NASA Contractor Report 181772

ICASE REPORT NO. 89-4

ICASE

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Contract No. NAS1-18107, NAS1-18605
January 1989

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

National Aeronautics and
Space Administration

Langley Research Center
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OPTIMAL FEEDBACK CONTROL OF INFINITE DIMENSIONAL PARABOLIC EVOLUTION SYSTEMS: APPROXIMATION TECHNIQUES*

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Abstract

We present a general approximation framework for computation of optimal feedback controls in linear quadratic regulator problems for nonautonomous parabolic distributed parameter systems. This is done in the context of a theoretical framework using general evolution systems in infinite dimensional Hilbert spaces. We discuss conditions for preservation under approximation of stabilizability and detectability hypotheses on the infinite dimensional system. The special case of periodic systems is also treated.

*Dedicated to Prof. E.J. McShane on the occasion of his 85th birthday

†Research supported in part under grants NSF MCS8504316, NASA NAG-1-517, AFOSR-84-0398, and AFOSR-F49620-86-C-0111. Part of this research was carried out while the first author was a visiting scientist at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA, which is operated under NASA Contracts NAS1-18107 and NAS1-18605.

1 Introduction

In this paper we present a theoretical approximation framework for computation of optimal feedback controls in linear quadratic regulator problems governed by parabolic partial differential equations with time dependent coefficients. Our efforts were originally motivated by the desire to develop control strategies (distributed in nature) for insect dispersal models (see Chapter 1 of [BK2] and the references therein) which have been shown to involve time dependent coefficients.

The presentation below is somewhat in the spirit of that for autonomous parabolic systems in [BK1] and [LT] in that we attempt to develop a convergence theory in which uniform stabilizability of the original system is preserved under approximation. It differs substantially from [BK1] and [LT] since we don't use directly sectorial properties of the operators and resolvent and spectral set arguments to establish preservation of stabilizability and detectability. (Indeed, the time dependent nature of our system prevents this.) Nor do we use the Trotter-Kato theorem (which is not well suited for use with nonautonomous control systems) in our convergence arguments.

In section 2 we summarize previous results for abstract LQR problems on infinite time intervals and formulate these in a form readily used in our subsequent discussions. This formulation is based on the abstract frameworks found in [CP], [G], [BK1] and [Da], [DI1], [DI2]; we rely heavily on the ideas of Da Prato and Ichikawa which guarantee uniqueness of solutions of the associated Riccati integral equations under certain stabilizability and detectability assumptions.

An approximation framework for abstract evolution systems in the spirit of [G], [BK1] is given in section 3; convergence of the approximate Riccati operators (and, of course, the corresponding controls and trajectories) is established under uniform stabilizability and detectability hypotheses on the approximate evolution control systems.

Our major contributions are given in section 4, along with the presentations in section 5 and 6, where we show how the hypotheses of section 4 can be verified for rather wide classes of problems of interest. In section 4 we focus our attention on parabolic systems described by time dependent sesquilinear forms (in the spirit of the autonomous system frameworks in [BK1], [BI1], [BI2]) and associated evolution equations. We make substantial use of the results of Tanabe [T] to formulate our problems in a weak (V^*) sense. Our fundamental convergence results (Theorem 4.4) for the uncontrolled systems rely on a sesquilinear or variational formulation of the systems, strong V-ellipticity of the parabolic evolution systems, approximation properties for the spaces approximating the state space, and the Gronwall inequality. (Certain aspects of this approach can be relaxed to allow us to treat weakly damped hyperbolic systems–see [BKS], [BKW].) We are then able to reduce convergence questions for the controlled systems (e.g., convergence of Riccati variables, optimal controls and feedback evolution systems) to conditions of uniform stabilizability and uniform detectability of the approximate systems (Theorem 4.5).

We show in section 5 that we can obtain these uniform stabilizability/detectability conditions by preservation under approximation of dissipative inequalities for certain classes of evolution control systems. Sufficient conditions that are readily checked in many examples are given and several special cases are noted.

An alternative approach is presented in section 6 where we restrict our considerations to parabolic systems for which the domain V of the generator of the evolution system embeds compactly in the state space H. In this case, it is shown that stabilizability/detectability of the original system is preserved under approximation.

Finally in section 7 we give an example of a class of parabolic partial differential equation control problems for which all the hypotheses of our theoretical framework can be easily verified.

We have used the ideas presented in this paper to develop and test computational packages for solving nonautonomous parabolic control problems of the type discussed in section 7. However, since our presentation here is already quite long and since a presentation of our detailed numerical findings would entail lengthy discussions, we will not discuss the numerical examples. A separate manuscript is under preparation; the interested reader can also consult [W].

