, we will write only what normally is needed in applications: the kinetic and strain-energy functionals, the damping functional and the actuator influence operator.

Remark 3.1 applies to this example, and to most examples with complex structures. The generalized displacement vector is

(3.9.1) $x = (\theta, w, w_1) \in H_0 = R \times L_2(0,1) \times R.$

The kinetic energy in the system is

(3.9.2)   Kinetic Energy = $1/2 \langle x, \dot{x} \rangle_H$

where H is H_0 with the inner product

$$(3.9.3) \quad \langle x, \hat{x} \rangle_H = m_b \int_0^1 [w + (r+s)\theta][\hat{w} + (r+s)\hat{\theta}]\, ds + I_0\, \theta\hat{\theta}$$

$$+ m_1[w_1 + (r+1)\theta][\hat{w}_1 + (r+1)\hat{\theta}] \, .$$

As in most applications we need not write the mass operator explicitly, but there exists a unique selfadjoint linear operator M_0 on H_0 such that

$$(3.9.4) \quad \langle x, x \rangle_H = \langle M_0 x, \hat{x} \rangle_{H_0} \, .$$

It is easy to see that M_0 is bounded and coercive. Hence H_0 and H have equivalent norms.

The input operator for $(3.1.1')$ (which maps R to H_0) is

$$(3.9.5) \quad B_0 = (1,0,0).$$

Since we multiply $(3.1.1')$ by M_0^{-1} to get $(3.1.1)$, the input operator for $(3.1.1)$ is $(M_0^{-1}B_0)$. Note that

$$(3.9.6) \quad (M_0^{-1}B_0)^{*H} = B_0^* \, ,$$

where $(M_0^{-1}B_0)^{*H}$ is the H-adjoint of $(M_0^{-1}B_0)$ and B_0^* is the H_0-adjoint of B_0.

Remark 3.2 also applies here. The only strain energy is in the beam and is given by

$$(3.9.7) \quad \text{Strain Energy} = 1/2\ a(x,x)$$

with

$$(3.9.8) \quad a(x,\hat{x}) = EI \int_0^1 w'' \, \hat{w}'' \, ds,$$

where $(\cdot)'' = d^2(\cdot)/ds^2$. To make $a(\cdot,\cdot)$ into an inner product, we must account for rigid-body rotation. Thus we set

(3.9.9) $\langle x, \hat{x} \rangle_V = a(x, \hat{x}) + \hat{\theta}\theta$

and define

(3.9.10) $V = \{x = (\theta, \phi, \phi(1)) : \phi \in H^2(0,1), \phi(0) = \phi'(0) = 0\}.$

Also, we have

(3.9.11) $\langle x, \hat{x} \rangle_V = a(x, \hat{x}) + \langle B_0 B_0^* x, \hat{x} \rangle_{H_0} = a(x, \hat{x}) + \langle (M_0^{-1} B_0)(M_0^{-1} B_0)^{*H} x, \hat{x} \rangle_H,$

so that $A_1 = B_0 B_0^*$, or $(M_0^{-1} B_0)(M_0^{-1} B_0)^{*H}$, depending on whether the H_0 or the H-inner product is used in computing the V-inner product. *But we need neither A_1 nor A_0 explicitly. We need only (3.9.8) and (3.9.9), along with (3.9.3), to compute the required inner products.*

As mentioned in Remark 3.2, the operator $\tilde{A}_0$ can be defined now by (3.2.5), and the stiffness operator is $A_0 = \tilde{A}_0 - A_1$. Using the H_0-inner product in (3.2.5) yields the A_0 for (3.1.1'), and using the H-inner product yields the A_0 for (3.1.1), which is M_0^{-1} following the A_0 for (3.1.1'). The A_0 for (3.1.1') is simple, and the reader might write it out. We will not, because we do not .need it.

We will point out that $D(A_0)$ requires both the geometric boundary conditions in V and the natural boundary condition $w''(t,1) = 0$; i.e., zero moment on the right end. That the geometric boundary conditions

(3.9.12) $w(t,0) = w'(t,0) = 0$

and

(3.9.13) $w(t,1) = w_1(t)$

are imposed in V but not in H -- i.e., on the generalized displacement but not on the generalized velocity -- is common in distributed models of flexible

structures. The natural norm for expressing the kinetic energy of distributed components is the L_2 norm, which cannot preserve constraints on sets of zero measure. Because the strain energy involves spatial derivatives, the stronger strain-energy norm can preserve the geometric boundary conditions (although, as for the boundary slope of an elastic plate, the V-norm may impose some of these boundary conditions in an L_2 rather than a pointwise sense). The strain-energy norm is based on the material model of the distributed components of the system, and it should not be surprising that such a norm is required to connect the various structural components.

We assume that the beam has Voigt-Kelvin viscoelastic damping, so that the damping operator in (3.1.1) is

$$(3.9.14) \qquad D_0 = c_0 A_0$$

where c_0 is a constant. This means that the damping functional is

$$(3.9.15) \quad d_0(x,\hat{x}) = c_0 a(x,\hat{x}) , \qquad\qquad x,\hat{x} \in V.$$

4. Approximation of the Distributed Model of the Structure

4.1 Abstract Finite Element Approximation

We begin this chapter by discussing a Ritz-Galerkin approximation framework that accommodates most common finite element schemes for flexible structures. The only hypotheses for this abstract approximation theory (in addition to those given in Chapter 3 for the infinite dimensional structural model) are the requirements stated in Hypothesis 4.1 for the basis vectors. As indicated in Section 4.1.2, modal approximation fits easily into this framework.

We assume that the basic space H is given, that the various operators and functionals associated with (3.1.1) satisfy the hypotheses stated in Chapter 3 and that the energy spaces V and $E = V \times H$ are defined as in Section 3.2 of Chapter 3.

Hypothesis 4.1. There exists a sequence of finite dimensional subspaces V_n of V such that the sequence of orthogonal projections P_{V_n} converges V-strongly to the identity, where P_{V_n} is the V-projection onto V_n. Also, each V_n is the span of n linearly independent vectors e_j. $\nabla\nabla\nabla$

Since it should cause no confusion, we will omit the subscript n and write just e_j, keeping in mind that the basis vectors may change from one V_n to another, as in most finite element schemes. Also, we will refer to the Hilbert space $E_n = V_n \times V_n$, which has the same inner product as $E = V \times H$.

4.1.1 Approximating Equations of Motion

For $n \geq 1$, we approximate $x(t)$ (the solution to (3.1.1)) by

(4.1.1) $\qquad x_n(t) = \displaystyle\sum_{j=1}^{n} \xi_j(t)e_j$

where $\xi(t) = [\xi_1(t)\ \xi_2(t)\ \ldots\ \xi_n(t)]^T$ satisfies

(4.1.2) $\qquad M^n\ddot{\xi} + D^n\dot{\xi} + K^n\xi = B_0^n u,$

and the mass matrix M^n, damping matrix D^n, stiffness matrix K^n, and actuator influence matrix B_0^n are given by

$$M^n = [\langle e_i, e_j\rangle_H], \qquad D^n = [d_0(e_i, e_j)],$$

(4.1.3) $\qquad K^n = [\langle A_0^{1/2}e_i, A_0^{1/2}e_j\rangle_H] = [\langle e_i, e_j\rangle_V] - [\langle A_1 e_i, e_j\rangle_H],$

$$B_0^n = [\langle e_i, b_j\rangle_H].$$

Of course, (4.1.2) can be written as

(4.1.4) $\qquad \dot{\eta} = A^n\eta + B^n u$

where

(4.1.5) $\qquad \eta = \begin{pmatrix} \xi \\ \dot{\xi} \end{pmatrix}$

and

(4.1.6) $\qquad A^n = \begin{bmatrix} 0 & I \\ -M^{-n}K^n & -M^{-n}K^n \end{bmatrix}, \qquad B^n = \begin{bmatrix} 0 \\ M^{-n}B_0^n \end{bmatrix}$

Note 4.2. Throughout this text, we use the superscript n in the designation of matrices in the nth approximating system and control problem, like A^n, B^n, M^n, etc. Hence the superscript n indicates the order of approximation -- and it *never* indicates a power of the matrix. By M^{-n}, we denote the inverse of the

mass matrix M^n. In the designation of a linear operator in the nth approximation, we use the subscript n. For example, A_n and B_n are the operators whose matrix representations are A^n and B^n, respectively. $\nabla\nabla\nabla$

In the class of approximation schemes considered here, the approximation to the equivalent measurement equations (3.7.1) and (3.7.3) can be written

$$(4.1.7) \qquad y_n = C_0 u + C^n \eta = C_0 u + C^n \binom{\xi}{\dot{\xi}}$$

with

$$(4.1.8) \qquad C^n = \begin{bmatrix} C_1^n & C_2^n \end{bmatrix}$$

where the ith column of the p x n matrix C_1^n is the p-vector equal to $C_1 e_i$ and the ith column of the p x n matrix C_2^n is the p-vector equal to $C_2 e_i$. The y_n in (4.1.7) would be equal to the exact measurement if the true generalized displacement vector x(t) were a linear combination of the first n basis vectors for all t (i.e., if x(t) were equal to the $x_n(t)$ in (4.1.1)) so that we could take $E = E_n$.

4.1.2 Modal Approximation

The equations in Section 4.1.1 include the important case where the basis vectors e_j are the mode shapes of undamped, free vibration (i.e., the eigenvectors of the stiffness operator A_0; recall (3.1.2) and (3.1.3)). In this case, the mode shapes usually are normalized with respect to mass so that the mass matrix M^n is the n x n identity matrix. The corresponding stiffness matrix then is

$$(4.1.9) \qquad K^n = \begin{bmatrix} \omega_1^2 & & & 0 \\ & \omega_2^2 & & \\ & & \ddots & \\ 0 & & & \omega_n^2 \end{bmatrix}$$

where the ω_j's are the natural frequencies of undamped, free vibration (i.e., the diagonal elements of K^n are the first n eigenvalues of A_0). A zero frequency corresponds to a rigid-body mode.

That Hypothesis 4.1 holds for modal approximation follows from the fact that the operator $\tilde{A}_0 |_V$ is a compact selfadjoint operator on V, since $\tilde{A}_0$ and $\tilde{A}_0^{-1}$ have the same eigenvectors.

4.2 Convergence

It is useful to note that (4.1.1) and (4.1.2) or (4.1.4) are equivalent to

$$(4.2.1) \qquad \dot{z}_n(t) = A_n z_n(t) + B_n u(t),$$

where $z_n = (x_n, \dot{x}_n) \in E_n$ and $A_n \in L(E_n)$ and $B_n \in L(R^m, E_n)$ are the operators whose matrix representations are given in (4.1.6). Also, for any real λ,

$$(4.2.2) \qquad (\lambda - A_n)\begin{pmatrix} v_n^1 \\ v_n^2 \end{pmatrix} = \begin{pmatrix} h_n^1 \\ h_n^2 \end{pmatrix}$$

is equivalent to

$$(4.2.3) \qquad (\lambda^2 M^n + \lambda D^n + K^n)\alpha^1 = (\lambda M^n + D^n)\beta^1 + M^n \beta^2$$

and

$$(4.2.4) \qquad \alpha^2 = \lambda \alpha^1 - \beta^1$$

if

42

$$(4.2.5) \qquad v_n^j = \sum_{i=1}^{n} \alpha_i^j e_i \quad \text{and} \quad h_n^j = \sum_{i=1}^{n} \beta_i^j e_i, \qquad j = 1,2.$$

(Substituting A^n and (4.2.5) into (4.2.2) yields (4.2.3) and (4.2.4).)

Next, we will prepare to invoke the Trotter-Kato semigroup approximation theorem to show how (4.1.2), (4.1.4) and (4.2.1) approximate (3.1.1) and (3.5.6). For this, we will treat only the case in which A_0 is coercive (no rigid-body modes), so that $A_1 = 0$ and $\tilde{A}_0 = A_0$; the general case is a straight-forward extension. In the present case, then, the open-loop semigroup generator A is maximal dissipative, and for each n, A^n is dissipative on E^n. The main idea here is to project $(\lambda - A)^{-1}$ onto V_n in a certain inner product and observe that the result is exactly $(\lambda - A_n)^{-1}$, where A_n is the operator on V_n in (4.2.1) and (4.2.2). Of course, we need only do this for real $\lambda > 0$.

For real $\lambda > 0$, then, we define an inner product on V by

$$(4.2.6) \qquad \langle \cdot, \cdot \rangle_\lambda = \lambda^2 \langle \cdot, \cdot \rangle_H + \lambda d_0(\cdot, \cdot) + \langle \cdot, \cdot \rangle_V.$$

Under the hypotheses in Chapter 2 on d_0, $\langle \cdot, \cdot \rangle_\lambda$ is clearly norm-equivalent to $\langle \cdot, \cdot \rangle_V$. For $n > 1$, we let $P_n(\lambda)$ be the projection of V onto V_n in the inner product $\langle \cdot, \cdot \rangle_\lambda$. If h^1, $h^2 \in H$,

$$(4.2.7) \qquad (\lambda - A)\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} h^1 \\ h^2 \end{pmatrix}$$

is equivalent to

$$(4.2.8) \qquad \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = A^{-1}\left[\lambda \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} - \begin{pmatrix} h^1 \\ h^2 \end{pmatrix}\right].$$

43

With A^{-1} from (3.5.1), (4.2.8) is equivalent to

$$(4.2.9) \qquad (I + \lambda D_V + \lambda^2 A_0^{-1})v^{-1} = (\lambda A_0^{-1} + D_V)h^1 + A_0^{-1}h^2$$

and

$$(4.2.10) \qquad v^2 = \lambda v^1 - h^1.$$

If

$$(4.2.11) \qquad v_n^1 = P_n(\lambda)v^1 \quad \text{and} \quad v_n^2 = P_n(\lambda)v^2,$$

it follows from (4.2.6) and (4.2.9) that

$$(4.2.12) \qquad \langle e_i, v_n^1 \rangle_\lambda = \langle e_i, v^1 \rangle_\lambda$$

$$= \lambda^2 \langle e_i, A_0 A_0^{-1} v^1 \rangle_H + \lambda \langle e_i, D_V v^1 \rangle_V + \langle e_i, v^1 \rangle_V$$

$$= \langle e_i, (\lambda^2 A_0^{-1} + \lambda D_V + I)v^1 \rangle_V$$

$$= \langle e_i, (\lambda A_0^{-1} + D_V)h^1 + A_0^{-1}h^2 \rangle_V \ ,$$

and from (4.2.10) that

$$(4.2.13) \qquad \langle e_i, v_n^2 \rangle = \langle e_i, v^2 \rangle = \lambda \langle e_i, v^1 \rangle_\lambda - \langle e_i, h^1 \rangle_\lambda.$$

Now, for $h^1 = h_n^1 \in V_n$, $h^2 = h_n^2 \in V_n$, and v_n^1, v_n^2, h_n^1 and h_n^2 written as in (4.2.5), (4.2.12) and (4.2.13) yield (4.2.3) and (4.2.4) again.

This shows that

$$(4.2.14) \qquad \begin{bmatrix} P_n(\lambda) & 0 \\ 0 & P_n(\lambda) \end{bmatrix} (\lambda - A)^{-1}|_{E_n} = (\lambda - A_n)^{-1},$$

which yields

$$(4.2.15) \qquad \begin{bmatrix} P_n(\lambda) & 0 \\ 0 & P_n(\lambda) \end{bmatrix} (\lambda - A)^{-1} P_{E_n} = (\lambda - A_n)^{-1} P_{E_n},$$

where P_{E_n} is the E-projection of E onto E_n. The projection P_{E_n} can be written

$$(4.2.16) \qquad P_{E_n} = \begin{bmatrix} P_{V_n} & 0 \\ 0 & P_{H_n} \end{bmatrix},$$

where P_{V_n} is the V-projection onto V_n, as before, and P_{H_n} is the H-projection onto V_n. Since the V-norm is stronger than the H-norm, it follows from Hypothesis 4.1 that $(\lambda - A_n)^{-1} P_{E_n}$ converges E-strongly to $(\lambda - A)^{-1}$ as $n \longrightarrow \infty$. Now, with A_n extended to $E_n^\perp$ as, say, $n(P_{E_n} - I)$, Trotter-Kato [Ka2, page 504, Theorem 2.16] yields the following.

Theorem 4.3. For A_0 coercive, let $T_n(t)$ be the (contraction) semigroup generated on E_n by A_n. Then, for each $t > 0$, $T_n(t) P_{E_n}$ converges strongly to $T(t)$, uniformly in t for t in any bounded interval. ▽▽▽

In the general case, when A_0 is not coercive, the open-loop generator A is obtained from the dissipative A by the bounded perturbation (3.5.4), so that [Gi3, Theorem 6.6] yields the following generalization of Theorem 4.3.

Corollary 4.4. Let $T_n(t)$ be the semigroup generated on E_n by A_n. Then, for each $t > 0$, $T_n(t) P_{E_n}$ converges strongly to $T(t)$, uniformly in t for t in any bounded interval. ▽▽▽

Theorem 4.5. When A has compact resolvent, $(\lambda-A_n)^{-1}P_{E_n}$ converges in $L(E)$ to $(\lambda-A)^{-1}$.

Proof. This follows from (4.2.15) and a standard result that the projection of a compact linear operator onto a sequence of subspaces converges in norm if the projections converge strongly to the identity, as do P_{E_n} and $P_n(\lambda)$. $\nabla\nabla\nabla$

That the adjoint semigroups also converge strongly follows from an argument entirely analogous to the proof of Theorem 4.3. In particular, equations like (4.2.6)-(4.2.12) are used to show that

$$(4.2.17) \qquad \begin{bmatrix} P_n(\lambda) & 0 \\ 0 & P_n(\lambda) \end{bmatrix} (\lambda-A^*)^{-1}P_{E_n} = (\lambda-A_n^*)^{-1}P_{E_n}.$$

In showing this, A^{-*} is used as A^{-1} was used above. Also, care must be taken to calculate A_n^* with respect to the E-inner product. The result is

Theorem 4.6. Let $T_n(t)$ be the sequence of semigroups in Corollary 4.4. Then, for each $t > 0$, $T_n^*(t)P_{E_n}$ converges strongly to $T^*(t)$, uniformly in t for t in any bounded interval. $\nabla\nabla\nabla$

For the approximation to the actuator influence operator $B \in L(R^m,E)$, recall $B_n \in L(R^m,E_n)$, the operator whose matrix representation is the matrix B^n in (4.1.6). From (4.1.3), it follows that

$$(4.2.18) \qquad B_n = P_{E_n}B.$$

Since B has finite rank m, B_n and B_n^* converge in norm to B and B^*, respectively.

We define the approximating measurement operator $C_n \in L(E_n, R^p)$ to be the operator whose matrix representation is the matrix C^n in (4.1.8) (defined imme-

diately after (4.1.8)). This means that

(4.2.19) $C_n = C|_{E_n}$,

where C is the operator in (3.7.1)-(3.7.3). Since C is a bounded linear opera-
tor with finite rank, $C_n P_{E_n}$ and C_n^* converge in norm to C and C^*, respectively.

4.3 Approximation of the Hub-Beam-Tip-Mass Example

Our approximation of the distributed model of the structure in Section 9 of
Chapter 3 is based on a finite element approximation of the beam that uses cubic
Hermite splines as basis functions [Sc1, SF1]. Cubic Hermite splines and their
first derivatives are continuous at the nodes, which are evenly spaced for the
numerical results in Chapter 9. Because the basis vectors e_j in Hypothesis 4.1
must be in the space V defined in (3.9.10), we write them as

(4.3.1a) $e_1 = (1,0,0)$,
(4.3.1b) $e_j = (0, \phi_j, \phi_j(1))$, $j = 2, 3, \ldots, n$,

where the ϕ_j's are the cubic spline functions defined over the length of the beam
and n-1 is the number of splines. For n_e elements, there are $2n_e$ linearly inde-
pendent splines. The $2n_e$ elastic degrees of freedom are usually taken to be the
displacements and slopes at the nodes, relative to the rigid-body position.
With the rigid-body mode (hub rotation), the total number of degrees of freedom
is $n = 2n_e + 1$.

The matrices in (4.1.3) are calculated according to (3.9.3), (3.9.8) and
(3.9.9), with B_0 given by (3.9.5). In particular,

$$K^n = [a(e_i, e_j)], \qquad D^n = c_0 K^n,$$

(4.3.2)
$$B_0^n = [1 \ 0 \ 0 \ \dots \ 0]^T$$

$$= [\langle e_i, M_0^{-1}(1,0,0) \rangle_H] = [\langle e_i, (1,0,0) \rangle_{H_0}]$$

Note that the first row and column of K^n are zero. The matrices A^n and B^n are given by (4.1.6).

If the measurement is the rigid-body angle θ, then the measurement matrices in (4.1.7) are $C_0 = 0$ and

(4.3.3)
$$C_n = [1 \ 0 \ 0 \ 0 \ \dots \ ].$$

5. Order Reduction Using Balanced Realizations

5.1 Introduction

Almost any engineering analysis and design task requires a model of a physical system. The model should be complete enough to describe all of the relevant physical phenomena, but not so complicated that insight is lost or computational burdens become excessive. The development of an appropriate model is largely an art which depends on many factors such as the distribution and type of inputs and outputs, the magnitude and frequency content of inputs and outputs and the structure and properties of the physical components making up the system. Trial and error and engineering intuition are common ingredients in the modeling process. Much research has focused on methods for determining what is essential about a model and what can be disregarded. Most of this work, generally known as model reduction, applies to systems where the input-output relationship is linear. Fortunately, the structural systems considered here fall into that class.

