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CONSERVATION LAWS: SPECTRAL COLLOCATION APPROXIMATIONS

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DOMAIN DECOMPOSITION METHODS FOR SYSTEMS OF CONSERVATION LAWS: SPECTRAL COLLOCATION APPROXIMATIONS

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Abstract

We consider hyperbolic systems of conservation laws which are discretized in space by spectral collocation methods and advanced in time by finite difference schemes. At any time-level we introduce a domain decomposition method based on an iteration-by-subdomain procedure yielding at each step a sequence of independent subproblems (one for each subdomain) that can be solved simultaneously. The method is set for a general nonlinear problem in several space variables. The convergence analysis, however, is carried out only for a linear one-dimensional system with continuous solutions. A precise form of the error reduction factor at each iteration is derived. Although the method is applied here to the case of spectral collocation approximation only, the idea is fairly general and can be used in a different context as well. For instance, its application to space discretization by finite differences is straightforward.

Introduction

In this paper we propose an effective domain decomposition method for the numerical solution of quasi-linear hyperbolic systems of conservation laws.

Though our emphasys is on spatial discretization by spectral collocation schemes, the domain decomposition approach we advocate here can be adopted for finite difference discretizations as well. At every time-level the method applies after that the initial value problem has been advanced by either an implicit or an explicit time-stepping.

The spatial computational domain is subdivided into several adjoining, non intersecting subdomains; within each of them we look for a polynomial solution of degree N with respect to each component. If Γ_i denotes the interface between two of such subdomains, say Ω_i and Ω_{i+1}, at each gridpoint on Γ_i we enforce the conditions of continuity of the physical variables. Furthermore we require the fulfillment of the so-called compatibility equations, i.e., of those characteristic combinations of the original equations which express wave propagation across Γ_i from Ω_i toward Ω_{i+1} and viceversa.

The main reason for using the multidomain spectral method rather than the standard single-domain spectral approach stems from its capability of covering problems in complex geometry. Moreover, the spectral multidomain approach allows local refinement to resolve internal layers (or even discontinuities) maintaining however the spectral accuracy enjoyed by the classical spectral collocation method (e.g. [CHQZ], Ch.12).

We also propose here an iteration-by-subdomain algorithm which allows the decoupling of the subproblems arising from the multidomain approach making it possible to solve at each iteration as many independent subproblems as the number of subdomains. This algorithm requires that on Γ_i the values of the characteristic variables impinging Ω_i equate those outgoing from Ω_{i+1} at the previous iteration, and viceversa.

For the case of linear hyperbolic systems we prove that the above iteration-by- subdomain method is convergent. Furthermore, we find a close expression of the reduction factor per iteration and we show that it is *independent of N*, the number of gridpoints inside each subdomain. Furthermore, an algebraic interpretation of our iteration-by-subdomain algorithm is derived in terms of the Schur complement (or capacitance) matrix.

Finally, we show how the method proposed here can be adapted to the case in which an internal interface is a discontinuity surface (actually, a "k-shock" in the sense of Lax [L]). This typically occurs if a shock fitting approach is adopted ([CHQZ], sect.8.6). In such a case, the compatibility equations together with the Rankine-Hugoniot conditions allow us to compute first the flow field within the upstream subdomain then the one in the downstream subdomain.

Examples of domain decomposition methods of similar type have been proposed in the latest years. We recall for instance [K] and the references given in [CHQZ], Ch.13. We also refer to [MS] and [MSH] for applications of spectral multi-domain techniques to reacting flow whose shape and motion are generated at each time-level. In the frame of finite differences we refer to the earlier paper [CGVV] where the issue of the compatibility equations at subdomain interfaces was adressed. More recent applications of finite difference multidomain methods for compressible flow simulations can

be found in the reference [CGPW].

An outline of this paper is reported below. In section 1 we introduce the initial-boundary-value problem and the compatibility equations, while in section 2 we state the domain decomposition formulation of the Chebyshev collocation approximation to the problem. In section 3 we present an iteration-by-subdomain algorithm for an effective solution of the domain decomposition problem. Thereafter we confine to the case of one-dimensional hyperbolic systems. We write the domain decomposition problem as well as the iteration-by-subdomain algorithms. In section 5 we carry out the convergence analysis for the iterative algorithm, and in section 6 we derive the relationship with the Schur complement matrix associated with the interface unknowns. Finally, in section 7, we report some numerical results for a one-dimensional model problem.

1. The Hyperbolic system and the compatibility equations
2. Spectral collocation approximation to the system (1.1): a domain decomposition approach
3. An iteration by subdomain algorithm for the solution of the domain decomposition problem
4. A particular case: one dimensional problems
5. Convergence analysis for the spectral collocation problem
6. Capacitance matrix interpretation
7. Numerical Results.

1. The Hyperbolic System and the Compatibility Equations

We consider the following differential system

$$(1.1) \qquad \frac{\partial u}{\partial t} + \sum_{j=1}^{m} A_j(u) \frac{\partial u}{\partial x_j} = f \quad in \quad \Omega \times (0, T)$$

where $m = 1, 2$ or 3, Ω is an open bounded domain of $\mathbf{R}^m$, u and f are two vector functions: $u, f : \Omega \times (0, T) \to \mathbf{R}^p$, with $p \geq 1$, and $A_j(u)$ are $p \times p$ matrices possibly depending on u.
The system (1.1) is supplemented by initial conditions of the form:

$$(1.2) \qquad u(x, 0) = \varphi(x) \qquad \forall \, x \in \Omega$$

and by suitable boundary conditions at $(0, T) \times \partial\Omega$ that will be specified later on.

For any $\xi \in \mathbf{R}^m$ such that $|\xi| = \left(\sum_{j=1}^{m} \xi_j^2\right)^{1/2} = 1$, let us define the characteristic matrix for the ξ direction as follows:

$$(1.3) \qquad A(\xi) := \sum_{j=1}^{m} A_j(u) \, \xi_j$$

The system (1.1) is assumed to be hyperbolic in time: this means that for any such ξ, $A(\xi)$ has p real eigenvalues and moreover it is diagonalisible.

Let us denote by $\{\lambda^k, k = 1, ..., p\}$ the eigenvalues of $A(\xi)$, and by $\{v^k, k = 1, ..., p\}$ the set of the corresponding left eigenvectors, so that

$$(1.4) \qquad v^k A(\xi) = \lambda^k v^k \qquad k = 1, ..., p$$

We assume that $\lambda^k > 0$ for $k = 1, ..., q$ and $\lambda^k < 0$ for $k = q + 1, ..., p$ for some $0 \leq q \leq p$ (obviously, q depends on the particular direction ξ that we are considering). Let us take the inner product of v^k with (1.1); for each k we obtain the scalar equation

$$(1.5) \qquad v^k \cdot \left(\frac{\partial u}{\partial t} + \sum_{j=1}^{m} A_j(u) \frac{\partial u}{\partial x_j} \right) = v^k \cdot f \qquad k = 1, ..., p$$

Denote now by τ the direction orthogonal to ξ, so that $\tau \cdot \xi = 0, |\tau| = 1$. Since

$$(1.6) \qquad \frac{\partial u}{\partial x_j} = \frac{\partial u}{\partial \xi} \xi_j + \frac{\partial u}{\partial \tau} \tau_j \qquad j = 1, ..., m$$

from (1.5) and (1.4) it follows that:

$$(1.7) \qquad v^k \cdot \left(\frac{\partial u}{\partial t} + \lambda^k \frac{\partial u}{\partial \xi} \right) = v^k \cdot \left(f - \sum_{j=1}^{m} A_j(u) \frac{\partial u}{\partial \tau} \tau_j \right) \qquad k = 1, ..., p$$

Following a terminology proposed in [CGVV], we will refer to (4.7) as to the *compatibility equations* for the problem (1.1).

Remark 1.1 Let Γ denote a (m-1)-dimensional manifold of $\mathbf{R}^m$, and denote by ν the unit normal direction to Γ. Taking $\xi = \nu$, equations (1.7) restricted to Γ become:

$$(1.8) \qquad v^k \cdot \left(\frac{\partial u}{\partial t} + \lambda^k \frac{\partial u}{\partial \nu} \right) = v^k \cdot \left(f - \sum_{j=1}^{m} A_j(u) \frac{\partial u}{\partial \tau} \tau_j \right) \qquad k = 1, ..., p$$

The right hand side of (1.8) depends on the tangential derivatives of u on Γ, while the left hand side yields a combination (through the components of the eigenvector v^k) of transport equations along a direction which is normal to Γ.

Assume Γ is the boundary of Ω and ν is oriented outward Ω (see fig.1.1 a); then for $k = 1, ..., q$, (1.8) yield q transport equations according to which information are propagated from the inside to the outside of Ω. For such a reason, equations (1.8) for $k = 1, ..., q$ will be called the compatibility equations for the domain Ω.

If we now assume that Ω_1 and Ω_2 are two adjoining subdomains of Ω, and Γ is their common boundary, taking as ν the normal direction to Γ, oriented from Ω_1, to Ω_2 (see fig.1.1 b), then the first q equations of (1.8) are the compatibility equations for Ω_1 (they entail propagation of information from Ω_1 to Ω_2). Obviously, for $k = q + 1, ..., p$, (1.8) provide the compatibility equations for Ω_2. Such a distinction will be crucial in the definition of the multidomain method we are going to propose in the forthcoming sections.◊
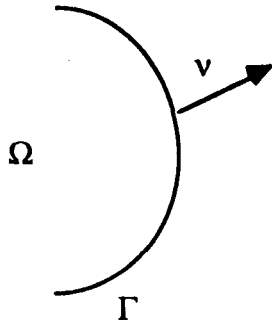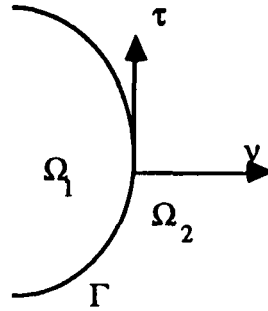

Fig. 1.1 a   Fig. 1.1 b

2. Spectral Collocation Approximation to the System (1.1): a Domain Decomposition Approach

We assume in this section that Ω is an m-dimensional hypercube. Let Ω_1 and Ω_2 be two open subregions so that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2, \overline{\Omega}_1 \cap \overline{\Omega}_2 = \Gamma, \Omega_1 \cap \Omega_2 = \phi$ where Γ is orthogonal to one cartesian direction. We assume here that the solution of (1.1) is continuous across Γ. The case of solutions that are discontinuous across Γ is faced in sect.2.1.