We believe that the present paper offers new results for time dependent infinite dimensional control systems. Moreover, our arguments are such that we offer an attractive alternative approach to those found in [G], [BK1], [LT] even in the case of autonomous parabolic systems.

2 The abstract linear quadratic regulator problem on an infinite time interval

In this section we formulate a linear quadratic regulator problem for evolution system dynamics in a Hilbert space. We present a collection of functional analytic and control theoretic results related to such problems. The results we give in this section are known and, while in some cases we have modified the statements to present the results in a form most suited to our purposes, the reader can easily refer to the literature for proofs. In particular, we use freely results found in [CP] and [G] and rely heavily on recent results of Da Prato and Ichikawa [Da], [DI1], [DI2].

We first recall results for evolutionary systems. Let H be a Hilbert space with inner product $< \cdot, \cdot >$. Let $\Delta(t_0, t_f) = \{(t,s) \mid t_0 \leq s \leq t \leq t_f\}$, $\Delta_\infty(t_0) = \{(t,s) \mid t_0 \leq s \leq t < \infty\}$ and $L(H)$ be the Banach algebra of bounded linear operators on H. We use $B_\infty([t_0, t_f]; L(H))$ to denote the set of operator valued functions that are bounded on $[t_0, t_f]$. We recall that $T(\cdot,\cdot) : \Delta(t_0, t_f) \mapsto L(H)$ is called an *evolution operator* if T satisfies the following conditions: (i) $T(t,s) = T(t,r)T(r,s)$, for $t_0 \leq s \leq r \leq t \leq t_f$; (ii) $T(t,t) = I$, for $t \in [t_0, t_f]$; and (iii) $T(t,s)$ is strongly continuous in s on $[t_0, t]$ and strongly continuous in t on $[s, t_f]$. We say that an evolution operator has *exponential growth* if there exists $M_1 \geq 1, \omega > 0$ such that $\|T(t,s)x\| \leq M_1 e^{\omega(t-s)} \|x\|$, for $(t,s) \in \Delta_\infty(t_0), x \in H$. An evolution operator is said to be *uniformly exponentially* *stable* if there exists $M \geq 1$ and $\alpha > 0$ such that $\|T(t,s)x\| \leq Me^{-\alpha(t-s)}\|x\|$, for $(t,s) \in \Delta_\infty(t_0), x \in H$. We have the following fundamental results of Datko which will be crucial to our presentation.

Lemma 2.1 [Dt] *Consider an evolution operator $T(\cdot,\cdot)$ with exponential growth. Then $T(\cdot,\cdot)$ is uniformly exponentially stable if and only if there exists an M_2 such that*

$$\int_s^\infty \|T(t,s)x\|^2 dt \leq M_2\|x\|^2, \quad \textit{for } s \geq t_0, x \in H.$$

Furthermore we can find constants $M_3 \geq 1, \alpha > 0$ depending only on M_1, M_2 and ω for which the following estimate holds:

$$\|T(t,s)\| \leq M_3 e^{-\alpha(t-s)}, \quad \textit{for } (t,s) \in \Delta_\infty(t_0).$$

The original statement and proof of this theorem are due to R.Datko. We have modified slightly (see Appendix A of [W]) the original proof in [Dt] to point out the relationship between the constants M_3, α and M_1, M_2 and ω. This will be essential for our subsequent use with approximation systems. In [W] it is shown that the constants M_3 and α can be chosen as:

$$M_3 = 2M_1 e^{4M_2 {M_1}^2 \omega(2\omega M_2+1)}, \quad \alpha = \frac{\log 2}{4M_2 {M_1}^2(2\omega M_2+1)}.$$

In our discussions of control systems, perturbations of evolution operators (see [CP]) will play an important role. Let $t_f < \infty$. Consider a uniformly bounded evolution operator $T(\cdot,\cdot)$ and $C(\cdot) \in B_\infty([t_0,t_f]; L(H))$. Then the integral equation for $S(t,s) \in L(H)$ given by

$$S(t,s)x = T(t,s)x + \int_s^t T(t,\eta)C(\eta)S(\eta,s)x\,d\eta, \quad \text{for } x \in H,$$

has a unique solution $S(\cdot,\cdot)$ in the class of strongly continuous operator valued functions. Moreover, $S(\cdot,\cdot)$ is an evolution operator and is called the *perturbed evolution operator* corresponding to the perturbation of $T(\cdot,\cdot)$ by $C(\cdot)$. In addition, $S(\cdot,\cdot)$ is also the unique solution of

$$S(t,s)x = T(t,s)x + \int_s^t S(t,\eta)C(\eta)T(\eta,s)x\,d\eta, \quad \text{for } x \in H.$$