In addition to generating a low order model of the system which is to be controlled, one may wish to reduce the order of the controller or compensator which has been synthesized to accomplish the control. If the controller is linear, one may often use the same methods which are used for reduction of the plant model. This chapter deals with an approach for model reduction of linear systems which has been widely used since its introduction in 1980, the method of balanced realizations [Mo1]. In its general form, the approach may be used for reduction of a plant model or controller. A simplified approximate version may be used when the equations have a special form corresponding to lightly damped mechanical systems. The method has intuitive appeal and is easy to apply.

A number of methods for model reduction have been used in the past with varying degrees of success. Modal truncation, for example, has been widely used in structures research. In this approach, the equations of motion are transformed into a set of uncoupled second order equations each of which describes the dynamics of one mode of vibration. Model reduction is accomplished by deleting those modes with the highest damped or undamped natural frequencies. The coupling of each mode to inputs or outputs does not play a role in the truncation procedure. The method is straightforward in its application and has intuitive appeal. It is an example of a method where reduction is accomplished by truncation after an appropriate coordinate transformation.

Direct model reduction procedures have been proposed which do not involve truncation. In these approaches, the analyst decides in advance what order of reduced model he wants and then uses a numerical optimization procedure to obtain the reduced model of the given order which minimizes a specified measure of model accuracy. Methods of this type have recently been developed by Hyland and Bernstein for application to reduced order controller design [HB1]. Their approach is called the Maximum Entropy Method, and it involves the simultaneous solution of two modified Riccati equations and two modified Liapunov equations. Numerical routines for solving these equations are not yet standard, but the authors have developed algorithms which appear to be effective.

The introduction of the method of Balanced Realizations in 1980 revitalized the field of model reduction and led to renewed research. Various modifications and extensions have been proposed and studied. Enns developed an extension for frequency weighted balanced realizations [En1] whereby the analyst could modify the basic balanced realization procedure to obtain relatively better reduced model accuracy in certain frequency ranges where model accuracy was critical.

50

Another approach to model reduction is the optimal Hankel norm approximation which is developed in great detail by Glover et al. in [GL1,Gl1]. Hankel norm approximations are closely related to approximations based on balanced realizations, and in fact begin by expressing the system as a balanced realization. Although a smaller error bound can be established for Hankel norm approximations, the computations required are substantially more involved than those for balanced realizations. Another feature of the Hankel norm approximation is that the reduced model may not be strictly proper even when the original model is. It is possible to alter the computations to ensure a strictly proper reduced model, but the required changes increase the error bound.

New techniques for model reduction continue to emerge. Five model reduction techniques (including balanced realizations, Hankel norm approximations, an approach by Davis and Skelton, an approach by Yousuff and Skelton and a new method by Liu and Anderson) have been compared recently by Liu and Anderson [LA1]. Although some of the methods generally performed better than others, there was no one method which was always superior. Thus, model reduction is still an area of active research. The present chapter does not attempt to examine and compare the many model reduction techniques currently available. Instead, its scope is limited to the method of balanced realizations. This approach has been found to compare favorably with other methods, and its computational simplicity makes it attractive for present applications.

5.2 *Observability and Controllability; Observability and Controllability Grammians*

Balanced realizations are based on the notions of controllability and observability of linear systems. Thus we begin by defining these terms and exploring their meaning. Suppose a physical system is described by the state space equations:

51

(5.2.1) $\dot{x} = Au + Bu$

(5.2.2) $y = Cx$

where

x = n x 1 state vector

y = r x 1 output/measurement vector

u = m x 1 input/disturbance vector

A, B and C are constant matrices of appropriate dimension.

To introduce the notion of controllability, one might ask the following question: "Can one find an input u(t) which drives the system (5.2.1) from an initial state $x(t_0) = 0$ to an arbitrary final state x_f in finite time t_f?" If the answer to this question is yes, the system is said to be *controllable*. Otherwise it is said to be *uncontrollable*. To determine whether a system is controllable, first note that if $x(t_0)$ is zero, then

$$(5.2.3) \qquad x(t) = \int_{t_0}^{t_f} e^{A(t-\tau)} B \, u(\tau) d\tau$$

Introduce a symmetric matrix $W_c(t_0,t)$ defined as:

$$(5.2.4) \qquad W_c(t_0,t) \triangleq \int_{t_0}^{t} e^{A(t-\tau)} BB^T e^{A^T(t-\tau)} d\tau$$

$W_c(t_0,t)$ is called the *controllability grammian*. If $W_c(t_0,t_f)$ is nonsingular, a control input $u_0(t)$ which drives x from the origin to x_f at time t_f is given by

$$(5.2.5) \qquad u_0(t) = B^T e^{A^T(t_f-t)} W_c^{-1}(t_0,t_f) x_f$$

This can be verified by substituting (5.2.5) into (5.2.3). Thus, if $W_c(t_0,t_f)$ is nonsingular, the system described by (5.2.1) is *controllable*. In the event

that $W_c(t_0, t_f)$ is singular, the system is said to be *uncontrollable.*

The control input given by (5.2.5) is not the only one that will drive x from the origin to x_f at time t_f. Let $u_1(t)$ be an input which drives x from the origin through an arbitrary trajectory and back to the origin at time t_f. Because of the linearity of (5.2.1), any input which drives x from 0 to x_f at time t_f can be written as

(5.2.6) $u(t) = u_0(t) + u_1(t)$

Among all such inputs, $u_0(t)$ is the one which minimizes the integral

(5.2.7) $E = \int_{t_0}^{t_f} u^T(\tau)u(\tau)d\tau$

E may be regarded as a measure of control effort, and $u_0(t)$ interpreted as the *minimum effort* input which accomplishes the desired change of state from 0 to x_f. Substituting $u_0(t)$ into equation (5.2.7) and evaluating the integral yields the minimum value of E.

(5.2.8) $E_{min} = x_f^T W_c^{-1}(t_0, t_f)x_f$

Three observations will be useful for subsequent work.

Observation 1

The control effort becomes small (i.e., the system becomes easier to control) when $W_c(t_0, t_f)$ becomes large.

Observation 2

$W_c(t_0, t_f)$ can be expressed as:

$$(5.2.9) \qquad W_c(t_0, t_f) = \int_0^{t_f - t_0} e^{A\lambda} BB^T e^{A^T\lambda} d\lambda$$

To see this, let $\lambda = t_f - \tau$ in (5.2.4).

Observation 3

If a system is controllable, it can be driven from an arbitrary initial state to an arbitrary final state. This can be seen by starting from an arbitrary initial state x_0, and expressing the final state as

$$(5.2.10) \qquad x(t_f) = e^{A(t_f - t_0)} x_0 + \int_{t_0}^{t_f} e^{A(t_f - \tau)} B\, u(\tau) d\tau$$

So,

$$(5.2.11) \qquad x(t_f) - e^{A(t_f - t_0)} x_0 = \int_{t_0}^{t_f} e^{A(t_f - \tau)} B\, u(\tau) d\tau$$

This shows that the control which drives the system from x_0 to $x(t_f)$ is the same as that which drives the system from the origin to the final state x_f where

$$(5.2.12) \qquad x_f = x(t_f) - e^{A(t_f - t_0)} x_0$$

The concept of controllability has several things in common with that of observability, but it has some differences as well. Assume A, B and C in Eqs. (5.2.1, 5.2.2) are known, and that the input $u(t)$ and the output $y(t)$ are known on $t_0 \leq t \leq t_f$. Then ask the question, "Can one determine the initial state x_0 from this information?" If the answer to this question is yes, the system is said to be *observable*. If the answer is no, the system is *unobservable*. Observability also implies that knowledge of the input and output over any interval $t_0 \leq t \leq t_f$ is sufficient to determine $x(t_f)$.

To develop the idea of observability, let

$$(5.2.13) \qquad \underset{\sim}{x}(t) \overset{\Delta}{=} x(t) - \int_{t_0}^{t} e^{A(t-\tau)}B\, u(\tau)d\tau$$

$$(5.2.14) \qquad \underset{\sim}{y}(t) \overset{\Delta}{=} C\, \underset{\sim}{x}(t)$$

Then,

$$(5.2.15) \qquad \underset{\sim}{y}(t) \overset{\Delta}{=} y(t) - \int_{t_0}^{t} C\, e^{A(t-\tau)}B\, u(\tau)d\tau$$

Since both terms on the right side of (5.2.15) are known, $\underset{\sim}{y}(t)$ may be regarded as known also. Now,

$$\dot{\underset{\sim}{x}} = \dot{x} - Bu - A\int_{t_0}^{t} e^{A(t-\tau)}B\, u(\tau)d\tau$$

$$= Ax + Bu - Bu - A\int_{t_0}^{t} e^{A(t-\tau)}B\, u(\tau)d\tau$$

$$= A\left[x - \int_{t_0}^{t} e^{A(t-\tau)}B\, u(\tau)d\tau \right]$$

$$(5.2.16) \qquad = A\, \underset{\sim}{x}$$

and

$$(5.2.17) \qquad \underset{\sim}{x}(t_0) = x(t_0)$$

Hence, the plant equations may be written in terms of $\underset{\sim}{x}$ and $\underset{\sim}{y}$ as

$$(5.2.18) \qquad \dot{\underset{\sim}{x}} = A\underset{\sim}{x}, \qquad \underset{\sim}{x}(t_0) = x(t_0)$$

$$(5.2.19) \qquad \underset{\sim}{y} = C\underset{\sim}{x}$$

The observability of (5.2.18) and (5.2.19) is the same as that of (5.2.13) and (5.2.14) since the ability to determine $\underset{\sim}{x}(t_0)$ given $\underset{\sim}{y}(t)$ on $[t_0, t_1]$ implies and

55

is implied by the ability to determine $x(t_0)$ given $y(t)$ on $[t_0, t_1]$ and $u(t)$ on $[t_0, t_1]$. In the subsequent development we will work with (5.2.18) and (5.2.19) because of their simpler form.

From (5.2.18) and (5.2.19), $\tilde{y}(t)$ may be written as

$$(5.2.20) \qquad \tilde{y}(t) = C\, e^{A(t-t_0)} \tilde{x}(t_0) \qquad\qquad t \le t \le t_f$$

Define the observability Grammian as

$$(5.2.21) \qquad W_0(t_0, t) \triangleq \int_{t_0}^{t} e^{A^T(\tau-t_0)} C^T C\, e^{A(\tau-t_0)}\, d\tau$$

Multiply (5.2.20) by $e^{A^T(t-t_0)} C$ and integrate from t_0 to t_f.

$$(5.2.22) \qquad \int_{t_0}^{t_f} e^{A^T(\tau-t_0)} C^T \tilde{y}(\tau)\,d\tau = \int_{t_0}^{t_f} e^{A^T(\tau-t_0)} C^T C\, e^{A(\tau-t_0)}\, d\tau\ \tilde{x}(t_0)$$

$$= W_0(t_0, t_f)\ \tilde{x}(t_0)$$

If $W_0(t_0, t_f)$ is nonsingular, and $\tilde{y}(t)$ is known on $[t_0, t_f]$, then (5.2.22) can be solved for $\tilde{x}(t_0)$.

$$(5.2.23) \qquad \tilde{x}(t_0) = W_0^{-1}(t_0, t_f) \int_{t_0}^{t_f} e^{A^T(\tau-t_0)} C^T \tilde{y}(\tau)\,d\tau$$

Hence, the system (5.2.18), (5.2.19) is *observable* if $W_0(t_0, t_f)$ is nonsingular. If $W_0(t_0, t_f)$ is singular, the system is unobservable.

For interpretation of results in the sequel, it will be useful to introduce a scalar measure of the output $y(t)$ analogous to the scalar measure of the input given in Eq. (5.2.7). Define a response function R as

$$(5.2.24) \qquad R \overset{\Delta}{=} \int_{t_0}^{t_f} \tilde{y}^T(\tau)\tilde{y}(\tau)d\tau$$

Substituting from (5.2.20) and (5.2.21),

$$\int_{t_0}^{t_f} \tilde{y}^T(\tau)\;\tilde{y}(\tau)\;d\tau = \tilde{x}^T(t_0) \int_{t_0}^{t_f} e^{A^T(\tau-t_0)}\;C^TC\;e^{A(\tau-t_0)}\;d\tau\;\tilde{x}(t_0)$$

$$(5.2.25) \qquad\qquad\qquad = \tilde{x}^T(t_0)\;W_0(t_0,t_f)\tilde{x}(t_0)$$

Thus, $W_0(t_0,t_f)$ provides a measure of the size of the response corresponding to an initial condition $\tilde{x}(t_0)$.

If the A matrix in Eq. (5.2.1) is stable (i.e., all of the eigenvalues of A are in the left half plane), then $W_c(t_0,t_f)$ and $W_0(t_0,t_f)$ remain finite as $t_f \to \infty$. Define

$$(5.2.26) \qquad W_c = W_c(0,\infty) = \int_0^\infty e^{A\tau}BB^Te^{A^T\tau}\;d\tau$$

$$(5.2.27) \qquad W_0 = W_0(0,\infty) = \int_0^\infty e^{A^T\tau}\;C^TCe^{A\tau}\;d\tau$$

It is not difficult to show that W_c and W_0 satisfy the Liapunov Equations:

$$(5.2.28) \qquad A\;W_c + W_c\;A^T + BB^T = 0$$

$$(5.2.29) \qquad W_0\;A + A^T\;W_0 + C^TC = 0$$

Since well documented numerical procedures are available for solving these equations [BS1], finding W_c and W_0 for the case where $t_f \to \infty$ is most easily accomplished by solving (5.2.28) and (5.2.29) rather than attempting direct evaluation of the integrals given in Eqs. (5.2.9) and (5.2.21). If the given A

matrix is not completely stable, then A must be partitioned into a stable part and an unstable part. Eqs. (5.2.28) and (5.2.29) may be used to find the controllability and observability grammians associated with the stable part of A in the infinite time case.

5.3 Balanced Realizations

It is always possible to express the system equations (5.2.1) and (5.2.2) in a different but equivalent form through the use of a coordinate transformation:

(5.3.1) $\bar{x} = T x$

where T is a nonsingular matrix. This transformation leads to

(5.3.2) $\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B} u$

(5.3.3) $y = \bar{C}\bar{x}$

where

(5.3.4) $\bar{A} = T^{-1}A\,T, \quad \bar{B} = T^{-1}B, \quad \bar{C} = C\,T$

There are many objectives one might have in performing such a transformation, e.g., to put the transformed system into diagonal or block diagonal form, to put it in controllability canonical form [Ka1, pp. 49-55], etc. When the transformed model is to be used for control system design, the *Balanced Realization* of Moore [Mo1] is appealing because expressing equations in this form allows one to compare the relative controllability and observability of individual states. This information is useful in deciding which states are most important to retain in a reduced model.

Suppose the original system matrix A is stable. (If this is not the case, then A must be partitioned into a stable and unstable part as noted above. The

discussion to follow applies to the stable part.) To cast the equations into balanced form, we seek a transformation matrix T such that W_c and W_o in Eqs. (5.2.26) and (5.2.27) are diagonal and equal. Suppose W_c and W_o are found by solving the Liapunov Equations (5.2.28) and (5.2.29) for the untransformed system. $\bar{W}_c$ and $\bar{W}_o$ for the transformed system then become

$$(5.3.5) \qquad \bar{W}_c = T^{-1}W_c T^{-T}; \qquad \bar{W}_o = T^T W_o T$$

The transformation matrix T is found by considering the eigenvalue problem

$$(5.3.6) \qquad [W_o - \lambda W_c^{-1}]\gamma = 0$$

Assuming distinct eigenvalues, the eigenvectors satisfy

$$(5.3.7) \qquad \gamma_j^T W_c^{-1} \gamma_i = 0; \qquad \gamma_j^T W_o \gamma_i = 0 \qquad\qquad i \neq j$$

Form a square matrix, Γ, from the eigenvectors of (5.3.6). The matrix Γ will satisfy

$$(5.3.8) \qquad \Gamma^T W_o \Gamma = \text{diagonal}, \qquad \Gamma^{-1} W_c \Gamma^{-T} = \text{diagonal}$$

We have reached part of our goal by finding a matrix which diagonalizes both W_c and W_o. However, the columns of Γ are not yet uniquely defined because we have not specified a normalization. This remaining degree of freedom is used to make W_c and W_o equal. Normalize the columns of Γ such that

$$(5.3.9) \qquad \gamma_i W_o \gamma_i = (\lambda_i)^{\frac{1}{2}} \qquad\qquad i = 1,2,\ldots,n$$

Then,

$$(5.3.10) \qquad W_c = W_o = \begin{bmatrix} \lambda_1 & & 0 \\ & \cdot & \cdot & \\ 0 & & \lambda_n \end{bmatrix}^{\frac{1}{2}}$$

The λ_i's are called Hankel singular values. Thus, when the original system is transformed using $T = \Gamma$, observability and controllability grammians satisfy (5.3.10). The transformed system is said to be a balanced realization.

5.4 Model Reduction Using Balanced Realizations

We now consider the interpretation of W_c and W_0 and examine how these matrices can be used to motivate a model reduction procedure. For a balanced realization, W_c and W_0 are diagonal and satisfy (5.3.10). For this situation, the control effort function E and the response function R introduced in section 5.2 become

$$(5.4.1) \qquad E = \int_0^\infty u^T u \, d\tau = \sum_{i=1}^n \frac{x_{fi}^2}{\lambda_i} \; ; \qquad R = \int_0^\infty y^T y \, d\tau = \sum_{i=1}^n x_{0i}^2 \lambda_i$$

If $\lambda_i > \lambda_j$, we can argue that less effort is required to move from the origin to x_{fi} than to move from the origin to x_{fj}. Thus, x_{fi} is more strongly affected by u than x_{fj}, i.e., x_{fi} is more controllable. Similarly, we can argue that an initial condition on x_{0i} results in a larger contribution to the response than an initial condition on x_{0j}, i.e., x_{0i} is more observable than x_{0j}. Taking these observations together, we see that a state x_i with a large λ_i is more controllable and observable than a state x_j with a small λ_j. If model truncation is contemplated, it is intuitively appealing to retain highly controllable and observable states, i.e., states with large λ's. Thus states corresponding to small values of λ are candidates for truncation. If there is no gap separating the states with large λ's from those with small λ's, it may not be clear where the division should occur between the retained states and the discarded states. This issue may be resolved by considering the convergence of control and estimator gains. See Chapters 6-9 and [MG1].

5.5 Asymptotic Expansions for the Balanced Singular Values of Lightly Damped Mechanical Systems

As explained in Section 4.2, calculating W_c and W_0 involves the solution of Liapunov equations. When Eqs. (5.2.1) and (5.2.2) describe a stable, lightly

damped mechanical system, the approximate solution of these Liapunov Equations may be accomplished using perturbation methods. This section concerns the development of first and second order perturbation solutions for W_c, W_o and their balanced singular values.

The mechanical systems addressed in this section are described by second order equations of the form:

(5.5.1) $M \ddot{q} + \epsilon D \dot{q} + K q = bu$

(5.5.2) $y = c_p q + c_v \dot{q}$

M, D and K are symmetric and positive definite n x n matrices with M being the mass matrix, D the damping matrix and K the stiffness matrix. (At the expense of some additional complexity, the assumptions that D and K are symmetric could be relaxed in the development to follow. Ref. [BM1] addresses the case where a skew symmetric gyroscopic matrix G is added to D and a skew symmetric cir- culatory matrix F is added to K. The present development is restricted to the case where G and F are absent.) The matrix b is the input distribution matrix, and u is the input vector representing either the control or disturbance input. The output vector is y, and c_p and c_v are the position and velocity distribution matrices, respectively. The scalar ϵ is a small parameter reflecting the light damping. By an appropriate coordinate transformation, these equations can be cast into a more convenient modal form:

(5.5.3) $\ddot{\eta} + \epsilon \Delta \dot{\eta} + \Omega^2 \eta = \beta u$

(5.5.4) $y = C_p \dot{\eta} + C_v \eta$

In Eqs. (5.5.3) and (5.5.4), Ω is a diagonal matrix of modal frequencies. The modal frequencies are assumed to be distinct. The damping matrix, Δ, is sym-

metric and positive definite, but it is not presumed to be diagonal. The matrices β, C_p and C_v are the input and output distribution matrices for the normalized system. For the development to follow, we wish to express Eqs. (5.5.3) and (5.5.4) in state space form. Let

$$(5.5.5) \qquad x = \left\{ \begin{matrix} \eta \\ \dot{\eta} \end{matrix} \right\} ; \qquad A = \begin{bmatrix} 0 & I \\ -\Omega^2 & -\epsilon\Delta \end{bmatrix} ; \qquad B = \begin{bmatrix} 0 \\ \beta \end{bmatrix}$$

$$C = [C_p \ C_v]$$

Then the state space version of (5.5.3) and (5.5.4) becomes:

$$(5.5.6) \qquad \dot{x} = Ax + Bu$$

$$(5.5.7) \qquad y = Cx$$

Consider first the solution of the Liapunov Equation (5.2.28) for W_c. Because of the special structure of A and B, it is useful to express W_c in the partitioned form:

$$(5.5.8) \qquad W_c = \begin{bmatrix} \dfrac{W_{11}}{\epsilon} & W_{12} \\ \\ W_{12}^T & \dfrac{W_{22}}{\epsilon} \end{bmatrix}$$

The scalar ϵ is introduced in Eq. (5.5.8) for convenience in the subsequent development. Substitution of (5.5.5) and (5.5.8) into (5.2.28) leads to three matrix equations:

$$(5.5.9) \qquad W_{12}^T + W_{12} = 0$$

$$(5.5.10) \qquad W_{22} - W_{11}\,\Omega^2 = \epsilon^2\,W_{12}\Delta$$

$$(5.5.11) \qquad -(\Omega^2 W_{12} + W_{12}^T\Omega^2) - (\Delta\,W_{22} + W_{22}\Delta) + \beta\beta^T = 0$$

62

These equations lead to some interesting observations. First we see that W_{12} is skew symmetric. This results from the fact that the state vector x has the special form $x = [\eta^T \; \dot{\eta}^T]^T$, i.e., the bottom half of the state vector is the derivative of the top half.

