Modeling and Control of Flexible Structures

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University of California at Los Angeles

December 15, 1988

National Aeronautics and Space Administration
Jet Propulsion Laboratory
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ABSTRACT

This monograph presents integrated modeling and controller design methods for flexible structures. The controllers, or compensators, developed are optimal in the linear-quadratic-Gaussian sense. The performance objectives, sensor and actuator locations and external disturbances influence both the construction of the model and the design of the finite dimensional compensator. The modeling and controller design procedures are carried out in parallel to ensure compatibility of these two aspects of the design problem. Model reduction techniques are introduced to keep both the model order and the controller order as small as possible.

A linear distributed, or infinite dimensional, model is the theoretical basis for most of the text, but finite dimensional models arising from both lumped-mass and finite element approximations also play an important role. A central purpose of the approach here is to approximate an optimal infinite dimensional controller with an implementable finite dimensional compensator. Both convergence theory and numerical approximation methods are given. Simple examples are used to illustrate the theory.
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J.S. Gibson and D.L. Mingori
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1. Introduction

Recent years have seen increasing research in active control of flexible structures. The primary motivation for this research is control of large flexible aerospace structures, which are becoming larger and more flexible at the same time that their performance requirements are becoming more stringent. For example, in tracking and other applications, satellites with large antennae, solar collectors and other flexible components must perform fast slew maneuvers while maintaining tight control over the vibrations of their flexible elements. Both of these conflicting objectives can be achieved only with a sophisticated controller. There are applications also to control of robotic manipulators with flexible links, and possibly to stabilization of large civil engineering structures such as long bridges and tall buildings.

A fundamental question that often arises with regard to designing a controller for a flexible structure is whether a finite dimensional model is sufficient as a basis for a controller that will produce the required performance, or is a distributed model necessary? The philosophy of this text is that, for almost any structure, a near optimal controller can be based on some finite dimensional approximate model but that, for large flexible space structures, the necessary order of the approximate model can be determined only in the controller design process, with some reference to the distributed model of the structure.

The main topic of this monograph is the design of LQG compensators based on finite dimensional approximations of distributed models of flexible structures. A primary objective of the approximation methods in subsequent chapters is convergence criteria that indicate the appropriate order of a finite dimensional
compensator for an infinite dimensional flexible structure. The basic idea in
determination of the compensator order and gains is to approximate an ideal
infinite dimensional compensator with an implementable finite dimensional com-
pensator.

This monograph deals with linear-quadratic-Gaussian (LQG) compensators. The
LQG optimal control problem for distributed, or infinite dimensional, systems is
a generalization to Hilbert space of the now classical LQG problem for finite
dimensional systems. The solution to the infinite dimensional problem yields an
infinite dimensional state-estimator-based compensator, which is optimal in the
context of this monograph. By a separation principle [Ba1, CP2], the problem
reduces to a deterministic linear-quadratic optimal control problem and an opti-
mal estimation, or filtering, problem with gaussian white noise. In infinite
dimensions, the control system dynamics are represented by a semigroup of
bounded linear operators instead of the matrix exponential operators in finite
dimensions, and the plant noise process may be an infinite dimensional random
process. The solutions to both the control and filtering problems involve
Riccati operator equations, which are generalizations of the Riccati matrix
equations in the finite dimensional case. Current results on the infinite
dimensional LQG problem are most complete for problems where the input and
measurement operators are bounded, as this monograph requires throughout. This
boundedness also permits the strongest approximation results. For related
control problems with unbounded input and measurement, see [Cu1, CS1, LT1, LT2,
LT3].

Our primary objective is to approximate the optimal infinite dimensional LQG
compensator for a distributed model of a flexible structure with finite dimen-
sional compensators based on approximations to the structure, and to have these
finite dimensional compensators produce near optimal performance of the closed-loop system. We discuss how the gains that determine the finite dimensional compensators converge to the gains that determine the infinite dimensional compensator, and we examine the sense in which the finite dimensional compensators converge to the infinite dimensional compensator. With this analysis, we can predict the performance of the closed-loop system consisting of the distributed plant and a finite dimensional compensator that approximates the infinite dimensional compensator.

Our design philosophy is to let the convergence of the finite dimensional compensators indicate the order of the compensator that is required to produce the desired performance of the structure. The two main factors that govern rate of convergence are the desired performance (e.g., fast response) and the structural damping. We should note that any one of our compensators whose order is not sufficient to approximate the infinite dimensional compensator closely will not in general be the optimal compensator of that fixed order; i.e., the optimal fixed-order compensator that would be constructed with the design philosophy in [BH1, BH2]. But as we increase the order of approximation to obtain convergence, our finite dimensional compensators become essentially identical to the compensator that is optimal over compensators of all orders.

An important question, of course, is how large a finite dimensional compensator we must use to approximate the infinite dimensional compensator. In [GM1, GM2, GM3, MG1], we have found that our complete design strategy yields compensators of reasonable size for distributed models of complex space structures. This strategy in general requires two steps to obtain an implementable compensator that is essentially identical to the optimal infinite dimensional compensator: the first step determines the optimal compensator by letting the finite
dimensional compensators converge to it; the second step reduces, if possible, the order of a large (converged) approximation to the optimal compensator. The first step, which is the one involving control theory and approximation theory for distributed systems, is the subject of Chapters 7-9. For the second step, a simple modal truncation of the large compensator sometimes is sufficient, but there are more sophisticated methods in finite dimensional control theory for order reduction. For example [GM2, MG1], we have found balanced realizations, discussed in Chapter 5, to work well for reducing large compensators.

The approximation theory in this monograph follows from the application of approximation results in [BK1, Gi3, Gi4] to a sequence of finite dimensional optimal LQG problems based on a Ritz-Galerkin approximation of the flexible structure. For the optimal linear-quadratic control problem, the approximation theory here is a substantial improvement over that in [Gi1] because here we allow rigid-body modes, more general structural damping (including damping in the boundary), and much more general finite element approximations. These generalizations are necessary to accommodate common features of complex space structures and the most useful finite element schemes. For example, we write the equations for constructing the approximating control and estimator gains and finite dimensional compensators in terms of matrices that are built directly from typical mass, stiffness and damping matrices for flexible structures, along with actuator influence matrices and measurement matrices. This means that the numerical methods in this text for compensator design can be used for almost any finite element model of a flexible structure, without reference to the infinite dimensional theory that establishes the validity of the numerical methods for a distributed model of the structure.

For the estimator problem, this monograph presents rigorous approximation theory that has evolved from less complete results in previous research [GM1,
GM2, GM3, MG1]. As in the finite dimensional case, the infinite dimensional optimal estimation problem is the dual of the infinite dimensional optimal control problem, and the solutions to both problems have the same structure. Because we exploit this duality to obtain the approximation theory for the estimation problem from the approximation theory for the optimal control problem, the analysis in this monograph is almost entirely deterministic. We discuss the stochastic interpretation of the estimation problem and the approximating state estimators briefly, but we are concerned mainly with deterministic questions about the structure and convergence of approximations to an infinite dimensional compensator and the performance -- especially stability -- of the closed-loop systems produced by the approximating compensators.

Now we will outline the technical contents of the monograph. Chapter 2 discusses a finite dimensional model for the forced linear vibrations of a flexible structure. Although the main goal is to design compensators for infinite dimensional flexible structures, the finite dimensional approximations upon which implementable compensators are based amount to finite dimensional models of flexible structures. These approximate models have the form of the finite dimensional model in Chapter 2. Important features of flexible structures, like natural modes and damping, are discussed in Chapter 2. Extensions of these notions for flexible structures with gyroscopic forces are also discussed. Gyroscopic terms appear whenever the structure rotates or contains rotating elements.

Chapter 3 defines the abstract distributed model of an infinite dimensional flexible structure and the energy spaces to be used in most of the subsequent chapters. We assume a finite number of actuators, since this is the case in all applications, and we assume that the actuator influence operator is bounded.
Chapter 3 also establishes certain mathematical properties of the open-loop system that are useful in control and approximation.

We develop the approximation of the distributed model of the structure in Chapter 4 and prove convergence of the approximating open-loop systems. The approximation scheme is essentially a Ritz-Galerkin method that includes modal, including component modal, approximations and most finite element approximations of flexible structures. For convergence, we use the Trotter-Kato semigroup approximation theorem, which was used in optimal open-loop control of hereditary systems in [BB1] and has been used in optimal feedback control of hereditary, hyperbolic and parabolic systems in [BK1, Gi1, Gi3] and other papers. The usual way to invoke Trotter-Kato is to prove that the resolvents of the approximating semigroup generators converge strongly. To prove this, we introduce an inner product that involves both the strain-energy inner product and the damping functional, and show that the resolvent of each finite dimensional semigroup generator is the projection, with respect to this special inner product, of the resolvent of the original semigroup generator onto the approximation subspace. The idea works as well for the adjoints of the resolvents, and when the open-loop semigroup generator has compact resolvent, it follows from our projection that the approximating resolvent operators converge in norm.

The speed of convergence to the optimal feedback law and optimal estimator is affected by the manner in which the model order is increased. It is useful to have a method of identifying important modes of the structure so they can be added to the model first. This issue is considered in Chapter 5. Moore's Balanced Realization Theory is introduced as a method for ordering the importance of states of a structural model. The approach takes into account input and output coupling as well as frequency and damping. An efficient approxima-
tion scheme is developed for structures which are lightly damped. Balanced Realization Theory may also be used to reduce the order of a general matrix transfer function, and in this context it provides a tool for reducing the order of the compensator.

Chapter 6 discusses the LQG optimal control problem for the distributed model of the structure and establishes some estimates involving bounds on solutions to infinite dimensional Riccati equations and open-loop and closed-loop decay rates. We need these estimates for the subsequent approximation theory. To get the approximation theory for the estimation problem, we have to give certain results on the control problem in a more general form than would be necessary were we interested only in the control problem for flexible structures. Therefore, in Chapter 6, we first give some generic results applicable to the LQG problem for a variety of distributed systems and then apply the generic results to the control of flexible structures.

For closing the loop on the control system, we assume a finite number of bounded linear measurements and construct the optimal state estimator, which is infinite dimensional in general. The gains for this estimator are obtained from the solution to an infinite dimensional Riccati equation that has the same form as the infinite dimensional Riccati equation in the control problem.

Since the approximation issues that this book treats are fundamentally deterministic, we make the book self-contained by defining the infinite dimensional estimator as an observer, although the only justification for calling this estimator and the corresponding compensator optimal is their interpretation in the context of stochastic estimation and control. We discuss the stochastic interpretation but do not use it. We say estimator and observer interchangeably to emphasize the deterministic definition of the estimator here.
Because we assume a finite number of actuators and a bounded input operator, the optimal feedback control law consists of a finite number of bounded linear functionals on the state space, which is a Hilbert space. This means that the feedback law can be represented in terms of a finite number of vectors, which we call functional control gains, whose inner products with the generalized displacement and velocity vectors define the control law. For any finite-rank, bounded linear feedback law for a control system on a Hilbert space, the existence of such gains is obvious and well known. A functional control gain for a flexible structure will have one or more distributed components, or kernels, corresponding to each distributed component of the structure and scalar components corresponding to each rigid component of the structure.

Analogous to the functional control gains are functional estimator gains corresponding to the finite number of sensor measurements. The functional gains play a prominent role in our analysis. They give a concrete representation of the infinite dimensional compensator and provide a criterion for convergence of the approximating finite dimensional compensators.

The optimal LQG compensator is infinite dimensional in general. The transfer function of this compensator is irrational, but it is still an \( m \times p \) matrix function of a complex variable, as in finite dimensional control theory. The optimal closed-loop system consists of the distributed model of the structure controlled by the optimal compensator.

We develop the approximation scheme for the LQG control problem in Chapter 7. We define a sequence of finite dimensional LQG problems, whose solutions approximate the solution to the infinite dimensional problem of Chapter 6. The solution to each finite dimensional problem is based on the solutions to two
Riccati matrix equations, and we give formulas for using these solutions to com-
pute approximations to the functional control and estimator gains as linear com-
binations of the basis vectors. The approximating functional gains indicate how
closely the finite dimensional compensators approximate the infinite dimensional
compensator. The brief discussion in Section 7.6 is the only place in the
monograph where stochastic estimation theory is necessary, and none of the
analysis in the rest of the monograph depends on this discussion.

Chapter 7 presents the finite dimensional equations to be used in designing
the finite dimensional compensators. This chapter contains no convergence
analysis. The equations in Chapter 7 can be used for numerical design of the
finite dimensional compensators, without worrying about the meaning of these
compensators with respect to the infinite dimensional control problem.

Chapter 8 contains convergence theory that gives conditions under which the
approximating compensators in Chapter 7 converge to the infinite dimensional
compensator in Chapter 6. This theory also describes the sense in which the
finite dimensional compensators converge. The compensator convergence is
discussed in terms of both the convergence of the approximating functional gains
and the convergence of the transfer functions of the finite dimensional compen-
sators.

In Chapter 9, we compute approximating finite dimensional compensators for a
compound structure that consists of an Euler-Bernoulli beam attached on one end
to a rotating rigid hub and on the other end to a lumped mass. We emphasize the
fact that we do not solve, or even write down, the coupled partial and ordinary
differential equations of motion. For both the definition and numerical solu-
tion of the problem, only the kinetic and strain energy functionals and a dissi-
pation functional for the damping are required. We show the approximating
functional control gains obtained by using a standard finite element approxima-
tion of the beam, and we discuss the effect on convergence of structural damping
and of the ratio of state weighting to control weighting in the performance
index. As suggested by a theorem in Chapter 8, the functional gains do not con-
verge when no structural damping is modeled.

We study the convergence of the finite dimensional compensators by examining
convergence of the approximating functional gains and the frequency responses of
the finite dimensional compensators. Also, we compute the eigenvalues of the
closed-loop system consisting of a finite dimensional compensator and larger-
order model of the structure. These eigenvalues indicate how many modes are
controlled how much by the infinite dimensional compensator and by the finite
dimensional compensators that are essentially identical to the infinite dimen-
sional compensator as far as the input/output map. The functional gains and the
compensator frequency response indicate the order of a finite dimensional com-
pensator necessary to approximate the infinite dimensional compensator closely,
and the closed-loop eigenvalues confirm this order.

Chapter 10 discusses several further issues important for implementation of
compensators designed by the methods in this book, and suggests further research
to make the ideas presented here useful for application to other control
problems for large flexible structures.
2. A Finite Dimensional Model of a Flexible Structure

2.1 Forced Linear Vibrations

The forced vibrations of a large class of flexible structures may be described approximately by the second-order finite dimensional differential equation

\[ M \ddot{x} + G \dot{x} + D \dot{x} + K x + F x = f_0 \]  

where the generalized displacement \( x(t) \) and the generalized forcing function \( f_0(t) \) are \( n \)-vectors, the mass matrix \( M \), damping matrix \( D \) and stiffness matrix \( K \) are real symmetric \( n \times n \) matrices, and the gyroscopic matrix \( G \) and circulatory matrix \( F \) are real skew-symmetric \( n \times n \) matrices. The mass matrix is positive definite, and the damping matrix is nonnegative. In most cases the stiffness matrix is also nonnegative, but some gyroscopic systems can have stiffness matrices which are indefinite. Any zero eigenvalue of \( K \) corresponds to a rigid-body mode. The gyroscopic matrix \( G \) results from rotating components, and the circulatory force matrix \( F \) may result from either follower forces or damping described in rotating reference frames. The terms \( M \ddot{x}, G \dot{x} \) and \( K x \) are conservative terms in (2.1.1) while \( D \dot{x} \) and \( F x \) are nonconservative. The forcing function \( f_0 \) is often expressed as

\[ f_0 = B_0 u(t) \]

In this expression, \( B_0 \) is a real \( n \times m \) input distribution matrix and \( u(t) \) is an \( m \times 1 \) input vector which could represent feedback control forces, external forces or both. In the present development, \( u \) will represent control forces or moments. The input distribution matrix \( B_0 \) depends on where the control or external forces act on the system, and it determines the relative effect of the inputs on the various coordinates.
The first-order form of (2.1.1) is

(2.1.3) \[ \dot{z} = Az + f \]

where the 2n x 2n state vector \( z \) is \( z(t) = [x^T \dot{x}^T]^T \) and

(2.1.4) \[ A = \begin{bmatrix} 0 & I \\ -M^{-1}(K+F) & -M^{-1}(G+D) \end{bmatrix}, \quad f = \begin{bmatrix} 0 \\ f_0 \end{bmatrix}. \]

Linear measurements of the state vector have the form

(2.1.5) \[ y = Cz + C_0u = [C_p \; C_v] (\begin{bmatrix} x \\ \dot{x} \end{bmatrix}) + C_0u \]

where \( C, C_0, C_p \) and \( C_v \) are real matrices of appropriate dimensions.

2.2 Natural Modes for Nongyroscopic Systems

Natural modes and natural frequencies may be defined by considering free vibrations of the conservative part of (2.1.1), i.e.,

(2.2.1) \[ M \ddot{x} + G \dot{x} + Kx = 0 \]

We first examine the case where \( G \) is zero and \( K \) is nonnegative. In Laplace transform notation, (2.2.1) reduces to:

(2.2.2) \[ [s^2M + K] x = 0 \]

Equation (2.2.2) defines an eigenvalue problem which may be solved for \( s \) and \( x \).

Since \( M \) and \( K \) are real symmetric matrices with \( M \) positive definite and \( K \) nonnegative, the solutions for \( s \) consist of 2n purely imaginary complex conjugates, \( s_r = \pm j\omega_r, \ r=1,2,...,n \) where the \( \omega_r \)'s are called the natural frequencies. The corresponding eigenvectors \( x_r \) are real and are called the natural mode shapes.
The mode shapes are complete in $\mathbb{R}^n$ and mode shapes corresponding to distinct frequencies are orthogonal with respect to both $M$ and $K$.

If $U$ is the matrix whose columns are the natural mode shapes normalized with respect to the mass matrix $M$, then substituting

\[(2.2.3) \quad x(s) = U a(s)\]

in (2.2.2) and multiplying the equation on the left by $U^T$ yields

\[(2.2.4) \quad s^2 a + \Omega^2 a = U^T f_0\]

where

\[(2.2.5) \quad \Omega = U^T K U = \begin{bmatrix}
\omega_1^2 & 0 & \cdots \\
0 & \omega_2^2 & \cdots \\
& \cdots & \ddots \\
0 & \cdots & \omega_n^2
\end{bmatrix}\]

The components of the n-vector $a(t)$ are called the modal coordinates of the structure, and the n scalar equations in (2.2.4) are called the modal equations of motion.

\[2.3 \quad \text{Modal Damping}\]

If a damping matrix $D$ is added to (2.2.2), then (2.2.4) becomes

\[(2.3.1) \quad s^2 a + s U^T D U a + \Omega^2 a = U^T f_0\]

The damping represented by the matrix $D$ is called modal damping if $U^T D U$ is diagonal. In this case, the damping does not couple the modes of undamped free
vibration and the eigenvectors of the damped structure are the same as the
eigenvectors of the undamped structure. The damping matrix $U^T DU$ is diagonal if
and only if $M^{-1}D$ commutes with $M^{-1}K$, or equivalently, if and only if $M^{-1}D$ and
$M^{-1}K$ have the same eigenvectors. In particular, $U^T DU$ is diagonal if $M^{-1}D$ is
equal to a convergent power series in $M^{-1}K$ or if $M^{-1}D$ is equal to a linear com-
bination of fractional powers of $M^{-1}K$. Although an assumption of modal damping
greatly simplifies the analysis of vibrating structures, it is not always easy
to reconcile this assumption with physical reality. (See Section 2.5.) In the
present discussion, we are thinking primarily of the damping matrix $D$ as
resulting from some material damping model for the flexible components of the
structure and/or models of joint damping or passive dampers attached to the
structure. In practice it is common to assume that the matrix $U^T DU$ is diagonal
and the values of modal damping ratios are estimated from experimental data.
The only justification for this questionable assumption is that for sufficiently
light damping, the eigenvectors of the finite dimensional model approach the
natural mode shapes from (2.2.2). Hence, whatever the damping matrix $D$, as long
as it is sufficiently small, a damping matrix $\tilde{D}$ for which $U^T \tilde{D} U$ is diagonal will
produce both eigenvalues and eigenvectors for the structure model that are close
to those produced by the correct $D$.

2.4 Gyroscopic Systems

If $G$ is not zero in (2.2.1), then the procedure for finding natural frequen-
cies and natural modes is conceptually and computationally more complicated.
Meirovitch [Me1] has suggested a procedure which casts the equations in a
form analogous to (2.2.2) where the coefficient matrices are symmetric. Once in
this form, it is possible to determine the natural frequencies and natural modes
as above (with some differences in interpretation). The procedure depends on $K$ being positive definite. If $K$ is only positive semi-definite the procedure can still be applied, but first the rigid body modes must be removed from (2.2.1). The system which remains is then treated as follows. First, rewrite (2.2.1) in an augmented form which has a symmetric and a skew symmetric coefficient matrix:

\[(2.4.1) \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix} (\ddot{x}) + \begin{bmatrix} G & K \\ -K & 0 \end{bmatrix} (\dot{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

Take the Laplace transform of (2.4.1) with $(\ddot{x}^T x^T)^T$ replaced by $q$.

\[(2.4.2) \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix} q + \begin{bmatrix} G & K \\ -K & 0 \end{bmatrix} q = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

The eigenvalue problem defined by (2.4.2) may be solved for $s$ and $q$. The solutions for $s$ consist of $2n$ purely imaginary complex conjugates, $s_r = \pm j\omega_r$, $r=1,2,...,n$. As before, the $\omega_r$'s are called the natural frequencies. The corresponding solutions for $q$ also occur in complex conjugate pairs and have the form $q_r = v_r + j\omega_r$, $r=1,2,...,n$. Setting $s_r = j\omega_r$ and $q_r = v_r + j\omega_r$ in (2.4.2) and equating the real and imaginary parts to zero separately yields two equations for $v_r$ and $\omega_r$.

\[(2.4.3) \omega_r \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix} v_r + \begin{bmatrix} G & K \\ -K & 0 \end{bmatrix} \omega_r = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

and

\[(2.4.4) -\omega_r \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix} \omega_r + \begin{bmatrix} G & K \\ -K & 0 \end{bmatrix} v_r = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

These equations may be rearranged in the form:
The appeal of (2.4.5) and (2.4.6) is that in each equation, the left hand matrix is symmetric and positive definite and the right hand matrix is symmetric as in the case where \( G \) is zero (see Eq. (2.2.2)). These features permit the use of simpler numerical algorithms, and the eigenvalue analysis proceeds as in the nongyroscopic case. There are, however, some differences in the interpretation of the eigenvectors since these vectors now include velocities as well as position coordinates. These eigenvectors are not in general referred to as natural modes. Refs. [Me1, Me2, Me3] discuss the solution and interpretation of (2.4.5) and (2.4.6) at some length.

It will also be noted that Eqs. (2.4.5) and (2.4.6) are identical. This does not mean that \( v_r \) and \( w_r \) are identical. From (2.4.3) and (2.4.4) one can show that \( v_r \) and \( w_r \) must satisfy the equation

\[
\begin{align*}
(2.4.7) \quad w_r &= \begin{bmatrix} 0 & \omega_r I \\ -\omega_r^{-1} I & 0 \end{bmatrix} v_r 
\end{align*}
\]

Thus only one eigenvalue problem must be solved to obtain both \( w_r \) and \( v_r \). If (2.4.5) is used to obtain \( v_r \), then (2.4.7) can be used to calculate \( w_r \). The \( w_r \) obtained in this way will satisfy (2.4.6).
2.5 A Lumped Mass Example

As noted in section 2.3, there may be some difficulty in justifying the assumption of modal damping. This will now be illustrated by means of the simple example shown in Figure 2.1. This structure consists of three masses, two springs and three dampers. The three degrees of freedom include one rigid-body mode and two elastic modes. It is instructive to consider the implications of assuming modal damping with the same damping ratio in each mode. Because we want the damping in each mode to be proportional to the undamped frequency of that mode, there should be no damping in the rigid-body mode, so there are no dampers attached to the ground.