We turn next to our formulation of the regulator problem for an evolution system. We let H, U be real Hilbert spaces with inner products $< \cdot, \cdot >_H$, $< \cdot, \cdot >_U$; H, U are the state space and the control space respectively. Consider an evolution operator $T(\cdot, \cdot)$ defined on $\Delta_\infty(t_0)$. The control system is described as follows: for any $u \in L^2([t_0, \infty); U)$, the corresponding trajectories satisfy the equation:

$$(2.1) \qquad x(t) = T(t,s)x(s) + \int_s^t T(t,\tau)B(\tau)u(\tau)d\tau, \quad \text{for } (t,s) \in \Delta_\infty(t_0).$$

The cost functional is given by

$$(2.2) \qquad J_\infty(u; t_0, x_0) = \int_{t_0}^\infty \{< W(t)x(t), x(t) >_H + < R(t)u(t), u(t) >_U\} \, dt,$$

where $x(\cdot)$ is the trajectory corresponding to u with $x(t_0) = x_0$. For each given t_0, x_0, the optimal control problem is to find a control u^* which minimizes (2.2) over all $u \in L^2([t_0, \infty); U)$.

We can consider (2.2) as the limit as $t_k \to \infty$ of

$$(2.3) \qquad \begin{aligned} J(u; t_0, t_k, x_0) =& < Gx(t_k), x(t_k) >_H \\ &+ \int_{t_0}^{t_k} \{< W(t)x(t), x(t) >_H + < R(t)u(t), u(t) >_U\} \, dt, \end{aligned}$$

with $G = 0$. Here we shall summarize existence results for optimal controls in the infinite time interval, existence and uniqueness of the solutions of the Riccati integral equation on an infinite time interval, and stability of the feedback system.

We make the following **standing assumptions** for all subsequent discussions of (2.1), (2.2): (i) The evolution operator $T(\cdot, \cdot)$ has exponential growth. (Thus, in particular, $T(t, s)$ is uniformly bounded for s, t in any bounded sub-interval of $[t_0, \infty)$); (ii) The operator valued function $B(\cdot) : [t_0, \infty) \mapsto L(U, H)$ is uniformly bounded in $[t_0, \infty)$, i.e. there exists M_B such that $\|B(t)\|_{L(U,H)} \leq M_B$ for all $t \in [t_0, \infty)$; (iii) The operator valued function $W(\cdot) : [t_0, \infty) \mapsto L(H)$ is uniformly bounded in the interval $[t_0, \infty)$, and $W(t)$ is nonnegative definite self-adjoint for all $t \in [t_0, \infty)$; (iv) The operator valued function $R(\cdot) : [t_0, \infty) \mapsto L(U)$ is uniformly bounded in the interval

$[t_0, \infty)$, and $R(t)$ is positive definite self-adjoint for all $t \in [t_0, \infty)$. Furthermore, there exists a constant $r > 0$ such that $< R(t)u, u >_U \geq r\|u\|_U^2$, for all $u \in U$.

Under these assumptions we consider the linear quadratic control problem in the interval $[t_0, t_k]$ for $t_k < \infty$. That is, we consider the cost functional (2.3) with our system (2.1). Then for any bounded self-adjoint nonnegative definite linear operator G, it is well known that for each given $x_0 \in H$, there exists a unique control u such that

$$J(u; t_0, t_k, x_0) \leq \min_{v \in L^2([t_0,t_k];U)} J(v; t_0, t_k, x_0).$$

This control u can be written in a feedback form $u(t) = -R^{-1}(t)B^*(t)Q(t)x(t)$, for $t \in [t_0, t_k]$, where $x(\cdot)$ is the corresponding trajectory and $Q(\cdot) : [t_0, t_f] \mapsto L(H)$, is the unique self-adjoint solution of the Riccati integral equation (RIE)

$$Q(t)x = T^*(t_k, t)GT(t_k, t)x + \int_t^{t_k} T^*(\eta, t)W(\eta)T(\eta, t)x d\eta - \int_t^{t_k} T^*(\eta, t)Q(\eta)B(\eta)R^{-1}(\eta)B^*(\eta)Q(\eta)T(\eta, t)x d\eta \tag{2.4}$$

for all $t \in [t_0, t_k]$ and $x \in H$.