Let N be a given positive integer; we denote by

$$(2.1) \qquad \Sigma := \left\{ x_k = \cos\frac{\pi k}{N} \ , \ k = (k_1, ..., k_m) \ , \ 0 \leq k_i \leq N \ , \ i = 1, ..., m \right\}$$

the set of Chebyshev-Lobatto points of the reference hypercube $[-1, 1]^m$. Further we denote by Σ^1 and Σ^2 the corresponding set of points in Ω_1 and Ω_2 respectively. Note that a point of Σ^1 (resp.Σ^2) lies on the boundary of Ω_1 (resp.Ω_2) if at least one of its subindices k_j is either 0 or N. Since we are using the same number of points in Ω_1 and Ω_2, the points of $\Sigma^1 \cup \Gamma$ and those of $\Sigma^2 \cup \Gamma$ are exactly the same. We will denote by Σ_Γ this common set of interface points.

A (continuous in time) spectral domain decomposition method for problem (1.1) is defined as follows. At each time $t > 0$ we look for $u_N^i(t) \in (\mathbf{P}_N(\Omega_i))^p$ that satisfies the following set of equations.

(a) At each point of Σ^1 internal to Ω_1

$$(2.2) \qquad \frac{\partial u_N^1}{\partial t} + \sum_{j=1}^m A_j(u_N^1) \frac{\partial u_N^1}{\partial x_j} = f \;\; ;$$

(b) at each point of Σ^2 internal to Ω_2

$$(2.3) \qquad \frac{\partial u_N^2}{\partial t} + \sum_{j=1}^m A_j(u_N^2) \frac{\partial u_N^2}{\partial x_j} = f \;\; ;$$

(c) at each point of Σ_Γ

$$(2.4) \qquad u_N^1 = u_N^2$$

Moreover we require that

$$(2.5) \qquad v^k \cdot \left(\frac{\partial u_N^1}{\partial t} + \lambda^k \frac{\partial u_N^1}{\partial \nu} \right) = v^k \cdot \left(f - \sum_{j=1}^m A_j(u_N^1) \frac{\partial u_N^1}{\partial \tau} \tau_j \right) \qquad k = 1, ..., q$$

ν is the outward unit normal direction to Ω_1 on Γ (see fig.1.1 b), λ^k and v^k are the eigenvalues and the (left) eigenvectors of the matrix $A(\nu)$ (with $\lambda^k > 0$ for $k = 1, ..., q$), and τ is the tangential direction on Γ.
Furthermore we satisfy the equations:

$$(2.6) \qquad v^k \cdot \left(\frac{\partial u_N^2}{\partial t} + \lambda^k \frac{\partial u_N^2}{\partial \nu} \right) = v^k \cdot \left(f - \sum_{j=1}^m A_j(u_N^2) \frac{\partial u_N^2}{\partial \tau} \tau_j \right) \qquad k = q+1, ..., p$$

Following the definitions given in the Remark 1.1, (2.4) are the q compatibility equations for Ω_1 on Γ, while, by virtue of (2.4), (2.5) are the $p - q$ ones for Ω_2.

(d) at each point of Σ^1 belonging to a "face" Φ of $\partial\Omega_1$ (excluding the interface Γ) we set

$$(2.7) \qquad v^k \cdot \left(\frac{\partial u_N^1}{\partial t} + \lambda^k \frac{\partial u_N^1}{\partial \nu} \right) = v^k \cdot \left(f - \sum_{j=1}^{m} A_j(u_N^1) \frac{\partial u_N^1}{\partial \tau} \tau_j \right) \qquad k = 1, ..., q$$

where ν is now the outward normal direction to Ω_1 on Φ, τ is the tangential direction on Φ, while λ^k and v^k are the eigenvalues and the (left) eigenvectors of the matrix $A(\nu)$ (note that here q is not necessarily the same of (2.5)).

The remaining $p - q$ equations that are needed at each collocation point of Φ must be provided by the physical boundary conditions that supplement (1.1) and (1.2).

(e) at each point of Σ^2 belonging to a "face" Φ of $\partial \Omega_2$ (excluding the interface Γ) we enforce

$$(2.8) \qquad v^k \cdot \left(\frac{\partial u_N^2}{\partial t} + \lambda^k \frac{\partial u_N^2}{\partial \nu} \right) = v^k \cdot \left(f - \sum_{j=1}^{m} A_j(u_N^2) \frac{\partial u_N^2}{\partial \tau} \tau_j \right) \qquad k = 1, ..., q$$

where ν is the outward normal direction to Ω_2 on Φ, τ the tangential direction on Γ, λ^k and v^k are the eigenvalues and the (left) eigenvectors of the matrix $A(\nu)$. (Again, $\lambda^k > 0$ for $k = 1, ..., q$, where q here is not necessarily the same as before).

The remaining $p - q$ equations are prescribed as boundary conditions.

Remark 2.1 In the frame of classical <u>single-domain</u> spectral collocation methods for hyperbolic systems, the use of compatibility equations for collocation <u>boundary</u> points was firstly advocated in [GGT] and in [CQ]. For the same method, a stability and convergence analysis were developed in [GLT1] and [GLT2] for the case of dissipative boundary conditions. No convergence analysis (with respect to N) is available so far for spectral multidomain approximations of hyperbolic systems.◊

2.1 The case of discontinuous solutions across the subdomain interface

We consider now the case in which Γ not an arbitrary surface anymore, but rather it is precisely the front shock propagating throughout the computational domain Ω. This case typically occurs when a shock fitting technique is adopted with the purpose of determining the shape and the motion law of the shock front at each time-level (see, e.g. [CHQZ], sect.8.6). We still denote by Ω_1 and Ω_2 the adjoining subdomains separated by Γ, by ν the outward normal vector to Ω_1 on Γ and with w the speed with which Γ propagates in the direction ν.

We will assume that there exists a vector function F: $\Omega \times (0, T) \to \mathbb{R}^p$ such that $A_j(u) = \frac{\partial F_j(u)}{\partial u}$ so that (1.1) can be written in conservation form as follows:

$$(2.9) \qquad \frac{\partial u}{\partial t} + \sum_{j=1}^{m} \frac{\partial F_j(u)}{\partial u} = f \quad in \ \Omega \times (0, T)$$

The weak solutions to (2.9) (e.g., [L]) are allowed to be discontinuous across the interface Γ, but should satisfy the following jump conditions (*Rankine-Hugoniot conditions*)

$$(2.10) \qquad w[u] - \sum_{j=1}^{m} \nu_j [F_j(u)] = 0$$

Here $[\cdot]$ denotes the difference between values in the brackets on the two sides of Γ. In the context of the above spectral Chebyshev approximation, the above equations read as follows:

$$(2.11) \qquad w(u_N^2 - u_N^1) - \sum_{j=1}^{m} \nu_j \{F_j(u_N^2) - F_j(u_N^1)\} = 0$$

At each collocation point Σ_Γ on Γ (2.11) yields p equations for the $2p + 1$ unknowns: u_N^1, u_N^2 and w. We need therefore $p + 1$ further independent conditions which will be provided by the compatibility equations, which, in the current situations, can be determined as follows. Let us define the characteristic matrix for Ω_1 at the point $P \in \Gamma$:

$$(2.12) \qquad A(\nu^1) = \sum_{j=1}^{m} \nu_j A_j(u_N^1)$$

and denote by $\lambda_{(1)}^k$ and $v_{(1)}^k$ the eigenvalues and left eigenvectors, respectively, of $A(\nu^1)$. Similarly, let us denote by $\lambda_{(2)}^k$ and $v_{(2)}^k$ respectively the eigenvalues and left eigenvectors of the matrix

$$(2.13) \qquad A(\nu^2) = \sum_{j=1}^{m} (-\nu_j) A_j(u_N^2)$$

(note that this time $\lambda_{(1)}^k \neq -\lambda_{(2)}^k$ and $v_{(1)}^k \neq v_{(2)}^k$ in general, since u_N^1 and u_N^2 are not coincident on Γ anymore). Assume that the (real) eigenvalues are ordered as follows

$$\lambda_{(1)}^j < \lambda_{(1)}^{j+1} \quad , \quad \lambda_{(2)}^i < \lambda_{(2)}^{i+1} \quad , \quad i,j = 1,...,p-1$$

We assume that Γ is the surface of propagation of a k-shock, so that the following entropy conditions are verified (e.g.[S] p.261): there exists an integer $k \in \{1,...,p\}$ such that:

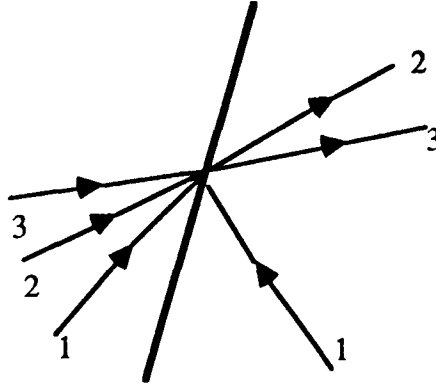$$(2.14) \qquad \lambda_{(1)}^{k-1} < w < \lambda_{(1)}^k \quad , \quad \lambda_{(2)}^k < w < \lambda_{(2)}^{k+1}$$

Fig.2.1   A k-shock with p=3 and k=1

Figure 2.1 illustrates an example of a k-shock for a one-dimensional problem. For a fixed time t we have drawn in bold the straight line with slope = w (the speed of the shock front). On its left (resp. wright) we have labelled with j the straight line whose slope is equal to the characteristic speed $\lambda_1^{(1)}$ (resp. $\lambda_2^{(j)}$), for $j = 1, ..., p$.