![Figure 2.1. Lumped Mass Example](image)

For the coordinates $x_1$, $x_2$ and $x_3$ shown in Figure 2.1, the mass stiffness and damping matrices are

\[
M = \begin{bmatrix}
m_1 & 0 & 0 \\
0 & m_2 & 0 \\
0 & 0 & m_3 \\
\end{bmatrix}, \quad K = \begin{bmatrix}
k & -k & 0 \\
-k & 2k & -k \\
0 & -k & k \\
\end{bmatrix}
\]

and

(2.5.1)
The damping coefficients $c_1$, $c_2$ and $c_3$ are nonnegative. Suppose that we have the (dimensionless) mass and spring constants

$$m_1 = 0.1 \quad m_2 = 1.0 \quad m_3 = 0.1 \quad k = 1.0.$$ 

For the matrix $U^TDU$ in the modal equation (2.3.1) to be equal to, say, .02 times the square root of the matrix on the right hand side of (2.2.5) with $n = 3$ (i.e., for one percent critical damping in each mode) the damping matrix $D$ must be

$$(2.5.3) \quad D = \begin{bmatrix} 0.010379 & -0.014434 & 0.004055 \\ -0.014434 & 0.028868 & -0.014434 \\ 0.004055 & -0.014434 & 0.010379 \end{bmatrix}.$$ 

No combination of the damping coefficients in Eq. (2.5.2) produces this damping matrix.

With more elaborate combinations of pulleys and dampers, infinitely many damping configurations are possible, so it may be possible to realize the damping matrix in (2.5.3) by some arrangement of dampers. But even if a realization of the required damping exists, it may be a very unlikely physical configuration. The difficulty of physically realizing constant-damping-ratio modal damping in this example suggests that assuming a form, particularly a diagonal form, for the damping matrix in the modal equations of motion, without considering the physical sources of the damping, can produce improbable damping models.
3. An Infinite Dimensional Model

3.1 Infinite Dimensional Equation of Motion

Now we will describe the abstract model that we will use throughout the book to represent an infinite dimensional flexible structure. The generalized displacement vector $x(t)$ is in a real Hilbert space $H$, the control vector $u(t)$ is in $\mathbb{R}^m$ for some finite $m$ and $x(t)$ is a mild solution to

\begin{equation}
\dot{x}(t) + D_0x(t) + A_0x(t) = B_0u(t), \quad t > 0.
\end{equation}

In this equation of motion, the linear stiffness operator $A_0$ is densely defined and selfadjoint with compact resolvent and at most a finite number of negative eigenvalues. The stiffness operators for structures involving beams, plates and membranes will satisfy the hypotheses here on $A_0$. Any nonpositive eigenvalues of $A_0$ will represent rigid-body modes. For now, we will assume that the damping operator $D_0$ is symmetric, nonnegative and bounded relative to $A_0$; this includes linear material damping in continuous structural components and linear damping from a viscous fluid surrounding the structure (not likely in space applications). In Section 3.4, we will discuss a more general type of damping that we will allow from then on, except where otherwise noted. The input operator $B_0$ is a linear operator from $\mathbb{R}^m$ to $H$, and hence bounded (so that this model does not include boundary control).

For the distributed model of a flexible structure, we will not discuss a gyroscopic operator corresponding to the matrix $G$ in Chapter 2. It is straightforward to add a bounded skew selfadjoint operator to the damping operator $D_0$ and generalize our infinite dimensional system theory and approximation theory appropriately, but we do not want to complicate the exposition of this and subsequent chapters by carrying this extra detail along.
Remark 3.1. Our analysis includes the system

\[
(3.1.1') \quad M_0 \ddot{x}(t) + D_0 x(t) + A_0 x(t) = B_0 u(t), \quad t > 0,
\]

where the mass operator \( M_0 \) is a selfadjoint, bounded and coercive linear operator on a real Hilbert space \( H_0 \). The operators \( A_0, B_0 \) and \( D_0 \) in (3.1.1') have the same properties with respect to \( H_0 \) that the corresponding operators in (3.1.1) have with respect to \( H \). To include (3.1.1') in our analysis, we need only take \( H \) to be \( H_0 \) with the norm-equivalent inner product \( \langle \cdot, \cdot \rangle_H = \langle M_0^{-1} \cdot, \cdot \rangle_{H_0} \), and multiply (3.1.1') on the left by \( M_0^{-1} \). In \( H \), the operator \( M_0^{-1}A_0 \) is self-adjoint with compact resolvent, and \( M_0^{-1}D_0 \) is symmetric and nonnegative. With no loss of generality, then, we will refer henceforth only to (3.1.1) and assume that the \( H \)-inner product accounts for the mass distribution.

By natural modes of a structure represented by (3.1.1), we will mean the eigenvectors \( \phi_j \) of the eigenvalue problem

\[
(3.1.2) \quad \lambda_j \phi_j = A_0 \phi_j.
\]

From our hypotheses on \( A_0 \), it follows that these eigenvalues form an infinitely increasing sequence of real numbers, of which all but a finite number are positive. Also, the corresponding eigenvectors are complete in \( H \) and satisfy

\[
(3.1.3) \quad \langle \phi_i, \phi_j \rangle_H = \langle A_0 \phi_i, \phi_j \rangle_H = 0, \quad i \neq j.
\]

(For these standard properties of the eigenvalue problem, see [Bal, Ka2] and other texts.) For \( \lambda_j > 0 \), \( \omega_j = \sqrt{\lambda_j} > 0 \) is a natural frequency.

3.2 The Energy Spaces

To discuss the damping in (3.1.1) more precisely, to derive the first-order form of (3.1.1) and to specify the class of measurements that we will consider,
we need to define two additional Hilbert spaces, the strain energy space \( V \) and the total energy space \( E \), in terms of the basic space \( H \) and the stiffness operator \( A_0 \). We choose a bounded selfadjoint linear operator \( A_1 \) on \( H \) such that \( A_1 \) is positive definite on the eigenspace of \( A_0 \) corresponding to nonpositive eigenvalues of \( A_0 \) and the null space of \( A_1 \) is the closed span of the eigenvectors of \( A_0 \) corresponding to positive eigenvalues. Thus \( A_0 = A_0 + A_1 \) is coercive; i.e., there exists \( p > 0 \) such that

\[
\langle A_0 x, x \rangle \geq p |x|^2, \quad x \in D(A_0) = D(A_0).
\]

We define the Hilbert space \( V \) to be the completion of \( D(A_0) \) with respect to the inner product

\[
\langle v_1, v_2 \rangle_V = \langle A_0 v_1, v_2 \rangle_H, \quad v_1, v_2 \in D(A_0).
\]

Equivalently, \( V = D(\tilde{A}_0) \) and \( \langle v_1, v_2 \rangle_V = \langle \tilde{A}_0 v_1, \tilde{A}_0 v_2 \rangle_H \).

In the usual way, we will use the imbedding

\[
V \subset H = H' \subset V',
\]

where the injection from \( V \) into \( H \) and from \( H' \) into \( V' \) are continuous with dense ranges. We denote by \( \Lambda_V \) the Riesz map from \( V \) onto its dual \( V' \); i.e.,

\[
\langle v, v_1 \rangle_V = (\Lambda_V v_1)v, \quad v_1, v \in V.
\]

Then \( \tilde{A}_0 \) is the restriction of \( \Lambda_V \) to \( D(A_0) \) in the sense that

\[
(\Lambda_V v_1)v = \langle v, \tilde{A}_0 v_1 \rangle_H, \quad v_1 \in D(A_0), v \in V.
\]

Remark 3.2. We began our description of the control system model with (3.1.1) because its form is familiar in the context of flexible structures. The stiffness operator \( A_0 \), for example, is the infinite dimensional analogue of the stiffness matrix in Chapter 2. In applications, though, it is often easier to begin with a strain-energy functional from which the correct strain-energy inner
product is obvious. There is a one-to-one correspondence between the stiffness operator $\tilde{A}_0$ and the strain-energy space $V$. We have seen how $V$ is defined in terms of $\tilde{A}_0$. If $V$ is specified first, then $\tilde{A}_0$ is defined in terms of the Riesz map for $V$ by (3.2.5) with $D(\tilde{A}_0) = \Lambda_V^{-1}H$ (see [Shl] also for this approach). Either way, the relationship between $\tilde{A}_0$ and $V$ is the same.

This means that if the space $H$ and its inner product (determined by kinetic energy; recall Remark 3.1) and the space $V$ and its inner product (determined by strain energy and geometric boundary conditions) are written down, then the operators $\tilde{A}_0$ and $\Lambda_V$ are determined implicitly. For both our theoretical analysis and our numerical computation, only the $H$ and $V$ inner products are needed; neither $\tilde{A}_0$, $\Lambda_V$ nor $\Lambda_V^{-1}$ need be written down (although it is instructive to write the partial differential operator $A_0$ for simple examples, as in Section 3.3).

Now we define the total energy space $E = V \times H$, noting that when $A_0$ is coercive and $x(t)$ is the solution to (3.1.1), then $\left| \langle x(t), \dot{x}(t) \rangle \right|^2_E$ is twice the total energy (kinetic plus potential) in the system. The total energy space is the natural space for the infinite dimensional state-space model of the flexible structure; i.e., the first order form of (3.1.1), which we will derive in Section 3.5.

3.3 Clamped-Free Beam Example

For a simple example, we consider the clamped-free Euler-Bernoulli beam in Figure 3.1. The length of the beam is 1 and the product of the modulus of elasticity and the second moment of the cross section is $EI$. The single control force $u$ is spread uniformly over the last five percent of the beam at the right end.
For this example, the most natural Hilbert space $H$ is $L^2(0,1)$, and the stiffness operator is

\begin{equation}
A_0 = EI \frac{d^4}{ds^4}, \quad D(A_0) = \{g \in H^4(0,1): g(0) = g'(0) = g''(1) = g'''(1) = 0\}.
\end{equation}

We model Kelvin-Voigt viscoelastic damping in the beam [CP1], which means that the damping operator is

\begin{equation}
D_0 = c_0 A_0
\end{equation}

where $c_0$ is a positive constant. The actuator influence operator is given by

---

\textsuperscript{1}We denote by $H^k(0,1)$ the $k$th-order Sobolev space of functions on $(0,1)$. A function $f$ is in this Hilbert space if $f$, along with its derivatives of orders up through $k-1$, are absolutely continuous on $[0,1]$ and $f^{(k)} \in L^2(0,1)$. The square of the $H^k(0,1)$ norm of $f$ is

\[
|f|_k^2 = \sum_{j=1}^{k} \int_0^1 [d^j f(t)/dt^j]^2 dt.
\]
0, 0 < s < .95,

\[(3.3.3) \quad b_0(u) = u b_0, \quad b_0(s) = 1, \quad .95 < s < 1.\]

Since \(A_0\) in this example is coercive, \(A_1 = 0\) and \(\tilde{A}_0 = A_0\). The strain-energy space \(V\) in this example is \(\{v \in H^2(0,1): v(0) = v'(0) = 0\}\), which follows from defining the inner product

\[(3.3.4) \quad \langle v_1, v_2 \rangle_V = \langle A_0 v_1, v_2 \rangle_H = \int_0^1 v_1'' v_2'' ds, \quad v_1, v_2 \in D(A_0),\]

according to (3.2.2) and completing \(D(A_0)\) with respect to the norm induced by this inner product. Although the geometric boundary conditions \(v(0) = v'(0) = 0\) are the only boundary conditions retained explicitly in \(V\), the boundary conditions \(v''(0) = v'''(1) = 0\) are retained implicitly by the imbedding of \(V\) in \(H\) and the Riesz map for \(V\) because specifying \(H\) and \(V\) is equivalent to specifying \(H\) and \(A_0\), as discussed in Remark 3.2. That \(H\) and \(V\) determine all of the boundary conditions for \(A_0\) should remind readers experienced in the structural dynamics of applying Hamilton's principle to derive the equations of motion for a distributed model of a flexible structure: once the energy functionals and geometric boundary conditions are defined, the natural boundary conditions follow from integrating by parts in the spatial domain.

We should note that the coercive, selfadjoint operator \(A_0^{\frac{1}{2}}\) exists for this example, but that it is not a differential operator. This square-root operator cannot be written down in closed form, but its infinite dimensional matrix representation with respect to its eigenvectors is diagonal with the natural frequencies of the clamped-free beam on the diagonal. The operators \(A_0\) and \(A_0^{\frac{1}{2}}\) have the same eigenvectors, which are complete and mutually orthogonal in
both $H$ and $V$.

### 3.4 The Damping Functional and Operator

Before we discuss the first-order form of (3.1.1), we will state the damping hypothesis that we will use from here on and discuss the representation of the damping admitted under this hypothesis. To construct the first-order form of (3.1.1), we do not require an operator $D_0$ defined from some subset of $H$ into $H$. Rather, we need only the weaker assumption that there exists a damping (or dissipation) functional

\[(3.4.1)\quad d_0(\cdot,\cdot) : V \times V \to \mathbb{R}\]

such that $d_0$ is bilinear, symmetric, continuous on $V \times V$ and nonnegative. Under this hypothesis on $d_0$, there is a unique nonnegative, selfadjoint operator $D_V \in \mathcal{L}(V)$ such that

\[(3.4.2)\quad d_0(v_1,v_2) = \langle D_v v_1, v_2 \rangle_V = \langle v_1, D_v v_2 \rangle_V, \quad v_1, v_2 \in V.
\]

If we have a symmetric, nonnegative damping operator $D_0$ defined from $D(A_0)$ into $H$ such that $D_0$ is bounded relative to $A_0$, then $\langle D_0 \cdot, \cdot \rangle_H$ defines a bilinear, symmetric, continuous, nonnegative functional on a dense subset of $V \times V$. In this case, $d_0$ is the unique extension of this functional to $V \times V$ and

\[(3.4.3)\quad D_V|_{D(A_0)} = A_0^{-1}D_0.
\]

(That $D_0$ being $A_0$-bounded implies continuity of $\langle D_0 \cdot, \cdot \rangle_H$ with respect to the $V$ norm follows from [Ka2, Theorem 4.12, page 292].)

In applications, either $D_0$ or $d_0$ should be straightforward to write from the physics of the structure, but $D_V$ is difficult to determine except for simple structures or damping that is a simple function of stiffness.
Remark 3.3. For our theoretical analysis and our numerical work, the damping functional \( d_0 \) is sufficient; it is not necessary to write down either of the damping operators \( D_V \) and \( D_0 \). (As with the stiffness operator, it is instructive to write the damping operators for simple examples; see Section 3.7.)

3.5 The Semigroup Generator and the First-Order Form of the Equation of Motion

We want to write (3.1.1) as a well-posed first-order evolution equation on the total energy space \( E = V \times H \). To do this, we will determine the appropriate semigroup generator for the open-loop system, so that \( C_0 \)-semigroup theory will guarantee the existence and uniqueness of solutions to the state-space differential equation under appropriate conditions on \( u(t) \). We will derive the semigroup generator by constructing its inverse explicitly, and then we will try to convince the reader that we do have the appropriate semigroup generator. This approach seems mathematically efficient, and we will need the inverse of the generator for approximation theory.

We define \( \tilde{A}^{-1} \in L(E,E) \) by

\[
\tilde{A}^{-1} = \begin{bmatrix}
-D_V & -\tilde{A}_0^{-1} \\
I & 0
\end{bmatrix}.
\]

(3.5.1)

This operator is clearly one-to-one, and its range is dense, since \( V \) is dense in \( H \) and \( D(A_0) \) is dense in \( V \). Now, we take

\[
\tilde{A} = (\tilde{A}^{-1})^{-1}.
\]

(3.5.2)

Direct calculation of the inner product shows

\[
\langle \tilde{A}^{-1}(\nu_h), (\nu_h) \rangle_E = -d_0(h,h),
\]

(3.5.3)
so that $\tilde{A}$ is dissipative with dense domain. Also, since $D(\tilde{A}^{-1}) = E$, $\tilde{A}$ is maximal dissipative by [Gil, Theorem 2.1]. Therefore, $\tilde{A}$ generates a $C_0$-contraction semigroup on $E$.

Finally, the open-loop semigroup generator is

$$
A = \tilde{A} + \begin{bmatrix} 0 & 0 \\ A_1 & 0 \end{bmatrix}, \quad D(A) = D(\tilde{A}),
$$

(3.5.4)

where $A_1$ is the bounded linear operator discussed in Section 3.2. With

$$
B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix} \in L(R^n, E),
$$

(3.5.5)

the first-order form of (3.1.1) is

$$
\dot{z}(t) = A z(t) + B u(t), \quad t > 0,
$$

(3.5.6)

where $z = (x, \dot{x}) \in E$.

To see that $A$ is indeed the appropriate open-loop semigroup generator, suppose that $A_0$ is coercive (so that $A_1 = 0$) and that we have a symmetric, non-negative $A_0$-bounded damping operator $D_0$. Then the appropriate generator should be a maximal dissipative extension of the operator

$$
A = \begin{bmatrix} 0 & I \\ -A_0 & -D_0 \end{bmatrix}, \quad D(\tilde{A}) = D(A_0) \times D(A_0).
$$

(3.5.7)

It is shown in [Gil, Section 2] that $\tilde{A}$ has a unique maximal dissipative extension, and after noting (3.4.3), it can be shown easily that the $A(= \tilde{A})$ defined
by (3.5.1) and (3.5.2) is an extension of $A$.

We should note that Showalter [Shl, Chapter VI] elegantly derives a
semigroup generator for a class of second-order systems that includes the
flexible-structure model here. The presentation here is most useful for our
approximation theory because of the explicit construction of the inverse of the
semigroup generator. For the purposes of this paper, we do not need to charac-
terize the operator $A$ itself more explicitly, but we should make the following
points.

First, from $\tilde{A}^{-1}$ we see

$\text{(3.5.8)} \quad D(A) = \{(x, \dot{x}) : \dot{x} \in V, x + D_V \dot{x} \in D(A_0)\}$.

In applications, the "natural boundary conditions" can be determined from
(3.5.8) and the boundary conditions included in the definition of $D(A_0)$ (see
Section 3.8), although we will not need the natural boundary conditions. In the
case of a damping operator that is bounded relative to $A_0^{1/2}$, $D(A) = D(A_0) \times V$.
If the damping operator is bounded relative to $A_0^\mu$ for $\mu < 1$, then $A$ has compact
resolvent.

In many structural applications, the open-loop semigroup is analytic,
although this has been proved only for certain important cases. Showalter
obtains an analytic semigroup when the damping functional is $V$-coercive; for
example, when there exists a damping operator that is both $A_0$-bounded and as
strong as $A_0$. Such a damping operator results from the Voigt-Kelvin viscoelastic
material model. Also, it can be shown that the semigroup is analytic for a
damping operator equal to $c_0 A_0^\mu$ for $\frac{1}{2} < \mu < 1$ and $c_0$ a positive scalar. The
case $\mu = \frac{1}{2}$, which produces the same damping ratio in all modes, is especially
common in structural models, and Chen and Russell [CR1] have shown that the
semigroup is analytic for a more general class of damping operators involving \( \tilde{A}_0^{\frac{\alpha}{2}} \).

We can guarantee that the open-loop semigroup generator is a spectral operator (i.e., its eigenvectors are complete in \( E \)) only for a damping operator that is a linear combination of an \( H \)-bounded operator and a fractional power of \( A_0 \). However, nowhere do we use or assume anything about the eigenvectors of either the open-loop or the closed-loop semigroup generator. The natural modes -- of undamped free vibration -- in (3.1.2) are always complete in both \( H \) and \( V \).

The Adjoint of the Semigroup Generator. Since \( D_V \) is selfadjoint on \( V \), direct calculation shows that \( \tilde{A}^{-*} = (\tilde{A}^{-1})^* \) -- the adjoint of \( \tilde{A}^{-1} \) with respect to the \( E \)-inner product -- is

\[
(3.5.9) \quad \tilde{A}^{-1} = \begin{bmatrix} -D_V & A_0^{-1} \\ -I & 0 \end{bmatrix}.
\]

Then \( \tilde{A}^* = (\tilde{A}^{-*})^{-1} \). Having \( \tilde{A}^{-*} \) explicitly facilitates proving strong convergence for approximating adjoint semigroups.

Exponential Stability of the Open-Loop System. According to the following theorem, a sufficient condition for the open-loop system to be uniformly exponentially stable is that there be no rigid-body modes and the damping be coercive. Coercive damping means, basically, that all structural components have positive damping. That the decay rate given for the energy norm depends only on the lower bound for the stiffness operator and the upper and lower bounds for the damping functional is essential for convergence results for the approximating optimal control problems of subsequent chapters.

**Theorem 3.4.** Suppose that \( A_0 \) and \( d_0 \) are \( H \)-coercive. Let \( \rho \) be the positive constant in (3.2.1), and let \( \delta_0 \) and \( \delta_1 \) be positive constants such that
(3.5.10) \( \delta_0 \| v \|_H^2 \leq d_0(v, v) \leq \delta_1 \| v \|_V^2, \quad v \in V. \)

Then

(3.5.11) \[ |T(t)| \leq \left( 1 + \frac{\delta_0}{\sqrt{\rho}} + \delta_1 \delta_0 / 2 \right)^{1/2} \exp \left[ \frac{-t}{2/\delta_0 + 2/\sqrt{\rho + \delta_1}} \right], \quad t \geq 0. \]

For a proof of this theorem, see [GA1]. The proof uses an explicit Liapunov functional for the open-loop system.

3.6 Representation of the Open-Loop Semigroup in the Case of Uncoupled Modes

Since the eigenvectors \( \phi_j \) of \( A_0 \) are complete in \( V \), \( x(t) \) can be written

(3.6.1) \[ x(t) = \sum_{j=1}^{\infty} a_j(t) \phi_j \]

where \( a_j(t) \) is the modal amplitude for the \( j^{th} \) mode. The energy space \( E = V \times H \) is isomorphic to the space \( L^2_E \) of sequences

(3.6.2) \[ a = \begin{pmatrix} a_1 \\ a_1 \\ a_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{pmatrix} \]

that are square summable in the sense of

(3.6.3) \[ |a|_E^2 = \sum_{j=1}^{\infty} \left( \omega_j^2 a_j^2 + a_j^2 \right) < \infty \]

where \( \omega_j \) is the \( j^{th} \) natural frequency.
When a damping operator $D_0$ exists and has the same eigenvectors as $A_0$, the damping leaves the natural modes of free vibration uncoupled and (3.1.1) is equivalent to the infinite set of modal equations

$$(3.6.4) \quad \ddot{a}_j(t) + 2\zeta_j \omega_j \dot{a}_j(t) + \omega_j^2 a_j(t) = B_{0j} u(t)$$

where

$$(3.6.5) \quad \zeta_j = \frac{\langle D_0 \phi_j, \phi_j \rangle_H}{2\omega_j |\phi_j|_H^2}$$

is the damping ratio of the $j$th mode and $B_{0j}$ is a $1 \times m$ matrix. On $\mathbb{R}_E$, the semigroup $T(t)$ is represented by an infinite dimensional matrix with all zeros except for $2 \times 2$ blocks $T_j(t)$ centered on the main diagonal and given by

$$(3.6.6) \quad T_j(t) = \exp \begin{bmatrix} 0 & 1 \\ -\omega_j^2 & -2\zeta_j \omega_j \end{bmatrix} t.$$

### 3.7 The Measurement Equation and Operators

Recall that the second-order equation (3.1.1) and the first-order equation (3.5.6) are equivalent. We assume that the measurement has the form

$$(3.7.1) \quad y(t) = C_0 u(t) + C z(t)$$

where $y(t)$ is a $p$-vector, $C_0$ is a real $p \times m$ matrix and $C$ is a bounded linear operator from $E$ to $\mathbb{R}^p$. Since $E = V \times H$, $C$ must have the form

$$(3.7.2) \quad C = [C_1 \quad C_2]$$

where $C_1$ and $C_2$ are bounded linear operators from $V$ and $H$, respectively, to $\mathbb{R}^p$. This means that (3.7.1) can be written
(3.7.3) \[ y(t) = C_0 u(t) + C_1 x(t) + C_2 \dot{x}(t). \]

Also, according to Riesz, if \((C z(t))_i\) is the \(i\)th component of the \(p\)-vector \(C z(t)\), then

(3.7.4) \[ (C z(t))_i = \langle c_{i1}, x \rangle_V + \langle c_{i2}, \dot{x} \rangle_H, \quad i = 1, \ldots, p, \]

where \(c_{i1} \in V\) and \(c_{i2} \in H\).