A second observation is that Eqs. (5.5.9-5.5.11) contain ϵ^2 but not ϵ. This means that W_{11}, W_{12} are functions of ϵ^2 and expansions of these matrices should be in terms of ϵ^2 and not ϵ.

With these observations in mind, we assume an expansion for W_{ij} in the form:

$$(5.5.12) \qquad W_{ij} = u_{ij} + \epsilon^2 v_{ij} + \epsilon^4 w_{ij} + \ldots$$

Substituting this expansion into Eqs. (5.5.9-5.5.11) and setting coefficients of like powers of ϵ^2 to zero leads to a set of equations which can be solved sequentially.

ϵ^0 terms:

$$(5.5.13) \qquad u_{12}^T + u_{12} = 0$$

$$(5.5.14) \qquad u_{22} - u_{11}\Omega^2 = 0$$

$$(5.5.15) \qquad -(\Omega^2 u_{12} + u_{12}^T \Omega^2) - (\Delta u_{22} + u_{22}\Delta) + \beta\beta^T = 0$$

ϵ^2 terms:

$$(5.5.16) \qquad v_{12}^T + v_{12} = 0$$

(5.5.17) $v_{22} - v_{11}\Omega^2 = u_{12}\Delta$

(5.5.18) $-(\Omega^2 v_{12} + v_{12}^T\Omega^2) - (\Delta v_{22} + v_{22}\Delta) = 0$

ϵ^4 terms:

(5.5.19) $w_{12}^T + w_{12} = 0$

(5.5.20) $w_{22} - w_{11}\Omega^2 + w_{12}\Delta = 0$

(5.5.21) $-(\Omega^2 w_{12} + w_{12}^T\Omega^2) - (\Delta w_{22} + w_{22}\Delta) = 0$

etc.

Define ω_i^2 as the ith diagonal element of Ω^2, and $(\beta\beta^T)_{ij}$, Δ_{ij} and $(u_{k\ell})_{ij}$ as the ij elements of $\beta\beta^T$, Δ and $u_{k\ell}$, respectively. Then, using Eqs. (5.5.13 - 5.5.15) and the assumption that the modal frequencies are distinct, the elements of u_{11} become:

(5.5.22) $(u_{11})_{ii} = (\beta\beta^T)_{ii}/(2\omega_i^2\Delta_{ii})$; $(u_{11})_{ij} = 0$ for $i \neq j$

(u_{11} is diagonal.) From (5.5.22) and (5.5.14), u_{22} becomes:

(5.5.23) $u_{22} = u_{11}\Omega^2$

(u_{22} is also diagonal.)

With the elements of u_{22} known, introduce a matrix P defined as:

(5.5.24) $P \triangleq \beta\beta^T - (\Delta u_{22} + u_{22}\Delta)$

(Note that $P_{ii} = 0$.) The elements of u_{12} then become

(5.5.25) $(u_{12})_{ij} = P_{ij}(\omega_i^2 - \omega_j^2)$; $i \neq j$

(5.5.26) $(u_{12})_{ii} = 0$

Next introduce matrices S and T based on the known solution for u_{12}.

(5.5.27) $S \stackrel{\Delta}{=} u_{12}\Delta - \Delta u_{12}^T$

(5.5.28) $T \stackrel{\Delta}{=} -(\Delta^2 u_{12}^T + u_{12}\Delta^2)$

Then, from (5.5.16 - 5.5.18), the elements of v_{11} become

(5.5.29) $(v_{11})_{ij} = S_{ij}/(\omega_i^2 - \omega_j^2)$; $i \neq j$

(5.5.30) $(v_{11})_{ii} = \dfrac{T_{ii} - 2\displaystyle\sum_{k=1}^{i-1}(\Delta_{ik}\omega_k^2(v_{11})_{ki}) - 2\displaystyle\sum_{k=i+1}^{n}(\Delta_{ik}\omega_k^2(v_{11})_{ki})}{2\Delta_{ii}\omega_i^2}$

Using this expression together with (5.5.17) we obtain

(5.5.31) $v_{22} = v_{11}\Omega^2 + u_{12}\Delta$

The expansion will not be carried beyond this point because the expressions become increasingly unwieldy. However, the expressions obtained so far permit one to write out the first three terms in the expansion for W_c. To see this, recall that

(5.5.32) $W_c = \begin{bmatrix} \dfrac{W_{11}}{\epsilon} & W_{12} \\[2mm] W_{12}^T & \dfrac{W_{22}}{\epsilon} \end{bmatrix} = \begin{bmatrix} \frac{1}{\epsilon}[u_{11} + \epsilon^2 v_{11} + \dots] & [u_{12} + \epsilon^2 v_{12} + \dots] \\[2mm] [u_{12}^T + \epsilon^2 v_{12}^T + \dots] & \frac{1}{\epsilon}[u_{22} + \epsilon^2 v_{22} + \dots] \end{bmatrix}$

65

$$= \frac{1}{\epsilon} \begin{bmatrix} u_{11} & 0 \\ 0 & u_{22} \end{bmatrix} + \begin{bmatrix} 0 & u_{12} \\ u_{12}^T & 0 \end{bmatrix} + \epsilon \begin{bmatrix} v_{11} & 0 \\ 0 & v_{22} \end{bmatrix} + \dots$$

The expressions for u_{11}, u_{12}, u_{22}, V_{11} and V_{22} are given in Eqs. (5.5.22)-(5.5.31).

For lightly damped structures (small ϵ), the formulas given above provide an efficient way to compute an approximation of the controllability grammian. Using a similar approach, it is possible to develop approximate formulas for the observability grammian (see Appendix A). The resulting expressions for W_c and W_o may be used to determine the balanced realization. Examination of (5.5.32) and the corresponding formula for the observability grammian reveals that as ϵ gets smaller, causing the first terms in the expansions to dominate, W_c and W_o tend to become diagonal. Thus, the product $(W_c W_o)$ tends to become diagonal, and the diagonal elements approximate the squares of the balanced singular values. Using the results of this chapter and Appendix A, one can write these approximate expressions as

$[i^{th}$ balanced singular value$]$

$$(5.5.33) \qquad \doteq \frac{1}{\epsilon} \left\{ \frac{(\beta \beta^T)_{ii} \ [(C_p^T C_p)_{ii} + \omega_i^2 (C_v^T C_v)_{ii}]}{4 \ \omega_i^2 \ \Delta_{ii}^2} \right\}^{\frac{1}{2}}$$

Thus when ϵ is small, the modal coordinates themselves constitute a balanced realization, and the ith balanced singular value is a measure of the importance of the ith mode of the model. A similar result was obtained by Gregory [Gr1] using a different approach.

Subsequent chapters will address the question of how many modes and which modes must be included in a structural model in order for the compensator to converge and stabilize the nominal plant. The procedure for doing this involves beginning with a model of finite size and gradually adding modes to the model until the compensator converges. The speed of convergence and the ultimate size of the required model are dependent in part on how good the analyst is in selecting the modes to be added. The approximate formulas given above allow one to identify which modes are likely to be the most important (the highly controllable and observable ones with large balanced singular values) so they can be added to the model first. This procedure helps keep the size of the model from getting too large.

5.6 Internal Balancing in Infinite Dimensions

The idea of internal balancing in finite dimensions can be extended to infinite dimensions. This extension will yield *balanced states* for distributed systems, and these states will be vectors in the abstract state spaces (Hilbert spaces) in which the distributed models are formulated.

Consider the control system

$$(5.6.1) \qquad \dot{x} = A x + B u , \qquad y = C x$$

where x is in a Hilbert space H, A generates an exponentially stable semigroup on H, and B and C are bounded linear operators of finite rank. In applications, B and C have finite rank because the number of actuators and the number of sensors are finite. The controllability and observability grammian operators W_c and W_o are the solutions to the operator equations

$$(5.6.2) \qquad A^*W_c + W_cA + BB^* = 0 \quad \text{and} \quad A W_o + W_oA^* + C^*C = 0 .$$

Since B and C have finite rank, both W_c and W_o are compact and selfadjoint. It can be shown, then, that the eigenvalue problem

$$(5.6.3) \qquad W_oW_c z_i = s_i^2 z_i$$

has a countable number of solutions, that the singular values s_i approach zero as i increases, and that the eigenvectors z_i are mutually orthogonal with respect to both W_c and W_o and complete in H. Note that these properties are quite similar to those of the eigenvalue problem for the natural modes of a flexible structure. The numerical problem of computing the balanced states of an infinite dimensional control system is also analogous to that of computing the natural models of a structure.

The results alluded to here could be developed more fully and explored with respect to developing efficient numerical methods for computing the balanced states of the distributed system. The approximation theory for computing balanced states will be especially important. To the authors' knowledge, work has not been carried out on this topic.

6. The Infinite Dimensional LQG Problem

Section 6.1 of this chapter presents some preliminary definitions and results for the optimal linear-quadratic regulator problem on an arbitrary real Hilbert space E. These results are generic in the sense that E is not necessarily the Energy space of Chapters 3 and 4, and the operators A, B, etc., do not necessarily represent a flexible structure. In subsequent chapters, such generic results facilitate derivation of approximation theory for the infinite dimensional state estimator from the analogous approximation theory for the control problem.

Section 6.2 discusses an infinite dimensional state estimator for the generic plant in Section 6.1, an infinite dimensional compensator and the corresponding closed-loop system. The purpose of Section 6.2 is to establish certain properties of the estimator, compensator and closed-loop system that do not depend on the control and estimator gains being optimal. Therefore, the gains in Section 6.2 are allowed to be arbitrary bounded linear operators between appropriate spaces, and neither the estimator nor the compensator is necessarily optimal.

Section 6.3 defines the LQG-optimal infinite dimensional estimator and compensator, still for a generic plant. Finally, Section 6.4 gives some important implications of the general results for the case where the plant is the infinite dimensional flexible structure model defined in Chapter 3.

Most of the results in this chapter are analogous to results for the finite dimensional LQG optimal control problem, with infinite dimensional operators corresponding to finite dimensional matrices. In particular, instead of the control and estimator gain matrices in finite dimensions, here we have gain

operators. The functional gains in Section 6.4 can be thought of as a generalization of the transposes of finite dimensional gain matrices, although the relationship between functional gains and infinite dimensional gain operators is not so simple.

6.1 The Generic Optimal Regulator Problem

Let a linear operator A generate a C_0-semigroup $T(t)$ on a real Hilbert space E, and suppose that $B \in L(R^m, E)$, that $Q \in L(E)$ is nonnegative and selfadjoint and that R is a positive definite, symmetric real m x m matrix. The *optimal control problem on E* is to choose the control function $u \in L_2(0, \infty; R^m)$ to minimize the performance index

$$(6.1.1) \qquad J(z(0),u) = \int_0^\infty (\langle Q\ z(t),\ z(t) \rangle_E + u(t)^T R\ u(t))\ dt$$

where the state z(t) satisfies

$$(6.1.2) \quad z(t) = T(t)\ z(0) + \int_0^t T(t-s)Bu(s)\ ds, \qquad\qquad t \geq 0.$$

Definition 6.1. A function $u \in L_2(0, \infty; R^m)$ is an *admissible control for the initial state z*, or simply an *admissible control for z*, if J(z,u) is finite; i.e., if the state z(t) corresponding to the control u(t) and the initial condition z(0) = z is in $L_2(0, \infty; E)$.

Definition 6.2. Let the operators A, B, Q and R be as defined above. An operator Π in L(E) is a solution of the Riccati algebraic equation if Π maps the domain of A into the domain of A^* and satisfies the Riccati algebraic equation

$$(6.1.3) \qquad A^* \Pi + \Pi A - \Pi BR^{-1}B^* \Pi + Q = 0. \quad \nabla\nabla\nabla$$

Theorem 6.3 (Theorems 4.6 and 4.11 of [Gi4]). There exists a nonnegative selfadjoint solution of the Riccati algebraic equation if and only if, for each z ∈ E, there is an admissible control for the initial state z. If Π is the minimal nonnegative selfadjoint solution of (6.1.3), then the unique control u(·) that minimizes J(z,u) and the corresponding optimal trajectory z(·) are given by

$$(6.1.4) \qquad u(t) = - R^{-1}B^{\star}\Pi \, z(t)$$

and

$$(6.1.5) \qquad z(t) = S(t) \, z,$$

where S(t) is the semigroup generated by $(A - BR^{-1}B^{\star}\Pi)$. Also,

$$(6.1.6) \qquad J(z,u) = \min_{v} J(z,v) = \langle \Pi \, z, \, z \rangle_E .$$

If, for each initial state and admissible control,

$$(6.1.7) \qquad \lim_{t \to \infty} |z(t)| = 0,$$

there exists at most one nonnegative selfadjoint solution of (6.1.3). If Q is coercive, (6.1.7) holds for each initial state and admissible control and S(t) is uniformly exponentially stable. ▽▽▽

We will refer to T(t) as the *open-loop semigroup* and to S(t) as the *closed-loop semigroup.*

6.2 An Infinite Dimensional Estimator and the Corresponding Compensator and Closed-Loop System

The differential equation corresponding to (6.1.2) is, of course,

$$(6.2.1) \qquad \dot{z}(t) = Az(t) + Bu(t), \quad t > 0.$$

While (6.1.2) is valid for any $u \in L_2(0, \infty; R^m)$, the sense in which (6.2.1) holds -- when it holds -- depends on how smooth u is. Thus it is more precise to write (6.1.2) for arbitrary L_2 controls, but continuing to write (6.1.2) in this section, along with the similar integral equations for the estimator and closed-loop system, would be too cumbersome. Therefore, we will write the differential equations with the understanding that when necessary they can be interpreted as representing the appropriate integral equations.

We assume that we have a p-dimensional measurement vector y(t) given by

$$(6.2.2) \qquad y(t) = C_0 u(t) + Cz(t),$$

where $C_0 \in L(R^m, R^p)$ and $C \in L(E, R^p)$ for some positive integer p.

Definition 6.4. For any $\hat{F} \in L(R^p, E)$, the system
$$(6.2.3) \qquad \dot{\hat{z}}(t) = A\hat{z}(t) + Bu(t) + \hat{F}[y(t) - C_0 u(t) - C\hat{z}(t)], \qquad t > 0,$$

will be called an estimator, or observer, for the system (6.2.1)-(6.2.2). Let S(t) be the semigroup generated by A-$\hat{F}$C. The observer in (6.2.3) is strongly (uniformly exponentially) stable if $\hat{S}(t)$ is strongly (uniformly exponentially) stable. ∇∇∇

To justify this definition, we write

$$(6.2.4) \qquad e(t) = z(t) - \hat{z}(t)$$

and, with (6.2.1)-(6.2.3), obtain

$$(6.2.5) \qquad e(t) = \hat{S}(t) e(0), \qquad\qquad\qquad t > 0.$$

Of course, an estimator is necessary because the full state z(t) will not be available for direct feedback, and the feedback control must be based on an estimate such as $\hat{z}(t)$. When the desired control law has the form

(6.2.6) $u(t) = Fz(t)$

for some $F \in L(E,R^m)$, the observer in (6.2.3) can be used to construct $z(t)$ from the measurement in (6.2.2) and then the control law in (6.2.6) can be applied to $\hat{z}(t)$. The control applied to the system is then

(6.2.7) $u(t) = F\hat{z}(t)$,

and the resulting closed-loop system satisfies

(6.2.8) $\begin{pmatrix} z(t) \\ \hat{z}(t) \end{pmatrix} = S_{\infty\infty}(t) \begin{pmatrix} z(0) \\ \hat{z}(0) \end{pmatrix}$ $t \geq 0$,

where $S_{\infty\infty}(t)$ is the semigroup generated on $E \times E$ by the operator

(6.2.9) $A = \begin{bmatrix} A & -BF \\ \hat{F}C & [A-BF-F\hat{C}] \end{bmatrix}$, $D(A_{\infty\infty}) = D(A) \times D(A)$.

With the estimator error $e(t)$ defined by (6.2.4), it is easy to show that (6.2.8) is equivalent to (6.2.5) and

(6.2.10) $\dot{z}(t) = (A-BF) z(t) + BF\, e(t)$, $t > 0$,

where $(A-BF)$ generates a semigroup $S(t)$ on E. Also, it is easy to prove the following.

Theorem 6.5. Suppose that there exist positive constants M_1, M_2, a_1 and a_2 such that

(6.2.11) $|S(t)| \leq M_1 e^{-a_1 t}$, $|S(t)| \leq M_2 e^{-a_2 t}$, $t > 0$.

Then, for each real $a_3 < \min \{a_1, a_2\}$, there exists a constant M_3 such that

(6.2.12) $|S_{\infty\infty}(t)| \leq M_3 e^{-a_3 t}$, $t \geq 0$.

73

Also,

$$(6.2.13) \qquad \sigma(A_{\infty\infty}) = \sigma(A-BF) \cup \sigma(A-\hat{F}C) ,$$

where $\sigma(A_{\infty\infty})$ is the spectrum of $A_{\infty\infty}$.

The observer in (6.2.3) and the control law in (6.2.7) constitute a compensator for the control system in (6.2.1) and (6.2.2). The transfer function of this compensator is

$$(6.2.14) \qquad \Phi(s) = -F(sI - [A-BF+\hat{F}(C_0F-C)])^{-1} \hat{F},$$

which is an m x p matrix function of the complex variable s. When E has infinite dimension, the compensator transfer function is irrational, except in degenerate, usually unimportant cases. Figure 6.1 shows the block diagram of the closed-loop system produced by applying this compensator to the plant in (6.2.1) and (6.2.2).

$$\dot{z} = A\,z + B\,u$$

$$y = C_0\,u + C\,z$$

$$y = (C_0 + C\,[sI - A]^{-1}\,B)\,u$$

Infinite Dimensional Plant

$$\dot{\hat{z}} = [A-BF+\hat{F}(C_0F-C)]\,\hat{z} + \hat{F}\,y$$

$$u = -\,F\,\hat{z}$$

$$u = -\,F(sI - [A-BF+\hat{F}(C_0F-C)])^{-1}\,\hat{F}\,y$$

Infinite Dimensional Compensator
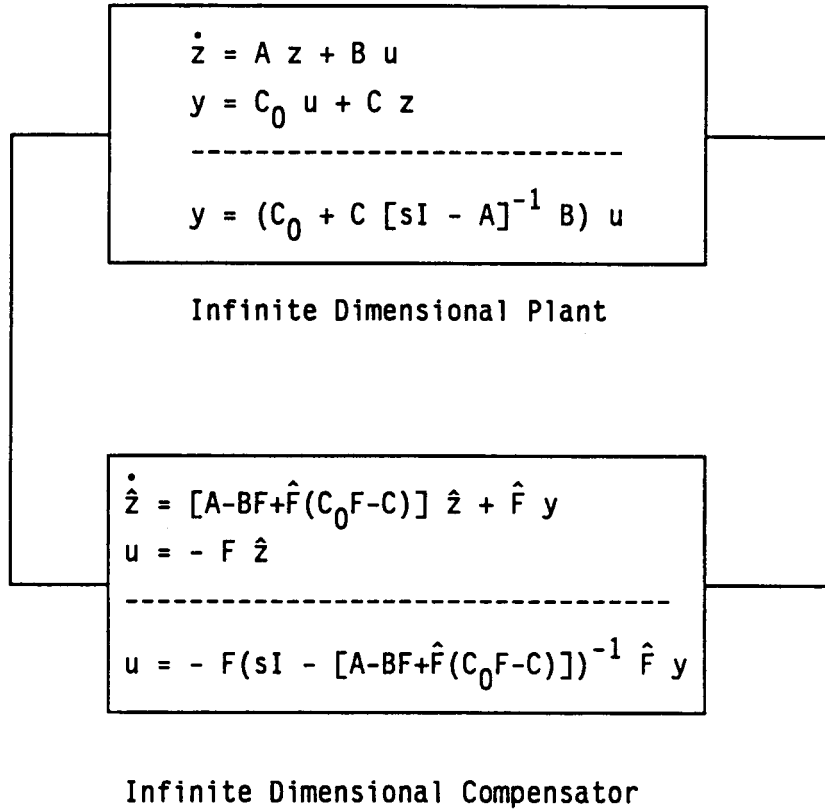
Figure 6.1 Closed-loop System with Infinite Dimensional Plant
and Infinite Dimensional Compensator

The foregoing definitions of this section and Theorem 6.5 are straightforward generalizations to infinite dimensions of observer-controller results in finite dimensions. Balas [Ba2, Ba3] and Schumacher [Sc2] have used similar extensions.