We note that in the case $G = 0$, the above equation reduces to

$$Q(t)x = \int_t^{t_k} T^*(\eta, t)[W(\eta) - Q(\eta)B(\eta)R^{-1}(\eta)B^*(\eta)Q(\eta)]T(\eta, t)x d\eta, \tag{2.5}$$

for all $t \in [t_0, t_k]$ and $x \in H$. We also note that the equation (2.4) is equivalent to

$$\begin{aligned} Q(s)x &= T^*(t, s)Q(t)T(t, s)x + \int_s^t T^*(\eta, s)W(\eta)T(\eta, s)x d\eta \\ &\quad - \int_s^t T^*(\eta, s)Q(\eta)B(\eta)R^{-1}(\eta)B^*(\eta)Q(\eta)T(\eta, s)x d\eta \\ Q(t_k)x &= Gx \end{aligned} \tag{2.6}$$

for all $t_0 \leq s \leq t \leq t_k$ and $x \in H$. Solutions of this latter equation have a representation that is often used in control theoretic arguments. Consider any $u(\cdot) \in L^2([t_0, t_k]; U)$, and for $x \in H$, define a H-valued function $y(\cdot)$ by

$$y(t) = T(t, s)x + \int_s^t T(t, \tau)B(\tau)u(\tau)d\tau, \quad \text{for } t \in [s, t_k].$$

If $Q(\cdot)$ is a self-adjoint solution of (2.6), then

$$
\begin{aligned}
<Q(s)x, x>_H &= \; <Gy(t_k), y(t_k)>_H \\
&\quad + \int_s^{t_k} \{<W(t)y(t), y(t)>_H + <R(t)u(t), u(t)>_U\}\, dt \\
&\quad - \int_s^{t_k} <R(t)z(t), z(t)>_U \, dt,
\end{aligned}
\tag{2.7}
$$

where $z(t) = u(t) + R^{-1}(t)B^*(t)Q(t)y(t)$. This can be used to show that (2.6) has a unique self-adjoint solution.

Before continuing our discussion, let us introduce additional notation. Let

$$
\Sigma^+ = \Big\{E \mid E \in L(H), E \text{ self-adjoint, nonnegative definite.}\Big\}
$$
$$
\mathcal{C}_s([t_0, t_k]; \Sigma^+) = \Big\{K : [t_0, t_k] \mapsto \Sigma^+ \mid K \text{ strongly continuous.}\Big\}
$$

By the uniqueness of the solution of equation (2.6), we can define a mapping Λ : $\Sigma^+ \mapsto \mathcal{C}_s([t_0, t_k]; \Sigma^+)$ as following: for each $G \in \Sigma^+$, ΛG is the unique nonnegative definite self-adjoint solution of equation (2.6). Under our general assumptions, it is easily seen that for fixed G the map Λ depends only on t_k; if we consider the linear quadratic regulator problem on two bounded intervals $[t_0, t_1]$ and $[t_0, t_2]$, we will use Λ_1, Λ_2 to denote the maps associated with each interval respectively.

Now consider a increasing sequence $\{t_k\}_{k=1}^{\infty}$, with $t_k < \infty$. The map Λ_k associates with each finite interval problem the Riccati equation on $[t_0, t_k]$. Let $G = 0$ and $Q_k(\cdot) = \Lambda_k G$. For simplicity, consider a bounded interval $[a, b] \subset [t_0, t_1]$, and for each $t \in [a, b], x \in H$, we assume that there exists a constant $M(t, x)$ such that for all k

$$
<Q_k(t)x, x>_H \;\; \leq \;\; M(t, x). \tag{2.8}
$$

The following theorem (see [Da], [G]) establishes the connection between control problems on a finite time interval and problems on an infinite time interval.

Theorem 2.1 *Under the above assumptions, we can conclude:*

(i) *For each* $t \in [a,b]$, *there exists a unique operator* $Q(t) \in \Sigma^+$ *such that* $Q_k(t) \to Q(t)$ *strongly and the convergence is uniform in* $[a,b]$. *Therefore* $Q(\cdot)$ *is strongly continuous, so uniformly bounded in* $[a,b]$.

(ii) *As a consequence of (i), we can define the perturbed evolution systems* $S_k(\cdot,\cdot), S(\cdot,\cdot)$ *corresponding to the perturbation of* $T(\cdot,\cdot)$ *by* $-BR^{-1}B^*Q_k$ *and* $-BR^{-1}B^*Q$ *respectively. We have* $S_k(t,s)x \to S(t,s)x$, *for all* $x \in H$, *and* $a \le s \le t \le b$. *Furthermore the convergence is uniform in* t *for* $t \in [s,b]$. *If* $T(\cdot,\cdot)$ *is jointly strongly continuous, then the convergence is uniform for all* $a \le s \le t \le b$.