3.8 Further Comments on the Clamped-Free Beam

According to Section 3.5, once the basic space \(H\) was defined in Section 3.3 to be \(L_2(0,1)\), the first-order form of the equation of motion for the beam in Section 3.3 was determined implicitly by the stiffness operator in (3.3.1), the damping operator in (3.3.2) and the actuator influence operator in (3.3.3). It follows from Theorem 3.4 that the open-loop beam (i.e., the free response of the beam) is uniformly exponentially stable for any positive damping coefficient \(c_0\) in (3.3.2).

As discussed in Remark 3.2, specifying the strain-energy inner product in (3.3.4) and the boundary conditions on the left end of the beam would have been equivalent to specifying \(A_0\). Also, according to Section 3.4, specifying the dissipation functional to be

(3.8.1) \[ d_0(v_1, v_2) = c_0 \langle v_1, v_2 \rangle_V, \quad v_1, v_2 \in V, \]

would have been equivalent to specifying the damping operator \(D_0\).

To see the advantage of being able to use the damping functional instead of the damping operator, suppose that we attach a linear damper with positive coefficient \(c_1\) between the right end of the beam and ground. Then there is no damping operator \(D_0\), but it is easy to write the dissipation functional.
(3.8.2) \( d_0(v_1, v_2) = c_0 \langle v_1, v_2 \rangle_V + c_1 v_1(1)v_2(1), \quad v_1, v_2 \in V, \)

since this means that the rate of energy dissipation for the open-loop beam is

\[
(3.8.3) \quad \dot{\mathcal{E}} = d_0(\dot{x}(t), \dot{x}(t)) = -c_0 \int_0^1 \dot{x}''(t,s)^2 \, ds - c_1 \dot{x}(t,1)^2.
\]

For none of our purposes in subsequent chapters do we need to go further in characterizing the damping. Nonetheless, it is instructive to look at the natural boundary conditions that the damping produces and how these boundary conditions can be determined with the results discussed so far. In this example (and probably only a few others), it is easy to write the operator \( D_V \) discussed in Section 3.4. The part of \( D_V \) corresponding to the Kelvin-Voigt damping is \( c_0 \) times the identity in \( V \), which follows from (3.4.3). The part of \( D_V \) corresponding to the linear damper on the end of the beam is less obvious, but it can be determined with (3.4.2) and integration by parts. The result (for the total damping functional in (3.8.2)) is

\[
(3.8.4) \quad D_V v = c_0 v + v(1) g_0
\]

where \( g_0 \in V \) is

\[
(3.8.5) \quad g_0(s) = (1 - s)^{3/6} + s/2 - 1/6, \quad 0 \leq s \leq 1.
\]

The natural boundary conditions for the equation of motion for the beam follow from (3.5.8). With the \( D(A_0) \) and \( V \) defined in Section 3.3 and \( D_V \) given by (3.8.4) and (3.8.5), (3.5.8) says that the domain of the semigroup generator contains those function pairs \((x, \dot{x})\) that satisfy \( x \in V, x + c_0 \dot{x} \in V \subset H^4(0,1) \),

\[
(3.8.6) \quad (x + c_0 \dot{x})''(1) = 0
\]

and

\[
(3.8.7) \quad (x + c_0 \dot{x})'''(1) + c_1 \dot{x}(1) = 0.
\]

These last two equations are the natural boundary conditions.
3.9 Hub-Beam-Tip-Mass Example

One end of the Euler-Bernoulli beam in Figure 3.2 is attached rigidly (cantilevered) to a rigid disc which is free to rotate about its center, point 0, which is fixed. Also, a point mass $m_1$ is attached to the other end of the beam. The control is a torque $u$ applied to the disc, and all motion is in the plane.

![Figure 3.2. Flexible Structure](image)

Table 3.1. Structural Data

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$ = hub radius</td>
<td>10 in</td>
</tr>
<tr>
<td>$l$ = beam length</td>
<td>100 in</td>
</tr>
<tr>
<td>$I_0$ = hub moment of inertia</td>
<td>100 slug in$^2$</td>
</tr>
<tr>
<td>about axis perpendicular to page</td>
<td></td>
</tr>
<tr>
<td>through 0</td>
<td></td>
</tr>
<tr>
<td>$m_b$ = beam mass per unit length</td>
<td>.01 slug/in</td>
</tr>
<tr>
<td>$m_1$ = tip mass</td>
<td>1 slug</td>
</tr>
<tr>
<td>EI = product of elastic modulus</td>
<td>13,333 slug in$^3$/sec$^2$</td>
</tr>
<tr>
<td>and second moment of cross section for beam</td>
<td></td>
</tr>
<tr>
<td>Fundamental frequency of undamped structure</td>
<td>.9672 rad/sec</td>
</tr>
</tbody>
</table>
The angle $\theta$ represents the rotation of the disc (the rigid-body mode), $w(t, \cdot)$ is the elastic deflection of the beam from the rigid-body position, and $w_1(t)$ is the displacement of $m_1$ from the rigid-body position. For technical reasons, we do not yet impose the condition $w_1(t) = w(t, 1)$; more on this later.

The control problem is to stabilize rigid-body motions and linear (small) transverse elastic vibrations about the state $\theta = 0$ and $w = 0$. Our linear model assumes not only that the elastic deflection of the beam is linear but also that the axial inertial force produced by the rigid-body angular velocity has negligible effect on the bending stiffness of the beam. The rigid-body angle need not be small.

For this example, it is a straightforward exercise to derive the three coupled differential equations (one partial and two ordinary differential equations) of motion in $\theta$, $w$ and $w_1$, and they do have the form (3.1.1'). However, to emphasize the fact that we do not use the explicit differential equations, we will not write these equations here. Rather, we will write only what normally is needed in applications: the kinetic and strain-energy functionals, the damping functional and the actuator influence operator.

Remark 3.1 applies to this example, and to most examples with complex structures. The generalized displacement vector is

$$ (3.9.1) \quad x = (\theta, w, w_1) \in H_0 = R \times L^2(0, 1) \times R. $$

The kinetic energy in the system is

$$ (3.9.2) \quad \text{Kinetic Energy} = \frac{1}{2} \langle x, x \rangle_H $$

where $H$ is $H_0$ with the inner product
\[(3.9.3) \quad \langle x, \dot{x} \rangle_H = m_b \int_0^1 \left[ w + (r+s)\dot{\theta} \right] \left[ \ddot{w} + (r+s)\ddot{\theta} \right] \, ds + I_0 \dot{\theta} \ddot{\theta} + m_1 [w_1 + (r+1)\dot{\theta}][\ddot{w}_1 + (r+1)\ddot{\theta}] . \]

As in most applications we need not write the mass operator explicitly, but there exists a unique selfadjoint linear operator \( M_0 \) on \( H_0 \) such that

\[(3.9.4) \quad \langle x, x \rangle_H = \langle M_0 x, \dot{x} \rangle_{H_0} . \]

It is easy to see that \( M_0 \) is bounded and coercive. Hence \( H_0 \) and \( H \) have equivalent norms.

The input operator for \((3.1.1')\) (which maps \( R \) to \( H_0 \)) is

\[(3.9.5) \quad B_0 = (1, 0, 0) . \]

Since we multiply \((3.1.1')\) by \( M_0^{-1} \) to get \((3.1.1)\), the input operator for \((3.1.1)\) is \( (M_0^{-1}B_0) \). Note that

\[(3.9.6) \quad (M_0^{-1}B_0)^*_H = B_0^* , \]

where \((M_0^{-1}B_0)^*_H\) is the \( H \)-adjoint of \((M_0^{-1}B_0)\) and \( B_0^* \) is the \( H_0 \)-adjoint of \( B_0 \).

Remark 3.2 also applies here. The only strain energy is in the beam and is given by

\[(3.9.7) \quad \text{Strain Energy} = \frac{1}{2} a(x, x) \]

with

\[(3.9.8) \quad a(x, \dot{x}) = EI \int_0^1 w'' \dot{w}'' \, ds , \]

where \((\cdot)'' = d^2(\cdot)/ds^2\). To make \( a(\cdot, \cdot) \) into an inner product, we must account for rigid-body rotation. Thus we set
and define

\[(3.9.10) \quad \mathcal{V} = \{x = (\theta, \phi, \phi(1)) : \phi \in \mathcal{H}^2(0,1), \phi(0) = \phi'(0) = 0\}.\]

Also, we have

\[(3.9.11) \quad \langle x, \dot{x} \rangle_\mathcal{V} = a(x, \dot{x}) + \langle B_0 \dot{B}_0^* x, \dot{x} \rangle_{\mathcal{H}_0} = a(x, \dot{x}) + \langle (M_0^{-1}B_0)(M_0^{-1}B_0)^*H x, \dot{x} \rangle_{\mathcal{H}},\]

so that \(A_1 = B_0 \dot{B}_0^*\), or \((M_0^{-1}B_0)(M_0^{-1}B_0)^*H\), depending on whether the \(H_0\) or the \(H\)-inner product is used in computing the \(\mathcal{V}\)-inner product. But we need neither \(A_1\) nor \(A_0\) explicitly. We need only \((3.9.8)\) and \((3.9.9)\), along with \((3.9.3)\), to compute the required inner products.

As mentioned in Remark 3.2, the operator \(\tilde{A}_0\) can be defined now by \((3.2.5)\), and the stiffness operator is \(A_0 = \tilde{A}_0 - A_1\). Using the \(H_0\)-inner product in \((3.2.5)\) yields the \(A_0\) for \((3.1.1')\), and using the \(H\)-inner product yields the \(A_0\) for \((3.1.1)\), which is \(M_0^{-1}\) following the \(A_0\) for \((3.1.1')\). The \(A_0\) for \((3.1.1')\) is simple, and the reader might write it out. We will not, because we do not need it.

We will point out that \(D(A_0)\) requires both the geometric boundary conditions in \(\mathcal{V}\) and the natural boundary condition \(w''(t,1) = 0\); i.e., zero moment on the right end. That the geometric boundary conditions

\[(3.9.12) \quad w(t,0) = w'(t,0) = 0\]

and

\[(3.9.13) \quad w(t,1) = w_1(t)\]

are imposed in \(\mathcal{V}\) but not in \(\mathcal{H}\) -- i.e., on the generalized displacement but not on the generalized velocity -- is common in distributed models of flexible
structures. The natural norm for expressing the kinetic energy of distributed components is the $L_2$ norm, which cannot preserve constraints on sets of zero measure. Because the strain energy involves spatial derivatives, the stronger strain-energy norm can preserve the geometric boundary conditions (although, as for the boundary slope of an elastic plate, the $V$-norm may impose some of these boundary conditions in an $L_2$ rather than a pointwise sense). The strain-energy norm is based on the material model of the distributed components of the system, and it should not be surprising that such a norm is required to connect the various structural components.

We assume that the beam has Voigt-Kelvin viscoelastic damping, so that the damping operator in (3.1.1) is

\[(3.9.14) \quad D_0 = c_0 A_0\]

where $c_0$ is a constant. This means that the damping functional is

\[(3.9.15) \quad d_0(x,\dot{x}) = c_0 a(x,\dot{x}), \quad x,\dot{x} \in V.\]
4. Approximation of the Distributed Model of the Structure

4.1 Abstract Finite Element Approximation

We begin this chapter by discussing a Ritz-Galerkin approximation framework that accommodates most common finite element schemes for flexible structures. The only hypotheses for this abstract approximation theory (in addition to those given in Chapter 3 for the infinite dimensional structural model) are the requirements stated in Hypothesis 4.1 for the basis vectors. As indicated in Section 4.1.2, modal approximation fits easily into this framework.

We assume that the basic space \( H \) is given, that the various operators and functionals associated with (3.1.1) satisfy the hypotheses stated in Chapter 3 and that the energy spaces \( V \) and \( E = V \times H \) are defined as in Section 3.2 of Chapter 3.

**Hypothesis 4.1.** There exists a sequence of finite dimensional subspaces \( V_n \) of \( V \) such that the sequence of orthogonal projections \( P_{V_n} \) converges \( V \)-strongly to the identity, where \( P_{V_n} \) is the \( V \)-projection onto \( V_n \). Also, each \( V_n \) is the span of \( n \) linearly independent vectors \( e_j \).

Since it should cause no confusion, we will omit the subscript \( n \) and write just \( e_j \), keeping in mind that the basis vectors may change from one \( V_n \) to another, as in most finite element schemes. Also, we will refer to the Hilbert space \( E_n = V_n \times V_n \), which has the same inner product as \( E = V \times H \).

4.1.1 Approximating Equations of Motion

For \( n \geq 1 \), we approximate \( x(t) \) (the solution to (3.1.1)) by
\( (4.1.1) \quad x^n(t) = \sum_{j=1}^{n} \xi_j(t)e_j \)

where \( \xi(t) = [\xi_1(t) \xi_2(t) \ldots \xi_n(t)]^T \) satisfies

\( (4.1.2) \quad M^n \ddot{\xi} + D^n \dot{\xi} + K^n \xi = B^0_n u, \)

and the mass matrix \( M^n \), damping matrix \( D^n \), stiffness matrix \( K^n \), and actuator influence matrix \( B^0_n \) are given by

\[ M^n = [\langle e_i, e_j \rangle_H], \quad D^n = [d_0(e_i, e_j)], \]

\[ K^n = [\langle A_0^{1/2}e_i, A_0^{1/2}e_j \rangle_H] = [\langle e_i, e_j \rangle_H] - [\langle A_1e_i, e_j \rangle_H], \]

\[ B^0_n = [\langle e_i, b_j \rangle_H]. \]

Of course, (4.1.2) can be written as

\( (4.1.4) \quad \dot{\eta} = A^n \eta + B^n u \)

where

\( (4.1.5) \quad \eta = (\xi) \)

and

\( (4.1.6) \quad A^n = \begin{bmatrix} 0 & I \\ -M^{-n}K^n & -M^{-n}K^n \end{bmatrix}, \quad B^n = \begin{bmatrix} 0 \\ M^{-n}B^0_n \end{bmatrix} \)

Note 4.2. Throughout this text, we use the superscript \( n \) in the designation of matrices in the \( n^{th} \) approximating system and control problem, like \( A^n, B^n, M^n, \) etc. Hence the superscript \( n \) indicates the order of approximation -- and it never indicates a power of the matrix. By \( M^{-n} \), we denote the inverse of the
mass matrix $M^n$. In the designation of a linear operator in the $n^{th}$ approximation, we use the subscript $n$. For example, $A_n$ and $B_n$ are the operators whose matrix representations are $A^n$ and $B^n$, respectively.

In the class of approximation schemes considered here, the approximation to the equivalent measurement equations (3.7.1) and (3.7.3) can be written

\begin{equation}
(4.1.7) \quad y_n = C_0 u + C^n_n = C_0 u + C^n(x, \xi)
\end{equation}

with

\begin{equation}
(4.1.8) \quad C^n = \begin{bmatrix} C_1^n & C_2^n \end{bmatrix}
\end{equation}

where the $i^{th}$ column of the $p \times n$ matrix $C_1^n$ is the $p$-vector equal to $C_1 e_i$ and the $i^{th}$ column of the $p \times n$ matrix $C_2^n$ is the $p$-vector equal to $C_2 e_i$. The $y_n$ in (4.1.7) would be equal to the exact measurement if the true generalized displacement vector $x(t)$ were a linear combination of the first $n$ basis vectors for all $t$ (i.e., if $x(t)$ were equal to the $x_n(t)$ in (4.1.1)) so that we could take $E = E_n$.

4.1.2 Modal Approximation

The equations in Section 4.1.1 include the important case where the basis vectors $e_j$ are the mode shapes of undamped, free vibration (i.e., the eigenvectors of the stiffness operator $A_0$; recall (3.1.2) and (3.1.3)). In this case, the mode shapes usually are normalized with respect to mass so that the mass matrix $M^n$ is the $n \times n$ identity matrix. The corresponding stiffness matrix then is

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\( \omega_j^2 \)

(4.1.9) \[ k^n = \begin{bmatrix} \omega_1^2 & 0 \\ \omega_2^2 & \ddots \\ 0 & \cdots & \omega_n^2 \end{bmatrix} \]

where the \( \omega_j \)'s are the natural frequencies of undamped, free vibration (i.e., the diagonal elements of \( k^n \) are the first \( n \) eigenvalues of \( A_0 \)). A zero frequency corresponds to a rigid-body mode.

That Hypothesis 4.1 holds for modal approximation follows from the fact that the operator \( \tilde{A}_0 |_V \) is a compact self-adjoint operator on \( V \), since \( \tilde{A}_0 \) and \( \tilde{A}_0^{-1} \) have the same eigenvectors.

4.2 Convergence

It is useful to note that (4.1.1) and (4.1.2) or (4.1.4) are equivalent to

\[ \dot{z}_n(t) = A_n z_n(t) + B_n u(t), \]

where \( z_n = (x_n, \dot{x}_n) \in E_n \) and \( A_n \in L(E_n) \) and \( B_n \in L(R^n, E_n) \) are the operators whose matrix representations are given in (4.1.6). Also, for any real \( \lambda \),

\[ (\lambda - A_n) \begin{pmatrix} v^1_n \\ v^2_n \end{pmatrix} = \begin{pmatrix} h^1_n \\ h^2_n \end{pmatrix} \]

is equivalent to

\[ (\lambda^2 M^n + \lambda D^n + K^n) \alpha^1 = (\lambda M^n + D^n) \beta^1 + M^n \beta^2 \]

and

\[ \alpha^2 = \lambda \alpha^1 - \beta^1 \]

if
(4.2.5) \[ v_n^j = \sum_{i=1}^{n} \alpha_{i}^{j} e_i \quad \text{and} \quad h_n^j = \sum_{i=1}^{n} \beta_{i}^{j} e_i, \quad j = 1, 2. \]

(Substituting \( A^n \) and (4.2.5) into (4.2.2) yields (4.2.3) and (4.2.4).)

Next, we will prepare to invoke the Trotter-Kato semigroup approximation theorem to show how (4.1.2), (4.1.4) and (4.2.1) approximate (3.1.1) and (3.5.6). For this, we will treat only the case in which \( A_0 \) is coercive (no rigid-body modes), so that \( A_1 = 0 \) and \( \tilde{A}_0 = A_0 \); the general case is a straightforward extension. In the present case, then, the open-loop semigroup generator \( A \) is maximal dissipative, and for each \( n \), \( A^n \) is dissipative on \( E^n \). The main idea here is to project \( (\lambda - A)^{-1} \) onto \( V_n \) in a certain inner product and observe that the result is exactly \( (\lambda - A_n)^{-1} \), where \( A_n \) is the operator on \( V_n \) in (4.2.1) and (4.2.2). Of course, we need only do this for real \( \lambda > 0 \).

For real \( \lambda > 0 \), then, we define an inner product on \( V \) by

(4.2.6) \[ \langle \cdot, \cdot \rangle_{\lambda} = \lambda^2 \langle \cdot, \cdot \rangle_H + \lambda \langle \cdot, \cdot \rangle_0 + \langle \cdot, \cdot \rangle_V. \]

Under the hypotheses in Chapter 2 on \( d_0 \), \( \langle \cdot, \cdot \rangle_{\lambda} \) is clearly norm-equivalent to \( \langle \cdot, \cdot \rangle_V \). For \( n > 1 \), we let \( P_n(\lambda) \) be the projection of \( V \) onto \( V_n \) in the inner product \( \langle \cdot, \cdot \rangle_{\lambda} \). If \( h^1, h^2 \in H \),

(4.2.7) \[ (\lambda - A) \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} h^1 \\ h^2 \end{pmatrix} \]

is equivalent to

(4.2.8) \[ \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = A^{-1} \left[ \lambda \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} - \begin{pmatrix} h^1 \\ h^2 \end{pmatrix} \right]. \]
With $A^{-1}$ from (3.5.1), (4.2.8) is equivalent to

\[(4.2.9) \quad (I + \lambda D_v + \lambda^2 A_0^{-1})v^{-1} = (\lambda A_0^{-1} + D_v)h^1 + A_0^{-1}h^2\]

and

\[(4.2.10) \quad v^2 = \lambda v^1 - h^1.\]

If

\[(4.2.11) \quad v^1_n = P_n(\lambda)v^1 \quad \text{and} \quad v^2_n = P_n(\lambda)v^2,\]

it follows from (4.2.6) and (4.2.9) that

\[(4.2.12) \quad \langle e_i, v^1_n \rangle_{\lambda} = \langle e_i, v^1 \rangle_{\lambda}
\]

\[= \lambda^2 \langle e_i, A_0 A_0^{-1} v^1 \rangle_H + \lambda \langle e_i, D_v v^1 \rangle_V + \langle e_i, v^1 \rangle_V \]

\[= \langle e_i, (\lambda^2 A_0^{-1} + \lambda D_v + I)v^1 \rangle_V \]

\[= \langle e_i, (\lambda A_0^{-1} + D_v)h^1 + A_0^{-1}h^2 \rangle_V,\]

and from (4.2.10) that

\[(4.2.13) \quad \langle e_i, v^2_n \rangle = \langle e_i, v^2 \rangle = \lambda \langle e_i, v^1 \rangle_{\lambda} - \langle e_i, h^1 \rangle_{\lambda}.\]

Now, for $h^1 = h^1_n \in V_n$, $h^2 = h^2_n \in V_n$, and $v^1_n$, $v^2_n$, $h^1_n$ and $h^2_n$ written as in (4.2.5), (4.2.12) and (4.2.13) yield (4.2.3) and (4.2.4) again.

This shows that
which yields
\[
(4.2.15) \quad \begin{bmatrix} P_n(\lambda) & 0 \\ 0 & P_n(\lambda) \end{bmatrix} (\lambda - A)^{-1} P_{E_n} = (\lambda - A_n)^{-1} P_{E_n},
\]
where $P_{E_n}$ is the $E$-projection of $E$ onto $E_n$. The projection $P_{E_n}$ can be written
\[
(4.2.16) \quad P_{E_n} = \begin{bmatrix} P_{V_n} & 0 \\ 0 & P_{H_n} \end{bmatrix},
\]
where $P_{V_n}$ is the $V$-projection onto $V_n$, as before, and $P_{H_n}$ is the $H$-projection onto $V_n$. Since the $V$-norm is stronger than the $H$-norm, it follows from Hypothesis 4.1 that $(\lambda - A_n)^{-1} P_{E_n}$ converges $E$-strongly to $(\lambda - A)^{-1}$ as $n \to \infty$.

Now, with $A_n$ extended to $E_n^1$ as, say, $n(P_{E_n} - I)$, Trotter-Kato [Ka2, page 504, Theorem 2.16] yields the following.

**Theorem 4.3.** For $A_0$ coercive, let $T_n(t)$ be the (contraction) semigroup generated on $E_n$ by $A_n$. Then, for each $t > 0$, $T_n(t)P_{E_n}$ converges strongly to $T(t)$, uniformly in $t$ for $t$ in any bounded interval. □□□

In the general case, when $A_0$ is not coercive, the open-loop generator $A$ is obtained from the dissipative $A$ by the bounded perturbation (3.5.4), so that [G13, Theorem 6.6] yields the following generalization of Theorem 4.3.

**Corollary 4.4.** Let $T_n(t)$ be the semigroup generated on $E_n$ by $A_n$. Then, for each $t > 0$, $T_n(t)P_{E_n}$ converges strongly to $T(t)$, uniformly in $t$ for $t$ in any bounded interval. □□□
Theorem 4.5. When \( A \) has compact resolvent, \( (\lambda - A_n)^{-1}P_{E_n} \) converges in \( L(E) \) to \( (\lambda - A)^{-1} \).