The compatibility equations for Ω_1 will be therefore given by the $p - k + 1$ equations

$$(2.15) \qquad v_{(1)}^r \cdot \left(\frac{\partial u_N^1}{\partial t} + \lambda_{(1)}^r \frac{\partial u_N^1}{\partial \nu^1} \right) = v_{(1)}^r \cdot \left(f - \sum_{j=1}^m A_j(u_N^1) \frac{\partial u_N^1}{\partial \tau} \tau_j \right) \qquad r = k, ..., p$$

while those for Ω_2 are given by the k equations:

$$(2.16) \qquad v_{(2)}^s \cdot \left(\frac{\partial u_N^2}{\partial t} + \lambda_{(2)}^s \frac{\partial u_N^2}{\partial \nu^2} \right) = v_{(2)}^s \cdot \left(f - \sum_{j=1}^m A_j(u_N^2) \frac{\partial u_N^2}{\partial \tau} \tau_j \right) \qquad s = p - k + 1, ..., p$$

As usual, τ is the unit vector tangential to Γ at the point P under consideration.

Notice that (2.14) can be rewritten in the form (see [L], p.25)

$$(2.17) \qquad \lambda_{(1)}^{k-1} < w < \lambda_{(2)}^{k+1} \quad , \quad \lambda_{(2)}^k < w < \lambda_{(1)}^k$$

which shows that there exists only one index k such that the shock speed w is intermediate to the characteristic speeds λ^k on both sides of the shock.

In the case of a flow regime supersonic in Ω_1 (upstream subdomain) and subsonic in Ω_2 (downstream subdomain) (see, e.g., Fig.2.1) the flow field within Ω_1 is uninfluenced by that within Ω_2, and it should therefore be computed firstly.

Remark 2.2 (Discontinuity due to unsmooth initial data)

For linear hyperbolic systems (A_j independent of u in (1.1)) with discontinuous initial data, there is propagation of discontinuity across fronts whose position and motion can be a priori determined in terms of the data. In these cases, w is given and $\lambda_{(1)}^k = -\lambda_{(2)}^{(k)}$, $v_{(1)}^k = v_{(2)}^{(k)}$ for all $k = 1, ..., p$, so that the p resulting compatibility equations together with the p Rankine-Hugoniot conditions (2.11) allow the calculation of the 2p unknowns u_N^1, u_N^2 at each collocation point on the discontinuity fronts.◊

3. An Iteration-By-Subdomain Algorithm for the Solution of the Domain Decomposition Problem

Let us go back to the case of solutions which are continuous across the subdomain interface Γ. The description of the domain decomposition method given in the previous section shows that the spectral collocation problems in Ω_1 and Ω_2 are coupled throughout the continuity conditions (2.4) at the interface Γ. To remove this coupling we propose the following iterative method. Assume the solution is available at the n-th step. Let ν, τ, λ^k and v^k be defined as at the point (c) of section 2 for each collocation point of Σ_Γ (clearly, λ^k and v^k depend on the value of the solution at the points of Γ).

Let $\Lambda(\nu)$ and $T(\nu)$ denote respectively the matrix of the eigenvalues of $A(\nu)$ and that of the corresponding (left) eigenvectors, so that:

$$(3.1) \qquad T(\nu) \, A(\nu) = \Lambda \, (\nu) \, T(\nu)$$

Then define at the point under consideration

$$(3.2) \qquad \chi^1 = (u_N^1)_{|\Gamma}^n \qquad \chi^2 = (u_N^2)_{|\Gamma}^n$$

Finally, we denote by $T_q(\nu)$ the $q \times p$ matrix given by the first q rows of $T(\nu)$, while $T_{p-q}(\nu)$ will denote the $(p - q) \times q$ matrix given by the remaining $p - q$ rows of $T(\nu)$. The solutions at the new step, say $(u_N^1)^{n+1}$ in Ω_1 and $(u_N^2)^{n+1}$ in Ω_2, can be obtained by solving two independent problems in Ω_1 and in Ω_2, respectively.

Precisely, $(u_N^1)^{n+1}$ satisfies the interior equations (2.2), the interface equations (2.5) together with

$$(3.3) \qquad T_{p-q}(\nu) \, (u_N^1)^{n+1} = T_{p-q}(\nu) \, \chi^2 \quad on \; \Gamma$$

and, finally, the boundary equations given at the point (d) of sect.2.

Similarly, $(u_N^2)^{n+1}$ satisfies the interior equations (2.3), the interface equations (2.6) together with

$$(3.4) \qquad T_q(\nu) \, (u_N^2)^{n+1} = T_q(\nu) \, \chi^1 \quad on \; \Gamma$$

and, finally, the boundary equations prescribed at the point (e) of sect.2. Note that at the limit of the convergence process, the (independent) conditions (3.3) and (3.4) will ensure the fulfillment of the continuity requirement (2.4).

Remark 3.1 The extension of the above method to decompositions with several subdomains is straightforward. At each step, one is left to solve with as many independent subproblems as the number of subdomains. All these subproblems can be solved simultaneously within a parallel computer environment.

4. A Particular Case: One Dimensional Problems

We consider the special case in which Ω is the one-dimensional interval (-1,1), and $\Omega_1 = (-1, \alpha)$, $\Omega_2 = (\alpha, 1)$ for some $0 < \alpha < 1$. In this case $\Gamma = \{\alpha\}$.
The differential system reads as follows

$$(4.1) \qquad \frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} = f \quad in \; \Omega \times (0, T)$$

Denoting with $\{\lambda^k, k = 1, ..., p\}$ and $\{v^k, k = 1, ..., p\}$ the eigenvalues and the left eigenvectors of $A(u)$, respectively, the compatibility equations (1.7) in the current case become

$$(4.2) \qquad v^k \cdot \left(\frac{\partial u}{\partial t} + \lambda^k \frac{\partial u}{\partial x}\right) = v^k \cdot f \quad k = 1, ..., p$$

Setting $\Lambda = \text{diag} \{\lambda^1, ..., \lambda^p\}$ and denoting by T the matrix whose k-th row contains v^k, we can write (4.2) as follows

$$(4.3) \qquad T \frac{\partial u}{\partial t} + \Lambda T \frac{\partial u}{\partial x} = T f$$

Obviously T and Λ depend on u if so does A.
Assume that at the interface point $x = \alpha$ the first q eigenvalues of A are positive. (Of course, A, Λ, T and q depend on the time level t). Let us denote by Λ_q and T_q the $q \times p$ matrices obtained suppressing the last $p - q$ rows of Λ and T, respectively. Then the following q equations

$$(4.4) \qquad T_q \frac{\partial u}{\partial t} + \Lambda_q T \frac{\partial u}{\partial x} = T_q f \quad at \; x = \alpha$$

provide the q compatibility equations for Ω_1. Similarly, denoting with Λ_{p-q} and T_{p-q} the lower part of the matrix Λ and T respectively, the equations

$$(4.5) \qquad T_{p-q} \frac{\partial u}{\partial t} + \Lambda_{p-q} T \frac{\partial u}{\partial x} = T_{p-q} f \quad at \ x = \alpha$$

provide the $p - q$ compatibility equations for Ω_2. The compatibility equations for Ω_2 at the right hand boundary point $x = 1$ are obtained in a similar way, and take the form (4.4), while at the left hand boundary $x = -1$ the compatibility equations for Ω_1 take the form (4.5).

The hyperbolic system (4.1) needs to be completed by the initial condition

$$(4.6) \qquad u(x, 0) = \varphi(x) \quad x \in \Omega$$

and by a set of boundary conditions (q equations at the point $x = -1$, and $p - q$ at the point $x = 1$) that we assume having the form

$$(4.7) \qquad \begin{aligned} B_1 u &= g^1 \quad at \ x = -1 \\ B_2 u &= g^2 \quad at \ x = 1 \end{aligned}$$

where B_1 is a $q \times p$ matrix, B_2 is a $(p - q) \times p$ matrix, and g^1 and g^2 are two given vector functions.

Remark 4.1 Since $\Lambda T = TA$ from (3.1), instead of (4.3) we can write

$$(4.3)' \qquad T \left[\frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} \right] = T f \quad \diamond$$

This is exactly the form that was used in [CQ] to derive the numerical boundary conditions at the physical boundary of the domain.

We also notice that if A is a constant matrix, then setting $z = Tu$ (characteristic variables), from (4.3) we obtain the following characteristic form of the compatibility equations

$$(4.3)'' \qquad \frac{\partial z}{\partial t} + \Lambda \frac{\partial z}{\partial x} = T f \qquad \diamond$$

We will now apply the multidomain spectral collocation method described in section two to the one dimensional problem (4.1). Let us define

$$x_j^1 = -1 + \frac{\alpha + 1}{2} (t_j + 1) \ , \quad x_j^2 = \alpha + \frac{1 - \alpha}{2} (t_j + 1) \ , \quad j = 0, ..., N \quad where \ t_j = \cos \frac{\pi j}{N}$$

The points t_j are the Chebyshev-Lobatto nodes in the reference interval [-1,1], while x_j^1 and x_j^2 are their images within the subdomains Ω_1 and Ω_2 respectively (note that $-1 < x_j^1 < \alpha < x_j^2 < 1$ for $1 \leq j \leq N - 1$).

At each time $t > 0$ we look for $u_N^1(t) \in (\mathbb{P}_N(\Omega_1))^p$, $u_N^2(t) \in (\mathbb{P}_N(\Omega_2))^p$ satisfying:

(a) internal equations

$$(4.8) \qquad \frac{\partial u_N^1}{\partial t} + A(u_N^1)\frac{\partial u_N^1}{\partial x} = f \quad at \ x_j^1 \ \ 1 \leq j \leq N - 1$$

$$(4.9) \qquad \frac{\partial u_N^2}{\partial t} + A(u_N^2)\frac{\partial u_N^2}{\partial x} = f \quad at \ x_j^2 \ \ 1 \leq j \leq N - 1 \ ;$$

(b) interface equations

$$(4.10) \qquad T_q\,\frac{\partial u_N^1}{\partial t} + \wedge_q T\,\frac{\partial u_N^1}{\partial x} = T_q\,f \quad at \ x = \alpha$$

$$(4.11) \qquad T_{p-q}\,\frac{\partial u_N^2}{\partial t} + \wedge_{p-q} T\,\frac{\partial u_N^2}{\partial x} = T_{p-q}\,f \quad at \ x = \alpha$$

$$(4.12) \qquad u_N^1 = u_N^2 \qquad at \ x = \alpha \ ;$$

(c) boundary equations

$$(4.13) \qquad T_q\,\frac{\partial u_N^2}{\partial t} + \wedge_q T\,\frac{\partial u_N^2}{\partial x} = T_q\,f \quad at \ x = 1$$

$$(4.14) \qquad T_{p-q}\,\frac{\partial u_N^1}{\partial t} + \wedge_{p-q} T\,\frac{\partial u_N^1}{\partial x} = T_{p-q}\,f \quad at \ x = -1$$

$$(4.15) \qquad B_2\,u_N^2 = g^2 \quad at \ x = 1$$

$$(4.16) \qquad B_1\, u_N^1 = g^1 \quad at\ x = -1$$

Note that (4.10) and (4.14) are the compatibility equations for Ω_1 at the points $x = \alpha$ and $x = -1$, respectively, while (4.11) and (4.13) are those of Ω_2 at the points $x = \alpha$ and $x = 1$, respectively. Finally, (4.15) and (4.16) are the boundary conditions (see (4.7)).