The only assumption on the sequence $\{t_k\}$ is that t_k increase as a function of k. In particular, the above theorem is valid when $t_k \to \infty$, as $k \to \infty$. Paralleling the usual approach to finite dimensional regulator problems, we can use these results to establish results for the control problem on an infinite time interval. To that end, consider a sequence $\{t_k\}_{k=1}^{\infty}$ with $t_k \to \infty$ as $k \to \infty$. Let $Q_k(\cdot), S_k(\cdot,\cdot)$ be defined as above. If for each $t \ge t_0$ we can find a constant $M(t)$ such that $< Q_k(t)x, x >_H \le M(t)\|x\|^2$, then by Theorem 2.1, we have $Q(\cdot), S(\cdot,\cdot)$ defined on $[t_0,\infty)$. Furthermore for any $(t,s) \in \Delta_\infty(t_0)$, Q satisfies

$$(2.9) \qquad \begin{aligned} Q(s)x &= T^*(t,s)Q(t)T(t,s)x + \int_s^t T^*(\eta,s)W(\eta)T(\eta,s)x\,d\eta \\ &\quad - \int_s^t T^*(\eta,s)Q(\eta)B(\eta)R^{-1}(\eta)B^*(\eta)Q(\eta)T(\eta,s)x\,d\eta. \end{aligned}$$

The equation (2.9) is called the Riccati integral equation (RIE) for the infinite time interval. We know from Theorem 2.1 that Q is strongly continuous and uniformly bounded in any bounded interval, but Q is not necessarily uniformly bounded in the entire interval $[t_0,\infty)$. If Q is not uniformly bounded, that implies the minimal cost for some initial state x will tend to infinity as t_k tends to infinity; that is, there is no control yielding finite cost for the infinite time interval problem. Let us state a condition which prohibits this situation.

Definition 2.1 (W-stabilizability) *We say that (2.1), (2.2) is* W-stabilizable *if there exists a constant* M *such that for any* $s \ge t_0$ *and* $x \in H$, *we can find a control*

$u \in L^2([t_0, \infty); U)$ *satisfying*

$$J_\infty(u; s, x) \leq M\|x\|^2. \tag{2.10}$$

One can then prove (see [DI1], Theorem 3.1) that $Q = \lim Q_k$ is a uniformly bounded solution of the Riccati integral equation (2.9) in $[t_0, \infty)$ if and only if (2.1), (2.2) is W-stabilizable. In this case we have $Q(t) \leq M \cdot I$ for $t \in [t_0, \infty)$. Furthermore, if $\hat{Q}$ is any other bounded self-adjoint solution of (2.9), we have that $Q(t) \leq \hat{Q}(t)$ for $t \in [t_0, \infty)$. It follows that using any sequence $\{t_k\}$ with $t_k \to \infty$, in the above limiting procedure yields the same solution Q to (2.9), which we shall refer to as the *minimal* bounded nonnegative self-adjoint solution of the Riccati integral equation on $[t_0, \infty)$ and denote by Q_{min}.

We note that if the system (2.1), (2.2) is W-stabilizable, then for any $s \geq t_0$ and $x \in H$, the unique optimal control for the infinite time interval problem is given by $u(t) = -R^{-1}(t)B^*(t)Q(t)S(t, s)x$.

Next we consider a uniformly bounded solution $\hat{Q}$ of (2.9) and let $\hat{S}$ be the evolution operator corresponding to the perturbation of T by $-BR^{-1}B^*\hat{Q}$. We say that $\hat{Q}$ is a *stability solution* of (2.9) if $\hat{S}(t, s)x \to 0$ as $t \to \infty$ for all $s \geq t_0$, $x \in H$.

It is shown in [DI1] that there is at most one stability solution of (2.9). Moreover, if $\hat{Q}$ is a stability solution satisfying $\hat{Q}(t) \leq M \cdot I$ and Q_k is the solution on $[t_0, t_k]$ with $Q_k(t_k) = M \cdot I$, then $\hat{Q}(t) \leq Q_k(t)$ for $t \in [t_0, t_k]$ and $Q_k(t)x \to \hat{Q}(t)x$ as $k \to \infty$ for each $x \in H$. In addition, if Q is any uniformly bounded solution, then $Q(t) \leq \hat{Q}(t)$, $t \in [t_0, \infty)$; that is, any stability solution is the *maximal* (uniformly bounded) solution. Finally, if the system (2.1), (2.2) is W-stabilizable and if the minimal solution Q_{min} of (2.9) is a stability solution, then it is the unique uniformly bounded solution of the RIE (2.9).