Proof. This follows from (4.2.15) and a standard result that the projection of a compact linear operator onto a sequence of subspaces converges in norm if the projections converge strongly to the identity, as do \( P_{E_n} \) and \( P_n(\lambda) \). \( \forall \forall \forall \)

That the adjoint semigroups also converge strongly follows from an argument entirely analogous to the proof of Theorem 4.3. In particular, equations like (4.2.6)-(4.2.12) are used to show that

\[
(4.2.17) \quad \begin{bmatrix} P_n(\lambda) & 0 \\ 0 & P_n(\lambda) \end{bmatrix} (\lambda - A_n^*)^{-1}P_{E_n} = (\lambda - A_n^*)^{-1}P_{E_n}.
\]

In showing this, \( A_n^* \) is used as \( A_n^{-1} \) was used above. Also, care must be taken to calculate \( A_n^* \) with respect to the \( E \)-inner product. The result is

Theorem 4.6. Let \( T_n(t) \) be the sequence of semigroups in Corollary 4.4. Then, for each \( t > 0 \), \( T_n(t)P_{E_n} \) converges strongly to \( T^*(t) \), uniformly in \( t \) for \( t \) in any bounded interval. \( \forall \forall \forall \)

For the approximation to the actuator influence operator \( B \in L(R^m, E) \), recall \( B_n \in L(R^m, E_n) \), the operator whose matrix representation is the matrix \( B_n \) in (4.1.6). From (4.1.3), it follows that

\[
(4.2.18) \quad B_n = P_{E_n}B.
\]

Since \( B \) has finite rank \( m \), \( B_n \) and \( B_n^* \) converge in norm to \( B \) and \( B^* \), respectively.

We define the approximating measurement operator \( C_n \in L(E_n, R^p) \) to be the operator whose matrix representation is the matrix \( C_n \) in (4.1.8) (defined imme-
diately after (4.1.8)). This means that

\[(4.2.19) \quad C_n = C|_{E_n},\]

where C is the operator in (3.7.1)-(3.7.3). Since C is a bounded linear operator with finite rank, \(C_n^P\) and \(C_n^*\) converge in norm to C and \(C^*\), respectively.

### 4.3 Approximation of the Hub-Beam-Tip-Mass Example

Our approximation of the distributed model of the structure in Section 9 of Chapter 3 is based on a finite element approximation of the beam that uses cubic Hermite splines as basis functions [Scl, SF1]. Cubic Hermite splines and their first derivatives are continuous at the nodes, which are evenly spaced for the numerical results in Chapter 9. Because the basis vectors \(e_j\) in Hypothesis 4.1 must be in the space \(V\) defined in (3.9.10), we write them as

\[\begin{align*}
(4.3.1a) \quad e_1 &= (1,0,0), \\
(4.3.1b) \quad e_j &= (0, \phi_j, \phi_j(1)), \\
& \quad j = 2, 3, \ldots, n,
\end{align*}\]

where the \(\phi_j\)'s are the cubic spline functions defined over the length of the beam and \(n-1\) is the number of splines. For \(n_e\) elements, there are \(2n_e\) linearly independent splines. The \(2n_e\) elastic degrees of freedom are usually taken to be the displacements and slopes at the nodes, relative to the rigid-body position. With the rigid-body mode (hub rotation), the total number of degrees of freedom is \(n = 2n_e + 1\).

The matrices in (4.1.3) are calculated according to (3.9.3), (3.9.8) and (3.9.9), with \(B_0\) given by (3.9.5). In particular,
\[ K^n = [a(e_i,e_j)], \quad B^n = c_0 K^n, \]

(4.3.2)

\[ B^n_0 = [1 \ 0 \ 0 \ 0 \ 0]^T \]

\[ = [\langle e_i, M_0^{-1}(1,0,0) \rangle_H] = [\langle e_i, (1,0,0) \rangle_{H_0}] \]

Note that the first row and column of \( K^n \) are zero. The matrices \( A^n \) and \( B^n \) are given by (4.1.6).

If the measurement is the rigid-body angle \( \theta \), then the measurement matrices in (4.1.7) are \( C_0 = 0 \) and

(4.3.3) \quad C_n = [1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0].
5. Order Reduction Using Balanced Realizations

5.1 Introduction

Almost any engineering analysis and design task requires a model of a physical system. The model should be complete enough to describe all of the relevant physical phenomena, but not so complicated that insight is lost or computational burdens become excessive. The development of an appropriate model is largely an art which depends on many factors such as the distribution and type of inputs and outputs, the magnitude and frequency content of inputs and outputs and the structure and properties of the physical components making up the system. Trial and error and engineering intuition are common ingredients in the modeling process. Much research has focused on methods for determining what is essential about a model and what can be disregarded. Most of this work, generally known as model reduction, applies to systems where the input-output relationship is linear. Fortunately, the structural systems considered here fall into that class.

In addition to generating a low order model of the system which is to be controlled, one may wish to reduce the order of the controller or compensator which has been synthesized to accomplish the control. If the controller is linear, one may often use the same methods which are used for reduction of the plant model. This chapter deals with an approach for model reduction of linear systems which has been widely used since its introduction in 1980, the method of balanced realizations [Mol]. In its general form, the approach may be used for reduction of a plant model or controller. A simplified approximate version may be used when the equations have a special form corresponding to lightly damped mechanical systems. The method has intuitive appeal and is easy to apply.
A number of methods for model reduction have been used in the past with varying degrees of success. Modal truncation, for example, has been widely used in structures research. In this approach, the equations of motion are transformed into a set of uncoupled second order equations each of which describes the dynamics of one mode of vibration. Model reduction is accomplished by deleting those modes with the highest damped or undamped natural frequencies. The coupling of each mode to inputs or outputs does not play a role in the truncation procedure. The method is straightforward in its application and has intuitive appeal. It is an example of a method where reduction is accomplished by truncation after an appropriate coordinate transformation.

Direct model reduction procedures have been proposed which do not involve truncation. In these approaches, the analyst decides in advance what order of reduced model he wants and then uses a numerical optimization procedure to obtain the reduced model of the given order which minimizes a specified measure of model accuracy. Methods of this type have recently been developed by Hyland and Bernstein for application to reduced order controller design [HB1]. Their approach is called the Maximum Entropy Method, and it involves the simultaneous solution of two modified Riccati equations and two modified Liapunov equations. Numerical routines for solving these equations are not yet standard, but the authors have developed algorithms which appear to be effective.

The introduction of the method of Balanced Realizations in 1980 revitalized the field of model reduction and led to renewed research. Various modifications and extensions have been proposed and studied. Enns developed an extension for frequency weighted balanced realizations [Enl] whereby the analyst could modify the basic balanced realization procedure to obtain relatively better reduced model accuracy in certain frequency ranges where model accuracy was critical.
Another approach to model reduction is the optimal Hankel norm approximation which is developed in great detail by Glover et al. in [GL1,Gl1]. Hankel norm approximations are closely related to approximations based on balanced realizations, and in fact begin by expressing the system as a balanced realization. Although a smaller error bound can be established for Hankel norm approximations, the computations required are substantially more involved than those for balanced realizations. Another feature of the Hankel norm approximation is that the reduced model may not be strictly proper even when the original model is. It is possible to alter the computations to ensure a strictly proper reduced model, but the required changes increase the error bound.

New techniques for model reduction continue to emerge. Five model reduction techniques (including balanced realizations, Hankel norm approximations, an approach by Davis and Skelton, an approach by Yousuff and Skelton and a new method by Liu and Anderson) have been compared recently by Liu and Anderson [LA1]. Although some of the methods generally performed better than others, there was no one method which was always superior. Thus, model reduction is still an area of active research. The present chapter does not attempt to examine and compare the many model reduction techniques currently available. Instead, its scope is limited to the method of balanced realizations. This approach has been found to compare favorably with other methods, and its computational simplicity makes it attractive for present applications.

5.2 Observability and Controllability; Observability and Controllability Grammians

Balanced realizations are based on the notions of controllability and observability of linear systems. Thus we begin by defining these terms and exploring their meaning. Suppose a physical system is described by the state space equations:
\[
\begin{align*}
(5.2.1) & \quad x &= Au + Bu \\
(5.2.2) & \quad y &= Cx \\
\end{align*}
\]

where

- \(x\) = \(n \times 1\) state vector
- \(y\) = \(r \times 1\) output/measurement vector
- \(u\) = \(m \times 1\) input/disturbance vector

A, B and C are constant matrices of appropriate dimension.

To introduce the notion of controllability, one might ask the following question: "Can one find an input \(u(t)\) which drives the system (5.2.1) from an initial state \(x(t_0) = 0\) to an arbitrary final state \(x_f\) in finite time \(t_f\)?" If the answer to this question is yes, the system is said to be controllable. Otherwise it is said to be uncontrollable. To determine whether a system is controllable, first note that if \(x(t_0)\) is zero, then

\[
(5.2.3) \quad x(t) = \int_{t_0}^{t_f} e^{A(t-\tau)} B u(\tau) d\tau
\]

Introduce a symmetric matrix \(W_c(t_0, t)\) defined as:

\[
(5.2.4) \quad W_c(t_0, t) = \int_{t_0}^{t} e^{A(t-\tau)} B B^T e^{A^T(t-\tau)} d\tau
\]

\(W_c(t_0, t)\) is called the controllability gramian. If \(W_c(t_0, t_f)\) is nonsingular, a control input \(u_0(t)\) which drives \(x\) from the origin to \(x_f\) at time \(t_f\) is given by

\[
(5.2.5) \quad u_0(t) = B^T e^{A^T(t_f-t)} W_c^{-1}(t_0, t_f) x_f
\]

This can be verified by substituting (5.2.5) into (5.2.3). Thus, if \(W_c(t_0, t_f)\) is nonsingular, the system described by (5.2.1) is controllable. In the event
that $W_c(t_0,t_f)$ is singular, the system is said to be uncontrollable.

The control input given by (5.2.5) is not the only one that will drive $x$ from the origin to $x_f$ at time $t_f$. Let $u_1(t)$ be an input which drives $x$ from the origin through an arbitrary trajectory and back to the origin at time $t_f$. Because of the linearity of (5.2.1), any input which drives $x$ from 0 to $x_f$ at time $t_f$ can be written as

$$u(t) = u_0(t) + u_1(t)$$

Among all such inputs, $u_0(t)$ is the one which minimizes the integral

$$(5.2.7) \quad E = \int_{t_0}^{t_f} u^T(\tau)u(\tau)\,d\tau$$

$E$ may be regarded as a measure of control effort, and $u_0(t)$ interpreted as the minimum effort input which accomplishes the desired change of state from 0 to $x_f$. Substituting $u_0(t)$ into equation (5.2.7) and evaluating the integral yields the minimum value of $E$.

$$(5.2.8) \quad E_{\text{min}} = x_f^T W_c^{-1}(t_0,t_f)x_f$$

Three observations will be useful for subsequent work.

Observation 1

The control effort becomes small (i.e., the system becomes easier to control) when $W_c(t_0,t_f)$ becomes large.

Observation 2

$W_c(t_0,t_f)$ can be expressed as:
To see this, let \( \lambda = t_f - \tau \) in (5.2.4).

**Observation 3**

If a system is controllable, it can be driven from an arbitrary initial state to an arbitrary final state. This can be seen by starting from an arbitrary initial state \( x_0 \), and expressing the final state as

\[
(5.2.10) \quad x(t_f) = e^{A(t_f-t_0)} x_0 + \int_{t_0}^{t_f} e^{A(t_f-\tau)} B \, u(\tau) \, d\tau
\]

So,

\[
(5.2.11) \quad x(t_f) - e^{A(t_f-t_0)} x_0 = \int_{t_0}^{t_f} e^{A(t_f-\tau)} B \, u(\tau) \, d\tau
\]

This shows that the control which drives the system from \( x_0 \) to \( x(t_f) \) is the same as that which drives the system from the origin to the final state \( x_f \) where

\[
(5.2.12) \quad x_f = x(t_f) - e^{A(t_f-t_0)} x_0
\]

The concept of controllability has several things in common with that of observability, but it has some differences as well. Assume \( A, B \) and \( C \) in Eqs. (5.2.1, 5.2.2) are known, and that the input \( u(t) \) and the output \( y(t) \) are known on \( t_0 \leq t \leq t_f \). Then ask the question, "Can one determine the initial state \( x_0 \) from this information?" If the answer to this question is yes, the system is said to be observable. If the answer is no, the system is unobservable. Observability also implies that knowledge of the input and output over any interval \( t_0 \leq t \leq t_f \) is sufficient to determine \( x(t_f) \).
To develop the idea of observability, let

\begin{equation}
(5.2.13) \quad x(t) \triangleq x(t) - \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) d\tau
\end{equation}

\begin{equation}
(5.2.14) \quad y(t) \triangleq C x(t)
\end{equation}

Then,

\begin{equation}
(5.2.15) \quad y(t) \triangleq y(t) - \int_{t_0}^{t} C e^{A(t-\tau)} B u(\tau) d\tau
\end{equation}

Since both terms on the right side of (5.2.15) are known, \( y(t) \) may be regarded as known also. Now,

\begin{align*}
\dot{x} &= \bar{x} - Bu - A \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) d\tau \\
&= Ax + Bu - Bu - A \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) d\tau \\
&= A \left[ x - \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) d\tau \right]
\end{align*}

\begin{equation}
(5.2.16) \quad = A \bar{x}
\end{equation}

and

\begin{equation}
(5.2.17) \quad \bar{x}(t_0) = x(t_0)
\end{equation}

Hence, the plant equations may be written in terms of \( x \) and \( y \) as

\begin{align*}
\dot{x} &= A\bar{x}, \quad \bar{x}(t_0) = x(t_0) \\
y &= C\bar{x}
\end{align*}

The observability of (5.2.18) and (5.2.19) is the same as that of (5.2.13) and (5.2.14) since the ability to determine \( \bar{x}(t_0) \) given \( \bar{y}(t) \) on \([t_0, t_1]\) implies and
is implied by the ability to determine $x(t_0)$ given $y(t)$ on $[t_0, t_1]$ and $u(t)$ on $[t_0, t_1]$. In the subsequent development we will work with (5.2.18) and (5.2.19) because of their simpler form.

From (5.2.18) and (5.2.19), $\ddot{y}(t)$ may be written as

$$\dot{y}(t) = C e^{A(t-t_0)} \ddot{x}(t_0)$$

$$t \leq t \leq t_f$$

Define the observability Grammian as

$$W_o(t_0, t) = \int_{t_0}^{t} e^{A^T(t-t_0)} \tilde{C} \tilde{C} e^{A(t-t_0)} dt$$

Multiply (5.2.20) by $e^{A^T(t-t_0)} \tilde{C}$ and integrate from $t_0$ to $t_f$.

$$\int_{t_0}^{t_f} e^{A^T(t-t_0)} \tilde{C} \tilde{y}(t) dt = \int_{t_0}^{t_f} e^{A^T(t-t_0)} \tilde{C} \tilde{C} \tilde{e}^{A(t-t_0)} dt \tilde{x}(t_0)$$

$$= W_o(t_0, t_f) \tilde{x}(t_0)$$

If $W_o(t_0, t_f)$ is nonsingular, and $\tilde{y}(t)$ is known on $[t_0, t_f]$, then (5.2.22) can be solved for $\tilde{x}(t_0)$.

$$\tilde{x}(t_0) = W_o^{-1}(t_0, t_f) \int_{t_0}^{t_f} e^{A^T(t-t_0)} \tilde{C} \tilde{y}(t) dt$$

Hence, the system (5.2.18), (5.2.19) is observable if $W_o(t_0, t_f)$ is nonsingular. If $W_o(t_0, t_f)$ is singular, the system is unobservable.

For interpretation of results in the sequel, it will be useful to introduce a scalar measure of the output $y(t)$ analogous to the scalar measure of the input given in Eq. (5.2.7). Define a response function $R$ as
Substituting from (5.2.20) and (5.2.21),

\[ \int_{t_0}^{t_f} \bar{y}^T(\tau) \bar{y}(\tau) d\tau = \bar{x}^T(t_0) \int_{t_0}^{t_f} e^{A(t-t_0)} C^T C e^{A(t-t_0)} d\tau \bar{x}(t_0) \]

(5.2.25)

\[ = \bar{x}^T(t_0) W_0(t_0, t_f) \bar{x}(t_0) \]

Thus, \( W_0(t_0, t_f) \) provides a measure of the size of the response corresponding to an initial condition \( \bar{x}(t_0) \).

If the \( A \) matrix in Eq. (5.2.1) is stable (i.e., all of the eigenvalues of \( A \) are in the left half plane), then \( W_c(t_0, t_f) \) and \( W_0(t_0, t_f) \) remain finite as \( t_f \to \infty \). Define

(5.2.26) \[ W_c = W_c(0, \infty) = \int_0^{\infty} e^{A\tau} B B^T e^{A^T \tau} d\tau \]

(5.2.27) \[ W_0 = W_0(0, \infty) = \int_0^{\infty} e^{A^T \tau} C^T C e^{A\tau} d\tau \]

It is not difficult to show that \( W_c \) and \( W_0 \) satisfy the Liapunov Equations:

(5.2.28) \[ A W_c + W_c A^T + B B^T = 0 \]

(5.2.29) \[ W_0 A + A^T W_0 + C^T C = 0 \]

Since well documented numerical procedures are available for solving these equations \([BS1]\), finding \( W_c \) and \( W_0 \) for the case where \( t_f \to \infty \) is most easily accomplished by solving (5.2.28) and (5.2.29) rather than attempting direct evaluation of the integrals given in Eqs. (5.2.9) and (5.2.21). If the given \( A \)
matrix is not completely stable, then $A$ must be partitioned into a stable part and an unstable part. Eqs. (5.2.28) and (5.2.29) may be used to find the controllability and observability grammians associated with the stable part of $A$ in the infinite time case.

5.3 Balanced Realizations

It is always possible to express the system equations (5.2.1) and (5.2.2) in a different but equivalent form through the use of a coordinate transformation:

\[ \dot{x} = T \dot{x} \]

where $T$ is a nonsingular matrix. This transformation leads to

\[ \dot{x} = \bar{A} \bar{x} + \bar{B} u \]
\[ y = \bar{C} \bar{x} \]

where

\[ \bar{A} = T^{-1}AT, \quad \bar{B} = T^{-1}B, \quad \bar{C} = CT \]

There are many objectives one might have in performing such a transformation, e.g., to put the transformed system into diagonal or block diagonal form, to put it in controllability canonical form [Kal, pp. 49-55], etc. When the transformed model is to be used for control system design, the Balanced Realization of Moore [Mol] is appealing because expressing equations in this form allows one to compare the relative controllability and observability of individual states. This information is useful in deciding which states are most important to retain in a reduced model.

Suppose the original system matrix $A$ is stable. (If this is not the case, then $A$ must be partitioned into a stable and unstable part as noted above. The
discussion to follow applies to the stable part.) To cast the equations into balanced form, we seek a transformation matrix $T$ such that $W_c$ and $W_0$ in Eqs. (5.2.26) and (5.2.27) are diagonal and equal. Suppose $W_c$ and $W_0$ are found by solving the Liapunov Equations (5.2.28) and (5.2.29) for the untransformed system. $\bar{W}_c$ and $\bar{W}_0$ for the transformed system then become

$$W_c = T^{-1}W_cT^{-T}; \quad W_0 = T^TW_0T$$

The transformation matrix $T$ is found by considering the eigenvalue problem

$$[W_0 - \lambda W_c^{-1}]\gamma = 0$$

Assuming distinct eigenvalues, the eigenvectors satisfy

$$\gamma_T W_c^{-1}\gamma_i = 0; \quad \gamma_T W_0 \gamma_i = 0 \quad i \neq j$$

Form a square matrix, $\Gamma$, from the eigenvectors of (5.3.6). The matrix $\Gamma$ will satisfy

$$\Gamma^TW_0\Gamma = \text{diagonal}, \quad \Gamma^{-1} W_c \Gamma^{-T} = \text{diagonal}$$

We have reached part of our goal by finding a matrix which diagonalizes both $W_c$ and $W_0$. However, the columns of $\Gamma$ are not yet uniquely defined because we have not specified a normalization. This remaining degree of freedom is used to make $W_c$ and $W_0$ equal. Normalize the columns of $\Gamma$ such that

$$\gamma_i W_0 \gamma_i = (\lambda_i)^{\frac{1}{2}} \quad i = 1,2,\ldots,n$$

Then,

$$W_c = W_0 = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \vdots \\ 0 & \ddots & \lambda_n \end{bmatrix}^{\frac{1}{2}}$$

The $\lambda_i$'s are called Hankel singular values. Thus, when the original system is transformed using $T = \Gamma$, observability and controllability grammians satisfy (5.3.10). The transformed system is said to be a balanced realization.
5.4 Model Reduction Using Balanced Realizations

We now consider the interpretation of $W_c$ and $W_o$ and examine how these matrices can be used to motivate a model reduction procedure. For a balanced realization, $W_c$ and $W_o$ are diagonal and satisfy (5.3.10). For this situation, the control effort function $E$ and the response function $R$ introduced in section 5.2 become

$$E = \int_0^\infty u^T u \, dt = \sum_{i=1}^{n} \frac{x_{f1}^2}{\lambda_i} \ ; \quad R = \int_0^\infty y^T y \, dt = \sum_{i=1}^{n} x_{0i}^2 \lambda_i$$

If $\lambda_i > \lambda_j$, we can argue that less effort is required to move from the origin to $x_{f1}$ than to move from the origin to $x_{fj}$. Thus, $x_{f1}$ is more strongly affected by $u$ than $x_{fj}$, i.e., $x_{f1}$ is more controllable. Similarly, we can argue that an initial condition on $x_{01}$ results in a larger contribution to the response than an initial condition on $x_{0j}$, i.e., $x_{01}$ is more observable than $x_{0j}$. Taking these observations together, we see that a state $x_i$ with a large $\lambda_i$ is more controllable and observable than a state $x_j$ with a small $\lambda_j$. If model truncation is contemplated, it is intuitively appealing to retain highly controllable and observable states, i.e., states with large $\lambda$'s. Thus states corresponding to small values of $\lambda$ are candidates for truncation. If there is no gap separating the states with large $\lambda$'s from those with small $\lambda$'s, it may not be clear where the division should occur between the retained states and the discarded states. This issue may be resolved by considering the convergence of control and estimator gains. See Chapters 6-9 and [MG1].

5.5 Asymptotic Expansions for the Balanced Singular Values of Lightly Damped Mechanical Systems

As explained in Section 4.2, calculating $W_c$ and $W_o$ involves the solution of Liapunov equations. When Eqs. (5.2.1) and (5.2.2) describe a stable, lightly
damped mechanical system, the approximate solution of these Liapunov Equations may be accomplished using perturbation methods. This section concerns the development of first and second order perturbation solutions for $W_c$, $W_0$ and their balanced singular values.