Remark 4.2 (Fully discrete approximation)
A fully discrete approximation to the problem (4.1), (4.6), (4.7) can be obtained using a time marching scheme in (4.8)-(4.16). Whatever scheme (either implicit or explicit) one adopts to advance from a known time level t^k to a new one t^{k+1}, the matching interface condition (4.12), as well as the boundary conditions (4.15) and (4.16) should be enforced at the new time level.
If an <u>explicit</u> time stepping is used, at the time level t^{k+1} the unknown vectors $\{u_N^1(x_j^1)\}$ and $\{u_N^2(x_j^2)\}, j = 1, ..., N - 1$, can be computed using solely the internal equations (4.8) and (4.9), respectively. Once these internal values are available, the interface equations (4.10)-(4.12), together with the boundary equations (4.13)-(4.16), can be solved to provide the remaining values $\{u_N^1(x_i^1)\}$ and $\{u_N^2(x_i^2)\}$ for $i = 0$ and N. Actually, we note that the presence of derivatives in space among boundary and interface equations relates boundary and interface values to each other. We also emphasize that the differential equations (4.10), (4.11), (4.13) and (4.14) ought to be advanced by the same explicit time stepping that was used for the equations at the internal points.
When an <u>implicit</u> time stepping is used, the internal unknowns are not decoupled from the remaining ones anymore. We will see an example in the next section.◊

We write now the *iteration-by-subdomain method* described in section 3 for the solution to the problem (4.8)-(4.16). Assume $(u_N^1)^n, (u_N^2)^n$ are available at the n-th step and define

$$(4.17) \qquad \chi^1 = (u_N^1)^n \quad , \quad \chi^2 = (u_N^2)^n \quad at\ x = \alpha$$

Then in Ω_1 we look for $(u_N^1)^{n+1}(t) \in (\mathbf{P}_N(\Omega_1))^p$ such that

$$(4.18) \qquad \frac{\partial}{\partial t}(u_N^1)^{n+1} + A\frac{\partial}{\partial x}(u_N^1)^{n+1} = f \quad at\ x_j^1 \ , \quad 1 \le j \le N - 1$$

$$(4.19) \qquad T_q\frac{\partial}{\partial t}(u_N^1)^{n+1} + \Lambda_q T\frac{\partial}{\partial x}(u_N^1)^{n+1} = T_q\, f \quad at\ x = \alpha$$

$$(4.20) \qquad T_{p-q}(u_N^1)^{n+1} = T_{p-q}\,\chi^2 \quad at\ x = \alpha$$

$$(4.21) \qquad T_{p-q} \frac{\partial}{\partial t} (u_N^1)^{n+1} + \Lambda_{p-q} T \frac{\partial}{\partial x} (u_N^1)^{n+1} = T_{p-q} f \quad at \ x = -1$$

$$(4.22) \qquad B_1 (u_N^1)^{n+1} = g^1 \quad at \ x = -1$$

where the matrices A, Λ and T depend on $(u_N^1)^{n+1}$.

In Ω_2 we solve for $(u_N^2)^{n+1}(t) \in (\mathbb{P}_N(\Omega_2))^p$ satisfying

$$(4.23) \qquad \frac{\partial}{\partial t} (u_N^2)^{n+1} + A \frac{\partial}{\partial x} (u_N^2)^{n+1} = f \quad at \ x_j^2 \ , \ \ 1 \leq j \leq N - 1$$

$$(4.24) \qquad T_{p-q} \frac{\partial}{\partial t} (u_N^2)^{n+1} + \Lambda_{p-q} T \frac{\partial}{\partial x} (u_N^2)^{n+1} = T_{p-q} f \quad at \ x = \alpha$$

$$(4.25) \qquad T_q (u_N^2)^{n+1} = T_q \chi^1 \quad at \ x = \alpha$$

$$(4.26) \qquad T_q \frac{\partial}{\partial t} (u_N^2)^{n+1} + \Lambda_q T \frac{\partial}{\partial x} (u_N^2)^{n+1} = T_q f \quad at \ x = 1$$

$$(4.27) \qquad B_2 (u_N^2)^{n+1} = g^2 \quad at \ x = 1$$

In (4.23)-(4.26) the matrices A, Λ and T depend now on $(u_N^2)^{n+1}$.

Note that the problem in Ω_1 is independent of that in Ω_2. The extension of the above iteration-by-subdomain method to the case of a subdivision by M subdomains ($M > 2$) is straightforward. At each iteration we obtain M independent subproblems that can be solved simultaneously.

Here above, the iteration-by-subdomain method has been applied to the semidiscrete (continuous in time) problem (4.8)-(4.16). To be effective, the iteration method should be applied at each time-step after that a time marching scheme has been used to get a full space-time discretization of the problem (see the previous remark). In this way, at the new time-level t^{k+1}, we can apply the iterative method until when we achieve convergence to the spectral multidomain solution $u_N^1(t^{k+1}), u_N^2(t^{k+1})$.

5. Convergence Analysis for the Iteration by Subdomain Method

In this section we will prove that the iteration by subdomain method introduced so far is convergent. Our analysis is confined to one-dimensional hyperbolic systems with constant matrix. The initial-boundary value problem under investigation takes the form (4.1), (4.6), (4.7) where A is a 2×2 matrix which is not restrictive to assume of the following form

$$(5.1) \qquad A = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix} \quad , \quad \text{with } |a| < 1$$

The eigenvalues of A, say λ and $-\mu$, have opposite sign, as $\lambda = a + 1 > 0$ and $-\mu = a - 1 < 0$. We notice that in this case

$$(5.2) \qquad A = T \wedge T, \text{ where } \wedge = \text{diag } \{\lambda, -\mu\} \text{ and } T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Among the various sets of boundary conditions that render this problem well posed we consider the following ones

$$(5.3) \qquad u_1(-1, t) = 0 \quad , \quad u_1(1, t) = 0 \quad \forall t \in (0, T)$$

We will require the initial value $\varphi(x)$ (see (1.2)) to be a continuous vector function, with a first component vanishing at $x = -1$ and $x = 1$. Under this assumption, initial and boundary conditions are compatible, hence the solution to our problem is continuous for all time. Assume that we are advancing in time the spectral multidomain problem (4.8)-(4.16) by an implicit method. For the sake of simplicity, let us consider the backward Euler method (no essential modification occur where considering another implicit method). When we advance from t^k to t^{k+1}, at the new level t^{k+1} the new functions $u^1 := (u_N^1)(t^{k+1})$ and $u^2 := (u_N^2)(t^{k+1})$ satisfy the following multidomain problem:

$$(5.4) \qquad \beta u^1 + A u_x^1 = f_{prev}^1 \quad at \ x_j^1, \ 1 \leq j \leq N - 1$$

$$(5.5) \qquad \beta u^2 + A u_x^2 = f_{prev}^2 \quad at \ x_j^2, \ 1 \leq j \leq N - 1$$

$$(5.6) \qquad T_q \beta u^1 + \lambda T_q u_x^1 = T_q f_{prev}^1 \quad at \ x = \alpha$$

$$(5.7) \qquad T_{p-q} \, \beta \, u^2 - \mu \, T_{p-q} \, u_x^2 = T_{p-q} \, f_{prev}^2 \quad at \; x = \alpha$$

$$(5.8) \qquad T_q \, u^2 = T_q \, u^1 \quad at \; x = \alpha$$

$$(5.9) \qquad T_{p-q} \, u^1 = T_{p-q} \, u^2 \quad x = \alpha$$

$$(5.10) \qquad T_q \, \beta \, u^2 + \lambda \, T_q \, u_x^2 = T_q \, f_{prev}^2 \quad at \; x = 1$$

$$(5.11) \qquad T_{p-q} \, \beta \, u^1 - \mu \, T_{p-q} \, u_x^1 = T_{p-q} \, f_{prev}^1 \quad at \; x = -1$$

$$(5.12) \qquad u_1^1 = 0 \quad at \; x = -1$$

$$(5.13) \qquad u_1^2 = 0 \quad at \; x = 1$$

In the previous equations β is the inverse of the time-step, $f_{prev}^1 = f^1 + \beta u_{prev}^1$ where u_{prev}^1 denotes the value of u_N^1 at the former level t^k, and f_{prev}^2 is defined similarly. Note that in the current case T_q and T_{p-q} denote the first and second row of the matrix T respectively. The two equations (5.8) and (5.9) correspond exactly to the continuity condition (4.12).

Let us apply now to the problem (5.4)-(5.13) the iteration-by subdomain method described in the previous section. At the (n+1)-step, the solutions $(u^1)^{n+1}$ and $(u^2)^{n+1}$ satisfy the equations (5.4)-(5.7) and (5.10)-(5.13); in accordance with (4.20) and (4.25) the matching interface conditions (5.8) and (5.9) are dealt with as follows

$$(5.14) \qquad T_q(u^2)^{n+1} = T_q(u^1)^n \quad at \; x = \alpha$$

$$(5.15) \qquad T_{p-q}(u^1)^{n+1} = T_{p-q}(u^2)^n \quad at \; x = \alpha$$

The main result we are going to prove in this section is that the above iterative procedure converges, i.e., $(u^1)^n \to u^1$ and $(u^2)^n \to u^2$ as $n \to \infty$. Furthermore, the convergence rate is independent of the polynomial degree N.