From the above remarks, it is clear that it is desirable to have conditions that guarantee a solution of the RIE be a stability solution. One such condition is a detectability condition.

Definition 2.2 (W-detectability) *Let* $V(t) = \sqrt{W(t)}$. *We say that the system (2.1), (2.2) is* W-detectable *if there exists a uniformly bounded function* $K(\cdot)$ *with* $K(t) \in L(H)$ *such that the evolution operator* T_{KV} *corresponding to the perturbation of* T *by* KV *is uniformly exponentially stable.*

We then have the following result.

Theorem 2.2 *Suppose that the system (2.1), (2.2) is W-stabilizable and W-detectable. Then the minimal solution* Q_{min} *of the RIE is the unique uniformly bounded solution of (2.9) and the evolution operator* S *defined by perturbation of* T *by* $-BR^{-1}B^*Q_{min}$ *is uniformly exponentially stable. In fact,*

$$\|S(t,s)\| \leq Me^{-\alpha(t-s)}, \quad (t,s) \in \Delta_\infty(t_0),$$

where the constants M *and* α *depend only on the bounds for* B, K, R^{-1}, Q_{min} *and* M_{KV}, β *in the bound* $\|T_{KV}(t,s)\| \leq M_{KV} \exp\{-\beta(t-s)\}$.

The first part of this theorem follows from [DI1] (Prop. 3.3). That the constants M and α depend only on the bounds indicated follows from use of the modified Datko lemma, Lemma 2.1 above. As we shall see in the next section on approximation, this dependence (or lack thereof) will allow us to infer a uniform exponential stability of the approximate feedback control systems whenever we have a uniform W-detectability condition satisfied by the approximate systems.

To conclude this section, we recall that an evolution operator is said to be θ-periodic if for any $(t,s) \in \Delta_\infty(t_0)$, we have $T(t+\theta, s+\theta)x = T(t,s)x$, for all $x \in H$. We note that any θ-periodic evolution operator satisfies the exponential growth assumption that is part of our standing assumptions in this paper. It is also easily argued that if the linear quadratic regulator problem is θ-periodic (i.e., B, W, R and T of (2.1), (2.2) are θ-periodic), then the minimal solution and the stability solution of the RIE are θ-periodic. Of course, we cannot argue that every uniformly bounded solution of the RIE is periodic under a periodicity assumption on the problem.

We turn next to approximation results for the abstract linear regulator problem on an infinite time interval.

3 Approximation of linear quadratic regulator problems on an infinite time interval

Let H^N and U^N be families of finite dimensional subspaces of the original state space and control space H, U respectively. For each N an approximate control system is described by

$$(3.1)\quad x^N(t) = T^N(t,s)x^N(s) + \int_s^t T^N(t,\eta)B^N(\eta)u^N(\eta)d\eta, \quad \text{for } (t,s) \in \Delta_\infty(t_0),$$

where $T^N(\cdot,\cdot) : \Delta_\infty(t_0) \mapsto L(H^N)$ is an evolution operator, and $B^N(\cdot) : U^N \mapsto H^N$. The cost functional is given by

$$(3.2)\qquad J_\infty^N(u^N; t_0, x_0^N) = \int_{t_0}^\infty < W^N(t)x^N(t), x^N(t) >_H dt + \int_{t_0}^\infty < R^N(t)u^N(t), u^N(t) >_U dt$$

where $x^N(\cdot)$ satisfies (3.1) and $x^N(t_0) = x_0^N$. Suppose that each of the approximate systems satisfies the standing assumptions for (2.1), (2.2) given above and that each is W-stabilizable. Then we can guarantee existence of $Q^N(\cdot)$, the minimal uniformly bounded solution of the associated Riccati integral equation on the infinite time interval $[t_0, \infty)$. Let $S^N(\cdot,\cdot)$ be the perturbed evolution operator corresponding to the perturbation of T^N by $-B^N(R^N)^{-1}B^{*N}Q^N$. In this section, we present results on the convergence of Q^N, S^N.

We need to make some basic assumptions on the approximate systems. Let $\{H^N\}_{N=1}^\infty$, $\{U^N\}_{N=1}^\infty$ be subspaces of H, U respectively, and P_H^N, P_U^N be projection operators which are assumed to satisfy $\|P_H^N x - x\|_H \to 0$, $\|P_U^N u - u\|_U \to 0$, as $N \to \infty$, for all $x \in H, u \in U$.