6.3 The Optimal Infinite Dimensional Estimator, Compensator and Closed-Loop System

Now suppose that $\hat{F}$ is chosen as

(6.3.1) $\qquad \hat{F} = \hat{\Pi}\,C^{*}\,\hat{R}^{-1}$

where $\hat{\Pi} \in L(E)$ is the minimal nonnegative selfadjoint solution to the Riccati

equation

(6.3.2) $A \hat{\Pi} + \hat{\Pi} A^* - \hat{\Pi} C^* \hat{R}^{-1} \hat{\Pi} + \hat{Q} = 0,$

with $\hat{Q} \in L(E)$ nonnegative and selfadjoint and $\hat{R}$ a positive definite symmetric p x p matrix. Theorem 6.3 (with A, B, Q, R, Π and S(t) replaced by A^*, C^*, $\hat{Q}$, $\hat{R}$, $\hat{\Pi}$ and $\hat{S}^*(t)$) gives sufficient conditions for $\hat{\Pi}$ to exist and for the semigroup $\hat{S}^*(t)$ -- and equivalently its adjoint, the $\hat{S}(t)$ generated by $A - \hat{\Pi}C^*\hat{R}^{-1}C$ -- to be uniformly exponentially stable.

Definition 6.6. When the control gain operator is

(6.3.3) $F = R^{-1}B^* \Pi,$

with Π the solution to the Riccati equation (6.1.3), and the estimator gain operator is given by (6.3.1) and (6.3.2), we call the compensator and the closed-loop system in Figure 6.1 the optimal infinite dimensional compensator and optimal closed-loop system, respectively.

The infinite dimensional estimator defined by (6.2.3), (6.3.1) and (6.3.2) is the optimal estimator for the stochastic version of (6.2.1) and (6.2.2) when (6.2.1) is disturbed by a stationary gaussian white noise process with zero mean and covariance operator $\hat{Q}$ and the measurement in (6.2.2) is contaminated by similar noise with covariance $\hat{R}$. For infinite dimensional stochastic estimation and control, see [Ba1, CP2]. When the state weighting operator Q in (6.1.1) is trace class, the optimal infinite dimensional compensator minimizes the time-average of the expected steady-state value of the integrand in (6.1.1). Existing theory for stochastic control of infinite dimensional systems requires trace-class $\hat{Q}$, but we have a well defined compensator for any bounded non-negative selfadjoint $\hat{Q}$ and Q, as long as the solutions to the Riccati equations exist. As the next chapter shows (without assuming trace-class $\hat{Q}$), the infi-

nite dimensional compensator is the limit of a sequence of finite dimensional compensators, each of which can be interpreted as an optimal LQG compensator for a finite dimensional model of the structure. Therefore, we do not require trace-class $\hat{Q}$ in our definition of the optimal compensator, even though this compensator solves a precise optimization problem only when $\hat{Q}$ is trace class.

This text is concerned primarily with how the finite dimensional compensators in Chapter 7 converge to the infinite dimensional compensator in this chapter, and the analysis of this convergence requires only the theory of infinite dimensional Riccati equations for deterministic optimal control problems and the corresponding approximation theory. While the stochastic interpretation of the infinite dimensional compensator and, in Section 4 of Chapter 7, of the finite dimensional estimators should be motivational, nothing in the rest of the paper depends on a stochastic formulation. We assume that the operators Q, R, $\hat{Q}$ and $\hat{R}$ are determined by some design criteria, either stochastic or deterministic. In many engineering applications, deterministic criteria such as the stability margin and robustness of the closed-loop system [B11, TS1], rather than a stochastic performance index and an assumed noise model, govern the choice of Q, R, $\hat{Q}$ and $\hat{R}$.

6.4 *Application to Optimal Control of Flexible Structures*

For the rest of this chapter, A_0, A_1, A, $T(t)$, B_0, B, C_1, C_2 and C are the operators defined in Chapter 3, and $E = V \times H$ is the energy space defined there.

The next theorem concerns a typical problem in control of aerospace structures: All elastic components have some structural damping; no damping is associated with the rigid-body modes but all rigid-body modes are controllable.

(This is the case with the structure in Section 9 of Chapter 3, for example.) For all rigid-body modes to be controllable, an actuator is required for each rigid-body mode.

Theorem 6.7. i) Suppose that $A_1 = B_0 B_0^*$ and that $\tilde{A}_0 = A_0 + A_1$ and $\tilde{d}_0 = d_0 + A_1$ are H-coercive, so that there exist positive constants ρ, γ and β such that, for all $v \in V$,

(6.4.1) $\qquad |v|_{V^2} \geq \rho |v|_{H^2}$

(6.4.2) $\qquad \tilde{d}_0(v,v) \geq \gamma |v|_{H^2}$

(6.4.3) $\qquad \tilde{d}_0(v,v) \leq \beta |v|_{V^2}$

and

(6.4.4) $\qquad \max\{|B_0|, |Q|, |R|\} \leq \beta.$

(The V-continuity of d_0 implies (6.4.3).) Then (6.1.3) has a minimal non-negative selfadjoint solution Π.

ii) Suppose also that

(6.4.5) $\qquad \langle Qz, z\rangle_E \geq \rho |z|_E^2.$

Then the optimal closed-loop semigroup satisfies

(6.4.6) $\qquad |S(t)| \leq M_2\, e^{-a_2 t}, \qquad\qquad\qquad t > 0,$

where M_2 and a_2 are positive constants depending on ρ, γ and β only.

Proof. See [GA2].

Now we will consider the structure of the optimal control law in more detail. Since $\Pi \in L(E)$ and $E = V \times H$, we can write

$$(6.4.7) \qquad \Pi = \begin{bmatrix} \Pi_0 & \Pi_1 \\ \Pi_1^* & \Pi_2 \end{bmatrix}$$

$\Pi_0 \in L(V)$, $\Pi_1 \in L(H,V)$, $\Pi_2 \in L(H)$, and Π_0 and Π_2 are nonnegative and self-adjoint. With $z = (x,\dot{x})$, as in Chapter 3, (6.1.4) becomes

$$(6.4.8) \qquad u(t) = - R^{-1}B_0^*[\Pi_1^* x(t) + \Pi_2 \dot{x}(t)].$$

Since $B_0 \in L(R^m,H)$, we must have vectors $b_i \in H$, $1 \leq i \leq m$, such that

$$(6.4.9) \qquad B_0 u = \sum_{i=1}^{m} b_i u_i$$

for

$$(6.4.10) \qquad u = [u_1 \; u_2 \; \ldots \; u_m]^T \in R^m.$$

Also, for $h \in H$,

$$(6.4.11) \qquad B_0^* h = [\langle b_1,h\rangle_H \; \langle b_2,h\rangle_H \; \ldots \; \langle b_m,h\rangle_H]^T.$$

Since $\Pi_1^* x(t)$ and $\Pi_2 \dot{x}(t)$ are elements of H, we see from (6.4.8) and (6.4.11) that the components of the optimal control have the feedback form

$$(6.4.12) \quad u_i(t) = - \langle f_i,x(t)\rangle_V - \langle g_i,\dot{x}(t)\rangle_H, \qquad\qquad i = 1, \ldots, m,$$

where $f_i \in V$ and $g_i \in H$ are given by

$$(6.4.13a) \qquad f_i = \sum_{j=1}^{m} (R^{-1})_{ij} \Pi_1 b_j,$$

$$(6.4.13b) \qquad g_i = \sum_{j=1}^{m} (R^{-1})_{ij} \Pi_2 b_j, \qquad\qquad i = 1,2,\ldots,m.$$

We call f_i and g_i *functional control gains.*

Since the measurement operator C in (6.2.2) now has the form discussed in Section 7 of Chapter 3, the estimator gain operator $\hat{F}$ has the form

$$(6.4.14) \qquad \hat{F}y = \sum_{i=1}^{p} (\hat{f}_i, \hat{g}_i) y_i$$

for $y = [y_1 \; y_2 \; \ldots \; y_p]^T \in R^p$, where the *functional estimator gains* $\hat{f}_i$ and $\hat{g}_i$ are elements of V and H, respectively.

For the optimal estimator gains, we can partition $\hat{\Pi}$ as

$$(6.4.15) \qquad \hat{\Pi} = \begin{bmatrix} \hat{\Pi}_0 & \hat{\Pi}_1 \\ \hat{\Pi}_1^* & \hat{\Pi}_2 \end{bmatrix}$$

and use (6.3.1) and (3.7.4) to get

$$(6.4.16a) \qquad \hat{f}_i = \sum_{j=1}^{p} (\hat{R}^{-1})_{ij} (\hat{\Pi}_0 c_{1j} + \hat{\Pi}_1 c_{2j}),$$

$$(6.4.16b) \qquad \hat{g}_i = \sum_{j=1}^{p} (\hat{R}^{-1})_{ij} (\hat{\Pi}_0^* c_{1j} + \hat{\Pi}_1 c_{2j}), \qquad i = 1,2,\ldots,p.$$

Now we partition $\hat{Q}$ as

$$(6.4.17) \qquad \hat{Q} = \begin{bmatrix} \hat{Q}_0 & \hat{Q}_1 \\ \hat{Q}_1^* & \hat{Q}_2 \end{bmatrix}.$$

In the optimal control problem, we almost always have a nonzero Q_0 because this operator penalizes the generalized displacement. For the results in this text, $\hat{Q}_0$ can be nonzero in the observer problem, and, as in the control problem, some of the strongest convergence results for finite dimensional approximations can be proved only for coercive $\hat{Q}$. However, if the observer is to be thought of as an optimal filter, then $\hat{Q}$ should be the covariance operator of the noise that disturbs (3.1.1). In this case, $\hat{Q}_0 = 0$ and $\hat{Q}_1 = 0$.

7. Approximation for the LQG Problem

7.1 Preliminaries

The approximating finite dimensional LQG problems to be solved numerically are defined for the nth approximating control system in (4.1.2) - (4.1.8). In approximating the solutions to the infinite dimensional LQR problem and the infinite dimensional state estimation problem, we will need the following $2n \times 2n$ grammian matrices:

$$(7.1.1) \qquad \widetilde{K}^n = [\langle e_i, e_j \rangle_V] = K^n + [\langle A_1 e_i, e_j \rangle_H],$$

$$(7.1.2) \qquad W^n = \begin{bmatrix} \widetilde{K}^n & 0 \\ 0 & M^n \end{bmatrix}.$$

(Recall the stiffness matrix K^n and the mass matrix M^n from (4.1.3).) *The superscript n on any matrix indicates the order of approximation, not a power of the matrix; the matrix W^{-n} will be the inverse of W^n.*

To construct the finite dimensional compensators, we must assume the following:

Hypothesis 7.1. There exist two sequences of symmetric, nonnegative $2n \times 2n$ matrices $\hat{Q}^n$ and $\widetilde{Q}^n$, n = 1, 2, ... ∇∇∇

The matrices $\hat{Q}^n$ and $\widetilde{Q}^n$ will be used in the Riccati matrix equations to be solved numerically for finite dimensional control and estimator gains. Of course, we want these matrices to be related to approximations of the operators Q and $\hat{Q}$, respectively, but we will postpone hypotheses related to operator convergence until the next chapter. For now we just will give the most common way of defining $\widetilde{Q}^n$ and $\overset{x}{Q}{}^n$.

82

The matrix $\tilde{Q}^n$ most often is defined to be the nonnegative, symmetric 2n x 2n matrix

$$(7.1.3) \qquad \tilde{Q}^n = \begin{bmatrix} \tilde{Q}_0^n & \tilde{Q}_1^n \\ \tilde{Q}_1^{nT} & \tilde{Q}_2^n \end{bmatrix}$$

whose n x n blocks are

$$\tilde{Q}_0^n = [\langle e_i, Q_0 e_j \rangle_V] ,$$

$$(7.1.4) \qquad \tilde{Q}_1^n = [\langle e_i, Q_1 e_j \rangle_V] ,$$

$$\tilde{Q}_2^n = [\langle e_i, Q_2 e_j \rangle_H] ,$$

where Q_0, Q_1 and Q_2 are the operators in Chapter 6, Section 4.

The most common way of defining the matrix $\tilde{\hat{Q}}^n$ is similar to that for $\tilde{Q}^n$, but there is an important difference. If the operator $\hat{Q}$ in Section 6.3 is partitioned as in (6.4.16) and a symmetric matrix $\tilde{\hat{Q}}^n$ is defined as in (7.1.4) with Q_0, Q_1 and Q_2 replaced by $\hat{Q}_0$, $\hat{Q}_1$ and $\hat{Q}_2$, then

$$(7.1.5) \qquad \tilde{\hat{Q}}^n = W^{-n} \tilde{\hat{Q}}^n W^{-n}.$$

Thus defined, the matrices $\tilde{Q}^n$ and $\tilde{\hat{Q}}^n$ are related to approximations of the operators Q and $\hat{Q}$ in the following way. We define $Q_n = P_{E_n} Q |_{E_n}$ and $\hat{Q}_n = P_{E_n} \hat{Q} |_{E_n}$ where P_{E_n} is the projection in (4.2.16), and we define Q^n and $\hat{Q}^n$ to be the matrix representations of Q_n and $\hat{Q}_n$, respectively. It follows, then, that

$$(7.1.6) \qquad \tilde{Q}^n = W^n Q^n$$

and

$$(7.1.7) \qquad \tilde{\hat{Q}}^n = \hat{Q}^n W^{-n}.$$

It is interesting to consider the case $Q = \hat{Q} = I$. The state weighting operator $Q = I$ (i.e., $Q_0 = I$, $Q_1 = 0$, $Q_2 = I$) in the infinite dimensional

83

control problem penalizes the sum of the total energy in the structure and the squares of any rigid-body displacements. The operator $\hat{Q} = I$ in the infinite dimensional estimator problem corresponds to *no physical noise*; nonetheless, our approximation theory allows $\hat{Q} = I$, and we often have found that a coercive $\hat{Q}$ produces desirable stability properties in the estimator.

For $\hat{Q} = Q = I$, (7.1.6) and (7.1.7) yield $\tilde{Q}^n = W^n$ and $\tilde{\hat{Q}}^n = W^{-n}$. This may appear dubious. Indeed, (7.1.6) and (7.1.7) may appear to violate the normal duality between the LQG control and estimator problems. However, examination of the performance index in (7.2.1) for the finite dimensional control problems and the stochastic interpretation in Section 7.6 of the finite dimensional estimators should demonstrate that (7.1.6) and (7.1.7) are natural from both the infinite dimensional and the finite dimensional perspectives.

While the $\tilde{Q}^n$ and $\tilde{\hat{Q}}^n$ in (7.1.3)-(7.1.7) often are used in the finite dimensional LQG problems, the rest of this chapter assumes only Hypothesis 7.1.

7.2 Approximation of the Optimal Control Law

7.2.1 The Finite Dimensional LQR Problems

The nth optimal LQR problem is: given $\eta(0) \in R^{2n}$ choose $u \in L_2(0,\infty;R^m)$ to minimize

$$(7.2.1) \qquad J_n(\eta(0),u) = \int_0^\infty [\eta(t)^T \tilde{Q}^n \eta(t) + u(t)^T R\, u(t)]dt$$

where $\eta(t) = [\xi(t)^T, \dot{\xi}(t)^T]^T$ satisfies (4.1.4). We assume:

Hypothesis 7.2. For each $n \geq 1$ and $\eta(0) \in R^{2n}$, there exists an admissible control (Definition 6.1) for (4.1.4) and (7.2.1).

A sufficient condition for Hypothesis 7.2 is that, for each n, the system (A^n, B^n) be stabilizable.

By Theorem 6.1 (or standard finite dimensional optimal control theory), the optimal control $u_n(t)$ has the feedback form

(7.2.2) $u_n(t) = -F^n \eta(t)$

where

(7.2.3) $F^n = R^{-1} B^{nT} \tilde{\Pi}^n$

and $\tilde{\Pi}^n$ is the minimal nonnegative, symmetric solution to the Riccati matrix equation

(7.2.4) $A^{nT} \tilde{\Pi}^n + \tilde{\Pi}^n A^n - \tilde{\Pi}^n B^n R^{-1} B^{nT} \tilde{\Pi}^n + \tilde{Q}^n = 0.$

7.2.2 Approximating Functional Control Gains

Now we define the $2n \times m$ matrix

(7.2.5) $G^n = W^{-n} F^{nT}$

and the following elements of V_n:

(7.2.6) $f_{in} = \sum_{j=1}^{n} G_{ji} e_j, \qquad g_{in} = \sum_{j=n+1}^{2n} G_{ji} e_j,$

$$i = 1, 2, \ldots, m.$$

Then, in view of (4.1.1) and (4.1.5), the ith component of the nth control law in (7.2.2) is

(7.2.7) $$u_{in}(t) = - [i^{th} \text{ column of } G^n] W^n \eta(t)$$

$$= - \sum_{j=1}^{n} \sum_{k=1}^{n} G_{ji} \langle e_j, e_k \rangle_V \xi_k(t)$$

$$- \sum_{j=n+1}^{2n} \sum_{k=n+1}^{2n} G_{ji} \langle e_j, e_k \rangle_H \dot{\xi}_k(t)$$

$$= - \langle f_{in}, x_n(t) \rangle_H - \langle g_{in}, \dot{x}_n(t) \rangle_V, \quad i = 1, 2, \dots, m.$$

We call f_{in} and g_{in} *approximating functional control gains*. In Chapter 8, we will see how f_{in} and g_{in} approximate the functional control gains f_i and g_i in (6.4.12).

7.3 Approximation of the Infinite Dimensional Estimator

7.3.1 The Finite Dimensional Estimators

The finite dimensional state estimator that is used on-line to approximate the optimal infinite dimensional estimator in Chapter 6 is

(7.3.1) $$\dot{\hat{\eta}} = A^n \hat{\eta} + B^n u + \hat{F} (y - C_0 u - C^n \hat{\eta})$$

where $\hat{\eta}(t) \in R^{2n}$ and C^n is the matrix in Section 1 of Chapter 4. The 2^n x p gain matrix $\hat{F}^n$ is

(7.3.2) $$\hat{F}^n = \tilde{\tilde{\Pi}}^n (C^n)^T R^{-1}$$

where $\tilde{\tilde{\Pi}}^n$ is the minimal nonnegative, symmetric solution to

(7.3.3) $$A^n \tilde{\tilde{\Pi}}^n + \tilde{\tilde{\Pi}}^n (A^n)^T - \tilde{\tilde{\Pi}}^n C^{nT} R^{-1} C^n \tilde{\tilde{\Pi}}^n + \tilde{Q} = 0$$

and $\tilde{Q}^n$ is a nonnegative, symmetric 2n x 2n matrix, as in Section 7.1. We assume

Hypothesis 7.3. For each $n \geq 1$, there exists a nonnegative, symmetric solution to (7.3.3).

A sufficient condition for Hypothesis 7.3 is that the pair (A^{nT}, C^{nT}) be stabilizable. If $(A^{nT}, \tilde{\hat{Q}}^n)$ is detectable, then (7.3.3) has at most one nonnegative symmetric solution.

7.3.2 The Approximating Functional Estimator Gains

We define

$$(7.3.4) \qquad \hat{f}_{in} = \sum_{j=1}^{n} \hat{F}^n_{ji} e_j \qquad \hat{g}_{in} = \sum_{j=n+1}^{2n} \hat{F}^n_{ji} e_j, \qquad i = 1, 2, \ldots, p,$$

where $\hat{F}^n$ is the estimator gain matrix in (7.3.2), and we call $\hat{f}_{in}$ and $\hat{g}_{in}$ *approximating functional estimator gains.* In Chapter 8, we will see how $\hat{f}_{in}$ and $\hat{g}_{in}$ approximate the functional estimator gains $\hat{f}_i$ and $\hat{g}_i$ in (6.4.13) and (6.4.15). We note that $\hat{f}_{in}$ and $\hat{g}_{in}$ are elements of V_n while $\hat{f}_i$ and $\hat{g}_i$ are elements of V and H, respectively.

7.4 The Finite Dimensional Compensators and the Realizable Closed-Loop Systems

The nth *compensator* consists of the control law in (7.2.2) applied to the output of the nth estimator in (7.3.1); i.e., the control law for the compensator is

$$(7.4.1) \qquad u_n = - F^n \hat{\eta}$$

where the control gain matrix F^n is given by (7.2.3). The block diagram in Figure 7.1 shows the realizable closed-loop system that results from the nth compensator. We will refer to this system as the nth *closed-loop system.*

$$\boxed{\begin{array}{l} \dot{z} = A\ z + B\ u \\ y = C_0\ u + C\ z \end{array}}$$

Control System

$$\boxed{\begin{array}{l} \dot{\hat{\eta}} = [A^n - B^n F^n + \hat{F}^n (C_0 F^n - C^n)]\hat{\eta} + \hat{F}^n y \\ u_n = -F^n \hat{\eta} \end{array}}$$
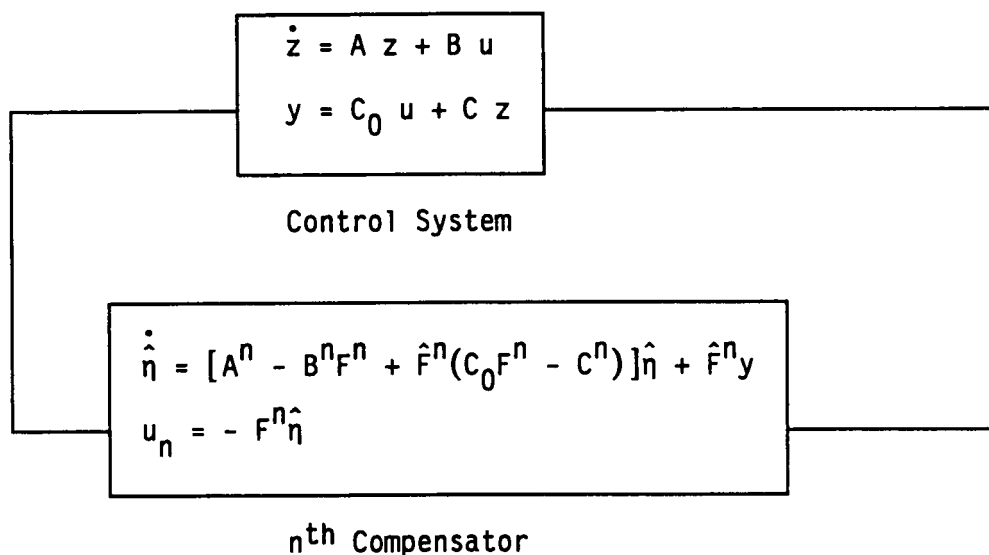
n^{th} Compensator

Figure 7.1. n^{th} Closed-Loop System

The transfer function of the nth compensator is

(7.4.2) $\Phi_n(s) = -F^n (sI - [A^n - B^n F^n + \hat{F}^n (C_0 F^n - C^n)])^{-1} \hat{F}^n$,

which is an m x p matrix function of the complex variable s for each n, as is the similar transfer function $\Phi(s)$ in (6.2.14) for the infinite dimensional compensator.