The proof is rather involved; it is convenient carrying out it using the characteristic form of the equations. For that, let us define the error functions (in characteristic form) at the $(n+1) - th$ iteration as follows

(5.16)
$$(e^i)^{n+1} := T[(u^i)^{n+1} - u^i] \quad for \; i = 1, 2$$

By comparing the equations (5.4)-(5.13) with the equations satisfied by the iterates $(u^i)^{n+1}, i = 1, 2$, it is readily seen that the following set of <u>error equations</u> hold:

<u>within Ω_1</u> (as usual, subindex denotes component):

(5.17.1)
$$\beta(e^1)^{n+1} + \wedge (e^1)_x^{n+1} = 0 \quad at \; x_j^1, \; 1 \le j \le N - 1$$

(5.17.2)
$$(e_1^1)^{n+1} + (e_2^1)^{n+1} = 0 \quad at \; x = -1$$

(5.17.3)
$$\beta(e_2^1)^{n+1} - \mu(e_2^1)_x^{n+1} = 0 \quad at \; x = -1$$

(5.17.4)
$$\beta(e_1^1)^{n+1} + \lambda(e_1^1)_x^{n+1} = 0 \quad at \; x = \alpha$$

(5.17.5)
$$(e_2^1)^{n+1} = (e_2^2)^n \quad at \; x = \alpha$$

<u>within Ω_2</u>:

(5.18.1)
$$\beta(e^2)^{n+1} + \wedge(e^2)_x^{n+1} = 0 \quad at \; x_j^2, \; 1 \le j \le N - 1$$

(5.18.2)
$$(e_1^2)^{n+1} + (e_2^2)^{n+1} = 0 \quad at \; x = 1$$

$$(5.18.3) \qquad \beta(e_1^2)^{n+1} + \lambda(e_1^2)_x^{n+1} = 0 \quad at \; x = 1$$

$$(5.18.4) \qquad \beta(e_2^2)^{n+1} - \mu(e_2^2)_x^{n+1} = 0 \quad at \; x = \alpha$$

$$(5.18.5) \qquad (e_1^2)^{n+1} = (e_1^1)^n \quad at \; x = \alpha$$

We start proving two important lemmas.

Lemma 5.1 *Let $\eta > 0$ and ε be two given real numbers, and consider the following "backward" collocation problem in the reference interval $[-1,1]$: find $u \in \mathbf{P}_N$ such that*

$$(5.19.1) \qquad u - \eta \, u_x = 0 \quad at \; t_j \, , \; j = 1, ..., N$$

$$(5.19.2) \qquad u = \varepsilon \quad at \; t_0 = 1$$

where, as usual, $t_j = \cos \frac{\pi j}{N}, j = 0, ..., N$ are the Chebyshev-Lobatto points in $[-1,1]$. There exists a constant $\sigma_N(\eta)$ whose absolute value is strictly less than 1 such that

$$(5.20) \qquad u(-1) = \sigma_N(\eta)\varepsilon$$

<u>Proof</u> Let $T_k(x)$ denote the Chebyshev polynomial of degree k. (We recall that $T_k(x) = \cos k\theta$, where $\theta = $ arcos x). Since for $j = 1, ..., N - 1$, t_j are the roots of $T_N'(x)$, from (5.21.1) we deduce the following identity:

$$(5.21) \qquad u - \eta \, u_x = \tau(\phi_{N-1} + \phi_N) \quad \forall x \in [-1, 1]$$

where $\phi_{N-1}(x) = T_N'(x)$, $\phi_N(x) = x T_N'(x)$, and τ is a constant that can be determined using the inflow boundary condition (5.19.2). Following [C] we can state that

$$(5.22) \qquad \text{degree } (\phi_n) = n, \text{parity of } \phi_n = \text{parity of } n \, , \; (\hat{\phi}_n)_k \geq 0$$

for $n = N - 1, N$ and $k = 0, ..., n$. Here $(\hat{\phi}_n)_k$ are the Chebyshev coefficients of ϕ_n, i.e.,

$$(5.23) \qquad (\hat{\phi}_n)_k = \frac{2}{\pi c_k} \int_{-1}^{1} \phi_n(x) T_k(x) \frac{1}{\sqrt{1-x^2}} dx \text{ with } c_0 = 2 \text{ and } c_k = 1 \text{ if } k \geq 1$$

Owing to (5.21) we can set

$$(5.24) \qquad u = \tau(\varphi_\eta + \psi_\eta)$$

where φ_η and ψ_η are the unique solutions of the following ordinary differential equations:

$$(5.25) \qquad \varphi_\eta \in \mathbb{P}_N \ : \ \varphi_\eta - \eta\, \varphi_{\eta,x} = \phi_{N-1}$$

$$(5.26) \qquad \psi_\eta \in \mathbb{P}_N \ : \ \psi_\eta - \eta\, \psi_{\eta,x} = \phi_N$$

If p is any polynomial of $\mathbb{P}_N$ we recall that (e.g., [CHQZ] Ch.2):

$$(5.27) \qquad p(x) = \sum_{k=0}^{N} \hat{p}_k\, T_k(x) \ , \ p_x(x) = \sum_{k=0}^{N-1} \hat{p}_k^{(1)}\, T_k(x) \ \text{ with } \ \hat{p}_k^{(1)} = \frac{2}{c_k} \sum_{\substack{m \geq k+1 \\ m \not\sim k}} m\, \hat{p}_m$$

where the symbol $m \not\sim k$ means that $|m - k|$ is odd. Taking into account (5.27), from (5.25) and (5.26) we obtain the following recurrence relations for the Chebyshev coefficients of φ_η and ψ_η:

$$(5.28) \qquad (\hat{\varphi}_\eta)_m = (\hat{\phi}_{N-1})_m + \eta\, \frac{2}{c_m} \sum_{\substack{k \geq m+1 \\ k \not\sim m}} k(\hat{\varphi}_\eta)_k \quad m = N, N-1, ..., 0$$

$$(5.29) \qquad (\hat{\psi}_\eta)_m = (\hat{\phi}_N)_m + \eta\, \frac{2}{c_m} \sum_{\substack{k \geq m+1 \\ k \not\sim m}} k(\hat{\psi}_\eta)_k \quad m = N, N-1, ..., 0$$

Owing to (5.22) we conclude that

$$(5.30) \qquad (\hat{\varphi}_\eta)_m \geq 0 \ , \ (\hat{\psi}_\eta)_m \geq 0 \quad for\ m = 0, ..., N \ , \ for\ all\ \eta > 0$$

Moreover, noting that $T_k(1) = 1$ for all k from (5.24) we obtain

$$(5.31) \qquad \tau = \varepsilon / \sum_{m=0}^{N} ((\hat{\varphi}_\eta)_m + (\hat{\psi}_\eta)_m) \quad \text{whence sign } \tau = \text{sign } \varepsilon$$

It follows that the solution to the collocation problem (5.19) has the following Chebyshev coefficients

$$(5.32) \qquad \hat{u}_m = ((\hat{\varphi}_\eta)_m + (\hat{\psi}_\eta)_m) \, \varepsilon / \sum_{m=0}^{N} ((\hat{\varphi}_\eta)_m + (\hat{\psi}_\eta)_m)$$

and therefore

$$(5.33) \qquad \text{sign } \hat{u}_m = \text{constant} = \text{sign } \varepsilon \quad , \quad m = 0, ..., N$$

Noticing that $u(-1) = \sum_{m=0}^{N} \hat{u}_m (-1)^m$ as $T_m(-1) = (-1)^m$, we deduce (5.20) from (5.32) by setting

$$(5.34) \qquad \sigma_N(\eta) := \sum_{m=0}^{N} (-1)^m ((\hat{\varphi}_\eta)_m + (\hat{\psi}_\eta)_m) / \sum_{k=0}^{N} ((\hat{\varphi}_\eta)_m + (\hat{\psi}_\eta)_m)$$

Finally, the property

$$(5.35) \qquad |\sigma_N(\eta)| < 1 \quad \forall \eta > 0$$

follows from (5.30). ◇

Lemma 5.2 *Let $\eta > 0$ and ε be two given real numbers, and consider the following "foreward" collocation problem in the reference interval $[-1,1]$: find $v \varepsilon \mathbf{P}_N$ such that*

$$(5.36.1) \qquad v + \eta v_x = 0 \quad at \ t_j \ , \ j = 0, ..., N-1$$

$$(5.36.2) \qquad v = \varepsilon \quad at \ t_N = -1$$

where the points t_j are defined as in Lemma 5.1.Then

(5.37)
$$v(1) = \sigma_N(\eta)\varepsilon$$

where σ_N is the same as in (5.34).
Proof We notice first of all that

(5.38)
$$v + \eta v_x = \rho(\phi_{N-1} - \phi_N) \quad \forall x \in [-1, 1] \,, \; \rho = \frac{\varepsilon + \eta v_x(-1)}{2(-1)^{N-1}N^2}$$

The constant ρ has been determined by the boundary condition (5.36.2) noting that $T_k'(1) = k^2$ and $T_k'(-1) = (-1)^{k+1}k^2$. Consider now the following "backward" problem associated with (5.36): look for $w \in \mathbb{P}_N$ s.t.

$$w - \eta \, w_x = 0 \quad at \quad t_j \,, \; j = 1, ..., N$$

$$w = \varepsilon \quad at \quad t_0 = 1$$

We want to show that

(5.39)
$$v(-x) = w(x) \quad \forall x \in [-1, 1]$$

As a matter of fact w verifies:

(5.40)
$$w - \eta \, w_x = \rho^*(\phi_{N-1} + \phi_N) \quad \forall x \in [-1, 1] \,, \; \rho^* = \frac{\varepsilon - \eta w_x(1)}{2N^2}$$

Writing (5.40) at the point $-x$ we obtain

$$w(-x) - \eta \, w_x(-x) = \rho^*(1 - x) \, T_N'(-x) \quad \forall x \in [-1, 1]$$

whence, in view of (5.39),

(5.41)
$$v(x) + \eta \, v_x(x) = [(-1)^{N-1}\rho^*] \, (1 - x)T_N'(x)$$

We have used the property that $T_k'(-x) = (-1)^{k+1}T_k'(x) \; \forall x \in [-1, 1]$ and the obvious relation: $w_x(-x) = -v_x(x)$. Since $(-1)^{N-1}\rho^* = \rho$ and $(1 - x)T_N'(x) = \phi_{N-1} - \phi_N$, we conclude from (5.41) that v satisfies (5.38), whence v is the only solution of (5.36).
We can now apply the previous lemma to get $w(-1) = \sigma_N(\eta)\varepsilon$ and therefore (5.37) follows owing to (5.39).