We note that the usual orthogonal projections of H and U onto H^N, U^N respectively satisfy these assumptions if H^N, U^N approximate H and U in an appropriate sense. (We shall specify approximation systems that satisfy these conditions in subsequent sections.) We make the further assumptions on our approximate systems.

Hypothesis 3.1 (Uniform boundedness)

(i) *There exist constants* $M \geq 1$ *and* $\omega > 0$ *such that*

$$\|T(t,s)\|_{L(H)} \leq Me^{\omega(t-s)}, \quad \|T^N(t,s)\|_{L(H^N)} \leq Me^{\omega(t-s)}$$

hold for all N *and* $(t,s) \in \Delta_\infty(t_0)$*;*

(ii) *There exists a constant* K_B *such that*

$$\|B(t)\|_{L(U,H)} \leq K_B, \quad \|B^N(t)\|_{L(U^N,H^N)} \leq K_B$$

for all N *and* $t \in [t_0, \infty)$*;*

(iii) *There exists a constant* K_W *such that*

$$\|W(t)\|_{L(H)} \leq K_W, \quad \|W^N(t)\|_{L(H^N)} \leq K_W$$

for all N *and* $t \in [t_0, \infty)$*. Furthermore* $W(t), W^N(t)$ *are nonnegative definite self-adjoint for all* $t \in [t_0, \infty)$*.*

(iv) *There exists a constant* K_R *such that*

$$\|R(t)\|_{L(U)} \leq K_R, \quad \|R^N(t)\|_{L(U^N)} \leq K_R$$

for all N *and* $t \in [t_0, \infty)$*. In addition,* $R(t), R^N(t)$ *are positive definite self-adjoint for all* $t \in [t_0, \infty)$*. There exists a constant* $r > 0$ *such that* $R(t) \geq r \cdot I$*,* $R^N(t) \geq r \cdot I$*, for all* $t \in [t_0, \infty)$*.*

Hypothesis 3.2 (Pointwise convergence) *The operators* $T^N(t,s)$*,* $T^{*N}(t,s)$*,* $B^N(t)$*,* $B^{*N}(t)$*,* G^N*,* $W^N(t)$*,* $R^N(t)$ *converge strongly to* $T(t,s)$*,* $T^*(t,s)$ *,* $B(t)$*,* $B^*(t)$*,* G*,* $W(t)$*,* $R(t)$ *for any* $t_0 \leq s \leq t < \infty$*, where* G, G^N *are nonnegative self-adjoint operators in* $L(H), L(H^N)$ *respectively.*

From arguments in [DI1] and [W], it is readily seen that W-stabilizability (i.e. condition (2.10)) is equivalent to the following: there exists a constant $M > 0$, and a uniformly bounded feedback operator $K(\cdot) : [t_0, \infty) \mapsto L(H, U)$ such that if $T_K(\cdot, \cdot)$ is the perturbed evolution operator corresponding to the perturbation of T by BK, then for any $s \geq t_0$, $x \in H$, the cost of the feedback control $u(t) = B(t)K(t)T_K(t, s)x$ satisfies $J_\infty(u; s, x) \leq M\|x\|^2$.

To guarantee the existence of uniformly bounded solutions of the Riccati integral equation on the infinite time interval for each of the approximate systems, we make a uniform W-stabilizability assumption.

Hypothesis 3.3 (Uniform W-stabilizability) *There exists a constant $M > 0$ such that for all N, there exist uniformly bounded feedback operators $K^N(\cdot) : [t_0, \infty) \mapsto L(H^N, U^N)$ satisfying the following: for all $s \geq t_0$ and $x^N \in H^N$, the feedback control $u^N(t) = B^N(t)K^N(t)T^N_{K^N}(t, s)x^N$ has a cost satisfying*