The mechanical systems addressed in this section are described by second order equations of the form:

\[
\begin{align*}
(5.5.1) & \quad M \ddot{q} + \epsilon D \dot{q} + K q = b u \\
(5.5.2) & \quad y = c_p q + c_v \dot{q}
\end{align*}
\]

$M$, $D$ and $K$ are symmetric and positive definite $n \times n$ matrices with $M$ being the mass matrix, $D$ the damping matrix and $K$ the stiffness matrix. (At the expense of some additional complexity, the assumptions that $D$ and $K$ are symmetric could be relaxed in the development to follow. Ref. [BM1] addresses the case where a skew symmetric gyroscopic matrix $G$ is added to $D$ and a skew symmetric circulatory matrix $F$ is added to $K$. The present development is restricted to the case where $G$ and $F$ are absent.) The matrix $b$ is the input distribution matrix, and $u$ is the input vector representing either the control or disturbance input. The output vector is $y$, and $c_p$ and $c_v$ are the position and velocity distribution matrices, respectively. The scalar $\epsilon$ is a small parameter reflecting the light damping. By an appropriate coordinate transformation, these equations can be cast into a more convenient modal form:

\[
\begin{align*}
(5.5.3) & \quad \ddot{\eta} + \epsilon \Delta \dot{\eta} + \Omega^2 \eta = \beta u \\
(5.5.4) & \quad y = c_p \dot{\eta} + c_v \eta
\end{align*}
\]

In Eqs. (5.5.3) and (5.5.4), $\Omega$ is a diagonal matrix of modal frequencies. The modal frequencies are assumed to be distinct. The damping matrix, $\Delta$, is sym-
metric and positive definite, but it is not presumed to be diagonal. The matrices $B$, $C_p$ and $C_v$ are the input and output distribution matrices for the normalized system. For the development to follow, we wish to express Eqs. (5.5.3) and (5.5.4) in state space form. Let

$$\begin{align*}
(5.5.5) & \quad x = \{\eta \} ; \\
& \quad A = \begin{bmatrix} 0 & I \\ -\Omega^2 & -\epsilon \Delta \end{bmatrix} ; \\
& \quad B = \begin{bmatrix} 0 \\ \beta \end{bmatrix} \\
& \quad C = [C_p \ C_v]
\end{align*}$$

Then the state space version of (5.5.3) and (5.5.4) becomes:

$$\begin{align*}
(5.5.6) & \quad \dot{x} = Ax + Bu \\
(5.5.7) & \quad y = Cx
\end{align*}$$

Consider first the solution of the Liapunov Equation (5.2.28) for $W_c$. Because of the special structure of $A$ and $B$, it is useful to express $W_c$ in the partitioned form:

$$W_c = \begin{bmatrix} W_{11} & W_{12} \\ \frac{\epsilon}{\epsilon} & W_{22} \end{bmatrix}$$

The scalar $\epsilon$ is introduced in Eq. (5.5.8) for convenience in the subsequent development. Substitution of (5.5.5) and (5.5.8) into (5.2.28) leads to three matrix equations:

$$\begin{align*}
(5.5.9) & \quad W_{12}^T + W_{12} = 0 \\
(5.5.10) & \quad W_{22} - W_{11} \Omega^2 = \epsilon^2 W_{12} \Delta \\
(5.5.11) & \quad -(\Omega^2 W_{12} + W_{12}^T \Omega^2) - (\Delta W_{22} + W_{22} \Delta) + \beta \beta^T = 0
\end{align*}$$
These equations lead to some interesting observations. First we see that $W_{12}$ is skew symmetric. This results from the fact that the state vector $x$ has the special form $x = [\eta^T \eta^T]^T$, i.e., the bottom half of the state vector is the derivative of the top half.

A second observation is that Eqs. (5.5.9-5.5.11) contain $\epsilon^2$ but not $\epsilon$. This means that $W_{11}, W_{12}$ are functions of $\epsilon^2$ and expansions of these matrices should be in terms of $\epsilon^2$ and not $\epsilon$.

With these observations in mind, we assume an expansion for $W_{ij}$ in the form:

\begin{equation}
W_{ij} = u_{ij} + \epsilon^2 v_{ij} + \epsilon^4 w_{ij} + \ldots
\end{equation}

Substituting this expansion into Eqs. (5.5.9-5.5.11) and setting coefficients of like powers of $\epsilon^2$ to zero leads to a set of equations which can be solved sequentially.

$\epsilon^0$ terms:

\begin{align}
(5.5.13) & \quad u_{12}^T + u_{12} = 0 \\
(5.5.14) & \quad u_{22} - u_{11} = 0 \\
(5.5.15) & \quad -(\Omega^2 u_{12} + u_{12}^T \Omega^2) - (Au_{22} + u_{22}A) + \beta \beta^T = 0
\end{align}

$\epsilon^2$ terms:

\begin{align}
(5.5.16) & \quad v_{12}^T + v_{12} = 0
\end{align}
(5.5.17) \[ v_{22} - v_{11} \Omega^2 = u_{12} \Delta \]

(5.5.18) \[ - (\Omega^2 v_{12} + v_{12}^T \Omega^2) - (\Delta v_{22} + v_{22} \Delta) = 0 \]

\$^4\$ terms:

(5.5.19) \[ w_{12}^T + w_{12} = 0 \]

(5.5.20) \[ w_{22} - w_{11} \Omega^2 + w_{12} \Delta = 0 \]

(5.5.21) \[ -(\Omega^2 w_{12} + w_{12}^T \Omega^2) - (\Delta w_{22} + w_{22} \Delta) = 0 \]

etc.

Define \( w_i \) as the \( i \)th diagonal element of \( \Omega^2 \), and \((\beta \beta^T)_{ij}, \Delta_{ij} \) and \((u_{kk})_{ij}\) as the \( ij \) elements of \( \beta \beta^T \), \( \Delta \) and \( u_{kk} \), respectively. Then, using Eqs. (5.5.13 - 5.5.15) and the assumption that the modal frequencies are distinct, the elements of \( u_{11} \) become:

(5.5.22) \[ (u_{11})_{ii} = (\beta \beta^T)_{ii}/(2w_i^2 \Delta_{ii}) \]; \( (u_{11})_{ij} = 0 \) for \( i \neq j \)

(\( u_{11} \) is diagonal.) From (5.5.22) and (5.5.14), \( u_{22} \) becomes:

(5.5.23) \[ u_{22} = u_{11} \Omega^2 \]

(\( u_{22} \) is also diagonal.)

With the elements of \( u_{22} \) known, introduce a matrix \( P \) defined as:

(5.5.24) \[ P \triangleq \beta \beta^T - (\Delta u_{22} + u_{22} \Delta) \]

(Note that \( P_{ii} = 0 \).) The elements of \( u_{12} \) then become
(5.5.25) \[(u_{12})_{1j} = p_{1j}(\omega_i^2 - \omega_j^2)\]; \(i \neq j\)

(5.5.26) \[(u_{12})_{11} = 0\]

Next introduce matrices \(S\) and \(T\) based on the known solution for \(u_{12}\).

\[
\begin{align*}
S &= u_{12}A - \Delta u_{12} \\
T &= -(\Delta^2 u_{12}^T + u_{12} \Delta^2)
\end{align*}
\]

Then, from (5.5.16 - 5.5.18), the elements of \(v_{11}\) become

(5.5.29) \[(v_{11})_{ij} = S_{ij}/(\omega_i^2 - \omega_j^2)\]; \(i \neq j\)

\[
(v_{11})_{ii} = \frac{\sum_{k=1}^{i-1} (\Delta_{ik} \omega_k^2 v_{11}(v_{11})_{k1}) - \sum_{k=i+1}^{n} (\Delta_{ik} \omega_k^2 v_{11}(v_{11})_{k1})}{2 \Delta_{ii} \omega_i^2}
\]

Using this expression together with (5.5.17) we obtain

(5.5.31) \[v_{22} = v_{11} \Omega^2 + u_{12} \Delta\]

The expansion will not be carried beyond this point because the expressions become increasingly unwieldy. However, the expressions obtained so far permit one to write out the first three terms in the expansion for \(W_c\). To see this, recall that

(5.5.32) \[
W_c = \begin{bmatrix}
\frac{W_{11}}{\epsilon} & W_{12} \\
W_{12}^T & \frac{W_{22}}{\epsilon}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\epsilon}[u_{11} + \epsilon^2 v_{11} + \ldots] & [u_{12} + \epsilon^2 v_{12} + \ldots] \\
[u_{12}^T + \epsilon^2 v_{12}^T + \ldots] & \frac{1}{\epsilon}[u_{22} + \epsilon^2 v_{22} + \ldots]
\end{bmatrix}
\]

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The expressions for $u_{11}$, $u_{12}$, $u_{22}$, $v_{11}$, and $v_{22}$ are given in Eqs. (5.5.22)-(5.5.31).

For lightly damped structures (small $\epsilon$), the formulas given above provide an efficient way to compute an approximation of the controllability gramian. Using a similar approach, it is possible to develop approximate formulas for the observability gramian (see Appendix A). The resulting expressions for $W_c$ and $W_o$ may be used to determine the balanced realization. Examination of (5.5.32) and the corresponding formula for the observability gramian reveals that as $\epsilon$ gets smaller, causing the first terms in the expansions to dominate, $W_c$ and $W_o$ tend to become diagonal. Thus, the product $(W_c W_o)$ tends to become diagonal, and the diagonal elements approximate the squares of the balanced singular values. Using the results of this chapter and Appendix A, one can write these approximate expressions as

\[
\begin{pmatrix}
1 + 1 + \epsilon \begin{pmatrix}
0 & u_{12} \\
0 & 0
\end{pmatrix}
\end{pmatrix}
\]

Thus when $\epsilon$ is small, the modal coordinates themselves constitute a balanced realization, and the $i^{th}$ balanced singular value is a measure of the importance of the $i^{th}$ mode of the model. A similar result was obtained by Gregory [Gr1] using a different approach.
Subsequent chapters will address the question of how many modes and which modes must be included in a structural model in order for the compensator to converge and stabilize the nominal plant. The procedure for doing this involves beginning with a model of finite size and gradually adding modes to the model until the compensator converges. The speed of convergence and the ultimate size of the required model are dependent in part on how good the analyst is in selecting the modes to be added. The approximate formulas given above allow one to identify which modes are likely to be the most important (the highly controllable and observable ones with large balanced singular values) so they can be added to the model first. This procedure helps keep the size of the model from getting too large.

5.6 Internal Balancing in Infinite Dimensions

The idea of internal balancing in finite dimensions can be extended to infinite dimensions. This extension will yield balanced states for distributed systems, and these states will be vectors in the abstract state spaces (Hilbert spaces) in which the distributed models are formulated.

Consider the control system

\begin{equation}
\dot{x} = Ax + Bu, \quad y = Cx
\end{equation}

where \(x\) is in a Hilbert space \(H\), \(A\) generates an exponentially stable semigroup on \(H\), and \(B\) and \(C\) are bounded linear operators of finite rank. In applications, \(B\) and \(C\) have finite rank because the number of actuators and the number of sensors are finite. The controllability and observability gramman operators \(W_c\) and \(W_o\) are the solutions to the operator equations
\[(5.6.2) \quad A^*W_c + W_cA + BB^* = 0 \quad \text{and} \quad A \, W_o + W_oA^* + C^*C = 0. \]

Since B and C have finite rank, both \( W_c \) and \( W_o \) are compact and selfadjoint. It can be shown, then, that the eigenvalue problem

\[(5.5.3) \quad W_oW_c z_i = s_i^2 z_i \]

has a countable number of solutions, that the singular values \( s_i \) approach zero as \( i \) increases, and that the eigenvectors \( z_i \) are mutually orthogonal with respect to both \( W_c \) and \( W_o \) and complete in \( H \). Note that these properties are quite similar to those of the eigenvalue problem for the natural modes of a flexible structure. The numerical problem of computing the balanced states of an infinite dimensional control system is also analogous to that of computing the natural models of a structure.

The results alluded to here could be developed more fully and explored with respect to developing efficient numerical methods for computing the balanced states of the distributed system. The approximation theory for computing balanced states will be especially important. To the authors' knowledge, work has not been carried out on this topic.
6. The Infinite Dimensional LQG Problem

Section 6.1 of this chapter presents some preliminary definitions and results for the optimal linear-quadratic regulator problem on an arbitrary real Hilbert space \( E \). These results are generic in the sense that \( E \) is not necessarily the Energy space of Chapters 3 and 4, and the operators \( A, B, \) etc., do not necessarily represent a flexible structure. In subsequent chapters, such generic results facilitate derivation of approximation theory for the infinite dimensional state estimator from the analogous approximation theory for the control problem.

Section 6.2 discusses an infinite dimensional state estimator for the generic plant in Section 6.1, an infinite dimensional compensator and the corresponding closed-loop system. The purpose of Section 6.2 is to establish certain properties of the estimator, compensator and closed-loop system that do not depend on the control and estimator gains being optimal. Therefore, the gains in Section 6.2 are allowed to be arbitrary bounded linear operators between appropriate spaces, and neither the estimator nor the compensator is necessarily optimal.

Section 6.3 defines the LQG-optimal infinite dimensional estimator and compensator, still for a generic plant. Finally, Section 6.4 gives some important implications of the general results for the case where the plant is the infinite dimensional flexible structure model defined in Chapter 3.

Most of the results in this chapter are analogous to results for the finite dimensional LQG optimal control problem, with infinite dimensional operators corresponding to finite dimensional matrices. In particular, instead of the control and estimator gain matrices in finite dimensions, here we have gain...
operators. The functional gains in Section 6.4 can be thought of as a generalization of the transposes of finite dimensional gain matrices, although the relationship between functional gains and infinite dimensional gain operators is not so simple.

6.1 The Generic Optimal Regulator Problem

Let a linear operator $A$ generate a $C_0$-semigroup $T(t)$ on a real Hilbert space $E$, and suppose that $B \in L(R^m, E)$, that $Q \in L(E)$ is nonnegative and selfadjoint and that $R$ is a positive definite, symmetric real $m \times m$ matrix. The optimal control problem on $E$ is to choose the control function $u \in L_2(0, \infty; R^m)$ to minimize the performance index

$$J(z(0), u) = \int_0^\infty (\langle Q z(t), z(t) \rangle_E + u(t)^T R u(t)) \, dt$$

where the state $z(t)$ satisfies

$$z(t) = T(t) z(0) + \int_0^t T(t-s) B u(s) \, ds, \quad t \geq 0.$$

Definition 6.1. A function $u \in L_2(0, \infty; R^m)$ is an admissible control for the initial state $z$, or simply an admissible control for $z$, if $J(z, u)$ is finite; i.e., if the state $z(t)$ corresponding to the control $u(t)$ and the initial condition $z(0) = z$ is in $L_2(0, \infty; E)$.

Definition 6.2. Let the operators $A$, $B$, $Q$ and $R$ be as defined above. An operator $\Pi$ in $L(E)$ is a solution of the Riccati algebraic equation if $\Pi$ maps the domain of $A$ into the domain of $A^*$ and satisfies the Riccati algebraic equation

$$A^* \Pi + \Pi A - \Pi B R^{-1} B^* \Pi + Q = 0.$$
Theorem 6.3 (Theorems 4.6 and 4.11 of [Gi4]). There exists a nonnegative selfadjoint solution of the Riccati algebraic equation if and only if, for each \( z \in E \), there is an admissible control for the initial state \( z \). If \( \Pi \) is the minimal nonnegative selfadjoint solution of (6.1.3), then the unique control \( u(\cdot) \) that minimizes \( J(z,u) \) and the corresponding optimal trajectory \( z(\cdot) \) are given by

\[
(6.1.4) \quad u(t) = -R^{-1}B^*\Pi z(t)
\]

and

\[
(6.1.5) \quad z(t) = S(t)z,
\]

where \( S(t) \) is the semigroup generated by \((A - BR^{-1}B^*\Pi)\). Also,

\[
(6.1.6) \quad J(z,u) = \min_v J(z,v) = \langle \Pi z, z \rangle_E.
\]

If, for each initial state and admissible control,

\[
(6.1.7) \quad \lim_{t \to \infty} |z(t)| = 0,
\]

there exists at most one nonnegative selfadjoint solution of (6.1.3). If \( Q \) is coercive, (6.1.7) holds for each initial state and admissible control and \( S(t) \) is uniformly exponentially stable. \VVV

We will refer to \( T(t) \) as the open-loop semigroup and to \( S(t) \) as the closed-loop semigroup.

6.2 An Infinite Dimensional Estimator and the Corresponding Compensator and Closed-Loop System

The differential equation corresponding to (6.1.2) is, of course,

\[
(6.2.1) \quad \dot{z}(t) = Az(t) + Bu(t), \quad t > 0.
\]
While (6.1.2) is valid for any \( u \in L_2(0, T; \mathbb{R}^m) \), the sense in which (6.2.1) holds -- when it holds -- depends on how smooth \( u \) is. Thus it is more precise to write (6.1.2) for arbitrary \( L_2 \) controls, but continuing to write (6.1.2) in this section, along with the similar integral equations for the estimator and closed-loop system, would be too cumbersome. Therefore, we will write the differential equations with the understanding that when necessary they can be interpreted as representing the appropriate integral equations.

We assume that we have a \( p \)-dimensional measurement vector \( y(t) \) given by

\[
y(t) = C_0 u(t) + Cz(t),
\]

where \( C_0 \in L(\mathbb{R}^m, \mathbb{R}^p) \) and \( C \in L(\mathbb{R}, \mathbb{R}^p) \) for some positive integer \( p \).

**Definition 6.4.** For any \( \hat{F} \in L(\mathbb{R}^p, \mathbb{E}) \), the system

\[
\dot{z}(t) = A_2(t) + Bu(t) + \hat{F}[y(t) - C_0 u(t) - Cz(t)], \quad t > 0,
\]

will be called an estimator, or observer, for the system (6.2.1)-(6.2.2). Let \( S(t) \) be the semigroup generated by \( A-FC \). The observer in (6.2.3) is strongly (uniformly exponentially) stable if \( \hat{S}(t) \) is strongly (uniformly exponentially) stable. \( \mathcal{W} \)

To justify this definition, we write

\[
e(t) = z(t) - \hat{z}(t)
\]

and, with (6.2.1)-(6.2.3), obtain

\[
e(t) = \hat{S}(t) e(0), \quad t > 0.
\]

Of course, an estimator is necessary because the full state \( z(t) \) will not be available for direct feedback, and the feedback control must be based on an estimate such as \( \hat{z}(t) \). When the desired control law has the form
(6.2.6) \[ u(t) = Fz(t) \]

for some \( F \in L(E, \mathbb{R}^m) \), the observer in (6.2.3) can be used to construct \( z(t) \) from the measurement in (6.2.2) and then the control law in (6.2.6) can be applied to \( \hat{z}(t) \). The control applied to the system is then

(6.2.7) \[ u(t) = F\hat{z}(t), \]

and the resulting closed-loop system satisfies

(6.2.8) \[ \begin{align*}
\dot{\hat{z}}(t) &= S_{-\infty}(t) \left( \hat{z}(0) \right) \\
\hat{z}(t) &= S_{-\infty}(t) \left( \hat{z}(0) \right)
\end{align*} \quad t \geq 0, \]

where \( S_{-\infty}(t) \) is the semigroup generated on \( E \times E \) by the operator

(6.2.9) \[ A = \begin{bmatrix}
A & -BF \\
\hat{F}C & [A-BF-F\hat{C}]
\end{bmatrix}, \quad D(A_{-\infty}) = D(A) \times D(A). \]

With the estimator error \( e(t) \) defined by (6.2.4), it is easy to show that (6.2.8) is equivalent to (6.2.5) and

(6.2.10) \[ \dot{\hat{z}}(t) = (A-BF) z(t) + BF e(t), \quad t > 0, \]

where \( (A-BF) \) generates a semigroup \( S(t) \) on \( E \). Also, it is easy to prove the following.

**Theorem 6.5.** Suppose that there exist positive constants \( M_1, M_2, a_1, a_2 \) such that

(6.2.11) \[ |S(t)| \leq M_1 e^{-a_1 t}, \quad |S(t)| \leq M_2 e^{-a_2 t}, \quad t > 0. \]

Then, for each real \( a_3 < \min \{a_1, a_2\} \), there exists a constant \( M_3 \) such that

(6.2.12) \[ |S_{-\infty}(t)| \leq M_3 e^{-a_3 t}, \quad t \geq 0. \]
Also,

\[ (6.2.13) \quad \sigma(A) = \sigma(A-BF) U \sigma(A-FC) , \]

where \( \sigma(A) \) is the spectrum of \( A \).

The observer in (6.2.3) and the control law in (6.2.7) constitute a compensator for the control system in (6.2.1) and (6.2.2). The transfer function of this compensator is

\[ (6.2.14) \quad \Phi(s) = -F(sI - [A-BF+F(C_0F-C)])^{-1} F, \]

which is an \( m \times p \) matrix function of the complex variable \( s \). When \( E \) has infinite dimension, the compensator transfer function is irrational, except in degenerate, usually unimportant cases. Figure 6.1 shows the block diagram of the closed-loop system produced by applying this compensator to the plant in (6.2.1) and (6.2.2).
\[
\dot{\hat{z}} = A\hat{z} + B\hat{u}
\]
\[
y = C_0\hat{u} + C\hat{z}
\]
\[
y = (C_0 + C[sI - A]^{-1}B)\hat{u}
\]

Infinite Dimensional Plant

\[
\dot{\hat{z}} = [A-BF+\hat{F}(C_0F-C)]\hat{z} + \hat{F}y
\]
\[
u = -F\hat{z}
\]
\[
u = -F(sI - [A-BF+\hat{F}(C_0F-C)])^{-1}\hat{F}y
\]

Infinite Dimensional Compensator

Figure 6.1 Closed-loop System with Infinite Dimensional Plant and Infinite Dimensional Compensator

The foregoing definitions of this section and Theorem 6.5 are straightforward generalizations to infinite dimensions of observer-controller results in finite dimensions. Balas [Ba2, Ba3] and Schumacher [Sc2] have used similar extensions.

6.3 The Optimal Infinite Dimensional Estimator, Compensator and Closed-Loop System

Now suppose that \( \hat{F} \) is chosen as

\[
(6.3.1) \quad \hat{F} = \hat{\Pi} C^* \hat{R}^{-1}
\]

where \( \hat{\Pi} \in L(E) \) is the minimal nonnegative selfadjoint solution to the Riccati
equation

\[(6.3.2) \quad A \hat{\Pi} + \hat{\Pi} A^* - \hat{\Pi} C^* \hat{R}^{-1} \hat{\Pi} + \hat{Q} = 0,\]

with \(\hat{Q} \in \mathcal{L}(E)\) nonnegative and selfadjoint and \(\hat{R}\) a positive definite symmetric \(p \times p\) matrix. Theorem 6.3 (with \(A, B, Q, R, \Pi\) and \(S(t)\) replaced by \(A^*, C^*, \hat{Q}, \hat{R}, \hat{\Pi}\) and \(\hat{S}^*(t)\)) gives sufficient conditions for \(\hat{\Pi}\) to exist and for the semigroup \(\hat{S}^*(t)\) -- and equivalently its adjoint, the \(\hat{S}(t)\) generated by \(A - \hat{\Pi}C^*\hat{R}^{-1}C\) -- to be uniformly exponentially stable.

**Definition 6.6.** When the control gain operator is

\[(6.3.3) \quad F = R^{-1}B^* \Pi,\]

with \(\Pi\) the solution to the Riccati equation (6.1.3), and the estimator gain operator is given by (6.3.1) and (6.3.2), we call the compensator and the closed-loop system in Figure 6.1 the optimal infinite dimensional compensator and optimal closed-loop system, respectively.

The infinite dimensional estimator defined by (6.2.3), (6.3.1) and (6.3.2) is the optimal estimator for the stochastic version of (6.2.1) and (6.2.2) when (6.2.1) is disturbed by a stationary gaussian white noise process with zero mean and covariance operator \(\hat{Q}\) and the measurement in (6.2.2) is contaminated by similar noise with covariance \(\hat{R}\). For infinite dimensional stochastic estimation and control, see [Bal, CP2]. When the state weighting operator \(Q\) in (6.1.1) is trace class, the optimal infinite dimensional compensator minimizes the time-average of the expected steady-state value of the integrand in (6.1.1).

Existing theory for stochastic control of infinite dimensional systems requires trace-class \(\hat{Q}\), but we have a well defined compensator for any bounded non-negative selfadjoint \(\hat{Q}\) and \(Q\), as long as the solutions to the Riccati equations exist. As the next chapter shows (without assuming trace-class \(\hat{Q}\)), the infi-
finite dimensional compensator is the limit of a sequence of finite dimensional compensators, each of which can be interpreted as an optimal LQG compensator for a finite dimensional model of the structure. Therefore, we do not require trace-class \( \hat{Q} \) in our definition of the optimal compensator, even though this compensator solves a precise optimization problem only when \( \hat{Q} \) is trace class.

This text is concerned primarily with how the finite dimensional compensators in Chapter 7 converge to the infinite dimensional compensator in this chapter, and the analysis of this convergence requires only the theory of infinite dimensional Riccati equations for deterministic optimal control problems and the corresponding approximation theory. While the stochastic interpretation of the infinite dimensional compensator and, in Section 4 of Chapter 7, of the finite dimensional estimators should be motivational, nothing in the rest of the paper depends on a stochastic formulation. We assume that the operators \( Q, R, \hat{Q} \) and \( \hat{R} \) are determined by some design criteria, either stochastic or deterministic. In many engineering applications, deterministic criteria such as the stability margin and robustness of the closed-loop system [Bll, TSl], rather than a stochastic performance index and an assumed noise model, govern the choice of \( Q, R, \hat{Q} \) and \( \hat{R} \).