7.5 Hilbert Space Representation of the Finite Dimensional Compensators and Associated Riccati Equations

Recall from Chapter 4 the relationships among $x_n(t)$, $z_n(t)$, $\xi(t)$ and $\eta(t)$. (See (4.1.1) - (4.1.6) and (4.2.1).) From here on the matrices $\tilde{Q}_n$ and $\dot{\tilde{\tilde{Q}}}_n$ will be defined as in (7.1.3) -(7.1.5).

We define Π_n to be the operator whose matrix representation is

(7.5.1) $\Pi^n = W^{-n} \tilde{\Pi}^n$

Since the matrix $\tilde{\Pi}^n$ is symmetric and nonnegative, the operator Π_n is selfadjoint

and nonnegative. The Riccati matrix equation (7.2.4) is equivalent to

$$(7.5.2) \qquad W^{-n}(A^n)^T W^n \Pi^n + \Pi^n A^n - \Pi^n B^n R^{-1}(B^n)^T W^n \Pi^n + Q^n = 0.$$

Since the matrix representations of the operators A_n and A_n^* are A^n and $W^{-n}(A^n)^T W^n$, respectively, and the matrix representation of B_n^* is $(B^n)^T W^n$, (7.5.2) is the matrix representation of the Riccati operator equation

$$(7.5.3) \qquad A_n^* \Pi_n + \Pi_n A_n - \Pi_n B_n R^{-1} B_n^* \Pi_n + Q_n = 0.$$

The control law in (7.2.2) is equivalent to

$$(7.5.4) \qquad u_n(t) = - F_n z_n(t)$$

with the gain operator F_n given by

$$(7.5.5) \qquad F_n = R^{-1} B_n^* \Pi_n.$$

Next, we define $\hat{\Pi}_n$ to be the nonnegative selfadjoint operator whose matrix representation is

$$(7.5.6) \qquad \hat{\Pi}^n = \tilde{\hat{\Pi}}_n W^n.$$

The Riccati matrix equation (7.3.3) is equivalent to

$$(7.5.7) \qquad A^n \hat{\Pi}^n + \hat{\Pi}^n W^{-n}(A^n)^T W^n - \hat{\Pi}^n W^{-n}(C^n)^T \hat{R}^{-1} C^n \hat{\Pi}^n + \hat{Q}^n = 0.$$

which is the matrix representation of the Riccati operator equation

$$(7.5.8) \qquad A_n \hat{\Pi}_n + \hat{\Pi}_n A_n^* - \hat{\Pi}_n C_n^* R^{-1} C_n \hat{\Pi}_n + \hat{Q}_n = 0.$$

(The matrix representation of the operator C_n^* is $W^{-n}(C^n)^T$.)

The estimator gain matrix $\hat{F}^n$ is the matrix representation of the operator

$$(7.5.9) \qquad \hat{F}_n = \hat{\Pi}_n C_n^* \hat{R}^{-1}.$$

C-2

89

If we write $\hat{z}_n(t) = \sum\limits_{i=1}^{2n} \hat{\eta}_i(t)e_i$, then the state estimator in (7.3.1) is equivalent to

$$(7.5.10) \qquad \dot{\hat{z}}_n = A_n \hat{z}_n + B_n u + \hat{F}_n(y - C_0 u - C_n \hat{z}_n).$$

For the convergence results in Chapter 8, it is useful to have the approximating operators that are equivalent to the various matrices in Sections 7.1 - 7.4. The finite dimensional Riccati operator equations (7.5.3) and (7.5.8) approximate, respectively, the infinite dimensional Riccati operator equations (6.1.3) and (6.3.2). The control and estimator gain operators in (7.5.5) and (7.5.9) approximate, respectively, the gain operators in (6.3.3) and (6.3.1). Also, the state estimator in (7.5.10), which is just the Hilbert space representation of the estimator in (7.3.1), approximates the optimal infinite dimensional estimator in Chapter 6.

7.6 Stochastic Interpretation of the Approximating Estimators

Our approximation of the infinite dimensional estimator is based on approximation of the infinite dimensional Riccati equation, whose structure is the same for both control and estimator problems, and stochastic properties of the optimal estimator problem never enter our approximation theory. Furthermore, using only the deterministic setting, in Chapter 8 we analyze the finite dimensional estimators and the compensators based upon them. Nonetheless, we should consider momentarily the sequence of finite dimensional stochastic estimation problems whose solution is given by (7.3.1) - (7.3.3) or equivalently by (7.5.8) - (7.5.10).

First, recall how the covariance operator of a Hilbert space-valued random variable is defined. The covariance operator of an E-valued random variable ω is the operator Q for which

(7.6.1)   expected value $\{<z,\omega>_E<\tilde{z},\omega>_E\} = <Qz,\tilde{z}>_E$,   z, $\tilde{z}$ ϵ E.

(See [Ba1, CP2].)

With $\hat{F}_n$ given by (7.5.9) and (7.5.8), (7.5.10) is the Kalman-Bucy filter for the system

(7.6.2) $\dot{z}_n = A_n z_n + B_n u + \omega_n$,

(7.6.3) $y = C_0 u + C_n z_n + \omega_0$,

where $\omega_n(t)$ is an E_n-valued white noise process with covariance operator $\hat{Q}_n$ and $\omega_0(t)$ is an R^p-valued white noise process with covariance operator (matrix) $\hat{R}$. Next, careful inspection will show that the filter defined by (7.3.1), (7.3.2) and (7.3.3) is the matrix representation of the filter defined by (7.5.10), (7.5.9) and (7.5.8).

With z_n and η related as in (4.1.1) and (4.1.5), (7.6.2) and (7.6.3) are equivalent to the system

(7.6.4) $\dot{\eta} = A_n \eta + B^n u + \nu$,

(7.6.5) $y = C_0 u + C^n \eta + \omega_0$,

where $\nu(t)$ is the R^{2n}-valued noise process related to $\omega_n(t)$ by

$$(7.6.6) \quad \omega_n(t) = \sum_{i=1}^{n} (\nu_i(t) e_i, \nu_{i+n}(t)e_i).$$

Certainly, a Kalman-Bucy filter for (7.6.4) and (7.6.5) has the form (7.3.1) with the filter gain given by (7.3.2) and (7.3.3). This particular filter is the matrix representation of the filter defined by (7.5.10), (7.5.9) and (7.5.8) if and only if the matrix $\tilde{\hat{Q}}^n$ defined by (7.1.7) is the covariance of the process $\nu(t)$. Since $\hat{Q}^n$ is the matrix representation of $\hat{Q}_n$, straightforward calculation using (7.1.7) and (7.6.1) shows that the $\tilde{\hat{Q}}^n$ in (7.1.7) is indeed the correct covariance matrix.

Of course, if $\omega_n(t)$ and $\nu(t)$ represent a physical disturbance to the structure, then $\omega_n(t)$ must have the form $(0,\omega_n^{(2)}(t))$ and the first n elements of $\nu(t)$ must be zero, but this is not necessary for our analysis.

Our finite dimensional observers can be interpreted now as a sequence of filters designed for the sequence of finite dimensional approximations to the flexible structure, with the nth approximate system disturbed by the noise process $\omega_n(t)$, whose covariance operator is $\hat{Q}_n$. According to Hypothesis 8.5 of Chapter 8, these covariance operators converge to the operator $\hat{Q}$ of Section 3 of Chapter 6. If we have a reliable model of a stationary, zero-mean gaussian disturbance for the structure, then we can take the covariance operator for this disturbance to be $\hat{Q}$ and think of the infinite dimensional observer as the optimal estimator. But this interpretation is not necessary for the rest of our analysis.

8. Convergence Theory for the Approximating LQG Problems

8.1 Convergence Results for Approximation of the Generic LQG Problem

Let the Hilbert space E and the linear operators A, T(t), B, Q and R be as in Section 1 of Chapter 6. Suppose that there exists a sequence of finite dimensional subspaces E_n, with the projection of E onto E_n denoted by P_{E_n}, and there exist sequences of operators $A_n \in L(E_n)$, $B_n \in L(R^m, E_n)$, $Q_n = Q_n^* \in L(E_n)$, $Q_n \geq 0$.

Hypothesis 8.1. For each $z \in E$,

(8.1.1) $\qquad P_{E_n} z \to z$ as $n \to \infty$,

(8.1.2) $\qquad \exp(A_n t) P_{E_n} z \to T(t) z$

and

(8.1.3) $\qquad \exp(A_n^* t) P_{E_n} z \to T^*(t) z$ as $n \to \infty$,

uniformly in t for t in bounded intervals; for each $u \in R^m$,

(8.1.4) $\qquad B_n u \to Bu$;

for each $z \in E$,

(8.1.5) $\qquad Q_n P_{E_n} z \to Qz$.

(In other words, P_{E_n}, $\exp(A_n t) P_{E_n}$, $\exp(A_n^* t) P_{E_n}$, B_n and $Q_n P_{E_n}$ converge strongly.)

Hypothesis 8.2. For each n, the system (A_n, B_n) is stabilizable and the system (Q_n, A_n) is detectable.

Hypothesis 8.2 guarantees that the Riccati equation (8.1.6) in the following theorem has a unique nonnegative, selfadjoint solution.

Theorem 8.3. For each n, let $\Pi_n \in L(E_n)$ be the nonnegative, selfadjoint solution to the Riccati operator equation

$$(8.1.6) \qquad A_n^* \Pi_n + \Pi_n A_n - \Pi_n B_n R_n^{-1} B_n^* \Pi_n = Q_n.$$

If $\|\Pi_n\|$ is bounded uniformly in n, then the Riccati algebraic equation (6.1.3) has a nonnegative, selfadjoint solution Π and $\Pi_n P_{E_n}$ converges weakly to Π; i.e.,

$$(8.1.7) \qquad \langle \Pi_n P_{E_n} z, \tilde{z} \rangle_E \to \langle \Pi z, \tilde{z} \rangle_E \qquad \forall z, \tilde{z} \in E.$$

If additionally there exist positive constants M and b, independent of n, such that

$$(8.1.8) \qquad \|\exp([A_n - B_n R^{-1} B_n^* \Pi_n]t)\| \le M e^{-bt}, \qquad t \ge 0,$$

then $\Pi_n P_{E_n}$ and $\exp([A_n - B_n R^{-1} B_n^* \Pi_n]t) P_{E_n}$ converge strongly to Π and $S(t)$, respectively; i.e.,

$$(8.1.9) \qquad \Pi_n P_{E_n} z \to P z \qquad \forall z \in E$$

and

$$(8.1.10) \qquad \exp([A_n - B_n R^{-1} B_n^* \Pi_n]t) P_{E_n} z \to S(t)z \qquad \forall z \in E,$$

uniformly in $t \ge 0$, where $S(t)$ is the semigroup generated by $A - BR^{-1} B^* P$. If there exists a positive constant δ, independent of n, such that

$$(8.1.11) \qquad Q_n \ge \delta,$$

then $\|P_n\|$ being bounded uniformly in n guarantees the existence of positive constants M and b for which (8.1.8) holds for all n.

Proof. The theorem follows from Theorem 5.3 of [Gi4] and Theorem 6.7 of [Gi3], whose proof is valid under the hypotheses here. In [Gi1, Section 4] the operators A_n, Q_n and Π_n are extended to all of E by defining them appropriately on E_n. Banks and Kunisch [BK1] have modified Theorem 5.3 of [Gi1] to obtain essentially the present theorem without using the artificial, and rather clumsy, extensions of E_n in the proof.

Theorem 8.4. The strong convergence in (8.1.9) implies uniform norm convergence of the optimal feedback laws:

$$(8.1.12) \qquad \| B_n^* \, \Pi_n P_{E_n} - B^* \, \Pi \| \to 0 \quad \text{as} \quad n \to \infty.$$

Proof. This follows from the selfadjointness of Π_n and P_{E_n} and the finite dimensionality of the control space R^m. See equations (4.23) and (4.24) of [Gi1].

Nonconvergence results for the case where no open-loop damping is modeled are given in [Gi1, Gi2, GA2].

For each n, the finite dimensional operator Riccati equation (8.1.6) is an approximation to the finite dimensional Riccati equation (6.1.3), but the solution to (8.1.6) also provides the solution to the optimal control problem defined by

$$(8.1.13) \qquad \dot{z}_n = A_n \, z_n + B_n \, u$$

and

$$(8.1.14) \qquad J_n = \int_0^\infty (\langle Q_n \, z_n, \, z_n \rangle_E + u^T R u) dt.$$

The optimal control law for this problem is

(8.1.15) $\qquad u_n = -F_n \, z_n$

where the operator $F_n \in L(E_n, R^m)$ is

(8.1.16) $\qquad F_n = R^{-1} B_n^* \, \Pi_n \, P_{E_n}.$

Hypothesis 8.5. There exist sequences $C_n \in L(E_n, R^P)$ and $\hat{Q}_n \in L(E_n)$ with $\hat{Q}_n = \hat{Q}_n^* \geq 0$ such that

(8.1.17) $\qquad C_n P_{E_n} \to C \qquad\qquad$ strongly

and

(8.1.18) $\qquad \hat{Q}_n P_{E_n} \to \hat{Q} \qquad\qquad$ strongly as $n \to \infty$.

As with B_n, (8.1.17) implies that $C_n P_{E_n}$ and C_n^* converge in norm to C and C^*, respectively.

Hypothesis 8.6. For each n, the system (A_n^*, C_n^*) is stabilizable. (In particular, any unstable modes of the system (A_n, C_n) are observable.) Also, the system $(A^*, \hat{Q}_n)$ is detectable.

We define the output of the system in (8.1.13) by

(8.1.19) $\qquad y_n = C_0 \, u + C_n \, z_n.$

The nth state estimator is

(8.1.20) $\qquad \dot{\hat{z}}_n = A_n \, \hat{z}_n + B_n \, u + \hat{F}_n(y_n - C_0 \, u - C_n \, \hat{z}_n)$

where the estimator gain is the operator

(8.1.21) $\qquad \hat{F}_n = \hat{\Pi}_n \, C_n^* \, \hat{R}^{-1}$

and Π_n is the nonnegative, selfadjoint solution to the Riccati operator equation

$$(8.1.22) \qquad A_n \hat{\Pi}_n + \hat{\Pi}_n A_n^* - \hat{\Pi}_n C_n^* \hat{R}^{-1} C_n \hat{\Pi}_n + \hat{Q}_n = 0.$$

Hypothesis 8.2 implies that such a solution exists and is unique.

As $n \to \infty$, the estimator defined by (8.1.20) - (8.1.22) converges to the infinite dimensional estimator in Chapter 6 as indicated by the next two theorems, which follow immediately from Theorems 8.3 and 8.4.

Theorem 8.7. Theorem 8.3 holds with A, B, Q, Π, S(t), A_n, B_n, Q_n and Π_n replaced, respectively, by A^*, C^*, $\hat{Q}$, $\hat{\Pi}$, $S^*(t)$, A_n^*, C_n^*, $\hat{Q}_n$ and $\hat{\Pi}_N$.

Theorem 8.8. If $\hat{\Pi}_n P_{E_n}$ converges strongly to $\hat{\Pi}$, then

$$(8.1.23) \qquad \|\hat{\Pi}_n C_n^* - \hat{\Pi} C^*\| \to 0 \qquad \text{as } n \to \infty.$$

8.2 Convergence of the Closed-Loop Systems

Now we will consider the sense in which the nth closed-loop system in Figure 7.1 approximates the optimal closed-loop system in Chapter 6. Recall from Chapter 4 how the approximating open-loop semigroups $T_n(\cdot)$ and their adjoints converge strongly and how the input operators B_n, the measurement operators C_n and their respective adjoints converge in norm. Section 8.1 has given sufficient conditions for the approximating control and estimator gains to converge to the gains for the optimal infinite dimensional compensator; i.e., (8.1.12) and (8.1.23) imply

$$(8.2.1) \qquad \|F_n P_{En} - F\| \to 0,$$

$$(8.2.2) \qquad \|\hat{F}_n - \hat{F}\| \to 0.$$

By identifying $\hat{\eta}$ with $\hat{z}_n$ as in Section 5 of Chapter 7, we can identify the closed-loop system in Figure 7.1 with a closed-loop system on the space $E \times E_n$. We denote the corresponding closed-loop semigroup and its generator by $S_{\infty n}(t)$ and $A_{\infty n}$, respectively. Recall $S_{\infty\infty}(t)$ and $A_{\infty\infty}$ from Chapter 6.

Proofs of the remaining theorems in this chapter are given in [GA2]. Since these proofs are rather technical and offer no particular insight into the compensator-design process, we do not repeat them here.

Theorem 8.9. For $t \geq 0$, $S_{\infty n}(t)P_{EEn}$ converges strongly to $S_{\infty\infty}(t)$, and the convergence is uniform in t for t in bounded intervals.

We should expect at least Theorem 8.9, but we need more. We should require, for example, that if $S(t)$ is uniformly exponentially stable, then $S_{\infty n}(t)$ must be also for n sufficiently large. Although numerical results for numerous examples with various kinds of damping and approximations suggest that this is usually true, we have been unable to prove it in general. We do have the result for the following important case.

Theorem 8.10.

i) Suppose that the basis vectors of the approximation scheme are the natural modes of undamped free vibration and that the structural damping does not couple the modes. Then $S_{\infty n}(t)$ converges in norm to $S_{\infty\infty}(t)$, uniformly in bounded t-intervals.

ii) If, additionally, $S_{\infty\infty}(t)$ is uniformly exponentially stable, then $S_{\infty n}(t)$ is uniformly exponentially stable for n sufficiently large.

This paper emphasizes using the convergence of the approximating control and estimator gain operators F_n and $\hat{F}_n$, and the convergence of the functional gains that can be used to represent these operators, to determine the finite dimensional compensator that will produce essentially optimal closed-loop performance. However, close examination of $A_{\infty\infty} - A_{\infty n}$ (see [GA2]) shows that, for $S_{\infty n}(t)$ to converge to $S_{\infty\infty}(t)$, we need the following differences to converge to zero:

(8.2.3) $B_n F_n P_{En} - BF = B_n(F_n P_{En} - F) + (B_n - B)F,$

(8.2.4) $\hat{F}_n C_n P_{En} - \hat{F}C = (\hat{F}_n - \hat{F})C_n P_{En} + \hat{F}(C_n P_{En} - C).$

The second term on the right hand side of each of these equations represents, respectively, control and observation spillover, which has been studied extensively by Balas [Ba2, Ba3]. Together, the control spillover and observation spillover couple the modes modeled in the compensator with the modes not modeled in the compensator. The spillover must go to zero -- as it does when B_n and C_n converge -- for $A_{\infty\infty} - A_{\infty n}$ to go to zero.

We should ask, then, whether there exists a correlation between the convergence of F_n and $\hat{F}_n$ and the elimination of spillover. The answer is yes if no modes lie in the null space of the state weighting operator Q in the performance index and if the assumed process noise, whose covariance operator is $\hat{Q}$, excites all modes, but this correlation is difficult to quantify. The two main factors that determine the convergence rates of the gains are the Q-to-R ratio and the damping, neither of which affects the convergence of B_n and C_n. On the other hand, when either factor (small Q/R or large damping) causes the gains to converge fast, it generally also causes the magnitude of F and $\hat{F}$ to be relatively small, thereby reducing the magnitude of the spillover terms in (8.2.3) and (8.2.4). Also, as n increases, the increasing frequencies of the truncated

modes usually reduce the coupling effect of spillover. This is well known, although it cannot be seen from the equations here. In examples that we have worked, we have found that when n is large enough to produce convergence of the control and estimator gains, the effect of any remaining spillover is negligible. But this may not always be true, and spillover should be remembered.

8.3 Convergence of the Compensator Transfer Functions

The transfer function of the nth compensator (shown in the bottom block of Figure 7.1) is given by (7.4.2) and the transfer function of the infinite dimensional compensator is given by (6.2.14). Each transfer function is an m x p matrix function of the complex variable s. We will denote the resolvent set of $[A - BF + \hat{F}(C_0F-C)]$ by $\rho([A - BF + \hat{F}(C_0F-C)])$.

Theorem 8.11. There exists a real number a_1 such that, if $Re(s) > a_1$, then $s \in \rho([A^n-B^n F^n+\hat{F}^n(C_0F^n-C^n)])$ for all n, and $\Phi_n(s)$ converges to $\Phi(s)$, uniformly in compact subsets of such s.

This result leaves much to be desired. For example, it does not guarantee that any subset of the imaginary axis will lie in $\rho([A^n-B^nF^n + \hat{F}^n(C_0F^n-C^n)])$ for sufficiently large n, even if all of the imaginary axis lies in $\rho([A-BF + \hat{F}(C_0F-C)])$. As with the convergence of the closed-loop systems, we can get more for certain important cases.