Furthermore, since the Chebyshev coefficients of w are given by (5.32), using again (5.39) we deduce that the Chebyshev coefficients of v are

$$
(5.42) \qquad \hat{v}_m = (-1)^m [(\hat{\varphi}_\eta)_m + (\hat{\psi}_\eta)_m] \, \varepsilon / \sum_{m=0}^{N} ((\hat{\varphi}_\eta)_m + (\hat{\psi}_\eta)_m)
$$

In view of (5.20) and (5.37), the factor defined in (5.34) will be called the *outflow/inflow ratio.*◇

We are now in the position to state the following result

Theorem 5.1 *If we define*

$$
(5.43) \qquad \lambda' = 2\lambda/\beta(1+\alpha) \,, \quad \mu' = 2\mu/\beta(1+\alpha)
$$

the solution to the problem (5.17) satisfies:

$$
(5.44) \qquad (e_1^1)^{n+1}(\alpha) = -\sigma_N(\lambda') \, \sigma_N(\mu') \, (e_2^2)^n(\alpha)
$$

<u>Proof</u> Let $t(x) = \frac{2}{\alpha+1}(x+1) - 1$ be the affine transformation from $\overline{\Omega}_1 = [-1, \alpha]$ to the reference interval $[-1,1]$. Let us define:

$$
(5.45) \qquad u \in \mathbb{P}_N : u(t(x)) = (e_2^1)^{n+1}(x) \quad \forall x \in \overline{\Omega}_1
$$

It is readily seen from (5.17.1), (5.17.3) and (5.17.5) that u is the solution to a backward collocation problem like (5.19), provided we set $\eta = \mu'$ and $\varepsilon = (e_2^2)^n(\alpha)$. By (5.20) we therefore deduce

$$
(5.46) \qquad (e_2^1)^{n+1}(-1) = u(-1) = \sigma_N(\mu') \, (e_2^2)^n(\alpha)
$$

Let us define now:

$$
(5.47) \qquad v \in \mathbb{P}_N : v(t(x)) = (e_1^1)^{n+1}(x) \quad \forall x \in \overline{\Omega}_1
$$

Owing to (5.17.1), (5.17.4) and (5.17.2), it follows that v is the solution in $[-1,1]$ of the foreward problem (5.36), provided we set $\eta = \lambda'$ and $\varepsilon = -(e_2^1)^{n+1}(-1)$. We can therefore apply the result (5.37), and owing to (5.47) and (5.46) we obtain

$$
(e_1^1)^{n+1}(\alpha) = v(1) = \sigma_N(\lambda')\varepsilon = -\sigma_N(\lambda')\sigma_N(\mu')(e_2^2)^n(\alpha)
$$

The following result is the counterpart of theorem 5.1 for the subdomain Ω_2.

Theorem 5.2 *Let us set*

(5.48) $$\lambda'' = 2\lambda/\beta(1 - \alpha) \quad and \quad \mu'' = 2\mu/\beta(1 - \alpha)$$

The solution to the problem (5.18) satisfies

(5.49) $$(e_2^2)^{n+1}(\alpha) = -\sigma_N(\lambda'')\,\sigma_N(\mu'')\,(e_1^1)^n(\alpha)$$

<u>Proof</u> Let $t(x) = \frac{2}{1-\alpha}(x - \alpha) - 1$ be the affine mapping from $\overline{\Omega}_2 = [\alpha, 1]$ into $[-1,1]$, and define:

(5.50) $$v \in \mathbb{P}_N : v(t(x)) = (e_1^2)^{n+1}(x) \quad \forall x \in \overline{\Omega}_2$$

Owing to (5.18.1), (5.18.3) and (5.18.5) we deduce that in $[-1,1]$ v satisfies a problem like (5.36) with η replaced by λ'' and ϵ by $(e_1^1)^n(\alpha)$. In view of (5.37) we have therefore

(5.51) $$(e_1^2)^{n+1}(1) = \sigma_N(\lambda'')\,(e_1^1)^n(\alpha)$$

On the other hand, from (5.18.1), (5.18.2) and (5.18.4) it follows that the function

(5.52) $$u \in \mathbb{P}_N : u(t(x)) = (e_2^2)^{n+1}(x) \quad \forall x \in \overline{\Omega}_2$$

satisfies upon $[-1,1]$ a problem like (5.19) provided η is replaced by μ'' and ϵ by $-(e_1^2)^{n+1}(1)$. Thus from (5.20) we obtain

$$(e_2^2)^{n+1}(\alpha) = -\sigma_N(\mu'')\,(e_1^2)^{n+1}(1)$$

Now (5.49) follows from (5.51). ◇

Let us define the following sequence of *interface errors*:

(5.53) $$E^n = [((e_1^1)^n(\alpha)]^2 + [(e_2^1)^n(\alpha)]^2 + [(e_1^2)^n(\alpha)]^2 + [(e_2^2)^n(\alpha)]^2 \quad for\ n \geq 1$$

From the previous theorems we deduce the following convergence result.

Theorem 5.3 *The interface error defined by (5.53) reduces at each iteration according to the law*

$$(5.54) \qquad E^{n+1} \le \sigma_N^*(\lambda, \mu; \alpha) E^n \ , \quad n \ge 2$$

where the reduction factor is defined as follows

$$(5.55) \qquad \sigma_N^*(\lambda, \mu; \alpha) := max\{\sigma_N^2(\lambda')\sigma_N^2(\mu'), \sigma_N^2(\lambda'')\sigma_N^2(\mu'')\} < 1$$

<u>Proof</u> Owing to (5.17.5), (5.18.5) and to the theorems 5.3 and 5.4 we obtain that the interface errors propagate according to the following relation

$$(5.56) \quad E^{n+1} = \{[(e_2^1)^n]^2 + [(e_2^2)^n]^2\}\sigma_N^2(\lambda')\sigma_N^2(\mu') + \{[(e_1^1)^n]^2 + [(e_1^2)^n]^2\}\sigma_N^2(\lambda'')\sigma_N^2(\mu'') \quad for\ n \ge 2$$

The inequality (5.54) follows easily. Note that $\sigma_N^*(\lambda, \mu; \alpha) < 1$ for all positive λ and μ as a consequence of (5.35). ◇

Remark 5.1 (Behaviour of the error reduction factor).
The behaviour of $\sigma_N^*(\lambda, \mu; \alpha)$ (and, by consequence, that of the interface error sequence (5.53)) is driven by the behaviour of the outflow/inflow ratio defined in (5.34).
From tables 5.1 and 5.2 below we see that for all values of N and η, $\sigma_N(\eta) > 0$. Moreover for any fixed value of η, $\sigma_N(\eta)$ is uniformly bounded from above by a constant strictly less than one as N increases (even better, the value of $\sigma_N(\eta)$ is substantially independent of N). This property is very important for it ensures that the error reduction factor does not approach one as N tends to infinity, yielding therefore a convergence rate for our iterative procedure which is (practically) independent of N. In table 5.2 we report the limit of $\sigma_N(\eta)$ as N tends to infinity, for several values of η. It is apparent that

$$(5.57) \qquad \lim_{N \nearrow \infty} \sigma_N(\eta) = e^{-2/\eta} \quad \forall \eta > 0$$

Thus, the limit is precisely the outflow/inflow ratio of the differential case, i.e., the value $v(-1)/v(1)$, where v is the solution to the ordinary differential equation $v - \eta v_x = 0$.
On the other hand, for fixed N, $\sigma_N(\eta)$ behaves like a monotonically increasing function of η (if N and/or η are not too small). In the current application, η takes the values of λ', λ'', μ' and μ'', which are all proportional to the time-step $\Delta t = \beta^{-1}$ (see (5.43) and (5.48)). Therefore, for large N we have approximately

$$(5.58) \qquad \sigma_N^*(\lambda, \mu; \alpha) \simeq exp\{-\frac{1}{\Delta t}2\alpha^*(\lambda + \mu)/\lambda\mu\} \ , \quad \alpha^* = min\,(1 + \alpha, 1 - \alpha)$$

We recall that λ and $-\mu$ are the eigenvalues of the transport matrix A (see (5.1)), while $x = \alpha$ is the abscissa of the interface between the two subdomains Ω_1 and Ω_2. Notice that α^* is the minimum measure of the subdomains.◊

N η	4	8	16	32
0.01	0.10195409	4.5329018E-2	2.5022730E-4	9.4749803E-11
0.05	7.6327433E-2	1.8829854E-2	1.9753051E-4	1.8740999E-11
0.1	5.8272078E-2	6.0076584E-3	1.7829169E-6	2.0611536E-9
0.5	2.8037383E-2	1.8321630E-2	1.8315638E-2	1.8315638E-2
1.	0.13687601	0.13533534	0.13533528	0.13533528
5.	0.67032260	0.67032005	0.67032005	0.67032005
10.	0.81873085	0.81873075	0.81873075	0.81873075
100.	0.98019867	0.98019867	0.98019867	0.98019867

Table 5.1 The value of the outflow/inflow ratio q_N (η) for several values of η and N

η	σ_∞ (η)
0.01	0.
0.05	4.24835425 E-18
0.1	2.0611536 E-9
0.5	1.8315638 E-2
1.	0.13533528
5.	0.67032005
10.	0.81873075
100.	0.98019867

Table 5.2 The value of σ_∞ $(\eta) := \lim_N \sigma_N (\eta)$ for several values of η

From the previous theorem we have that

$$(5.59) \qquad \qquad lim_{n \to \infty} E^n = 0$$

In order to conclude that our iteration-by-subdomain procedure converges it remains to prove that

the error functions $(e^i)^n(x)$, $i = 1, 2$, defined in (5.16) attain their maximum values at the interface point $x = \alpha$. This is stated by the following theorem.