6.4 Application to Optimal Control of Flexible Structures

For the rest of this chapter, \( A_0, A_1, A, T(t), B_0, B, C_1, C_2 \) and \( C \) are the operators defined in Chapter 3, and \( E = V \times H \) is the energy space defined there.

The next theorem concerns a typical problem in control of aerospace structures: All elastic components have some structural damping; no damping is associated with the rigid-body modes but all rigid-body modes are controllable.
(This is the case with the structure in Section 9 of Chapter 3, for example.)

For all rigid-body modes to be controllable, an actuator is required for each rigid-body mode.

**Theorem 6.7.** i) Suppose that $A_1 = B_0 B_0^*$ and that $\tilde{A}_0 = A_0 + A_1$ and $\tilde{d}_0 = d_0 + A_1$ are $H$-coercive, so that there exist positive constants $\rho$, $\gamma$ and $\beta$ such that, for all $v \in V$,

\begin{align*}
(6.4.1) \quad |v|_V^2 &\geq \rho |v|_H^2 \\
(6.4.2) \quad \tilde{d}_0(v,v) &\geq \gamma |v|_H^2 \\
(6.4.3) \quad \tilde{d}_0(v,v) &\leq \beta |v|_V^2
\end{align*}

and

\begin{align*}
(6.4.4) \quad \max\{|B_0|,|Q|,|R|\} &\leq \beta.
\end{align*}

(The $V$-continuity of $d_0$ implies (6.4.3).) Then (6.1.3) has a minimal non-negative selfadjoint solution $\Pi$.

ii) Suppose also that

\begin{align*}
(6.4.5) \quad \langle Qz, z \rangle_E &\geq \rho |z|_E^2.
\end{align*}

Then the optimal closed-loop semigroup satisfies

\begin{align*}
(6.4.6) \quad |S(t)| &\leq M_2 e^{-a_2 t}, & t > 0,
\end{align*}

where $M_2$ and $a_2$ are positive constants depending on $\rho$, $\gamma$ and $\beta$ only.

**Proof.** See [GA2].
Now we will consider the structure of the optimal control law in more detail. Since $\Pi \in L(E)$ and $E = V \times H$, we can write

\begin{equation} \tag{6.4.7} \Pi = \begin{bmatrix} \Pi_0 & \Pi_1 \\ \Pi_1^* & \Pi_2 \end{bmatrix} \end{equation}

$\Pi_0 \in L(V)$, $\Pi_1 \in L(H,V)$, $\Pi_2 \in L(H)$, and $\Pi_0$ and $\Pi_2$ are nonnegative and self-adjoint. With $z = (x, \dot{x})$, as in Chapter 3, (6.1.4) becomes

\begin{equation} \tag{6.4.8} u(t) = - R^{-1}B^*_0[\Pi_1 x(t) + \Pi_2 \dot{x}(t)]. \end{equation}

Since $B_0 \in L(R^m,H)$, we must have vectors $b_i \in H$, $1 \leq i \leq m$, such that

\begin{equation} \tag{6.4.9} B_0 u = \sum_{i=1}^{m} b_i u_i \end{equation}

for

\begin{equation} \tag{6.4.10} u = [u_1, u_2, \ldots, u_m]^T \in R^m. \end{equation}

Also, for $h \in H$,

\begin{equation} \tag{6.4.11} B_0^* h = [\langle b_1, h \rangle_H, \langle b_2, h \rangle_H, \ldots, \langle b_m, h \rangle_H]^T. \end{equation}

Since $\Pi_1 x(t)$ and $\Pi_2 \dot{x}(t)$ are elements of $H$, we see from (6.4.8) and (6.4.11) that the components of the optimal control have the feedback form

\begin{equation} \tag{6.4.12} u_i(t) = - \langle f_i, x(t) \rangle_V - \langle g_i, \dot{x}(t) \rangle_H, \quad i = 1, \ldots, m, \end{equation}

where $f_i \in V$ and $g_i \in H$ are given by
\[(6.4.13a) \quad f_i = \sum_{j=1}^{m} (R^{-1})_{ij} \Pi_1 b_j,\]

\[(6.4.13b) \quad g_i = \sum_{j=1}^{m} (R^{-1})_{ij} \Pi_2 b_j, \quad i = 1, 2, \ldots, m.\]

We call \(f_i\) and \(g_i\) functional control gains.

Since the measurement operator \(C\) in (6.2.2) now has the form discussed in Section 7 of Chapter 3, the estimator gain operator \(\hat{F}\) has the form

\[(6.4.14) \quad \hat{F} y = \sum_{i=1}^{p} (\hat{f}_i, \hat{g}_i) y_i\]

for \(y = [y_1, y_2, \ldots, y_p]^T \in \mathbb{R}^p\), where the functional estimator gains \(\hat{f}_i\) and \(\hat{g}_i\) are elements of \(V\) and \(H\), respectively.

For the optimal estimator gains, we can partition \(\hat{\Pi}\) as

\[(6.4.15) \quad \hat{\Pi} = \begin{bmatrix} \hat{\Pi}_0 & \hat{\Pi}_1 \\ \hat{\Pi}_1^T & \hat{\Pi}_2 \end{bmatrix}\]

and use (6.3.1) and (3.7.4) to get

\[(6.4.16a) \quad \hat{f}_i = \sum_{j=1}^{p} (\hat{R}^{-1})_{ij} (\hat{\Pi}_0 b_{1j} + \hat{\Pi}_1 c_{2j}),\]

\[(6.4.16b) \quad \hat{g}_i = \sum_{j=1}^{p} (\hat{R}^{-1})_{ij} (\hat{\Pi}_0 c_{1j} + \hat{\Pi}_1 c_{2j}), \quad i = 1, 2, \ldots, p.\]

Now we partition \(\hat{Q}\) as

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In the optimal control problem, we almost always have a nonzero $Q_0$ because this operator penalizes the generalized displacement. For the results in this text, $\hat{Q}_0$ can be nonzero in the observer problem, and, as in the control problem, some of the strongest convergence results for finite dimensional approximations can be proved only for coercive $\hat{Q}$. However, if the observer is to be thought of as an optimal filter, then $\hat{Q}$ should be the covariance operator of the noise that disturbs (3.1.1). In this case, $\hat{Q}_0 = 0$ and $\hat{Q}_1 = 0$. 

\[(6.4.17) \quad \hat{Q} = \begin{bmatrix} \hat{Q}_0 & \hat{Q}_1 \\ \hat{Q}_1^* & \hat{Q}_2 \end{bmatrix}.\]
7. Approximation for the LQG Problem

7.1 Preliminaries

The approximating finite dimensional LQG problems to be solved numerically are defined for the \( n^{th} \) approximating control system in (4.1.2) - (4.1.8). In approximating the solutions to the infinite dimensional LQR problem and the infinite dimensional state estimation problem, we will need the following 2n x 2n grammian matrices:

\[
\mathcal{K}^n = [\langle e_i, e_j \rangle] = k^n + [\langle A_i e_i, e_j \rangle],
\]

\[
W^n = \begin{bmatrix}
\mathcal{K}^n & 0 \\
0 & M^n
\end{bmatrix}.
\]

(Recall the stiffness matrix \( k^n \) and the mass matrix \( M^n \) from (4.1.3).) The superscript \( n \) on any matrix indicates the order of approximation, not a power of the matrix; the matrix \( W^{-n} \) will be the inverse of \( W^n \).

To construct the finite dimensional compensators, we must assume the following:

**Hypothesis 7.1.** There exist two sequences of symmetric, nonnegative 2n x 2n matrices \( Q^n \) and \( \tilde{Q}^n \), \( n = 1, 2, \ldots \)

The matrices \( Q^n \) and \( \tilde{Q}^n \) will be used in the Riccati matrix equations to be solved numerically for finite dimensional control and estimator gains. Of course, we want these matrices to be related to approximations of the operators \( Q \) and \( \hat{Q} \), respectively, but we will postpone hypotheses related to operator convergence until the next chapter. For now we just will give the most common way of defining \( Q^n \) and \( \tilde{Q}^n \).
The matrix $\tilde{Q}^n$ most often is defined to be the nonnegative, symmetric $2n \times 2n$ matrix

$$\tilde{Q}^n = \begin{bmatrix} \tilde{Q}_0^n & \tilde{Q}_1^n \\ \tilde{Q}_1^n & \tilde{Q}_2^n \end{bmatrix}$$

whose $n \times n$ blocks are

$$\tilde{Q}_0^n = [\langle e_i, Q_0 e_j \rangle],$$

$$\tilde{Q}_1^n = [\langle e_i, Q_1 e_j \rangle],$$

$$\tilde{Q}_2^n = [\langle e_i, Q_2 e_j \rangle],$$

where $Q_0$, $Q_1$ and $Q_2$ are the operators in Chapter 6, Section 4.

The most common way of defining the matrix $\tilde{Q}^n$ is similar to that for $Q^n$, but there is an important difference. If the operator $\hat{Q}$ in Section 6.3 is partitioned as in (6.4.16) and a symmetric matrix $\tilde{Q}^n$ is defined as in (7.1.4) with $Q_0$, $Q_1$ and $Q_2$ replaced by $\hat{Q}_0$, $\hat{Q}_1$ and $\hat{Q}_2$, then

$$\tilde{Q}^n = W^{-n} Q^n W^{-n}.$$ 

Thus defined, the matrices $\tilde{Q}^n$ and $\hat{Q}^n$ are related to approximations of the operators $Q$ and $\hat{Q}$ in the following way. We define $Q_n = P_{En} Q | E_n$ and $\hat{Q}_n = P_{En} \hat{Q} | E_n$ where $P_{En}$ is the projection in (4.2.16), and we define $Q^n$ and $\hat{Q}^n$ to be the matrix representations of $Q_n$ and $\hat{Q}_n$, respectively. It follows, then, that

$$Q^n = W^n Q^n$$

and

$$\tilde{Q}^n = \hat{Q}^n W^{-n}.$$ 

It is interesting to consider the case $Q = \hat{Q} = I$. The state weighting operator $Q = I$ (i.e., $Q_0 = I$, $Q_1 = 0$, $Q_2 = I$) in the infinite dimensional
control problem penalizes the sum of the total energy in the structure and the squares of any rigid-body displacements. The operator \( \hat{Q} = I \) in the infinite dimensional estimator problem corresponds to no physical noise; nonetheless, our approximation theory allows \( \hat{Q} = I \), and we often have found that a coercive \( \hat{Q} \) produces desirable stability properties in the estimator.

For \( \hat{Q} = Q = I \), (7.1.6) and (7.1.7) yield \( \hat{Q}^n = W^n \) and \( \tilde{Q}^n = W^{-n} \). This may appear dubious. Indeed, (7.1.6) and (7.1.7) may appear to violate the normal duality between the LQG control and estimator problems. However, examination of the performance index in (7.2.1) for the finite dimensional control problems and the stochastic interpretation in Section 7.6 of the finite dimensional estimators should demonstrate that (7.1.6) and (7.1.7) are natural from both the infinite dimensional and the finite dimensional perspectives.

While the \( \hat{Q}^n \) and \( \tilde{Q}^n \) in (7.1.3)-(7.1.7) often are used in the finite dimensional LQG problems, the rest of this chapter assumes only Hypothesis 7.1.

7.2 Approximation of the Optimal Control Law

7.2.1 The Finite Dimensional LQR Problems

The \( n^{th} \) optimal LQR problem is: given \( \eta(0) \in \mathbb{R}^{2n} \) choose \( u \in L^2(0,T;\mathbb{R}^m) \) to minimize

\[
J_n(\eta(0),u) = \int_0^T [\eta(t)^T\hat{Q}^n\eta(t) + u(t)^TRu(t)]dt
\]

where \( \eta(t) = [\xi(t)^T,\dot{\xi}(t)^T]^T \) satisfies (4.1.4). We assume:

Hypothesis 7.2. For each \( n \geq 1 \) and \( \eta(0) \in \mathbb{R}^{2n} \), there exists an admissible control (Definition 6.1) for (4.1.4) and (7.2.1).
A sufficient condition for Hypothesis 7.2 is that, for each n, the system \((A^n, B^n)\) be stabilizable.

By Theorem 6.1 (or standard finite dimensional optimal control theory), the optimal control \(u_n(t)\) has the feedback form

\[
(7.2.2) \quad u_n(t) = -F^n \eta(t)
\]

where

\[
(7.2.3) \quad F^n = R^{-1} B^n T \tilde{P}^n
\]

and \(\tilde{P}^n\) is the minimal nonnegative, symmetric solution to the Riccati matrix equation

\[
(7.2.4) \quad A^n T \tilde{P}^n + \tilde{P}^n A^n - \tilde{P}^n B^n R^{-1} B^n T \tilde{P}^n + \tilde{Q}^n = 0.
\]

### 7.2.2 Approximating Functional Control Gains

Now we define the \(2n \times m\) matrix

\[
(7.2.5) \quad G^n = W^{-n} F^n T
\]

and the following elements of \(V_n^e:\)

\[
(7.2.6) \quad f_{in} = \sum_{j=1}^{n} G_{ji} e_j, \quad g_{in} = \sum_{j=n+1}^{2n} G_{ji} e_j,
\]

\[
i = 1, 2, \ldots, m.
\]

Then, in view of (4.1.1) and (4.1.5), the \(i\)th component of the \(n\)th control law in (7.2.2) is
\[ u_{1n}(t) = - [i^{th} \text{ column of } G^n] \nu^n(t) \]
\[ = - \sum_{j=1}^{n} \sum_{k=1}^{n} G_{ji} \langle e_j, e_k \rangle \nu \xi_k(t) \]
\[ - \sum_{j=n+1}^{2n} \sum_{k=n+1}^{2n} G_{ji} \langle e_j, e_k \rangle \nu \xi_k(t) \]
\[ = - \langle f_{1n}, x_n(t) \rangle_H - \langle g_{1n}, \dot{x}_n(t) \rangle_V, \quad i = 1, 2, \ldots, m. \]

We call \( f_{1n} \) and \( g_{1n} \) approximating functional control gains. In Chapter 8, we will see how \( f_{1n} \) and \( g_{1n} \) approximate the functional control gains \( f_i \) and \( g_i \) in (6.4.12).

### 7.3 Approximation of the Infinite Dimensional Estimator

#### 7.3.1 The Finite Dimensional Estimators

The finite dimensional state estimator that is used on-line to approximate the optimal infinite dimensional estimator in Chapter 6 is

\[ \dot{\hat{n}} = A^n \hat{n} + B^n u + \hat{F} (y - C_0u - C^n \hat{n}) \]

where \( \hat{n}(t) \in \mathbb{R}^{2n} \) and \( C^n \) is the matrix in Section 1 of Chapter 4. The \( 2n \times p \) gain matrix \( \hat{F}^n \) is

\[ \hat{F}^n = \tilde{\hat{n}}^n (C^n)^T R^{-1} \]

where \( \tilde{\hat{n}}^n \) is the minimal nonnegative, symmetric solution to

\[ A^n \tilde{\hat{n}}^n + \tilde{\hat{n}}^n (A^n)^T - \tilde{\hat{n}}^n C^n T R^{-1} C^n \tilde{\hat{n}}^n + \tilde{Q} = 0 \]

and \( \tilde{Q}^n \) is a nonnegative, symmetric \( 2n \times 2n \) matrix, as in Section 7.1. We assume
Hypothesis 7.3. For each $n \geq 1$, there exists a nonnegative, symmetric solution to (7.3.3).

A sufficient condition for Hypothesis 7.3 is that the pair $(A^n^T, C^n^T)$ be stabilizable. If $(A^n^T, Q^n)$ is detectable, then (7.3.3) has at most one non-negative symmetric solution.

7.3.2 The Approximating Functional Estimator Gains

We define

\[ (7.3.4) \quad \hat{f}_{in} = \sum_{j=1}^{n} F_{ji} e_j \quad \hat{g}_{in} = \sum_{j=n+1}^{2n} F_{ji} e_j, \quad i = 1, 2, \ldots, p, \]

where $\hat{F}^n$ is the estimator gain matrix in (7.3.2), and we call $\hat{f}_{in}$ and $\hat{g}_{in}$ approximating functional estimator gains. In Chapter 8, we will see how $\hat{f}_{in}$ and $\hat{g}_{in}$ approximate the functional estimator gains $\hat{f}_i$ and $\hat{g}_i$ in (6.4.13) and (6.4.15). We note that $\hat{f}_{in}$ and $\hat{g}_{in}$ are elements of $V_n$ while $\hat{f}_i$ and $\hat{g}_i$ are elements of $V$ and $H$, respectively.

7.4 The Finite Dimensional Compensators and the Realizable Closed-Loop Systems

The $n^{th}$ compensator consists of the control law in (7.2.2) applied to the output of the $n^{th}$ estimator in (7.3.1); i.e., the control law for the compensator is

\[ (7.4.1) \quad u_n = - F^n_{\eta} \]

where the control gain matrix $F^n$ is given by (7.2.3). The block diagram in Figure 7.1 shows the realizable closed-loop system that results from the $n^{th}$ compensator. We will refer to this system as the $n^{th}$ closed-loop system.
The transfer function of the \( n^{th} \) compensator is

\[
\Phi_n(s) = -F^n(sI - [A^n - B^nF^n + \hat{F}^n(C_0F^n - C^n)])^{-1}\hat{F}^n,
\]

which is an \( m \times p \) matrix function of the complex variable \( s \) for each \( n \), as is the similar transfer function \( \Phi(s) \) in (6.2.14) for the infinite dimensional compensator.

### 7.5 Hilbert Space Representation of the Finite Dimensional Compensators and Associated Riccati Equations

Recall from Chapter 4 the relationships among \( x_n(t), z_n(t), \xi(t) \) and \( \eta(t) \). (See (4.1.1) - (4.1.6) and (4.2.1).) From here on the matrices \( \tilde{Q}_n \) and \( \hat{Q}_n \) will be defined as in (7.1.3) - (7.1.5).

We define \( \Pi_n \) to be the operator whose matrix representation is

\[
\Pi^n = W^{-n}\tilde{\Pi}^n
\]

Since the matrix \( \tilde{\Pi}^n \) is symmetric and nonnegative, the operator \( \Pi_n \) is selfadjoint.
and nonnegative. The Riccati matrix equation (7.2.4) is equivalent to

\[ W'(A)W + I^nA^n - R^{-1}B^*R \]

(7.5.2)

Since the matrix representations of the operators \( A_n \) and \( A_n^* \) are \( A^n \) and \( W^{-1}(A^n)^T W^n \), respectively, and the matrix representation of \( B_n^* \) is \( (B^n)^T W^n \), (7.5.2) is the matrix representation of the Riccati operator equation

\[ A^n \Pi_n + \Pi_n A^n - \Pi_n B_n R^{-1} B^* \Pi_n + Q_n = 0. \]

(7.5.3)

The control law in (7.2.2) is equivalent to

\[ u_n(t) = -F_n z_n(t) \]

(7.5.4)

with the gain operator \( F_n \) given by

\[ F_n = R^{-1} B^* \Pi_n. \]

(7.5.5)

Next, we define \( \hat{\Pi}_n \) to be the nonnegative selfadjoint operator whose matrix representation is

\[ \hat{\Pi}^n = \hat{\Pi}_n W^n. \]

(7.5.6)

The Riccati matrix equation (7.3.3) is equivalent to

\[ A^n \hat{\Pi} + \hat{\Pi} W^{-1}(A^n)^T W^n - \Pi_n W^{-1}(C^n)^T R^{-1} C^n \hat{\Pi} + Q^n = 0. \]

(7.5.7)

which is the matrix representation of the Riccati operator equation

\[ A_n \hat{\Pi} + \hat{\Pi} A_n^* - \hat{\Pi} C_n R^{-1} C_n^* \hat{\Pi} + Q_n = 0. \]

(7.5.8)

(The matrix representation of the operator \( C_n^* \) is \( W^{-1}(C^n)^T \).)

The estimator gain matrix \( \hat{F}_n \) is the matrix representation of the operator

\[ \hat{F}_n = \hat{\Pi}_n C_n R^{-1}. \]

(7.5.9)
If we write  \( \hat{z}_n(t) = \sum_{i=1}^{2n} \hat{\eta}_i(t) e_i \), then the state estimator in (7.3.1) is equivalent to

\[
(7.5.10) \quad \dot{\hat{z}}_n = A_n \hat{z}_n + B_n u + \hat{F}_n (y - C_0 u - C_n \hat{z}_n).
\]

For the convergence results in Chapter 8, it is useful to have the approximating operators that are equivalent to the various matrices in Sections 7.1 - 7.4. The finite dimensional Riccati operator equations (7.5.3) and (7.5.8) approximate, respectively, the infinite dimensional Riccati operator equations (6.1.3) and (6.3.2). The control and estimator gain operators in (7.5.5) and (7.5.9) approximate, respectively, the gain operators in (6.3.3) and (6.3.1). Also, the state estimator in (7.5.10), which is just the Hilbert space representation of the estimator in (7.3.1), approximates the optimal infinite dimensional estimator in Chapter 6.

7.6 Stochastic Interpretation of the Approximating Estimators

Our approximation of the infinite dimensional estimator is based on approximation of the infinite dimensional Riccati equation, whose structure is the same for both control and estimator problems, and stochastic properties of the optimal estimator problem never enter our approximation theory. Furthermore, using only the deterministic setting, in Chapter 8 we analyze the finite dimensional estimators and the compensators based upon them. Nonetheless, we should consider momentarily the sequence of finite dimensional stochastic estimation problems whose solution is given by (7.3.1) - (7.3.3) or equivalently by (7.5.8) - (7.5.10).
First, recall how the covariance operator of a Hilbert space-valued random variable is defined. The covariance operator of an $E$-valued random variable $\omega$ is the operator $Q$ for which

\[(7.6.1) \quad \text{expected value } \langle z, \omega \rangle_E \langle \tilde{z}, \omega \rangle_E = \langle Qz, \tilde{z} \rangle_E, \quad z, \tilde{z} \in E.\]

(See [Bal, CP2].)

With $\hat{F}_n$ given by (7.5.9) and (7.5.8), (7.5.10) is the Kalman-Bucy filter for the system

\[(7.6.2) \quad \dot{z}_n = A_n z_n + B_n u + \omega_n,\]

\[(7.6.3) \quad y = C_0 u + C_n z_n + \omega_0,\]

where $\omega_n(t)$ is an $E_n$-valued white noise process with covariance operator $\hat{Q}_n$ and $\omega_0(t)$ is an $R^p$-valued white noise process with covariance operator (matrix) $\hat{R}$. Next, careful inspection will show that the filter defined by (7.3.1), (7.3.2) and (7.3.3) is the matrix representation of the filter defined by (7.5.10), (7.5.9) and (7.5.8).

With $z_n$ and $\eta$ related as in (4.1.1) and (4.1.5), (7.6.2) and (7.6.3) are equivalent to the system

\[(7.6.4) \quad \dot{\eta} = A_n \eta + B^n u + \nu,\]

\[(7.6.5) \quad y = C_0 u + C^n \eta + \omega_0,\]

where $\nu(t)$ is the $R^{2n}$-valued noise process related to $\omega_n(t)$ by
Certainly, a Kalman-Bucy filter for (7.6.4) and (7.6.5) has the form (7.3.1) with the filter gain given by (7.3.2) and (7.3.3). This particular filter is the matrix representation of the filter defined by (7.5.10), (7.5.9) and (7.5.8) if and only if the matrix \( \tilde{Q}^n \) defined by (7.1.7) is the covariance of the process \( v(t) \). Since \( \tilde{Q}^n \) is the matrix representation of \( \hat{Q}_n \), straightforward calculation using (7.1.7) and (7.6.1) shows that the \( \tilde{Q}^n \) in (7.1.7) is indeed the correct covariance matrix.

Of course, if \( \omega_n(t) \) and \( v(t) \) represent a physical disturbance to the structure, then \( \omega_n(t) \) must have the form \( (0, \omega_n^{(2)}(t)) \) and the first \( n \) elements of \( v(t) \) must be zero, but this is not necessary for our analysis.