Remark 8.12. If the open-loop semigroup $T(\cdot)$ (whose generator is A) is an analytic semigroup, then there exist real numbers a, θ and M, with θ and M positive, such that $\rho([A-BF + \hat{F}(C_0 F-C)])$ contains the sector $\{s: |\arg(s-a)| < \frac{\pi}{2} + \theta\}$, and for each s in this sector,

$$(8.3.1) \qquad \|(sI - [A-BF + \hat{F}(C_0 F-C)])^{-1}\| \leq M/|s-a|.$$

Theorem 8.13. i) If the basis vectors of the approximation scheme are the natural modes of undamped free vibration and the structural damping does not couple the modes, then each s in $\rho([A-BF + \hat{F}(C_0 F-C)])$ is in $\rho([A^n-B^n F^n + \hat{F}^n(C_0 F^n-C^n)])$ for n sufficiently large and $\Phi_n(s)$ converges to $\Phi(s)$ as $n \to \infty$, uniformly in compact subsets of $\rho([A-BF + \hat{F}(C_0 F-C)])$. ii) If, additionally, $T(\cdot)$ is an analytic semigroup, then $\Phi_n(s)$ converges to $\Phi(s)$ uniformly in the sector described in Remark 8.12.

Theorem 8.14. If A has compact resolvent, then $\Phi_n(s)$ converges to $\Phi(s)$ for each $s \in \rho([A-BF + \hat{F}(C_0 F-C)])$, uniformly in compact subsets.

9. Application: Compensator Design for a Compound Structure

9.1 The LQG Problem for the Distributed Model of the Structure

Here we pose an infinite dimensional LQG problem for the hub-beam-tip mass structure in Section 9 of Chapter 3 and Section 3 of Chapter 4. In the infinite dimensional optimal control problem, we take $Q = I$ in the performance index in (6.1.1). This means that the state weighting term $\langle Qz,z \rangle_E$ is twice the total energy in the structure plus the square of the rigid-body rotation. Since there is one input, the control weighting R is a scalar.

According to (6.4.12), the optimal control has the full-state feedback form

$$(9.1.1) \qquad u(t) = - \langle f,x(t) \rangle_V - \langle g,\dot{x}(t) \rangle_H$$

where x(t) has the form (3.9.1) and

$$(9.1.2) \qquad f = (\alpha_f, \phi_f, \beta_f) \in V \ , \quad g = (\alpha_g, \phi_g, \beta_g) \in H.$$

In (9.1.2) α_f, β_f, α_g and β_g are scalars and ϕ_f and ϕ_g are functions defined over the length of the beam. Note that $\beta_f = \phi_f(\ell)$ is not used in the control law -- recall (3.9.8) and (3.9.9).

The single sensor measures the rigid-body angle θ, and we assume that this measurement has zero-mean Gaussian white noise with variance $R = 10^{-4}$. Also, we model a disturbance on the right side of (3.1.1) that is a zero-mean Gaussian white noise process distributed uniformly over the beam and having concentrated components acting on the hub and tip mass. For this disturbance, the covariance operator $\hat{Q}$ in (6.3.2) is

$$(9.1.3) \qquad \hat{Q} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} : V \times H \longrightarrow V \times H.$$

According to (6.4.14), with $p = 1$, the gain operator $\hat{F}$ for the infinite dimensional estimator has the form

$$(9.1.4) \qquad\qquad \hat{F}y = (\hat{f}, \hat{g})y,$$

and the functional estimator gains $\hat{f}$ and $\hat{g}$ have the form (9.1.2).

9.2 Approximation of the Optimal Control Gains

Now we solve the finite dimensional LQR problem in Chapter 7 for increasing approximation orders n. For each n, the solution to the finite dimensional Riccati equation (7.2.4) yields the matrix F^n in (7.2.3). We use this control gain matrix to compute the approximating functional control gains in (7.2.6) and (7.2.7) and to compute the feedback law (7.4.1) for the finite dimensional compensator in Figure 7.1. Recall that $n = 2n_e + 1$ where n_e is the number of elements used to approximate the beam.

For the numerical solution to the nth approximating optimal control problem, we begin with the n x n matrices K^n, D^n and B_0^n in (4.3.2) and the n x n mass matrix M^n computed with the inner product in (3.9.3) for the basis vectors in (4.3.1). With these matrices, we form the 2n x 2n matrices A^n and B^n in (4.1.6) and the n x n matrix K^n and the 2n x 2n matrix W^n in (7.1.1) and (7.1.2); $\tilde{K}^n$ is K^n with 1 added to the first element. Since the operator Q in the infinite dimensional control problem is the identity, the 2n x 2n matrix $\tilde{Q}^n$ in (7.1.3) and (7.1.6) is equal to W^n. With these matrices, we solve (7.2.4) numerically by the standard eigenvector decomposition of the Hamiltonian matrix [KS1], often called the Potter method.

The approximating functional control gains, computed with (7.2.6) with $m = 1$, have the form

(9.2.1) $\qquad f_n = (\alpha_{fn}, \phi_{fn}, \beta_{fn})$, $g_n = (\alpha_{gn}, \phi_{gn}, \beta_{gn})$.

As n increases, these functional gains should converge to the functional gains in (9.1.2), which are optimal for the infinite dimensional control problem.

For the damping coefficient $c_0 = 10^{-4}$ and the control weighting R = .05, Figures 9.2.1 and 9.2.2 show the functional gain kernels ϕ_{fn}'' and ϕ_{gn} computed with cubic Hermite splines and n_e = 4, 6, 8 and 10 beam elements. Table 9.2.1 lists the corresponding scalar components of the gains. We have plotted ϕ_{fn}'' because the second derivative appears in the strain-energy inner product and because ϕ_{fn} converges in $H^2(0,\ell)$, so that ϕ_{fn}'' converges in L_2. The numerical results indicate that ϕ_{fn}'' and ϕ_{gn} converge uniformly on $[0,\ell]$, even though the Hermite splines yield a ϕ_{fn}'' that is discontinuous at the nodes. We have seen this convergence for the functional gains corresponding to beams in all of our structure-control examples, and we suspect that it is the case generally, although the convergence theorems in [GA2] guarantee only L_2 convergence for ϕ_{fn}'' and ϕ_{gn}. Since β_{fn} is not used in the nth control law, we omit it from the tables.

n_e	α_{fn}	α_{gn}	β_{gn}
4	4.4721	1.2440	-133.87
6	4.4721	1.2973	-139.69
8	4.4721	1.3106	-141.15
10	4.4721	1.3141	-141.54

Table 9.2.1. Scalar Components of Functional Control Gains

$\qquad$ Damping coefficient $c_0 = 10^{-4}$; Control weighting R = .05

$\qquad$ Number of elements n_e = 4, 6, 8, 10

$\qquad$ Hermite Spline Approximation

Figure 9.2.2. Functional Control Gain Component ϕ_{gn}

Damping coefficient $c_0 = 10^{-4}$; control weighting $R = .05$

Number of elements $n_e = 4, 6, 8, 10$

Hermite Spline Approximation

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Figure 9.2.1. Functional Control Gain Component $\phi_{fn}^{\prime\prime}$

Damping coefficient $c_0 = 10^{-4}$; control weighting $R = .05$

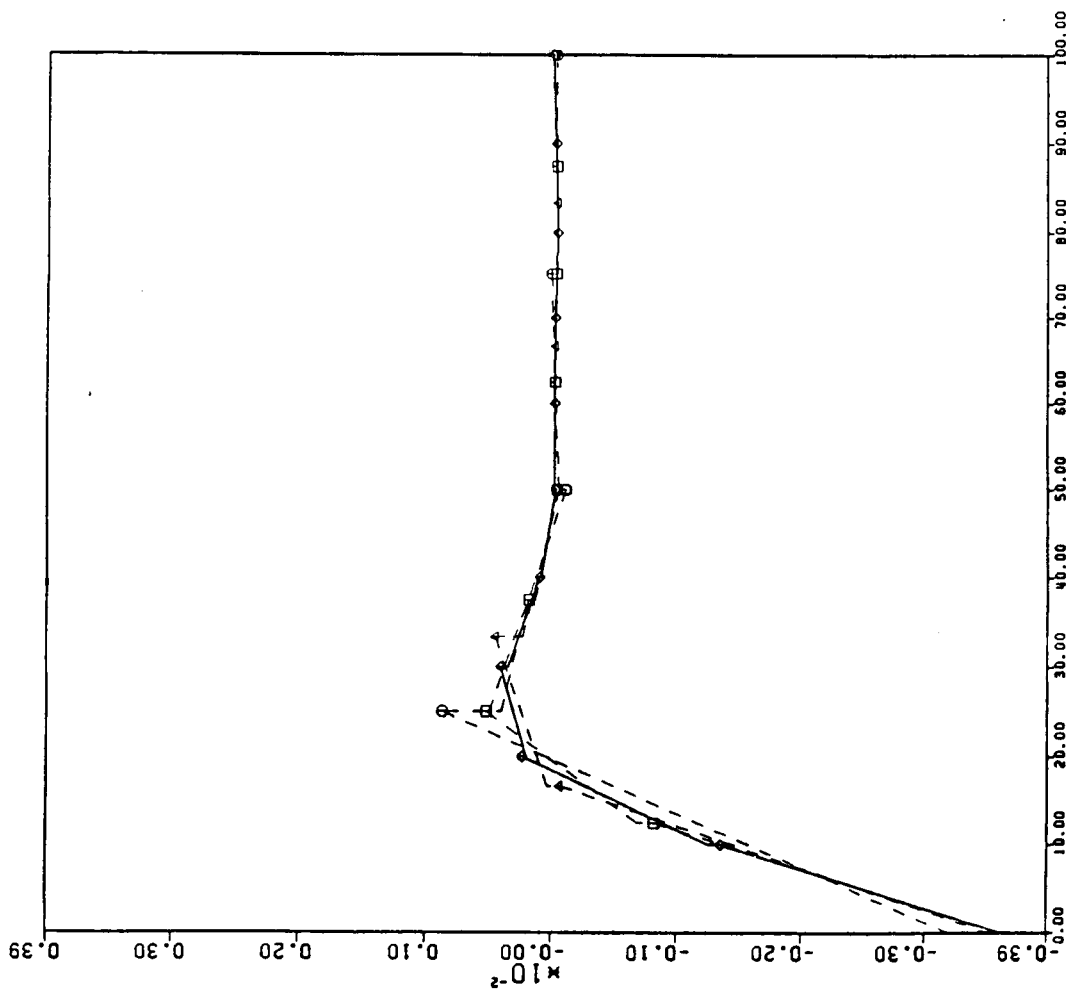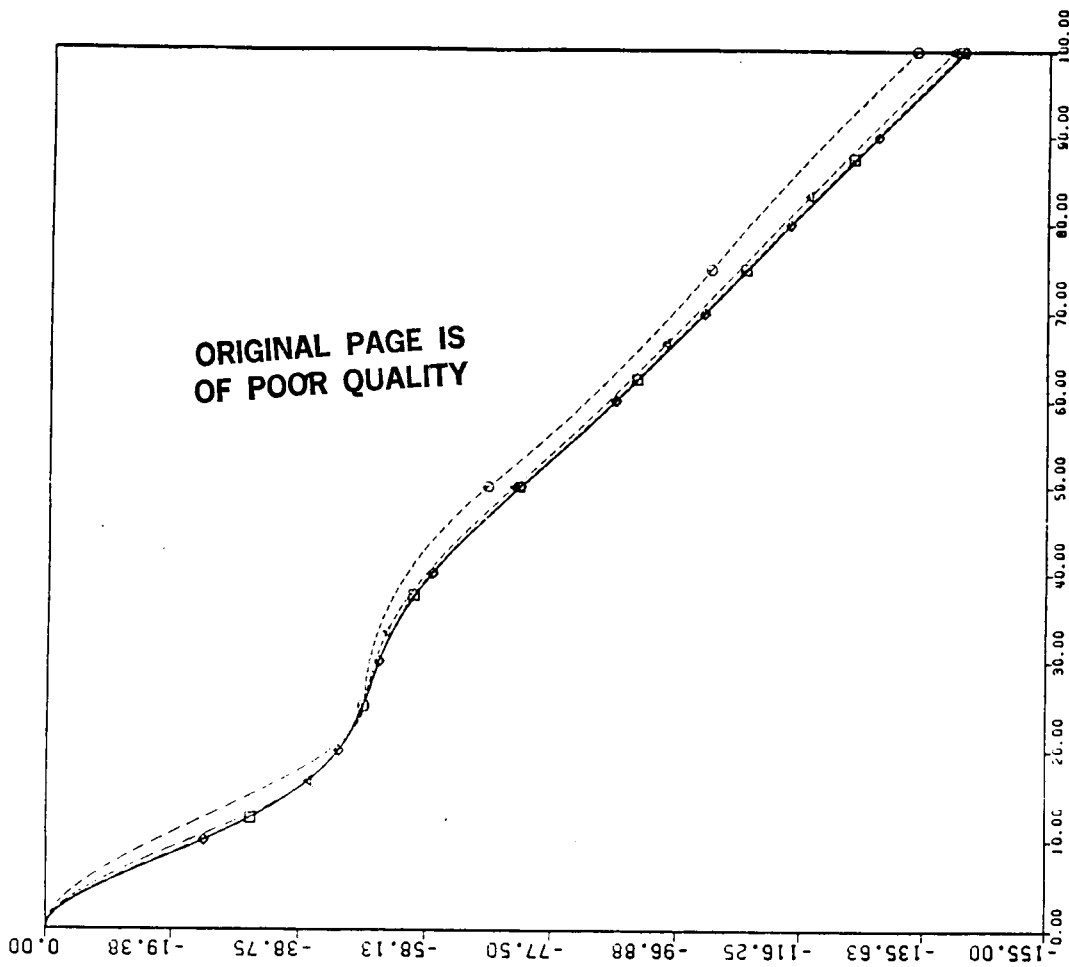Number of elements $n_e = 4, 6, 8, 10$

Hermite Spline Approximation

105

9.3 Approximation of the Optimal Estimator Gains

To compute the gain matrix $\hat{F}^n$ for the nth approximating estimator, we solve the Riccati equation (7.3.3), whose solution yields $\hat{F}^n$ according to (7.3.2). The 1 x 2n matrix C^n is given by (4.3.2), and 2n x 2n matrix $\tilde{\hat{Q}}^n$ is

$$(9.3.1) \qquad \tilde{\hat{Q}}^n = \begin{bmatrix} 0 & 0 \\ 0 & M^{-n} \end{bmatrix} \, ,$$

according to (9.1.3) and (7.1.5). (As always, M^{-n} is the inverse of the mass matrix.)

We compute the approximating functional estimator gains according to (7.3.4). Like the functional control gains, the functional estimator gains have the form

$$(9.3.2) \qquad \hat{f}_n = (\alpha_{fn}, \, \phi_{fn}, \, \beta_{fn}) \, , \, \hat{g}_n = (\alpha_{fn}, \, \phi_{fn}, \, \beta_{fn}) .$$

As in the control problem, our convergence theory establishes only L_2-convergence for ϕ_{fn}'' and ϕ_{gn}, but the numerical results show uniform convergence on $[0, \ell]$. Since $\phi_{fn}(0) = \phi_{fn}'(0) = 0$, the convergence of ϕ_{fn}'' implies the convergence of $\beta_{fn} = \phi_{fn}(\ell)$. Thus, β_{fn} is not an independent piece of information about the estimator gains while, as far as our convergence results go, β_{gn} is. We maintain analogy with the control problem and list only β_{gn} in the subsequent tables. Figures 9.3.1 and 9.3.2 show ϕ_{fn}'' and ϕ_{gn}, and Table 9.3.1 lists the scalars α_{fn}, α_{gn} and β_{gn}.

n_e	α_{fn}	α_{gn}	β_{gn}
4	5.3195	14.149	-1495.7
6	5.3567	14.347	-1517.5
8	5.3611	14.371	-1520.1
10	5.3623	14.377	-1520.8

Table 9.3.1. Scalar components of Functional Estimator Gains

Damping coefficient $c_0 = 10^{-4}$;

Estimator $\hat{R} = 10^{-4}$

Number of elements n_e = 4, 6, 8, 10

Hermite Spline Approximation

Figure 9.3.2. Functional Estimator Gain Component ϕ_{gn}

Damping $c_0 = 10^{-4}$; estimator $\hat{R} = 10^{-4}$

Number of elements $n_e = 4, 6, 8, 10$

Hermite Spline Approximation

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Figure 9.3.1. Functional Estimator Gain Component ϕ''_{fn}

Damping $c_0 = 10^{-4}$; estimator $\hat{R} = 10^{-4}$

Number of elements $n_e = 4, 6, 8, 10$

Hermite Spline Approximation

108
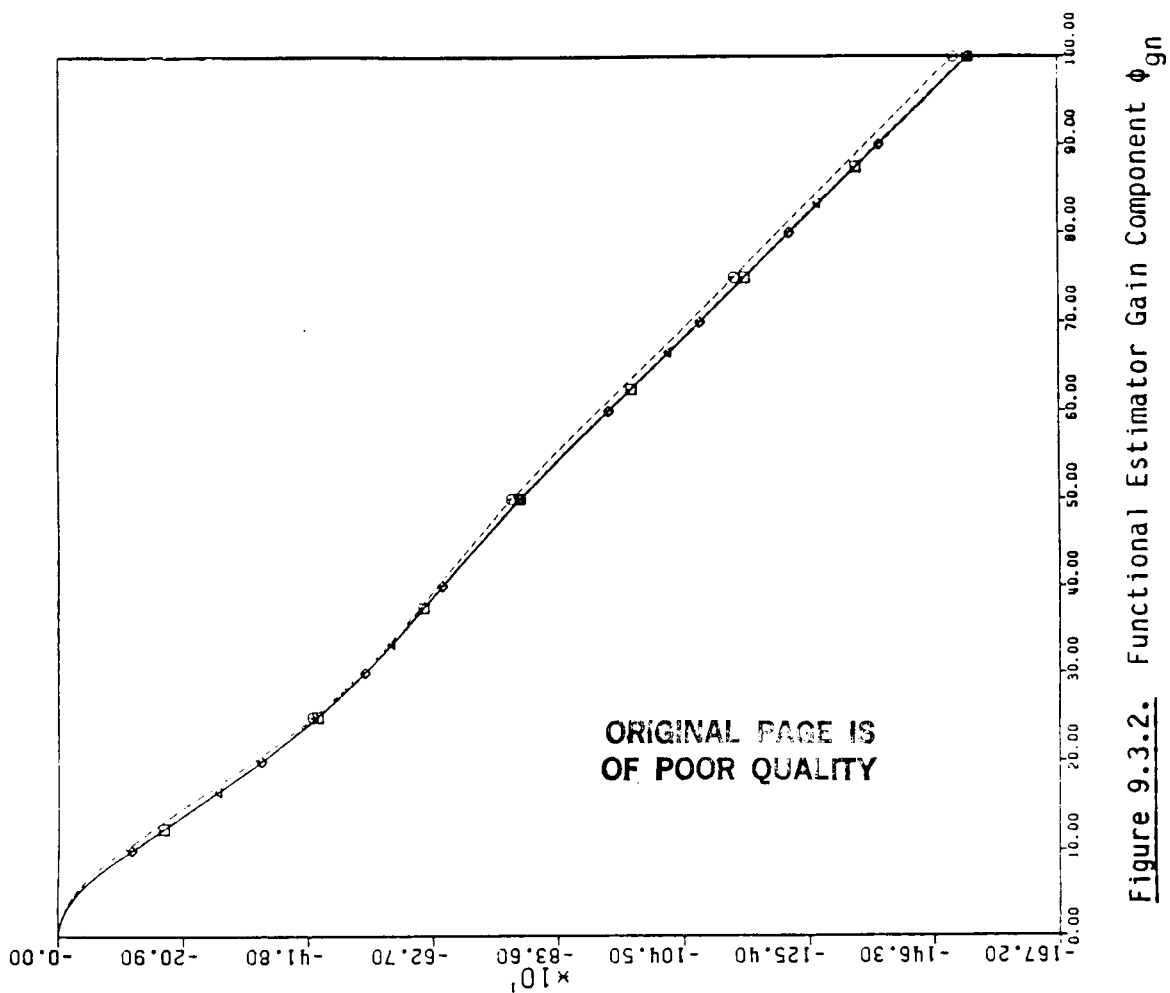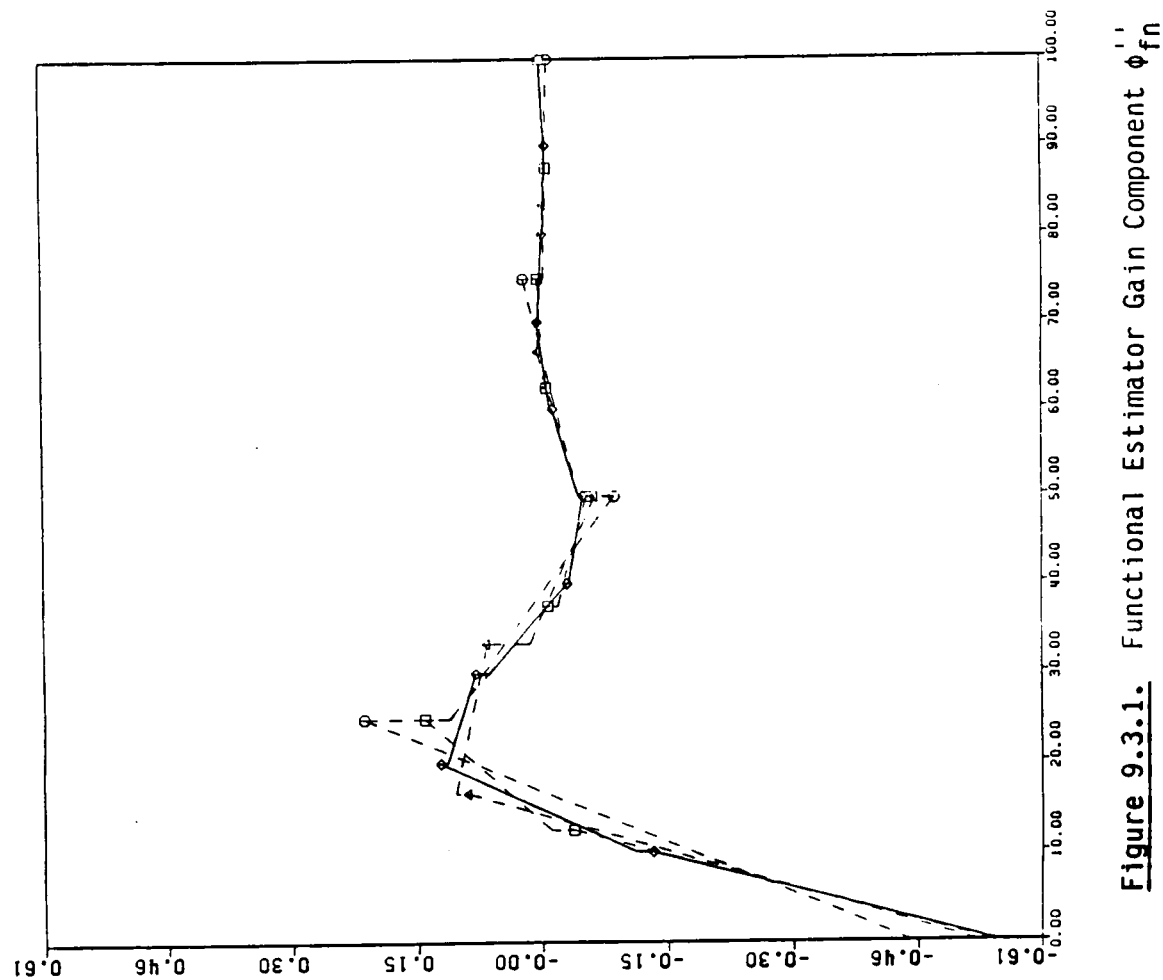
9.4 The Finite Dimensional Compensators

Now we construct the finite dimensional compensator in Figure 7.1 for 4, 6, 8 and 10 beam elements. Recall that (7.4.2) gives the transfer function of the nth compensator. Figure 9.4.1 shows the frequency responses (bode plots) of these compensators. The first graph in each figure contains the gain (magnitude) plots for all four orders of approximation, while the phase plot represents only the highest-order approximation. The finite dimensional compensators converge to the infinite dimensional compensator as n_e increases. This is consistent with the convergence of the functional control and estimator gains.