Theorem 5.4 *With the notation (5.16) and (5.23) we have*

(5.60)
$$
\begin{aligned}
max_{-1 \leq x \leq \alpha}\{[(e_1^1)^n(x)]^2 + [(e_2^1)^n(x)]^2\}+ \\
max_{\alpha \leq x \leq 1}\{[(e_1^2)^n(x)]^2 + [(e_2^2)^n(x)]^2\} \leq E^n \quad \forall\, n \geq 1
\end{aligned}
$$

Proof From the proof of theorem 5.1 we easily deduce that for all $x \in [-1, \alpha]$:

(5.61) $\quad (e_2^1)^{n+1}(x) = \sum_{m=0}^{N} \hat{u}_m \, T_m(t(x)) = \dfrac{(e_2^2)^n(\alpha)}{\sum_{k=0}^{N}[(\hat{\varphi}_{\mu'})_k + (\hat{\psi}_{\mu'})_k]} \sum_{m=0}^{N}[(\hat{\varphi}_{\mu'})_m + (\hat{\psi}_{\mu'})_m] \, T_m(t(x))$

and

(5.62)
$$
(e_1^1)^{n+1}(x) = \sum_{m=0}^{N} \hat{v}_m \, T_m(t(x)) = -\sigma_N(\mu') \, \dfrac{(e_2^2)^n(\alpha)}{\sum_{k=0}^{N}[(\hat{\varphi}_{\lambda'})_k + (\hat{\psi}_{\lambda'})_k]} \sum_{m=0}^{N}[(\hat{\varphi}_{\lambda'})_m + (\hat{\psi}_{\lambda'})_m] \, T_m(t(x))
$$

Owing to (5.30) and the fact that $|T_m(\xi)| \leq 1 \ \forall \xi \in [-1, 1]$, we deduce from the previous equalities and from (5.17.5) that for all $x \in [-1, \alpha]$

(5.63) $\qquad |(e_2^1)^{n+1}(x)| \leq |(e_2^1)^{n+1}(\alpha)| \quad , \quad |(e_1^1)^{n+1}(x)| \leq |\sigma_N(\mu')||(e_2^1)^{n+1}(\alpha)|$

In a similar way, using theorem 5.2 we prove that for all $x \in [\alpha, 1]$

(5.64) $\qquad |(e_1^2)^{n+1}(x)| \leq |(e_1^2)^{n+1}(\alpha)| \quad , \quad |(e_2^2)^{n+1}(x)| \leq |\sigma_N(\lambda'')||(e_1^2)^{n+1}(\alpha)|$

The inequality (5.60) follows easily from (5.61)-(5.64).◇

We can now state our final convergence result.

Corollary 5.1 *The iteration-by-subdomain procedure applied to the multidomain Chebyshev collocation problem (5.4)-(5.13) converges as $n \to \infty$.*

Proof The result follows from (5.60) and (5.59). Actually, since T is non singular, the convergence of the sequence $(e^i)^n$ implies that of $(u^i)^n - u^i$ owing to (5.16).◇

Remark 5.2 The same kind of result holds if the spectral Legendre collocation method is used instead of the Chebyshev one.

Remark 5.3 (Convergence for decomposition with several subdomains)
For a decomposition of [-1,1] using more than two subdomains we can essentially carry out the same type of convergence proof. Aside the technical difficulties, it is not hard to see that in this case the error reduction factor at each interface behaves like (5.58), where now α^* is the measure of the smallest subdomain of the decomposition.
If for instance, the computational domain [-1,1] is partitioned into M subdomains of equal length, the error reduction factor behaves like

$$(5.65) \qquad exp\left\{-\frac{1}{\Delta t}\frac{4}{M}\frac{\lambda+\mu}{\lambda\mu}\right\}$$

hence the convergence rate slows down as far as M increases.◊

6. Capacitance matrix interpretation

We construct now the capacitance (or influence) matrix associated with the multidomain problem (5.4)-(5.13). First of all we rewrite the problem in terms of the characteristic variables

$$(6.1) \qquad z^1 = Tu^1 \ , \ z^2 = Tu^2$$

If we define $h^1 = Tf^1_{prev}$ and $h^2 = Tf^2_{prev}$, from (5.4)-(5.13) we obtain:

$$(6.2.1) \qquad \beta\, z^1_1 + \lambda\, z^1_{1,x} = h^1_1 \quad at\ x^1_j\ ,\ j = 0, ..., N-1$$

$$(6.2.2) \qquad \beta\, z^1_2 - \mu\, z^1_{2,x} = h^1_2 \quad at\ x^1_j\ ,\ j = 1, ..., N$$

$$(6.2.3) \qquad z^1_1 + z^1_2 = 0 \quad at\ x^1_N = -1$$

$$(6.2.4) \qquad z^1_2 = z^2_2 \quad at\ x = \alpha$$

$$(6.2.5) \qquad z^2_1 = z^1_1 \quad at\ x = \alpha$$

$$(6.2.6) \qquad z_1^2 + z_2^2 = 0 \quad at \ x_0^2 = 1$$

$$(6.2.7) \qquad \beta \, z_1^2 + \lambda \, z_{1,x}^2 = h_1^2 \quad at \ x_j^2 \,, \ j = 0, ..., N-1$$

$$(6.2.8) \qquad \beta \, z_2^2 - \mu \, z_{2,x}^2 = h_2^2 \quad at \ x_j^2 \,, \ j = 1, ..., N$$

Note that (6.2.4) and (6.2.5) are equivalent to (5.9) and (5.8), respectively. The capacitance (or Schur complement) system is the one that allows the determination of the values of the two characteristic variables z_1^1 and z_2^1 (and henceforth, those of z_1^2 and z_2^2 owing to the continuity relations (6.2.4) and (6.2.5)) at the interface $x = \alpha$.

Consider the polynomial solutions to the two following multidomain collocation problems:

$$(6.3) \qquad \begin{cases} \beta \, U_1^1 + \lambda \, U_{1,x}^1 = 0 \quad at \ x_j^1 \,, \ j = 0, ..., N-1 \\ \beta \, U_2^1 - \mu \, U_{2,x}^1 = 0 \quad at \ x_j^1 \,, \ j = 1, ..., N \\ U_1^1 + U_2^1 = 0 \quad at \ x_N^1 = -1 \\ U_2^1 = 1 \quad at \ x = \alpha \end{cases}$$

$$(6.4) \qquad \begin{cases} \beta \, U_1^2 + \lambda \, U_{1,x}^2 = 0 \quad at \ x_j^2 \,, \ j = 0, ..., N-1 \\ \beta \, U_2^2 - \mu \, U_{2,x}^2 = 0 \quad at \ x_j^2 \,, \ j = 1, ..., N \\ U_1^2 + U_2^2 = 0 \quad at \ x_0^2 = 1 \\ U_1^2 = 1 \quad at \ x = \alpha \end{cases}$$

Furthermore, let us consider the two polynomial solutions to the multidomain collocation problems:

$$(6.5) \qquad \begin{cases} \beta \, V_1^1 + \lambda \, V_{1,x}^1 = h_1^1 \quad at \ x_j^1 \,, \ j = 0, ..., N-1 \\ \beta \, V_2^1 - \mu \, V_{2,x}^1 = h_2^1 \quad at \ x_j^1 \,, \ j = 1, ..., N \\ V_1^1 + V_2^1 = 0 \quad at \ x_N = -1 \\ V_2^1 = 0 \, at \ x = \alpha \end{cases}$$

$$(6.6) \qquad \begin{cases} \beta \, V_1^2 + \lambda \, V_{1,x}^2 = h_1^2 \quad at \ x_j^2 \,, \ j = 0, ..., N-1 \\ \beta \, V_2^2 - \mu \, V_{2,x}^2 = h_2^2 \quad at \ x_j^2 \,, \ j = 1, ..., N \\ V_1^2 + V_2^2 = 0 \quad at \ x = \alpha \\ V_1^2 = 0 \, at \ x = \alpha \end{cases}$$

Let us denote by ξ the (unknown) value of z_1^1 $(= z_1^2)$ at $x = \alpha$, and by η the (unknown) value of z_2^1 $(= z_2^2)$ at $x = \alpha$. The following relationships hold between the solution to the problem (6.2) and those of the auxiliary problems (6.3)-(6.6):

(6.7)
$$z^1 = V^1 + \eta\, U^1 \quad in\ \Omega_1 \ ,$$

(6.8)
$$z^2 = V^2 + \xi\, U^2 \quad in\ \Omega_2$$

The values of ξ and η are therefore determined by the two following equations that arise from (6.2.4) and (6.2.5):

(6.9)
$$V_1^1(\alpha) + \eta\, U_1^1(\alpha) = V_1^2(\alpha) + \xi\, U_1^2(\alpha) \ ,$$

(6.10)
$$V_2^1(\alpha) + \eta\, U_2^1(\alpha) = V_2^2(\alpha) + \xi\, U_2^2(\alpha)$$

Using the last equations of (6.3)-(6.6) we obtain the 2×2 system:

(6.11)
$$\begin{pmatrix} 1 & -U_1^1(\alpha) \\ -U_2^2(\alpha) & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} V_1^1(\alpha) \\ V_2^2(\alpha) \end{pmatrix}$$

The matrix in (6.11) is the <u>capacitance matrix</u>(it is also called influence or Schur complement matrix) ; hereafter it will be denoted by S.

In order to give a precise form to its entries, we note that by comparison of (5.17) with (6.3) U^1 and $(e^1)^{n+1}$ satisfy the same set of equations in Ω_1. The only difference regards the value attained at $x = \alpha$ by the second component of the solution (precisely, $U_2^1(\alpha) = 1$ and $(e_2^1)^{n+1}(\alpha) = (e_2^1)^n(\alpha)$). Owing to (5.44) we can therefore conclude that

(6.12)
$$-U_1^1(\alpha) = \sigma_N(\lambda')\, \sigma_N(\mu')$$

with λ' and μ' given in (5.43).