Our finite dimensional observers can be interpreted now as a sequence of filters designed for the sequence of finite dimensional approximations to the flexible structure, with the \( n^{th} \) approximate system disturbed by the noise process \( \omega_n(t) \), whose covariance operator is \( \hat{Q}_n \). According to Hypothesis 8.5 of Chapter 8, these covariance operators converge to the operator \( \hat{Q} \) of Section 3 of Chapter 6. If we have a reliable model of a stationary, zero-mean gaussian disturbance for the structure, then we can take the covariance operator for this disturbance to be \( \hat{Q} \) and think of the infinite dimensional observer as the optimal estimator. But this interpretation is not necessary for the rest of our analysis.
8. Convergence Theory for the Approximating LQG Problems

8.1 Convergence Results for Approximation of the Generic LQG Problem

Let the Hilbert space $E$ and the linear operators $A, T(t), B, Q$ and $R$ be as in Section 1 of Chapter 6. Suppose that there exists a sequence of finite dimensional subspaces $E_n$, with the projection of $E$ onto $E_n$ denoted by $P_{E_n}$, and there exist sequences of operators $A_n \in L(E_n), B_n \in L(R^m, E_n), Q_n = Q_n^* \in L(E_n), Q_n \geq 0$.

**Hypothesis 8.1.** For each $z \in E$,

\begin{align*}
(8.1.1) \quad P_{E_n}z & \to z \quad \text{as} \quad n \to \infty, \\
(8.1.2) \quad \exp(A_n t)P_{E_n}z & \to T(t)z
\end{align*}

and

\begin{align*}
(8.1.3) \quad \exp(A_n^* t)P_{E_n}z & \to T^*(t)z \quad \text{as} \quad n \to \infty,
\end{align*}

uniformly in $t$ for $t$ in bounded intervals; for each $u \in R^m$,

\begin{align*}
(8.1.4) \quad B_n u & \to Bu;
\end{align*}

for each $z \in E$,

\begin{align*}
(8.1.5) \quad Q_n P_{E_n}z & \to Qz.
\end{align*}

(In other words, $P_{E_n}, \exp(A_n t)P_{E_n}, \exp(A_n^* t)P_{E_n}, B_n$ and $Q_n P_{E_n}$ converge strongly.)

**Hypothesis 8.2.** For each $n$, the system $(A_n, B_n)$ is stabilizable and the system $(Q_n, A_n)$ is detectable.

Hypothesis 8.2 guarantees that the Riccati equation (8.1.6) in the following theorem has a unique nonnegative, selfadjoint solution.
Theorem 8.3. For each \( n \), let \( \Pi_n \in \mathcal{L}(E_n) \) be the nonnegative, selfadjoint solution to the Riccati operator equation

\[
A_n^* \Pi_n + \Pi_n A_n - \Pi_n B_n R_n^{-1} B_n^* \Pi_n = Q_n. \quad (8.1.6)
\]

If \( \|\Pi_n\| \) is bounded uniformly in \( n \), then the Riccati algebraic equation (6.1.3) has a nonnegative, selfadjoint solution \( \Pi \) and \( \Pi_n P_{E_n} \) converges weakly to \( \Pi \); i.e.,

\[
\langle \Pi_n P_{E_n} z, \tilde{z} \rangle_E \to \langle \Pi z, \tilde{z} \rangle_E \quad \forall z, \tilde{z} \in E. \quad (8.1.7)
\]

If additionally there exist positive constants \( M \) and \( b \), independent of \( n \), such that

\[
\|\exp([A_n - B_n R_n^{-1} B_n^* \Pi_n]t)\| \leq M e^{-bt}, \quad t \geq 0, \quad (8.1.8)
\]

then \( \Pi_n P_{E_n} \) and \( \exp([A_n - B_n R_n^{-1} B_n^* \Pi_n]t)P_{E_n} \) converge strongly to \( \Pi \) and \( S(t) \), respectively; i.e.,

\[
\Pi_n P_{E_n} z \to Pz \quad \forall z \in E \quad (8.1.9)
\]

and

\[
\exp([A_n - B_n R_n^{-1} B_n^* \Pi_n]t)P_{E_n} z \to S(t)z \quad \forall z \in E, \quad (8.1.10)
\]

uniformly in \( t \geq 0 \), where \( S(t) \) is the semigroup generated by \( A - BR^{-1}B^*P \). If there exists a positive constant \( \delta \), independent of \( n \), such that

\[
Q_n \geq \delta, \quad (8.1.11)
\]

then \( \|P_n\| \) being bounded uniformly in \( n \) guarantees the existence of positive constants \( M \) and \( b \) for which (8.1.8) holds for all \( n \).
Proof. The theorem follows from Theorem 5.3 of [Gi4] and Theorem 6.7 of [Gi3], whose proof is valid under the hypotheses here. In [Gi1, Section 4] the operators $A_n$, $Q_n$ and $I_n$ are extended to all of $E$ by defining them appropriately on $E_n$. Banks and Kunisch [BK1] have modified Theorem 5.3 of [Gi1] to obtain essentially the present theorem without using the artificial, and rather clumsy, extensions of $E_n$ in the proof.

Theorem 8.4. The strong convergence in (8.1.9) implies uniform norm convergence of the optimal feedback laws:

$$(8.1.12) \quad \|B_n^* I_n E_n - B^* I\| \to 0 \quad \text{as} \quad n \to \infty.$$  

Proof. This follows from the selfadjointness of $I_n$ and $P E_n$ and the finite dimensionality of the control space $R^m$. See equations (4.23) and (4.24) of [Gi1].

Nonconvergence results for the case where no open-loop damping is modeled are given in [Gi1, Gi2, GA2].

For each $n$, the finite dimensional operator Riccati equation (8.1.6) is an approximation to the finite dimensional Riccati equation (6.1.3), but the solution to (8.1.6) also provides the solution to the optimal control problem defined by

$$\dot{z}_n = A_n z_n + B_n u \quad (8.1.13)$$

and

$$J_n = \int_0^\infty (Q_n z_n, z_n)_E + u^T R u) dt. \quad (8.1.14)$$

The optimal control law for this problem is
where the operator $F_n \in L(E_n, R^m)$ is

(8.1.16) $F_n = R^{-1} B_n^* \Pi_n P E_n^*.$

**Hypothesis 8.5.** There exist sequences $C_n \in L(E_n, R^p)$ and $\hat{Q}_n \in L(E_n)$ with

\[ \hat{Q}_n = \hat{Q}_n^* \geq 0 \]

such that

(8.1.17) $C_n^* P E_n \rightarrow C$ strongly

and

(8.1.18) $\hat{Q}_n^* P E_n \rightarrow \hat{Q}$ strongly as $n \rightarrow \infty.$

As with $B_n$, (8.1.17) implies that $C_n P E_n$ and $C_n^*$ converge in norm to $C$ and $C^*$, respectively.

**Hypothesis 8.6.** For each $n$, the system $(A_n^*, C_n^*)$ is stabilizable. (In particular, any unstable modes of the system $(A_n, C_n)$ are observable.) Also, the system $(A_n^*, \hat{Q}_n)$ is detectable.

We define the output of the system in (8.1.13) by

(8.1.19) $y_n = C_0 u + C_n z_n.$

The $n$th state estimator is

(8.1.20) $\hat{z}_n = A_n \hat{z}_n + B_n u + \hat{F}_n (y_n - C_0 u - C_n \hat{z}_n)$

where the estimator gain is the operator

(8.1.21) $\hat{F}_n = \hat{\Pi}_n C_n^* R^{-1}$

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and \( \Pi_n \) is the nonnegative, selfadjoint solution to the Riccati operator equation

\[
(8.1.22) \quad A_n \hat{\Pi}_n + \hat{\Pi}_n A_n^* - \hat{\Pi}_n C_n^* \hat{R}^{-1} C_n \hat{\Pi}_n + \hat{Q}_n = 0.
\]

Hypothesis 8.2 implies that such a solution exists and is unique.

As \( n \to \infty \), the estimator defined by \((8.1.20) - (8.1.22)\) converges to the infinite dimensional estimator in Chapter 6 as indicated by the next two theorems, which follow immediately from Theorems 8.3 and 8.4.

**Theorem 8.7.** Theorem 8.3 holds with \( A, B, Q, \Pi, S(t), A_n, B_n, Q_n \) and \( \Pi_n \) replaced, respectively, by \( A^*, C^*, \hat{Q}, \hat{\Pi}, S^*(t), A_n^*, C_n^* \), \( \hat{Q}_n \) and \( \hat{\Pi}_n \).

**Theorem 8.8.** If \( \hat{\Pi}_n \|^2 E_n \) converges strongly to \( \hat{\Pi} \), then

\[
(8.1.23) \quad \| \hat{\Pi}_n C_n^* - \hat{\Pi} C^* \| \to 0 \quad \text{as} \ n \to \infty.
\]

8.2 Convergence of the Closed-Loop Systems

Now we will consider the sense in which the \( n^{th} \) closed-loop system in Figure 7.1 approximates the optimal closed-loop system in Chapter 6. Recall from Chapter 4 how the approximating open-loop semigroups \( T_n(\cdot) \) and their adjoints converge strongly and how the input operators \( B_n \), the measurement operators \( C_n \) and their respective adjoints converge in norm. Section 8.1 has given sufficient conditions for the approximating control and estimator gains to converge to the gains for the optimal infinite dimensional compensator; i.e., \((8.1.12)\) and \((8.1.23)\) imply

\[
(8.2.1) \quad \| F_n P_{E_n} - F \| \to 0,
\]

\[
(8.2.2) \quad \| \hat{F}_n - \hat{F} \| \to 0.
\]

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By identifying $\hat{\eta}$ with $\hat{z}_n$ as in Section 5 of Chapter 7, we can identify the closed-loop system in Figure 7.1 with a closed-loop system on the space $E \times E_n$. We denote the corresponding closed-loop semigroup and its generator by $S_{\infty}(t)$ and $A_{\infty}$, respectively. Recall $S_{\infty}(t)$ and $A_{\infty}$ from Chapter 6.

Proofs of the remaining theorems in this chapter are given in [GA2]. Since these proofs are rather technical and offer no particular insight into the compensator-design process, we do not repeat them here.

**Theorem 8.9.** For $t \geq 0$, $S_{\infty}(t)P_{EEn}$ converges strongly to $S_{\infty}(t)$, and the convergence is uniform in $t$ for $t$ in bounded intervals.

We should expect at least Theorem 8.9, but we need more. We should require, for example, that if $S(t)$ is uniformly exponentially stable, then $S_{\infty}(t)$ must be also for $n$ sufficiently large. Although numerical results for numerous examples with various kinds of damping and approximations suggest that this is usually true, we have been unable to prove it in general. We do have the result for the following important case.

**Theorem 8.10.**

i) Suppose that the basis vectors of the approximation scheme are the natural modes of undamped free vibration and that the structural damping does not couple the modes. Then $S_{\infty}(t)$ converges in norm to $S_{\infty}(t)$, uniformly in bounded $t$-intervals.

ii) If, additionally, $S_{\infty}(t)$ is uniformly exponentially stable, then $S_{\infty}(t)$ is uniformly exponentially stable for $n$ sufficiently large.
This paper emphasizes using the convergence of the approximating control and estimator gain operators $F_n$ and $\hat{F}_n$, and the convergence of the functional gains that can be used to represent these operators, to determine the finite dimensional compensator that will produce essentially optimal closed-loop performance. However, close examination of $A_{\infty} - A_n$ (see [GA2]) shows that, for $S_{\infty}(t)$ to converge to $S_n(t)$, we need the following differences to converge to zero:

$$B_n F_n P_{En} - BF = B_n(F_n P_{En} - F) + (B_n - B)F,$$

$$\hat{F}_n C_n P_{En} - \hat{F}C = (\hat{F}_n - \hat{F})C_n P_{En} + \hat{F}(C_n P_{En} - C).$$

The second term on the right hand side of each of these equations represents, respectively, control and observation spillover, which has been studied extensively by Balas [Ba2, Ba3]. Together, the control spillover and observation spillover couple the modes modeled in the compensator with the modes not modeled in the compensator. The spillover must go to zero -- as it does when $B_n$ and $C_n$ converge -- for $A_{\infty} - A_n$ to go to zero.

We should ask, then, whether there exists a correlation between the convergence of $F_n$ and $\hat{F}_n$ and the elimination of spillover. The answer is yes if no modes lie in the null space of the state weighting operator $Q$ in the performance index and if the assumed process noise, whose covariance operator is $\hat{Q}$, excites all modes, but this correlation is difficult to quantify. The two main factors that determine the convergence rates of the gains are the $Q$-to-$R$ ratio and the damping, neither of which affects the convergence of $B_n$ and $C_n$. On the other hand, when either factor (small $Q/R$ or large damping) causes the gains to converge fast, it generally also causes the magnitude of $F$ and $\hat{F}$ to be relatively small, thereby reducing the magnitude of the spillover terms in (8.2.3) and (8.2.4). Also, as $n$ increases, the increasing frequencies of the truncated
modes usually reduce the coupling effect of spillover. This is well known, although it cannot be seen from the equations here. In examples that we have worked, we have found that when \( n \) is large enough to produce convergence of the control and estimator gains, the effect of any remaining spillover is negligible. But this may not always be true, and spillover should be remembered.

### 8.3 Convergence of the Compensator Transfer Functions

The transfer function of the \( n^{th} \) compensator (shown in the bottom block of Figure 7.1) is given by (7.4.2) and the transfer function of the infinite dimensional compensator is given by (6.2.14). Each transfer function is an \( m \times p \) matrix function of the complex variable \( s \). We will denote the resolvent set of \([A - BF + \hat{F}(C_0F-C)]\) by \( \rho([A - BF + \hat{F}(C_0F-C)])\).

**Theorem 8.11.** There exists a real number \( a_1 \) such that, if \( \text{Re}(s) > a_1 \), then \( s \in \rho([A^n B^n F^n + \hat{F}_n(C_0F^n - C^n)]) \) for all \( n \), and \( \Phi_n(s) \) converges to \( \Phi(s) \), uniformly in compact subsets of such \( s \).

This result leaves much to be desired. For example, it does not guarantee that any subset of the imaginary axis will lie in \( \rho([A^n B^n F^n + \hat{F}_n(C_0F^n - C^n)]) \) for sufficiently large \( n \), even if all of the imaginary axis lies in \( \rho([A - BF + \hat{F}(C_0F-C)]) \). As with the convergence of the closed-loop systems, we can get more for certain important cases.
Remark 8.12. If the open-loop semigroup $T(\cdot)$ (whose generator is $A$) is an analytic semigroup, then there exist real numbers $a$, $\theta$ and $M$, with $\theta$ and $M$ positive, such that $\rho([A-BF + \hat{F}(C_0F-C)])$ contains the sector $\{s:|\text{arg}(s-a)| < \frac{\pi}{2} + \theta\}$, and for each $s$ in this sector,

$$(8.3.1) \quad \|(sI - [A-BF + \hat{F}(C_0F-C)])^{-1}\| \leq \frac{M}{|s-a|}.$$ 

Theorem 8.13. i) If the basis vectors of the approximation scheme are the natural modes of undamped free vibration and the structural damping does not couple the modes, then each $s$ in $\rho([A-BF + \hat{F}(C_0F-C)])$ is in $\rho([A^n-B^nF^n + \hat{F}^n(C_0F^n-C^n)])$ for $n$ sufficiently large and $\Phi_n(s)$ converges to $\Phi(s)$ as $n \to \infty$, uniformly in compact subsets of $\rho([A-BF + \hat{F}(C_0F-C)])$. ii) If, additionally, $T(\cdot)$ is an analytic semigroup, then $\Phi_n(s)$ converges to $\Phi(s)$ uniformly in the sector described in Remark 8.12.

Theorem 8.14. If $A$ has compact resolvent, then $\Phi_n(s)$ converges to $\Phi(s)$ for each $s \in \rho([A-BF + \hat{F}(C_0F-C)])$, uniformly in compact subsets.
9. Application: Compensator Design for a Compound Structure

9.1 The LQG Problem for the Distributed Model of the Structure

Here we pose an infinite dimensional LQG problem for the hub-beam-tip mass structure in Section 9 of Chapter 3 and Section 3 of Chapter 4. In the infinite dimensional optimal control problem, we take $Q = I$ in the performance index in (6.1.1). This means that the state weighting term $<Qz, z>_H$ is twice the total energy in the structure plus the square of the rigid-body rotation. Since there is one input, the control weighting $R$ is a scalar.

According to (6.4.12), the optimal control has the full-state feedback form

\begin{equation}
(9.1.1) \quad u(t) = - <f, x(t)>_V - <g, \dot{x}(t)>_H
\end{equation}

where $x(t)$ has the form (3.9.1) and

\begin{equation}
(9.1.2) \quad f = (a_f, \phi_f, \beta_f) \in V, \quad g = (a_g, \phi_g, \beta_g) \in H.
\end{equation}

In (9.1.2) $a_f$, $\beta_f$, $a_g$ and $\beta_g$ are scalars and $\phi_f$ and $\phi_g$ are functions defined over the length of the beam. Note that $\beta_f = \phi_f(\ell)$ is not used in the control law -- recall (3.9.8) and (3.9.9).

The single sensor measures the rigid-body angle $\theta$, and we assume that this measurement has zero-mean Gaussian white noise with variance $R = 10^{-4}$. Also, we model a disturbance on the right side of (3.1.1) that is a zero-mean Gaussian white noise process distributed uniformly over the beam and having concentrated components acting on the hub and tip mass. For this disturbance, the covariance operator $\hat{Q}$ in (6.3.2) is

\begin{equation}
(9.1.3) \quad \hat{Q} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} : V \times H \rightarrow V \times H.
\end{equation}
According to (6.4.14), with \( p = 1 \), the gain operator \( \hat{F} \) for the infinite dimensional estimator has the form

\[
\hat{F}y = (\hat{f}, \hat{g})y,
\]

and the functional estimator gains \( \hat{f} \) and \( \hat{g} \) have the form (9.1.2).

### 9.2 Approximation of the Optimal Control Gains

Now we solve the finite dimensional LQR problem in Chapter 7 for increasing approximation orders \( n \). For each \( n \), the solution to the finite dimensional Riccati equation (7.2.4) yields the matrix \( F^n \) in (7.2.3). We use this control gain matrix to compute the approximating functional control gains in (7.2.6) and (7.2.7) and to compute the feedback law (7.4.1) for the finite dimensional compensator in Figure 7.1. Recall that \( n = 2n_e + 1 \) where \( n_e \) is the number of elements used to approximate the beam.

For the numerical solution to the \( n^{th} \) approximating optimal control problem, we begin with the \( n \times n \) matrices \( K^n \), \( B^n \) and \( B_0^n \) in (4.3.2) and the \( n \times n \) mass matrix \( M^n \) computed with the inner product in (3.9.3) for the basis vectors in (4.3.1). With these matrices, we form the \( 2n \times 2n \) matrices \( A^n \) and \( B^n \) in (4.1.6) and the \( n \times n \) matrix \( K^n \) and the \( 2n \times 2n \) matrix \( W^n \) in (7.1.1) and (7.1.2); \( \tilde{K}^n \) is \( K^n \) with 1 added to the first element. Since the operator \( Q \) in the infinite dimensional control problem is the identity, the \( 2n \times 2n \) matrix \( \tilde{Q}^n \) in (7.1.3) and (7.1.6) is equal to \( W^n \). With these matrices, we solve (7.2.4) numerically by the standard eigenvector decomposition of the Hamiltonian matrix [KS1], often called the Potter method.

The approximating functional control gains, computed with (7.2.6) with \( m = 1 \), have the form
(9.2.1) \[ f_n = (\alpha_{fn}, \phi_{fn}, \beta_{fn}), \quad g_n = (\alpha_{gn}, \phi_{gn}, \beta_{gn}). \]

As \( n \) increases, these functional gains should converge to the functional gains in (9.1.2), which are optimal for the infinite dimensional control problem.

For the damping coefficient \( c_0 = 10^{-4} \) and the control weighting \( R = .05 \), Figures 9.2.1 and 9.2.2 show the functional gain kernels \( \phi_{fn}' \) and \( \phi_{gn} \) computed with cubic Hermite splines and \( n_e = 4, 6, 8 \) and 10 beam elements. Table 9.2.1 lists the corresponding scalar components of the gains. We have plotted \( \phi_{fn}' \) because the second derivative appears in the strain-energy inner product and because \( \phi_{fn} \) converges in \( H^2(0,a) \), so that \( \phi_{fn}' \) converges in \( L_2 \). The numerical results indicate that \( \phi_{fn}' \) and \( \phi_{gn} \) converge uniformly on \([0,a]\), even though the Hermite splines yield a \( \phi_{fn}' \) that is discontinuous at the nodes. We have seen this convergence for the functional gains corresponding to beams in all of our structure-control examples, and we suspect that it is the case generally, although the convergence theorems in [GA2] guarantee only \( L_2 \) convergence for \( \phi_{fn}' \) and \( \phi_{gn} \). Since \( \beta_{fn} \) is not used in the \( n^{th} \) control law, we omit it from the tables.

\[
\begin{array}{cccc}
\text{n_e} & \alpha_{fn} & \alpha_{gn} & \beta_{gn} \\
4 & 4.4721 & 1.2440 & -133.87 \\
6 & 4.4721 & 1.2973 & -139.69 \\
8 & 4.4721 & 1.3106 & -141.15 \\
10 & 4.4721 & 1.3141 & -141.54 \\
\end{array}
\]

Table 9.2.1. Scalar Components of Functional Control Gains

Damping coefficient \( c_0 = 10^{-4} \); Control weighting \( R = .05 \)
Number of elements \( n_e = 4, 6, 8, 10 \)
Hermite Spline Approximation
Figure 9.2.1. Functional Control Gain Component $\phi_{fn}$
Damping coefficient $c_0 = 10^{-4}$; control weighting $R = .05$
Number of elements $n_e = 4, 6, 8, 10$
Hermite Spline Approximation

Figure 9.2.2. Functional Control Gain Component $\phi_{gn}$
Damping coefficient $c_0 = 10^{-4}$; control weighting $R = .05$
Number of elements $n_e = 4, 6, 8, 10$
Hermite Spline Approximation
9.3 Approximation of the Optimal Estimator Gains

To compute the gain matrix \( \hat{F}_n \) for the \( n^{th} \) approximating estimator, we solve the Riccati equation (7.3.3), whose solution yields \( \hat{F}_n \) according to (7.3.2). The \( 1 \times 2n \) matrix \( C^n \) is given by (4.3.2), and \( 2n \times 2n \) matrix \( \tilde{Q}_n \) is

\[
(9.3.1) \quad \tilde{Q}_n = \begin{bmatrix} 0 & 0 \\ 0 & M^{-n} \end{bmatrix},
\]

according to (9.1.3) and (7.1.5). (As always, \( M^{-n} \) is the inverse of the mass matrix.)