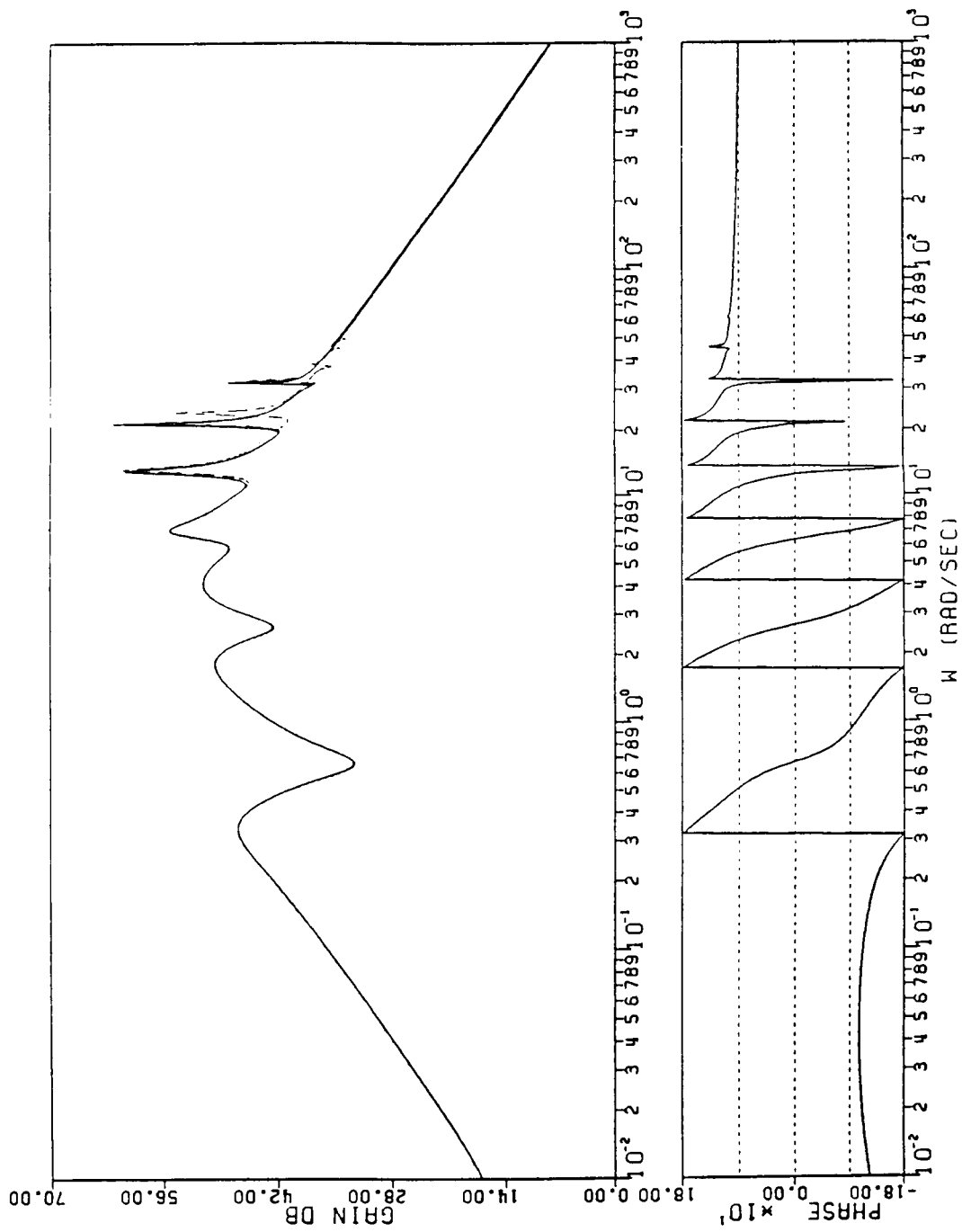**Figure 9.4.1.** Bode Plot for Compensator Based on Hermite Splines

(number of elements n_e = 4, 6, 8, 10)

110

9.5 The Closed-Loop Eigenstructure

As (6.2.13) says, the spectrum of the optimal closed-loop system is the union of the spectrum of (A-BF) and the spectrum of (A-$\hat{F}$C). If the damping operator D_0 in (3.1.1) is bounded relative to A_0^μ for some $\mu < 1$, then A has compact resolvent. In this case, (A-BF) and (A-$\hat{F}$C) also have compact resolvent because BF and $\hat{F}$C are bounded, so that the closed-loop spectrum consists only of isolated eigenvalues, each with finite multiplicity. If, as in our example, $D_0 = c_0 A_0$ with $c_0 > 0$, A does not have compact resolvent; if the eigenvalues of A_0 are ω_j^2, $\sigma(A)$ contains at most a finite number of complex eigenvalues and the sequences $\lambda_j = (-c_0 \pm \sqrt{c_0^2 \omega_j^2 - 4})\, \omega_j/2$, which approach $-\infty$ and the continuous spectrum $\{-1/c_0\}$ as j increases. When $D_0 = c_0 A_0^\mu$ for any $0 \le \mu \le 1$, the eigenvectors of A are the same as those when $D_0 = 0$, and hence are complete in E.

For the remainder of this discussion, let us assume that, as in our example, the spectrum of A (i.e., the open-loop spectrum) consists of isolated eigenvalues with finite multiplicity and possibly a finite number of limit points, and that the eigenvectors are complete in E. (This is the case in our example.) Then, since BF and $\hat{F}$C are bounded with finite rank, it follows from applying standard perturbation results [Ka2, pp. 208-214] that the asymptotic properties of both σ(A-BF) and σ(A-$\hat{F}$C) are identical to those of σ(A). This means that, beyond some number of eigenvalues, the eigenvalues of A-BF are virtually identical to those of A because the optimal compensator essentially controls only a finite number of modes. It also means that, although the optimal estimator contains copies of all the modes of the structure, essentially it observes only a finite number of modes and feeds virtually no sensor data into the copies of the rest. Of course, the infinite number of inactive estimator modes should be truncated before implementation.

We have not proved as much as we would like to about how the nth closed-loop spectrum -- i.e., the spectrum of the operator $A_{\infty n}$ discussed in Section 2 of Chapter 8 -- converges to the optimal closed-loop spectrum. In [GA2, Section 9], we showed that an extension of $A_{\infty n}$ converges in norm to $A_{\infty \infty}$ (the optimal closed-loop generator in (6.2.9)) when the damping operator D_0 does not couple the modes of free vibration and the natural mode shapes are the basis vectors for the approximation scheme. We also showed that, when A has compact resolvent, $(sI - A_{\infty n})^{-1}$ converges in norm to $(sI - A_{\infty \infty})^{-1}$. In either of these cases, it follows from the section of Kato cited above that the eigenvalues of $A_{\infty n}$ converge to those of $A_{\infty \infty}$. Our numerical results indicate that the eigenvalues of the nth closed-loop system based on the Hermite spline approximation also converge to those of the optimal closed-loop system, but we have not proved this.

To compute the closed-loop eigenvalues, we approximated the flexible structure (the plant) with 15 natural modes (obtained using 30 beam elements with Hermite splines) and formed the closed-loop system with the nth compensator for various values of n. The numerical results indicate that the optimal compensator ignores and does not affect any mode past the ninth (the eight flexible mode), and we are stretching matters to say that the ninth mode is controlled.

Table 9.5.1 lists the eigenvalues for the closed-loop system consisting of the 15-mode model of the structure and the compensator based on 10 elements with Hermite splines. For this compensator, n = 2*10 + 1 = 21, so that the estimator contains approximations to the first 21 modes of the structure. Most of these approximations have not converged to the true natural modes, as can be seen from the imaginary parts of the eigenvalues under A-$\hat{F}$C, which converge from above to slight perturbations of the open-loop frequencies.

The last twelve of the twenty-one modes in the 10-element Hermite spline compensator are inactive as far as the input/output map of the compensator. They do not couple with the rest of the compensator or with any modes in the structure, and they could be truncated without affecting the closed-loop response. The eigenvalues of the inactive compensator modes are equal to the corresponding open-loop eigenvalues of the ten-element approximation to the structure. The magnitudes of the real parts are larger than those for the corresponding true open-loop eigenvalues because the damping is proportional to the stiffness and the frequencies of the higher modes of the approximation are greater than the true frequencies. Since the inactive compensator modes reside in the nth estimator, we list the corresponding eigenvalues under $(A-\hat{F}C)$.

The closed-loop eigenvalues in Table 9.5.1 corresponding to modes 5, 6 and 7 are not equal to the corresponding eigenvalues of $(A_{21}-B_{21}F_{21})$ and $(A_{21}-\hat{F}_{21}C_{21})$. (For each of modes 5, 6 and 7, we listed one pair of closed-loop eigenvalues under $(A-BF)$ and one pair under $(A-\hat{F}C)$.) This means that the Hermite-spline compensator for ten elements is not as close to convergence as the preceding functional gain and bode plots might suggest. Still, the closed-loop response produced by the ten-element Hermite-spline compensator should be close to optimal, except possibly for the sixth mode. Table 9.5.1 shows that this closed-loop system is stable.

Table 9.5.1. Closed-loop Eigenvalues with Compensator Based on Hermite Splines for 10 Elements

OPEN-LOOP		A-BF		A-F̂C	
REAL	IMAG	REAL	IMAG	REAL	IMAG
0.0	0.0	-0.03305	0.0	-0.27579	0.33295
0.0	0.0	-0.00799	0.0	-0.27579	-0.33295
-0.00005	0.96522	-0.11498	0.96522	-1.07059	1.52394
-0.00052	3.21201	-0.17909	3.21192	-0.83359	3.27848
-0.00242	6.96055	-0.15092	6.95926	-0.33913	6.97026
-0.00840	12.96107	-0.09170	12.94415	-0.14130	12.98792
-0.02280	21.35301	-0.03957	21.33965	-0.09902	21.40999
-0.05149	32.08958	-0.05132	32.08852	-0.08468	32.23854
-0.10190	45.14356	-0.10176	45.14358	-0.11477	45.54628
-0.18305	60.50577	-0.18300	60.50578	-0.19281	61.43464
-0.30559	78.17711	-0.30557	78.17711	-0.32191	80.02459
-0.48184	98.16564	-0.48183	98.16564	-0.51073	100.99211
-0.72587	120.48589	-0.72586	120.48589	-0.87879	132.55647
-1.05361	145.15908	-1.05361	145.15908	-1.28294	160.16850
-1.48299	172.21366	-1.48299	172.21366	-1.89848	194.84366
-2.03407	201.68604	-2.03407	201.68604	-2.78066	235.80614
				-4.03103	283.90815
				-5.78013	339.95430
				-8.15257	403.71367

10. Other Applications Issues and Future Research Topics

The methods developed or applied in this monograph provide a mathematically sound yet practical framework for designing feedback control systems for distributed parameter structural systems. As in most research, development of the ideas presented here raises new questions as it answers current ones. In this chapter we mention some of the issues which have emerged during the course of this work. Some of these we have pursued to the extent of achieving promising but as yet incomplete results. Others we note as issues for future investigation. The ideas will fall into three areas: 1) discrete time modeling and control, 2) robustness, and 3) model reduction.

10.1 Discrete Time Modeling and Control

Recent research [GR1, GR2, GR3] has developed approximation methods and convergence results for discrete-time LQG control problems for distributed systems. Much of the discrete-time theory is analogous to that in this monograph for the continuous-time problem. The infinite dimensional discrete-time theory and approximation methods have been applied to control of flexible structures in [GR1, GR2]. Further research is needed on numerical difficulties encountered in approximate solution of infinite dimensional discrete-time Riccati operator equations. One respect in which the discrete-time problem differs from the continuous-time problem is that boundary control for flexible structures usually results in a bounded input operator for the discrete-time problem, whereas the input operators corresponding to boundary controls in continuous-time problems usually are unbounded. This is the subject of [GR2, GR3]. Further work is needed along these lines to expand the class of boundary operators and the class of unbounded measurement operators to which the discrete-time approximation theory applies.

Also, discrete-time input/output representation of infinite dimensional models of flexible structures has been investigated in [Ja1, GJ1]. Infinite dimensional auto-regressive-moving average (ARMA) models of both the plant and the compensator have been developed for simple structures. Preliminary approximation theory for such models has been developed, but considerably more work is needed to bring this point of view to the level of maturity and usefulness of the state-space methods for distributed systems. Such research should be particularly important for adaptive identification and control of flexible structures, as suggested by the large orders of finite dimensional ARMA models found necessary in [JG1, JG2] for adaptive identification of an experimental flexible structure at NASA Langley Research Center.

10.2 Robustness

An important feature of the approach developed in this monograph is that the order and modal composition of the model are automatically adjusted as a function of the performance objectives, disturbance environment, sensor locations and actuator locations. This is an effective way to deal with problems due to model truncation. However, there is nothing in the approach that explicitly addresses the issue of modeling errors due to poorly known parameter values (e.g., frequencies, damping constants, mode shapes, etc.). There are always some differences between a model and the physical system the model is supposed to describe. Since control system designs are based on models, one must always be concerned about the degree to which the theoretical performance is degraded due to differences between the model and the physical system. Hence, modeling errors of all types are a major concern in designing control systems.

In addressing the modeling error problem, it is important to describe the modeling errors in a way that facilitates dealing with them. One approach is

to use the structured uncertainty description of Doyle [Do1]. Structured uncertainty models represent specific modeling errors (frequency errors, damping errors, mode shape errors, etc.) in the form of auxiliary multivariable feedback loops appended to the plant model. These descriptions may be used as the starting point for different approaches to robust controller design, e.g., Doyle's μ synthesis approach [Do1] or the Linear Quadratic Gaussian (LQG) approach which is the focus of this monograph. In the context of the LQG approach, the forms of the auxiliary feedback loops motivate changes in the performance weighting matrix in the LQR problem and the disturbance noise covariance matrix in the K-B filter problem. These changes involve adding weighting matrices which reflect the specific nature of the uncertainty [BM1, TS1]. Making the additional terms large or small permits a controlled trade-off between performance and robustness. Thus one can arrive at a suitable balance between achieving a desired performance goal and minimizing the effect of structured uncertainties in the plant model.

Some results of using this approach are described in [BM1] and they show a marked increase in robustness with only a modest loss in performance. The robustness is not achieved by rolling off the loop gain before the first uncertain modes. Several of the uncertain modes are actively controlled. Numerical results in [BM1] focus on uncertain frequencies, but the methods used can be extended to other types of parameter errors. Work to study the nature and effectiveness of these extensions is currently in progress.

Another approach for improving robustness involves using nonlinear programming to achieve robustness by sensitivity optimization, which means minimizing the sensitivity of closed-loop eigenvalues with respect to uncertainties in plant parameters. The method can also be used to optimize the shape of

structural elements to reduce structural weight. An overall control/structure optimal design can be achieved by using both of these features simultaneously. Appropriate constraints are imposed to ensure that high performance is maintained. This approach to robustness is developed in papers [Ad1, AG1, AG2]. The idea is to use nonlinear programming to reduce the sensitivity of the closed-loop eigenvalues with respect to modeling errors, while maintaining sufficiently high performance of the closed-loop control system.

The paper [AG1] derives formulas for the sensitivities of closed-loop eigenvalues with respect to uncertain plant parameters and presents a numerical example that demonstrates the effect of these sensitivities on robustness in control of a flexible structure. The analysis in [AG1] indicates that the first-order sensitivities of the closed-loop eigenvalues approach infinity as a controller and an estimator eigenvalue approach each other and suggests that robustness can be improved by separating controller and estimator eigenvalues. The numerical results for the flexible structure example in [AG1, AG2] demonstrate the improved robustness achieved by moving the estimator eigenvalues to the left of the controller eigenvalues.

The work in [Ad1, AG1, AG2] has led to a general guideline for choosing the state weighting matrix and the process noise covariance (the Q matrices) in the LQG problem to improve robustness: After the state weighting for the control problem is chosen according to performance criteria, the Q matrix for the estimator design is chosen to move the estimator eigenvalues for the controlled modes with higher frequencies sufficiently to the left of the closed-loop controller eigenvalues to reduce the closed-loop eigenvalue sensitivities to acceptable levels. Examples of Q matrices that achieve this sensitivity reduction are given in [Ad1, AG1-AG2]. In general, the estimator Q for a modal

118

representation of a structure is diagonal and its diagonal elements increase as the corresponding structural frequencies increase.

References [Ad1, AG2] discuss optimal eigenvalue sensitivity reduction in conjunction with optimal weight reduction by structural shape optimization. The idea is to combine minimization of closed-loop eigenvalue sensitivity with optimization of structural mass distribution, subject to constraints on eigenvalue location, to produce a robust controller, a light structure and a closed-loop system with fast response. While the measure of robustness used in the design objective is the first-order sensitivity of the closed-loop eigenvalues, the final evaluation of the robustness of the design is based on large variations in the uncertain parameters. The numerical results in [Ad1, AG1, AG2] demonstrate the effectiveness of the method for producing both a robust control system and a light structure.

Recently, a significant body of literature has been devoted to Liapunov stability methods for robustness analysis and design of control systems with structured uncertainties [HB1, KB1, Ye1, YL1, ZK1]. These papers use quadratic Liapunov functions to estimate a region in parameter space for which a control system will remain stable. Such Liapunov robustness analysis is combined in [Be1, GB1, PH1] with nonstandard Riccati matrix equations for robust controller design.

The main advantage of these Liapunov methods is that they can be applied in a straightforward and computationally efficient manner to complex, realistic systems. However, the methods generally yield conservative robustness estimates because, with the possible exception of the method in [HB1], all of the Liapunov robustness methods in current literature are zero-order in the sense that a

single Liapunov function is used for all permissible variations in the uncertain parameters.

Recent research at UCLA [LG1] has produced a first-order Liapunov robustness analysis approach, which allows the Liapunov function to vary with the uncertain plant parameters. On most problems the robustness margins predicted by this new method are substantially larger than those predicted by the zero-order methods. If the first-order Liapunov robustness analysis continues to be as successful as it has been in early applications, an important line of research should be to develop related first-order robustness design methods motivated by the Riccati/Liapunov methods in [Be1, PH1]. Because of the theory and approximation methods presented in this text for infinite dimensional Riccati equations and the success of the applications to optimal controller design for large space structures, it seems reasonable to expect that this kind of theory and the associated approximation methods could be used to develop Riccati/Liapunov robust controller design methods for large space structures.