Exploiting a similar analogy between U^2 and the solution of problem (5.18) we easily obtain from (5.49) that

(6.13)
$$-U_2^2(\alpha) = \sigma_N(\lambda")\, \sigma_N(\mu")$$

where λ'' and μ'' are defined in (5.48). Then

$$(6.14) \qquad S = \begin{pmatrix} 1 & \sigma_N(\lambda')\sigma_N(\mu') \\ \sigma_N(\lambda'')\sigma_N(\mu'') & 1 \end{pmatrix}$$

Theorem 6.1 *The capacitance matrix is positive definite. Moreover, if $\alpha = 0$ (i.e., Ω_1 and Ω_2 have the same measure), then S is symmetric and its eigenvalues are*

$$(6.15) \qquad \lambda_{1,2} = 1 \pm \sigma_N(\lambda')\sigma_N(\mu')$$

<u>Proof</u> S is positive definite since $|\sigma_N(\eta)| < 1$ for any positive η. The eigenvalues of S are

$$(6.16) \qquad \lambda_{1,2} = 1 \pm \sqrt{\sigma_N(\lambda')\,\sigma_N(\mu')\,\sigma_N(\lambda'')\,\sigma_N(\mu'')}$$

If meas Ω_1 = meas Ω_2 (in our example, this means that $\alpha = 0$), we have $\lambda' = \lambda''(= 2\lambda/\beta)$ and $\mu' = \mu''(= 2\mu/\beta)$, whence

$$\sigma_N(\lambda') = \sigma_N(\lambda'') , \ \sigma_N(\mu') = \sigma_N(\mu'')$$

Thus S is symmetric, and its eigenvalues are given by (6.15) owing to (6.16).◇

We conclude this section by establishing an important relationship between the matrix S and the iteration-by-subdomain method we described in the previous section.
Let us consider the spectral multidomain problem (5.4)-(5.13), and denote by $\Xi = (\xi, \eta)^t$ the unknown vector of the characteristic solution at the interface point $x = \alpha$, i.e.,

$$(6.17) \qquad \xi = z_1^1(\alpha) = z_1^2(\alpha) , \ \eta = z_2^1(\alpha) = z_2^2(\alpha)$$

By a straightforward calculation it is not hard to see that one step of our iteration-by-subdomain procedure amounts to apply one step of the Richardson iterative method to the capacitance system (6.11). Precisely, going from the step n to the step $n+1$ produces a change on the interface variables Ξ which is ruled by the following relationship:

$$(6.18) \qquad \Xi^{n+1} = \Xi^n + (R - S\,\Xi^n) ,$$

Here we have denoted by R the right hand side of (6.11).
Using in (6.18) an acceleration parameter ω a priori different than 1, i.e., considering the following sequence

$$(6.19) \qquad \Xi^{n+1} = \Xi^n + \omega(R - S\Xi^n) \ , \ \ \omega > 0$$

amounts to replace the interface matching conditions (5.14)-(5.15) by the new ones

$$(6.20) \qquad T_q(u^2)^{n+1} = \omega T_q(u^1)^n + (1 - \omega)T_q(u^2)^n \quad at \ x = \alpha$$

$$(6.21) \qquad T_{p-q}(u^1)^{n+1} = \omega T_{p-q}(u^2)^n + (1 - \omega)T_{p-q}(u^1)^n \quad at \ x = \alpha$$

The other statements of the iterative algorithm hold unchanged. In (6.20)-(6.21) ω plays the role of a relaxation factor. In view of (6.19), it is well known that whenever S is symmetric and positive definite, the best choice of ω is (see [Y])

$$\omega_{opt} = 2/(\lambda_{min} + \lambda_{max})$$

where λ_{min} and λ_{max} are the minimum and maximum eigenvalues of S. Thus if meas (Ω_1) = meas (Ω_2), in view of theorem 5.5 we conclude that $\omega_{opt} = 1$.

7. Numerical Results

We show the potential capability of our domain decomposition method to yield effective solutions of hyperbolic systems by presenting some numerical results for the simple problem considered in section 5. We allow however non homogeneous right hand side of (5.3).

The time discretization method considered is the second order Crank-Nicolson scheme.
We considered the case of two different sets of initial and boundary data, so that the corresponding exact solutions are:

$$1^{st} \text{test function: } u^1(x, t) = e^t \cos \alpha x, u^2(x, t) = e^t \sin \alpha x$$

$$2^{nd} \text{test function: } u^1(x, t) = e^t arctg \ \beta(x + 0.5), u^2(x, t) = e^t \ arctg \ \beta(x - 0.5)$$

where α and β are some given parameters.
In all cases, we display the results obtained at the time $t = 1$ for several values of the time step Δt, the polynomial degree N used inside each subdomain, and the number M of subdomains. In tables 7.1 and 7.2 we report the maximum norm of the relative error between the approximate and exact solutions at the time $t = 1$ using $\Delta t = 0.01$. The tables refer to the first and second test functions respectively, for single domain approximations and for approximations with two subdomains of equal length.

N \ α	one domain			N \ α	two subdomains		
	1	4	10		1	4	10
4	1.361E-4	2.429E-2	0.184	4	1.599E-5	1.059E-2	0.747
8	1.472E-7	8.134E-3	3.741E-1	8	1.436E-7	1.311E-5	7.266E-2
20	1.472E-7	1.546E-7	3.718E-4	12	1.363E-7	1.548E-7	1.552E-3

Table 7.1 First test function, one and two subdomain approximations

N \ β	one domain			N \ β	two subdomains		
	1	4	10		1	4	10
4	4.767E-4	1.288E-2	2.812E-2	4	1.937E-4	6.119E-3	3.668E-2
8	1.094E-4	5.079E-3	2.345E-2	8	7.377E-7	1.024E-3	1.893E-2
20	2.572E-7	2.860E-4	1.005E-2	12	2.577E-7	1.943E-4	9.231E-3

Table 7.2 Second test function, one and two subdomain approximations

As we can see, the better accuracy we obtain using two subdomains rather than simply one is more relevant if the expected solution exhibits important oscillations (and/or variations) within the computational domain. Furthermore, since the matrices associated with the spectral collocation method are full, the use of several subdomains, accompanied with a sound tuning of the polynomial degree within each subdomain, allows the reduction of the overall complexity of the numerical problem.

We report now the results obtained by the iterative procedure presented in section 3 for the resolution of the multidomain problem.

In the tables below we report the number of iterations that are needed in order to damp the initial error by a factor of 10^{-5}. As usual M denotes the number of subdomains and N the polynomial degree of the discrete solution inside each subdomain.

Table 7.3 refers to the second test function when $\beta = 10$: it shows that the number of iterations is independent of N, while it grows at most linearly with M when Δt is "large". According to the convergence theory of section 5, the rate of convergence is as larger as the time-step Δt is smaller. Finally, in the table 7.4 we present the results for the first test function when $\alpha = 1$, obtained for $N = 16$ and several values of M and Δt.

N \ M	2	4	8	12
4	2 (2)	5 (4)	8 (5)	12 (5)
12	2 (2)	4 (4)	8 (5)	12 (5)
20	2 (2)	4 (4)	8 (4)	12 (4)

Table 7.3 Number of iterations for the second test function when $\beta = 10$ and $\Delta t = 0.1$ (within the brackets are the values for the case $\Delta t = 0.01$)

Δt \ M	2	4	8	12	16
0.1	2	4	8	12	16
0.02	2	4	4	4	5
0.01	2	4	4	4	4

Table 7.4

REFERENCES

[C] C.Canuto, Spectral methods and maximum principle, Math. Comput. (to appear).

[CGPW] T.F.Chan, R.Glowinski, J.Periaux and O.Widlund, Eds., Domain Decomposition Methods for P.D.E.s, II, SIAM, Philadelphia 1988 (in press).

[CGVV] L.Cambier, W.Ghazzi, J.P.Veuillot and H.Viviand, Une Approche par Domaines pour le Calcul d'Ecoulements Compressibles, in Comp.Meth.Appl.Sciences and Eng., V, R.Glowinski and J.L.Lions, Eds., North-Holland, Amsterdam, 1982, pp.423-446.

[CHQZ] C.Canuto, M.Y.Hussaini, A.Quarteroni and T.A.Zang, Spectral Methods in Fluid Dynamics, Springer-Verlag, Berlin, Heidelberg, New York, 1988.

[CQ] C.Canuto and A.Quarteroni, On the boundary treatment in spectral methods for hyperbolic systems, J.Comput.Phys. 71 (1987), pp.100-110.

[GLT1] D.Gottlieb, L.Lustman and E.Tadmor, Stability Analysis of Spectral Methods for Hyperbolic Initial-Boundary Value Systems, in [T], pp.63-92, 1988.

[GLT2] D.Gottlieb, L.Lustman and E.Tadmor, Convergence of Spectral Methods for Hyperbolic Initial-Boundary Value Systems, in [T], pp.51-62, 1988.

[GGT] D.Gottlieb, M.Gunzburger and E.Turkel, On Numerical Boundary Treatment for Hyperbolic Systems, SIAM J.Numer.Anal. 19 (1982), pp.671-697.

[K] D.A.Kopriva, A Spectral Multidomain Method for the Solution of Hyperbolic Systems, in Applied Numer.Math. 2 (1986), pp.221-241.

[L] P.D.Lax, Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves, Conf.Board.Math.Sci., 11, SIAM, Philadelphia, 1973.

[MS] M.Macaraeg and C.L.Streett, An analysis of artificial viscosity effects on reacting flows using a spectral multi-domain technique, in Computational Fluid Dynamics, G. de Vahl Davis and C.Fletcher, Eds., North Holland, Amsterdam, 1988, pp.503-514.

[MSH] M.G.Macaraeg and C.L.Streett, A Spectral Multi-Domain Technique Applied to High-Speed Chemically Reacting Flows, to appear in [CGPW].

[S] J.Smoller, Shock Waves and Reaction-Diffusion Equations, Springer, Berlin and New York, 1983.

[T] R.Teman, Methodes Spectrales, Collections de la Direction des Etudes et Recherches d'Electricite de France, n.68, EYROLLES, Paris, 1988.

[Y] D.M.Young, Iterative Solution of Large Linear Systems, Academic Press, New York, 1971.

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Final Report

16. Abstract

We consider hyperbolic systems of conversation laws which are discretized in space by spectral collocation methds and advanced in time by finite difference schemes. At any time-level we introduce a domain decomposition method based on an iteration-by-subdomain procedure yielding at each step a sequence of independent subproblems (one for each subdomain) that can be solved simultaneously. The method is set for a general nonlinear problem in several space variables. The convergence analysis, however, is carried out only for a linear one-dimensional system with continuous solutions. A precise form of the error reduction factor at each iteration is derived. Although the method is applied here to the case of spectral collocation approximation only, the idea is fairly general and can be used in a different context as well. For instance, its application to space discretization by finite differences is straightforward.

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