We compute the approximating functional estimator gains according to (7.3.4). Like the functional control gains, the functional estimator gains have the form

\[
(9.3.2) \quad \hat{f}_n = (\alpha_{fn}, \phi_{fn}, \beta_{fn}), \quad \hat{g}_n = (\alpha_{fn}, \phi_{fn}, \beta_{fn}).
\]

As in the control problem, our convergence theory establishes only \( L_2 \)-convergence for \( \phi''_{fn} \) and \( \phi_{gn} \), but the numerical results show uniform convergence on \( [0, \ell] \). Since \( \phi_{fn}(0) = \phi'_{fn}(0) = 0 \), the convergence of \( \phi''_{fn} \) implies the convergence of \( \beta_{fn} = \phi_{fn}(\ell) \). Thus, \( \beta_{fn} \) is not an independent piece of information about the estimator gains while, as far as our convergence results go, \( \beta_{gn} \) is. We maintain analogy with the control problem and list only \( \beta_{gn} \) in the subsequent tables. Figures 9.3.1 and 9.3.2 show \( \phi''_{fn} \) and \( \phi_{gn} \), and Table 9.3.1 lists the scalars \( \alpha_{fn}, \alpha_{gn} \) and \( \beta_{gn} \).
<table>
<thead>
<tr>
<th>$n_e$</th>
<th>$a_{fn}$</th>
<th>$a_{gn}$</th>
<th>$\beta_{gn}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5.3195</td>
<td>14.149</td>
<td>-1495.7</td>
</tr>
<tr>
<td>6</td>
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<td>8</td>
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</tr>
<tr>
<td>10</td>
<td>5.3623</td>
<td>14.377</td>
<td>-1520.8</td>
</tr>
</tbody>
</table>

Table 9.3.1. Scalar components of Functional Estimator Gains

Damping coefficient $c_0 = 10^{-4}$;
Estimator $\hat{R} = 10^{-4}$
Number of elements $n_e = 4, 6, 8, 10$
Hermite Spline Approximation
Figure 9.3.1. Functional Estimator Gain Component \( \Phi_{fn}'' \)
Damping \( c_0 = 10^{-4} \); estimator \( \hat{R} = 10^{-4} \)
Number of elements \( n_e = 4, 6, 8, 10 \)
Hermite Spline Approximation

Figure 9.3.2. Functional Estimator Gain Component \( \Phi_{gn}'' \)
Damping \( c_0 = 10^{-4} \); estimator \( \hat{R} = 10^{-4} \)
Number of elements \( n_e = 4, 6, 8, 10 \)
Hermite Spline Approximation
9.4 The Finite Dimensional Compensators

Now we construct the finite dimensional compensator in Figure 7.1 for 4, 6, 8 and 10 beam elements. Recall that (7.4.2) gives the transfer function of the nth compensator. Figure 9.4.1 shows the frequency responses (bode plots) of these compensators. The first graph in each figure contains the gain (magnitude) plots for all four orders of approximation, while the phase plot represents only the highest-order approximation. The finite dimensional compensators converge to the infinite dimensional compensator as ne increases. This is consistent with the convergence of the functional control and estimator gains.
Figure 9.4.1. Bode Plot for Compensator Based on Hermite Splines

(number of elements $n_e = 4, 6, 8, 10$)
9.5 The Closed-Loop Eigenstructure

As (6.2.13) says, the spectrum of the optimal closed-loop system is the union of the spectrum of \((A-BF)\) and the spectrum of \((A-F\hat{C})\). If the damping operator \(D_0\) in (3.1.1) is bounded relative to \(A_0^H\) for some \(\mu < 1\), then \(A\) has compact resolvent. In this case, \((A-BF)\) and \((A-F\hat{C})\) also have compact resolvent because \(BF\) and \(\hat{C}\) are bounded, so that the closed-loop spectrum consists only of isolated eigenvalues, each with finite multiplicity. If, as in our example, \(D_0 = c_0A_0\) with \(c_0 > 0\), \(A\) does not have compact resolvent; if the eigenvalues of \(A_0\) are \(\omega_j^2\), \(\sigma(A)\) contains at most a finite number of complex eigenvalues and the sequences \(\lambda_j = (-c_0 \pm \sqrt{c_0^2\omega_j^2 - 4}) \omega_j/2\), which approach \(-\) and the continuous spectrum \([-1/c_0]\) as \(j\) increases. When \(D_0 = c_0A_0^H\) for any \(0 \leq \mu \leq 1\), the eigenvectors of \(A\) are the same as those when \(D_0 = 0\), and hence are complete in \(E\).

For the remainder of this discussion, let us assume that, as in our example, the spectrum of \(A\) (i.e., the open-loop spectrum) consists of isolated eigenvalues with finite multiplicity and possibly a finite number of limit points, and that the eigenvectors are complete in \(E\). (This is the case in our example.) Then, since \(BF\) and \(\hat{C}\) are bounded with finite rank, it follows from applying standard perturbation results [Ka2, pp. 208-214] that the asymptotic properties of both \(\sigma(A-BF)\) and \(\sigma(A-F\hat{C})\) are identical to those of \(\sigma(A)\). This means that, beyond some number of eigenvalues, the eigenvalues of \(A-BF\) are virtually identical to those of \(A\) because the optimal compensator essentially controls only a finite number of modes. It also means that, although the optimal estimator contains copies of all the modes of the structure, essentially it observes only a finite number of modes and feeds virtually no sensor data into the copies of the rest. Of course, the infinite number of inactive estimator modes should be truncated before implementation.
We have not proved as much as we would like to about how the \( n \)th closed-loop spectrum -- i.e., the spectrum of the operator \( A_{\infty} \) discussed in Section 2 of Chapter 8 -- converges to the optimal closed-loop spectrum. In \[ \text{GA2, Section 91}, \] we showed that an extension of \( A_{\infty} \) converges in norm to \( A_{\infty} \) (the optimal closed-loop generator in (6.2.9)) when the damping operator \( D_0 \) does not couple the modes of free vibration and the natural mode shapes are the basis vectors for the approximation scheme. We also showed that, when \( A \) has compact resolvent, \((sI - A_{\infty})^{-1}\) converges in norm to \((sI - A_{\infty})^{-1}\). In either of these cases, it follows from the section of Kato cited above that the eigenvalues of \( A_{\infty} \) converge to those of \( A_{\infty} \). Our numerical results indicate that the eigenvalues of the \( n \)th closed-loop system based on the Hermite spline approximation also converge to those of the optimal closed-loop system, but we have not proved this.

To compute the closed-loop eigenvalues, we approximated the flexible structure (the plant) with 15 natural modes (obtained using 30 beam elements with Hermite splines) and formed the closed-loop system with the \( n \)th compensator for various values of \( n \). The numerical results indicate that the optimal compensator ignores and does not affect any mode past the ninth (the eight flexible mode), and we are stretching matters to say that the ninth mode is controlled.

Table 9.5.1 lists the eigenvalues for the closed-loop system consisting of the 15-mode model of the structure and the compensator based on 10 elements with Hermite splines. For this compensator, \( n = 2 \times 10 + 1 = 21 \), so that the estimator contains approximations to the first 21 modes of the structure. Most of these approximations have not converged to the true natural modes, as can be seen from the imaginary parts of the eigenvalues under \( A - FC \), which converge from above to slight perturbations of the open-loop frequencies.
The last twelve of the twenty-one modes in the 10-element Hermite spline compensator are inactive as far as the input/output map of the compensator. They do not couple with the rest of the compensator or with any modes in the structure, and they could be truncated without affecting the closed-loop response. The eigenvalues of the inactive compensator modes are equal to the corresponding open-loop eigenvalues of the ten-element approximation to the structure. The magnitudes of the real parts are larger than those for the corresponding true open-loop eigenvalues because the damping is proportional to the stiffness and the frequencies of the higher modes of the approximation are greater than the true frequencies. Since the inactive compensator modes reside in the \( n^{th} \) estimator, we list the corresponding eigenvalues under \((A-\hat{F}C)\).

The closed-loop eigenvalues in Table 9.5.1 corresponding to modes 5, 6 and 7 are not equal to the corresponding eigenvalues of \((A_{21}-B_{21}F_{21})\) and \((A_{21}-\hat{F}_{21}C_{21})\). (For each of modes 5, 6 and 7, we listed one pair of closed-loop eigenvalues under \((A-BF)\) and one pair under \((A-\hat{F}C)\).) This means that the Hermite-spline compensator for ten elements is not as close to convergence as the preceding functional gain and bode plots might suggest. Still, the closed-loop response produced by the ten-element Hermite-spline compensator should be close to optimal, except possibly for the sixth mode. Table 9.5.1 shows that this closed-loop system is stable.
Table 9.5.1. Closed-loop Eigenvalues with Compensator Based on Hermite Splines for 10 Elements

<table>
<thead>
<tr>
<th>OPEN-LOOP</th>
<th>A-BF</th>
<th>A-FC</th>
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<tr>
<td>REAL</td>
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</table>
10. Other Applications Issues and Future Research Topics

The methods developed or applied in this monograph provide a mathematically sound yet practical framework for designing feedback control systems for distributed parameter structural systems. As in most research, development of the ideas presented here raises new questions as it answers current ones. In this chapter we mention some of the issues which have emerged during the course of this work. Some of these we have pursued to the extent of achieving promising but as yet incomplete results. Others we note as issues for future investigation. The ideas will fall into three areas: 1) discrete time modeling and control, 2) robustness, and 3) model reduction.

10.1 Discrete Time Modeling and Control

Recent research [GR1, GR2, GR3] has developed approximation methods and convergence results for discrete-time LQG control problems for distributed systems. Much of the discrete-time theory is analogous to that in this monograph for the continuous-time problem. The infinite dimensional discrete-time theory and approximation methods have been applied to control of flexible structures in [GR1, GR2]. Further research is needed on numerical difficulties encountered in approximate solution of infinite dimensional discrete-time Riccati operator equations. One respect in which the discrete-time problem differs from the continuous-time problem is that boundary control for flexible structures usually results in a bounded input operator for the discrete-time problem, whereas the input operators corresponding to boundary controls in continuous-time problems usually are unbounded. This is the subject of [GR2, GR3]. Further work is needed along these lines to expand the class of boundary operators and the class of unbounded measurement operators to which the discrete-time approximation theory applies.
Also, discrete-time input/output representation of infinite dimensional models of flexible structures has been investigated in [Jal, GJ]. Infinite dimensional auto-regressive-moving average (ARMA) models of both the plant and the compensator have been developed for simple structures. Preliminary approximation theory for such models has been developed, but considerably more work is needed to bring this point of view to the level of maturity and usefulness of the state-space methods for distributed systems. Such research should be particularly important for adaptive identification and control of flexible structures, as suggested by the large orders of finite dimensional ARMA models found necessary in [JG1, JG2] for adaptive identification of an experimental flexible structure at NASA Langley Research Center.

10.2 Robustness

An important feature of the approach developed in this monograph is that the order and modal composition of the model are automatically adjusted as a function of the performance objectives, disturbance environment, sensor locations and actuator locations. This is an effective way to deal with problems due to model truncation. However, there is nothing in the approach that explicitly addresses the issue of modeling errors due to poorly known parameter values (e.g., frequencies, damping constants, mode shapes, etc.). There are always some differences between a model and the physical system the model is supposed to describe. Since control system designs are based on models, one must always be concerned about the degree to which the theoretical performance is degraded due to differences between the model and the physical system. Hence, modeling errors of all types are a major concern in designing control systems.

In addressing the modeling error problem, it is important to describe the modeling errors in a way that facilitates dealing with them. One approach is
to use the structured uncertainty description of Doyle [Dol]. Structured uncertainty models represent specific modeling errors (frequency errors, damping errors, mode shape errors, etc.) in the form of auxiliary multivariable feedback loops appended to the plant model. These descriptions may be used as the starting point for different approaches to robust controller design, e.g., Doyle's \( \mu \) synthesis approach [Dol] or the Linear Quadratic Gaussian (LQG) approach which is the focus of this monograph. In the context of the LQG approach, the forms of the auxiliary feedback loops motivate changes in the performance weighting matrix in the LQR problem and the disturbance noise covariance matrix in the \( K-B \) filter problem. These changes involve adding weighting matrices which reflect the specific nature of the uncertainty [BM1, TS1]. Making the additional terms large or small permits a controlled trade-off between performance and robustness. Thus one can arrive at a suitable balance between achieving a desired performance goal and minimizing the effect of structured uncertainties in the plant model.

Some results of using this approach are described in [BM1] and they show a marked increase in robustness with only a modest loss in performance. The robustness is not achieved by rolling off the loop gain before the first uncertain modes. Several of the uncertain modes are actively controlled. Numerical results in [BM1] focus on uncertain frequencies, but the methods used can be extended to other types of parameter errors. Work to study the nature and effectiveness of these extensions is currently in progress.

Another approach for improving robustness involves using nonlinear programming to achieve robustness by sensitivity optimization, which means minimizing the sensitivity of closed-loop eigenvalues with respect to uncertainties in plant parameters. The method can also be used to optimize the shape of
structural elements to reduce structural weight. An overall control/structure optimal design can be achieved by using both of these features simultaneously. Appropriate constraints are imposed to ensure that high performance is maintained. This approach to robustness is developed in papers [Ad1, AG1, AG2]. The idea is to use nonlinear programming to reduce the sensitivity of the closed-loop eigenvalues with respect to modeling errors, while maintaining sufficiently high performance of the closed-loop control system.

The paper [AG1] derives formulas for the sensitivities of closed-loop eigenvalues with respect to uncertain plant parameters and presents a numerical example that demonstrates the effect of these sensitivities on robustness in control of a flexible structure. The analysis in [AG1] indicates that the first-order sensitivities of the closed-loop eigenvalues approach infinity as a controller and an estimator eigenvalue approach each other and suggests that robustness can be improved by separating controller and estimator eigenvalues. The numerical results for the flexible structure example in [AG1, AG2] demonstrate the improved robustness achieved by moving the estimator eigenvalues to the left of the controller eigenvalues.

The work in [Ad1, AG1, AG2] has led to a general guideline for choosing the state weighting matrix and the process noise covariance (the Q matrices) in the LQG problem to improve robustness: After the state weighting for the control problem is chosen according to performance criteria, the Q matrix for the estimator design is chosen to move the estimator eigenvalues for the controlled modes with higher frequencies sufficiently to the left of the closed-loop controller eigenvalues to reduce the closed-loop eigenvalue sensitivities to acceptable levels. Examples of Q matrices that achieve this sensitivity reduction are given in [Ad1, AG1-AG2]. In general, the estimator Q for a modal
representation of a structure is diagonal and its diagonal elements increase as the corresponding structural frequencies increase.

References [Ad1, AG2] discuss optimal eigenvalue sensitivity reduction in conjunction with optimal weight reduction by structural shape optimization. The idea is to combine minimization of closed-loop eigenvalue sensitivity with optimization of structural mass distribution, subject to constraints on eigenvalue location, to produce a robust controller, a light structure and a closed-loop system with fast response. While the measure of robustness used in the design objective is the first-order sensitivity of the closed-loop eigenvalues, the final evaluation of the robustness of the design is based on large variations in the uncertain parameters. The numerical results in [Ad1, AG1, AG2] demonstrate the effectiveness of the method for producing both a robust control system and a light structure.

Recently, a significant body of literature has been devoted to Liapunov stability methods for robustness analysis and design of control systems with structured uncertainties [HB1, KB1, Ye1, YL1, ZK1]. These papers use quadratic Liapunov functions to estimate a region in parameter space for which a control system will remain stable. Such Liapunov robustness analysis is combined in [Be1, GB1, PH1] with nonstandard Riccati matrix equations for robust controller design.

The main advantage of these Liapunov methods is that they can be applied in a straightforward and computationally efficient manner to complex, realistic systems. However, the methods generally yield conservative robustness estimates because, with the possible exception of the method in [HB1], all of the Liapunov robustness methods in current literature are zero-order in the sense that a
Recent research at UCLA [LG1] has produced a first-order Liapunov robustness analysis approach, which allows the Liapunov function to vary with the uncertain plant parameters. On most problems the robustness margins predicted by this new method are substantially larger than those predicted by the zero-order methods. If the first-order Liapunov robustness analysis continues to be as successful as it has been in early applications, an important line of research should be to develop related first-order robustness design methods motivated by the Riccati/Liapunov methods in [Bel, PH1]. Because of the theory and approximation methods presented in this text for infinite dimensional Riccati equations and the success of the applications to optimal controller design for large space structures, it seems reasonable to expect that this kind of theory and the associated approximation methods could be used to develop Riccati/Liapunov robust controller design methods for large space structures.

10.3 Model Reduction

The model reduction work described in this monograph deals with systems described by linear constant coefficient differential equations. This is a feature that is shared with most of the model reduction work reported in the literature. Equations of this type arise when one is studying small vibrations about a state of rest in inertial space and this is a situation of great interest for applications. Constant coefficient equations may also arise for some systems performing small vibrations about a state of steady motion (e.g., steady spin). In many cases of practical interest, however, the nominal motion either is periodic or consists of a single maneuver over a finite time interval.
If special inertial symmetry conditions are not satisfied, the linearized equations describing small vibrations about periodic motions generally have periodic coefficients, while a general finite time maneuver often leads to linear equations with time varying coefficients of a general form. A Dual-Spin Spacecraft where both bodies are flexible and neither body is inertially symmetric is an example of a system where small variations about steady spin lead to linear equations with periodic coefficients. During steady state operation, the equations governing transverse attitude motion of such a spacecraft assume the form:

\[(10.3.1) \quad X = A(t)X ; \quad X(0) = X_0\]

\[(10.3.2) \quad A(t+T) = A(t)\]

where \(T = 2\pi/\omega_{\text{rel}}\) and \(\omega_{\text{rel}}\) is the relative rate between the two bodies.

If one wishes to use the methods of this monograph to design a control system for such a spacecraft, it would be desirable to have a scheme similar to that described in Ch. 5 for obtaining lower order models of the system.

Of the various methods available for model reduction, balanced realization theory is perhaps the best suited for extension to the time varying case. The application of balanced realization theory for time varying systems has been considered in the literature [SS1, VK1], but work in this area has not been extensive. Examples worked out so far have been simple and nonphysical. (In particular, these examples were not from the field of structural dynamics.) The potential of this approach for time varying structural systems is largely untapped, and it would make a suitable topic for further investigation.
REFERENCES


APPENDIX A. Asymptotic Expansion for the Observability Grammian

The observability grammian satisfies the equation

\[ W_0 A + A^T W_0 + C^T C = 0 \]  

For the lightly damped mechanical systems of Section 5.5,

\[ A = \begin{bmatrix} 0 & I \\ -\Omega^2 & -\epsilon \Delta \end{bmatrix}, \quad C = [C_p \ C_v] \]

The development of an expansion for \( W_c \) in Section (5.5) is simplified somewhat because the matrix \( B B^T \) is zero except for the lower right hand quadrant. In contrast, the matrix \( C C^T \) is generally full, so the development of an expansion for \( W_o \) is more complicated. For this reason, we will develop an expression only for the first two terms in this expansion. When \( C C^T \) is full, there is no particular advantage in assuming a form like that of Eq. (5.5.8) for \( W_o \). Let

\[ A_0 = \begin{bmatrix} 0 & I \\ -\Omega^2 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\Delta \end{bmatrix} \]

\[ W_o = \frac{1}{\epsilon} U + V + \epsilon W + \ldots \]

Substituting (A.2-A.4) into (A.1) yields

\[ \frac{1}{\epsilon}[UA_0 + A_0^T U] + [VA_0 + A_0^T V + UA_1 + A_1^T U + C^T C] \]

\[ + \epsilon[WA_0 + A_0^T W + VA_1 + A_1^T V] + \ldots = 0 \]

Set the coefficients of like powers in \( \epsilon \) equal to zero to obtain

\[ UA_0 + A_0^T U = 0 \]
\[(A.7) \quad VA_0 + A_0^T V + UA_1 + A_1^T U + C^T C = 0\]

\[(A.8) \quad WA_0 + A_0^T W + VA_1 + A_1^T V = 0\]

e tc. Writing \((A.6)\) in partitioned form leads to

\[(A.9) \quad u_{12} \Omega^2 + \Omega^2 u_{12}^T = 0\]

\[(A.10) \quad u_{11} - \Omega^2 u_{22} = 0 ; \quad u_{11} - u_{22} \Omega^2 = 0\]

\[(A.11) \quad U_{12} + u_{12} = 0\]

It follows from \((A.9-A.11)\) and the assumption that the frequencies in \(\Omega^2\) are distinct that

\[(A.12) \quad u_{12} = 0\]

\[(A.13) \quad u_{11}, u_{22} \text{ are diagonal}\]

Thus,

\[(A.14) \quad U = \begin{bmatrix} \Omega^2 u_{22} & 0 \\ 0 & u_{22} \end{bmatrix}\]

To determine the elements of the diagonal matrix \(u_{22}\), write \((A.7)\) in partitioned form

\[(A.15) \quad v_{12} + v_{12}^T - \Delta u_{22} - u_{22} \Delta + C_v^T C_v = 0\]

\[(A.16) \quad -\Omega^2 v_{22} + v_{11} + C_p^T C_v = 0 ; \quad v_{11} - v_{22} \Omega^2 + C_v^T C_p = 0\]

\[(A.17) \quad -\Omega^2 v_{12}^T - v_{12} \Omega^2 + C_p^T C_p = 0\]
The diagonal elements of $v_{12}$ can be easily found from (A.17)

\[(A.18) \quad (v_{12})_{ii} = \frac{(C_p^T C_p)_{ii}}{2 \omega_i^2} \]

Using (A.18) and considering the diagonal terms in (A.15) one may derive explicit expressions for the diagonal elements of $u_{22}$.

\[(A.19) \quad (u_{22})_{ii} = \frac{(C_p^T C_p)_{ii} + \omega_i^2(C_v^T C_v)_{ii}}{2 \omega_i^2 \Delta_{ii}} \]

Equations (A.14) and (A.19) define the dominant first term in the expansion for $W_0$. To determine the second term in (A.4), we proceed to examine the off diagonal terms in (A.15)-(A.17). This leads to

\[(A.20) \quad (v_{12})_{ij} = \frac{(C_p^T C_p)_{ij} + \omega_i^2(C_v^T C_v)_{ij} - \omega_j^2(\Delta u_{22} + u_{22} \Delta)_{ij}}{\omega_j^2 - \omega_i^2}; \quad i \neq j \]

\[(A.21) \quad (v_{11})_{ij} = \frac{\omega_j^2(C_p^T C_v)_{ij} - \omega_i^2(C_p^T C_v)_{ij}}{\omega_i^2 - \omega_j^2}; \quad i \neq j \]

\[(A.22) \quad (v_{22})_{ij} = \frac{(C_p^T C_v)_{ij} - (C_v^T C_v)_{ij}}{\omega_i^2 - \omega_j^2}, \quad i \neq j \]

Equations (A.18) and (A.20) define $v_{12}$ completely, and (A.21) and (A.22) define the off diagonal terms of $v_{11}$ and $v_{22}$. To find the diagonal terms of $v_{11}$ and $v_{22}$, Eq. (A.8) must be written in partitioned form.

\[(A.23) \quad w_{12} + w_{12}^T - \Delta v_{22} - v_{22} \Delta = 0 \]

\[(A.24) \quad -\Omega^2 w_{22} + w_{11} - v_{12} \Delta = 0; \quad w_{11} - w_{22} \Omega^2 - \Delta v_{12}^T = 0 \]

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(A.25) \[-\Omega^2 w_{12}^T - w_{12} \Omega^2 = 0\]

From (A.25) we find that

(A.26) \[ (w_{12})_{ii} = 0 \]

Setting the diagonal elements of (A.23) equal to zero and using (A.26) yields

(A.27) \[ (v_{22})_{ii} = \frac{1}{2\Delta_{ii}} \sum_{k=1}^{n} \left[ \Delta_{ki} (v_{22})_{ki} + (v_{22})_{ik} \Delta_{ki} \right] \]

\[ = \frac{1}{\Delta_{ii}} \sum_{k=1}^{n} \left[ \left( c_p^T c_v \right)_{ki} - \left( c_p^T c_v \right)_{ik} \right] \left( \frac{\omega_i^2}{\omega_k^2} - \frac{\omega_i^2}{\omega_k^2} \right) \]

Then, considering the diagonal terms in (A.16),

(A.28) \[ (v_{11})_{ii} = (v_{22})_{ii} - (c_p^T c_v) \]

Thus Eqs. (A.14, A.19, A.20-A.22, A.27, A.28) represent the first two terms in the expansion for $W_0$. 

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16. Abstract

This monograph presents integrated modeling and controller design methods for flexible structures. The controllers, or compensators, developed are optimal in the linear-quadratic-Gaussian sense. The performance objectives, sensor and actuator locations and external disturbances influence both the construction of the model and the design of the finite dimensional compensator. The modeling and controller design procedures are carried out in parallel to ensure compatibility of these two aspects of the design problem. Model reduction techniques are introduced to keep both the model order and the controller order as small as possible.

A linear distributed, or infinite dimensional, model is the theoretical basis for most of the text, but finite dimensional models arising from both lumped-mass and finite element approximations also play an important role. A central purpose of the approach here is to approximate an optimal infinite dimensional controller with an implementable finite dimensional compensator. Both convergence theory and numerical approximation methods are given. Simple examples are used to illustrate the theory.