10.3 Model Reduction

The model reduction work described in this monograph deals with systems described by linear constant coefficient differential equations. This is a feature that is shared with most of the model reduction work reported in the literature. Equations of this type arise when one is studying small vibrations about a state of rest in inertial space and this is a situation of great interest for applications. Constant coefficient equations may also arise for some systems performing small vibrations about a state of steady motion (e.g., steady spin). In many cases of practical interest, however, the nominal motion either is periodic or consists of a single maneuver over a finite time interval.

If special inertial symmetry conditions are not satisfied, the linearized equations describing small vibrations about periodic motions generally have periodic coefficients, while a general finite time maneuver often leads to linear equations with time varying coefficients of a general form. A Dual-Spin Spacecraft where both bodies are flexible and neither body is inertially symmetric is an example of a system where small variations about steady spin lead to linear equations with periodic coefficients. During steady state operation, the equations governing transverse attitude motion of such a spacecraft assume the form:

(10.3.1) $X = A(t)X$; $X(0) = X0$

(10.3.2) $A(t+T) = A(t)$

where $T = 2\pi/\omega_{rel}$ and ω_{rel} is the relative rate between the two bodies. If one wishes to use the methods of this monograph to design a control system for such a spacecraft, it would be desirable to have a scheme similar to that described in Ch. 5 for obtaining lower order models of the system.

Of the various methods available for model reduction, balanced realization theory is perhaps the best suited for extension to the time varying case. The application of balanced realization theory for time varying systems has been considered in the literature [SS1, VK1], but work in this area has not been extensive. Examples worked out so far have been simple and nonphysical. (In particular, these examples were not from the field of structural dynamics.) The potential of this approach for time varying structural systems is largely untapped, and it would make a suitable topic for further investigation.

REFERENCES

Ad1   A. Adamian, "Integrated Control/Structure Design and Robustness," UCLA Ph.D. Dissertation, September 1986.

AG1   A. Adamian and J.S. Gibson, "Sensitivity of Closed-Loop Eigenvalues and Robustness," *AIAA J. Guidance and Control*, Vol. 10, No. 6, pp. 588-592, November-December 1987.

AG2   A. Adamian and J.S. Gibson, "Integrated Control/Structure Design and Robustness," submitted to *Automatica*.

Ba1   A.V. Balakrishnan, *Applied Functional Analysis*, Second Edition, Springer-Verlag, New York, 1981.

Ba2   M.J. Balas, "Feedback Control of Flexible Systems," *IEEE Trans. AC*, AC-23, pp. 673-679, 1978.

Ba3   M.J. Balas, "Modal Control of Certain Flexible Systems", *SIAM J. Contr. Opt.*, Vol. 16, pp. 450-462, 1978.

Be1   D.S. Bernstein, "Robust Static and Dynamic Output Feedback Stabilization: Deterministic and Stochastic Perspectives," *IEEE Trans. AC*, Vol. AC-32, No. 12, pp. 1076-1084, December 1987.

Bl1   P.A. Blelloch, "A Modified Loop Transfer Recovery Approach for Robust Control of Flexible Spacecraft," UCLA Ph.D. Dissertation, June 1986.

BB1   H.T. Banks and J.A. Burns, "Hereditary Control Problems: Numerical Methods Based on Averaging Approximations," *SIAM J. Contr. Opt.*, Vol. 16, pp. 169-208, 1978.

BH1   D.S. Bernstein and D.C. Hyland, "The Optimal Projection Approach to Designing Finite-Dimensional Controllers for Distributed Parameter Systems," 23rd IEEE Conference on Decision and Control, Las Vegas, NV, December 1984.

BH2   D.S. Bernstein and D.C. Hyland, "The Optimal Projection Equations for Finite-Dimensional Fixed-Order Dynamic Compensation of Infinite-Dimensional Systems," *SIAM J. Contr. Opt.*, to appear.

BK1   H.T. Banks and K. Kunisch, "The Linear Regulator Problem for Parabolic Systems," *SIAM J. Contr. Opt.*, Vol. 22, No. 5, pp. 684-698, September 1984.

BM1   P. Blelloch and D.L. Mingori, "The Design of Robust LTR Compensators for Flexible Structures," Proceedings of the AIAA Guidance, Navigation and Control Conference, pp. 314-318, Williamsburg, VA, August 18-22, 1986.

BS1   R.H. Bartels and G.W. Stewart, "Solution of the Matrix Equation AX + XB = C," *Communications of the ACM*, Vol. 15, No. 9, pp. 820-826, September 1972.

Cu1   R.F. Curtain, "On Stabilizability of Linear Spectral Systems via State Boundary Feedback," *SIAM J. Contr. Opt.*, Vol. 23, No. 1, pp. 144-152, 1985.

CP1   R.W. Clough and J. Penzien, *Dynamics of Structures*, McGraw-Hill, New York, 1975.

CP2   R.F. Curtain and A.J. Pritchard, *Infinite Dimensional Linear Systems Theory*, Springer-Verlag, New York, 1978.

CR1   G. Chen and D.L. Russell, "A Mathematical Model for Linear Elastic Systems with Structural Damping," *Quarterly of Applied Mathematics*, pp. 433-454, January 1982.

CS1   R.F. Curtain and D. Salamon, "Finite Dimensional Compensators for Infinite Dimensional Systems with Unbounded Input Operators," *SIAM J. Contr. Opt.*, Vol. 24, No. 4, pp. 797-816, July 1986.

Do1   J.C. Doyle, "Structured Uncertainty in Control System Design," Proceedings of the 24th IEEE Conference on Decision and Control, pp. 260-265, 1985.

En1   D. Enns, "Model Reduction for Control System Design," Ph.D. Dissertation, Dept. of Aeronautics and Astronautics, Stanford University, Stanford, CA, 1984.

Gi1   J.S. Gibson, "An Analysis of Optimal Modal Regulation: Convergence and Stability," *SIAM J. Contr. Opt.*, Vol. 19, No. 5, pp. 686-707, September 1981.

Gi2   J.S. Gibson, "A Note on Stabilization of Infinite Dimensional Linear Oscillators by Compact Linear Feedback," *SIAM J. Contr. Opt.*, Vol. 18, pp. 311-316, 1980.

Gi3   J.S. Gibson, "Linear-quadratic Optimal Control of Hereditary Differential Systems: Infinite Dimensional Riccati Equations and Numerical Approximations," *SIAM J. Contr. Opt.*, Vol. 21, No. 1, pp. 95-139, January 1983.

Gi4   J.S. Gibson, "The Riccati Integral Equations for Optimal Control Problems on Hilbert Spaces," *SIAM J. Contr. Opt.*, Vol. 17, No. 4, pp. 537-565, July 1979.

Gl1   K. Glover, "All Optimal Hankel-Norm Approximations of Linear Multivariable Systems and Their L-Error Bounds," *Int'l. Contr.*, Vol. 39, No. 6, pp. 1115-1193, 1984.

Gr1   C.Z. Gregory, "Reduction of Large Flexible Spacecraft Models Using Internal Balancing Theory, "*AIAA J. Guidance and Control*, Vol. 7, pp. 17-32, 1984.

GA1   J.S. Gibson and A. Adamian, "A Comparison of Three Approximation Schemes for Optimal Control of a Flexible Structure," *SIAM Frontiers Edition on Control of Distributed Systems*, to appear.

GA2   J.S. Gibson and A. Adamian, "Approximation Theory for LQG Optimal Control of Flexible Structures," ICASE Report No. 88-48, Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, 1988.

GB1   A.R. Galimidi and B.R. Barmish, "The Constrained Liapunov Problem and Its Application to Robust Output Feedback Stabilization," *IEEE Trans. AC*, Vol. AC-31, No. 5, pp. 410-419, May 1986.

GJ1   J.S. Gibson and F. Jabbari, "A Digital Input/Output Model for Trace-Class Systems," to appear in *J. Math. Anal. and Appl.*

GL1   K. Glover and D.J.N. Limebeer, "Robust Multivariable Control System Design Using Optimal Reduced Order Plant Models," Proceedings of the 1983 Automatic Control Conference, San Francisco, CA, pp. 644-649, June 22-24, 1983.

GM1   J.S. Gibson and D.L. Mingori, "Approximation and Compensator Order Determination in Optimal Control of Flexible Structures," *Proc. IFAC Workshop on Model Error Concepts and Compensation*, Boston, MA, June 1985.

GM2   J.S. Gibson, D.L. Mingori, A. Adamian and F. Jabbari, "Approximation of Optimal Infinite Dimensional Compensators for Flexible Structures," JPL Workshop of Identification and Control of Space Structures, San Diego, CA, June 1984.

GM3   J.S. Gibson, D.L. Mingori, et al., HR Textron Report #956541-Final, to JPL, September 1984.

GR1   J.S. Gibson and I.G. Rosen, "Numerical Approximation for the Infinite-Dimensional Discrete-Time Optimal Linear-Quadratic Regulator Problem," *SIAM J. Contr. Opt.*, Vol. 26, pp. 428-451.

GR2   J.S. Gibson and I.G. Rosen, "Approximation in Discrete-Time Boundary Control of Flexible Structures," Proceedings of the 26th IEEE Conference on Decision and Control, Los Angeles, CA, December 9-11, 1987.

GR3   J.S. Gibson and I.G. Rosen, "Approximation of Discrete-Time LQG Compensators for Distributed Systems with Boundary Input and Unbounded Measurement," *Automatica*, July 1988.

HB1   D. Hyland and D. Bernstein, "The Majorant Liapunov Equation: A Nonnegative Matrix Equation for Robust Stability and Performance of Large Scale Systems," *IEEE Trans. AC*, Vol. AC-32, No. 11, pp. 1005-1013, November 1987.

Ja1   F. Jabbari, "Vector-Channel Lattice Filters and Identification of Flexible Structures," UCLA Ph.D. Dissertation, 1986.

JG1   F. Jabbari and J.S. Gibson, "Vector-Channel Lattice Filters and Identification of Flexible Structures," *IEEE Trans. on Automatic Control*, Vol. 33, No. 5, May 1988.

JG2   F. Jabbari and J.S. Gibson, "Adaptive Identification of a Flexible Structure by Lattice Filters," to appear in *AIAA J. Guidance, Control and Dynamics.*

Ka1   T. Kailath, *Linear Systems*, Prentice-Hall, pp. 49-55, 1980.

Ka2   T. Kato, *Perturbation Theory for Linear Operators*, Second Edition, Springer-Verlag, New York, 1984.

KB1   L.H. Keel, S.P. Bhattacharyya and J.W. Howze, "Robust Control with Structured Perturbations," *IEEE Trans. AC*, Vol. AC-33, No. 1, pp. 68-78, January 1988.

KS1   H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*, Wiley, 1972.

LA1   Y. Liu and B.D.O. Anderson, "Controller Reduction via Stable Factorization and Balancing," *Int'l. J. Contr.*, Vol. 44, No. 2, pp. 507-531, 1986.

LG1   M.A. Leal and J.S. Gibson, "Liapunov Robustness Bounds for Linear Systems with Uncertain Parameters," to appear.

LT1   I. Lasiecka and R. Triggiani, "Dirichlet Boundary Control Problem for Parabolic Equations with Quadratic Cost: Analyticity and Riccati's Feedback Synthesis," *SIAM J. Contr. Opt.*, Vol. 21, No. 1, pp. 41-67, January 1983.

LT2   I. Lasiecka and R. Triggiani, "Riccati Equations for Hyperbolic Partial Differential Equations with $L2(0,T; L2(\Gamma))$ - Dirichlet Boundary Terms," *SIAM J. Contr. Opt*, Vol. 24, No. 5, pp. 884-925, September 1986.

LT3   I. Lasiecka and R. Triggiani, "The Regulator Problem for Parabolic Equations with Dirichlet Boundary Control," Parts I and II, *Appl. Math. Optim.*, No. 16, pp. 147-168, 187-216, 1987.

Me1   L. Meirovitch, *Computational Methods in Structural Dynamics*, Sijthoff and Noordhoff, 1980.

Me2   L. Meirovitch, "A New Method of Solution of the Eigenvalue Problem for Gyroscopic Systems," *AIAA Journal*, Vol. 12, No. 10, pp. 1337-1442, October 1974.

Me3   L. Meirovitch, "A Modal Analysis of the Response of Linear Gyroscopic Systems," *J. Appl. Mech.*, Vol. 42, No. 2, pp. 446-450, June 1974.

Mo1   B.C. Moore, "Principal Component Analysis in Linear Systems: Controllability, Observability, and Model Reduction," *IEEE Trans. Automat. Contr.*, Vol. AC-26, pp. 17-32, February 1981.

MG1   D.L. Mingori, J.S. Gibson, P. Blelloch and A. Adamian, "Control of a Flexible Space Antenna: A Finite Dimensional Perspective Based on Distributed Parameter Theory," JPL Workshop on Identification and Control of Space Structures, San Diego, CA, June 1984.

PH1   I.R. Peterson and C.V. Hollot, "A Riccati Equation Approach to the Stabilization of Uncertain Linear Systems," *Automatica*, Vol. 22, pp. 397-411, 1986.

Sc1   M.H. Schultz, *Spline Analysis*, Prentice-Hall, Englewood Cliffs, N.J., 1973.

Sc2   J.M. Schumacher, "A Direct Approach to Compensator Design for Distributed Parameter Systems," *SIAM J. Contr. Opt.*, Vol. 21, No. 6, pp. 823-836, 1983.

Sh1   R.E. Showalter, *Hilbert Space Methods for Partial Differential Equations*, Pitman, London, 1977.

SF1   G. Strang and G.J. Fix, *An Analysis of the Finite Element Method*, Prentice-Hall, Englewood Cliffs, N.J., 1973.

SS1   S. Shokoohi, L.M. Silverman and P.M. Van Dooren, "Linear Time-Variable Systems: Balancing and Model Reduction," *IEEE Transactions on Automatic Control*, Vol. AC-28, pp. 810-822, August 1983.

TS1 M. Tahk and J. Speyer, "Modelling of Parameter Variations and Asymptotic LQG Synthesis," *IEEE Transactions on Automatic Control*, Vol. AC-32, No. 9, pp. 793-801, September 1987.

VK1 E.I. Verriest and T. Kailath, "On Generalized Balanced Realizations," *IEEE Transactions on Automatic Control*, Vol. AC-28, pp. 833-844, August 1983.

Ye1 R.K. Yedavalli, "Improved Measures of Stability Robustness of Linear State Space Models," *IEEE Trans. AC*, Vol. AC-30, No. 6, pp. 557-579, June 1985.

YL1 R.K. Yedavalli and Z. Liang, "Reduced Conservatism in Stability Robustness Bounds by State Transformation," *IEEE Trans. AC*, Vol. AC-31, No. 9, pp. 863-866, September 1986.

ZK1 K. Zhou and P.P. Khargonekar, "Stability Robustness Bounds for Linear State-space Models with Structured Uncertainty," *IEEE Trans. AC*, Vol. AC-32, No. 7, pp. 621-623, July 1987.

APPENDIX A. Asymptotic Expansion for the Observability Grammian

The observability grammian satisfies the equation

(A.1) $$W_0 A + A^T W_0 + C^T C = 0$$

For the lightly damped mechanical systems of Section 5.5,

(A.2) $$A = \begin{bmatrix} 0 & I \\ -\Omega^2 & -\epsilon\Delta \end{bmatrix}, \qquad C = [C_p \ C_v]$$

The development of an expansion for W_c in Section (5.5) is simplified somewhat because the matrix BB^T is zero except for the lower right hand quadrant. In contrast, the matrix CC^T is generally full, so the development of an expansion for W_0 is more complicated. For this reason, we will develop an expression only for the first two terms in this expansion. When CC^T is full, there is no particular advantage in assuming a form like that of Eq. (5.5.8) for W_0. Let

(A.3) $$A_0 \triangleq \begin{bmatrix} 0 & I \\ -\Omega^2 & 0 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\Delta \end{bmatrix}$$

(A.4) $$W_0 = \frac{1}{\epsilon} U + V + \epsilon W + \ldots$$

Substituting (A.2-A.4) into (A.1) yields

(A.5) $$\frac{1}{\epsilon}[UA_0 + A_0^T U] + [VA_0 + A_0^T V + UA_1 + A_1^T U + C^T C]$$
$$+ \epsilon[WA_0 + A_0^T W + VA_1 + A_1^T V] + \ldots = 0$$

Set the coefficients of like powers in ϵ equal to zero to obtain

(A.6) $$UA_0 + A_0^T U = 0$$

(A.7) $VA_0 + A_0^T V + UA_1 + A_1^T U + C^T C = 0$

(A.8) $WA_0 + A_0^T W + VA_1 + A_1^T V = 0$

etc. Writing (A.6) in partitioned form leads to

(A.9) $u_{12}\Omega^2 + \Omega^2 u_{12}^T = 0$

(A.10) $u_{11} - \Omega^2 u_{22} = 0$; $u_{11} - u_{22}\Omega^2 = 0$

(A.11) $u_{12}^T + u_{12} = 0$

It follows from (A.9-A.11) and the assumption that the frequencies in Ω^2 are distinct that

(A.12) $u_{12} = 0$

(A.13) u_{11}, u_{22} are diagonal

Thus,

(A.14) $U = \begin{bmatrix} \Omega^2 u_{22} & 0 \\ 0 & u_{22} \end{bmatrix}$

To determine the elements of the diagonal matrix u_{22}, write (A.7) in partitioned form

(A.15) $v_{12} + v_{12}^T - \Delta u_{22} - u_{22}\Delta + C_v^T C_v = 0$

(A.16) $-\Omega^2 v_{22} + v_{11} + C_p^T C_v = 0$; $v_{11} - v_{22}\Omega^2 + C_v^T C_p = 0$

(A.17) $-\Omega^2 v_{12}^T - v_{12}\Omega^2 + C_p^T C_p = 0$

128

The diagonal elements of v_{12} can be easily found from (A.17)

$$(A.18) \qquad (v_{12})_{ii} = \frac{(c_p^T c_p)_{ii}}{2 \omega_i^2}$$

Using (A.18) and considering the diagonal terms in (A.15) one may derive explicit expressions for the diagonal elements of u_{22}.

$$(A.19) \qquad (u_{22})_{ii} = \frac{(c_p^T c_p)_{ii} + \omega_i^2 (c_v^T c_v)_{ii}}{2 \omega_i^2 \Delta_{ii}}$$

Equations (A.14) and (A.19) define the dominant first term in the expansion for W_0. To determine the second term in (A.4), we proceed to examine the off diagonal terms in (A.15)-(A.17). This leads to

$$(A.20) \qquad (v_{12})_{ij} = \frac{(c_p^T c_p)_{ij} + \omega_i^2 (c_v^T c_v)_{ij} - \omega_i^2 (\Delta u_{22} + u_{22}\Delta)_{ij}}{\omega_j^2 - \omega_i^2} \ ; \qquad i \neq j$$

$$(A.21) \qquad (v_{11})_{ij} = \frac{\omega_j^2 (c_p^T c_v)_{ij} - \omega_i^2 (c_p^T c_v)_{ji}}{\omega_i^2 - \omega_j^2} \ ; \qquad i \neq j$$

$$(A.22) \qquad (v_{22})_{ij} = \frac{(c_p^T c_v)_{ij} - (c_p^T c_v)_{ji}}{\omega_i^2 - \omega_j^2} \ , \qquad i \neq j$$

Equations (A.18) and (A.20) define v_{12} completely, and (A.21) and (A.22) define the off diagonal terms of v_{11} and v_{22}. To find the diagonal terms of v_{11} and v_{22}, Eq. (A.8) must be written in partitioned form.

$$(A.23) \qquad w_{12} + w_{12}^T - \Delta v_{22} - v_{22}\Delta = 0$$

$$(A.24) \qquad -\Omega^2 w_{22} + w_{11} - v_{12}\Delta = 0 \ ; \qquad w_{11} - w_{22}\Omega^2 - \Delta v_{12}^T = 0$$

(A.25) $\quad -\Omega^2 w_{12}^T - w_{12}\Omega^2 = 0$

From (A.25) we find that

(A.26) $\quad (w_{12})_{ii} = 0$

Setting the diagonal elements of (A.23) equal to zero and using (A.26) yields

$$(A.27) \quad (v_{22})_{ii} = \frac{1}{2\Delta_{ii}} \sum_{\substack{k=1 \\ k\neq i}}^{n} [\Delta_{ki}(v_{22})_{ki} + (v_{22})_{ik}\Delta_{ki}]$$

$$= \frac{1}{\Delta_{ii}} \sum_{\substack{k=1 \\ k\neq i}}^{n} \left[\frac{(c_p^T c_v)_{ki} - (c_p^T c_v)_{ik}}{\omega_i^2 - \omega_k^2} \right]$$

Then, considering the diagonal terms in (A.16),

(A.28) $\quad (v_{11})_{ii} = (v_{22})_{ii} - (c_p^T c_v)$

Thus Eqs. (A.14, A.19, A.20-A.22, A.27, A.28) represent the first two terms in the expansion for W_0.

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15. Supplementary Notes

16. Abstract

This monograph presents integrated modeling and controller design methods for flexible structures. The controllers, or compensators, developed are optimal in the linear-quadratic-Gaussian sense. The performance objectives, sensor and actuator locations and external disturbances influence both the construction of the model and the design of the finite dimensional compensator. The modeling and controller design procedures are carried out in parallel to ensure compatibility of these two aspects of the design problem. Model reduction techniques are introduced to keep both the model order and the controller order as small as possible.

A linear distributed, or infinite dimensional, model is the theoretical basis for most of the text, but finite dimensional models arising from both lumped-mass and finite element approximations also play an important role. A central purpose of the approach here is to approximate an opitmal infinite dimensional controller with an implementable finite dimensional compensator. Both convergence theory and numerical approximation methods are given. Somple examples are used to illustrate the theory.

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