$$J^N_\infty(u^N; s, x^N) \leq M\|x^N\|^2_H.$$

Now consider $\{t_k\}_{k=1}^\infty$ with $t_k \to \infty$ as $k \to \infty$. For each N, let Σ^{+N} be the set of nonnegative self-adjoint linear operators in H^N; we define the map $\Lambda^N_k : \Sigma^{+N} \mapsto \mathcal{C}_s([t_0, t_k]; \Sigma^{+N})$ via the finite dimensional Riccati integral equation on $[t_0, t_k]$ as before. Let $G \in \Sigma^+$, and $G^N \in \Sigma^{+N}$. Define $Q^N_k(\cdot) = \Lambda^N_k G^N$, and $Q_k(\cdot) = \Lambda_k G$. Let the evolution operators S^N_k and S_k correspond to the perturbation of T^N, T by $-B^N(R^N)^{-1}B^{*N}Q^N_k$ and $-BR^{-1}B^*Q_k$ respectively. The theories of the approximation of linear quadratic control problems on a finite time interval (e.g. see [G], [BK1]) guarantee that under Hypothesis 3.1, 3.2, for each k, $Q^N_k(t)$ and $S^N_k(t, s)$ converge strongly to $Q_k(t)$ and $S_k(t, s)$ respectively as $N \to \infty$ for every $t_0 \leq s \leq t \leq t_k$. Furthermore the convergence is uniform in the interval $[t_0, t_k]$, if we replace Hypothesis 3.2 by the following assumptions.

Hypothesis 3.4 (Continuity and uniform convergence) *The operator valued functions $B(t)$, $B^*(t)$, $W(t)$, $R(t)$ are strongly piecewise continuous in t (with only a* *finite number of discontinuity points in any bounded interval); the evolution operators T, T^* are jointly strongly continuous. The convergences in Hypothesis 3.2 are uniform in t and (t,s) on any bounded interval.*

From the theory of the linear quadratic control problem for the infinite time intervals, there are two cases where Q_k^N converges strongly as $k \to \infty$. In the first case, let $Q_{min}^N(t)$ be the minimal uniformly bounded solution of the Riccati integral equation in H^N on the infinite time interval $[t_0, \infty)$. If $G = G^N = 0$, then for any N we have $Q_k^N(t)x^N \to Q_{min}^N(t)x^N$, as $k \to \infty$. Furthermore the convergence is uniform in t for t in any bounded interval $[t_0, t_f]$. In the second case, we assume the following conditions hold:

Hypothesis 3.5 *Assume that there exists a stability solution $Q_s(\cdot)$ of the Riccati integral equation for the infinite time interval infinite dimensional system and there exists a stability solution $Q_s^N(\cdot)$ of the Riccati integral equation on the infinite time interval for each approximate system. (Then the evolution operator S^N corresponding to the perturbation of T^N by $-B^N(R^N)^{-1}B^{*N}Q_s^N$ satisfies $S^N(t,s)x \to 0$, as $t \to \infty$, for $s \geq t_0, x \in H^N$.) Furthermore assume that there exists a constant M such that for each N, $Q_s^N(t) \leq M \cdot I$ for all $t \geq t_0$. Also assume $Q_s(t) \leq M \cdot I$ for all $t \geq t_0$.*

Assuming that Hypothesis 3.5 holds, we let $G = G^N = M \cdot I$ and Q_k^N, Q_k be the solutions of the RIE on $[t_0, t_k]$ satisfying $Q_k^N(t_k) = G^N$, $Q_k(t_k) = G$. Then from our results for stability solutions given in section 2 we have $Q_k^N(t)x^N \to Q_s^N(t)x^N$, $Q_k(t)x \to Q_s(t)x$ as $k \to \infty$ for all $x^N \in H^N$ and $x \in H$.

We note that if $S^N(t,s)x \to 0$ uniformly in N, then we have $Q_k(s)$ and $Q_k^N(s)$ uniformly bounded for all k and N. To see this, we consider Q_k, Q_s as given above. Then we have, using the relationship in (2.7) with $y(t) = S(t,s)x$ where S is the evolution operator corresponding to the perturbation of T by $-BR^{-1}B^*Q_s$,

$$\begin{aligned} < (Q_k(s) - Q_s(s))x, x >_H &\leq < (Q_k(t_k) - Q_s(t_k))y(t_k), y(t_k) >_H \\ &\leq 2M\|y(t_k)\|^2. \end{aligned}$$

Since $y(t_k) \to 0$, it follows using the uniform boundedness principle that $Q_k(s)$ is uniformly bounded for all k. Repeating this argument with $Q_k^N(s), Q_s^N(s)$ and $y^N(t) = S^N(t,s)x$, we see that the uniform (in N) decay of S^N yields the claimed uniform boundedness for $Q_k^N(s)$.

In each of the two cases above, we have the following situation:

$$\begin{array}{ccc} Q_k^N(t)P_H^N x & \overset{k\to\infty}{\to} & Q_s^N(t)P_H^N x \\ N\to\infty \downarrow & & \downarrow ? N\to\infty \\ Q_k(t)x & \overset{k\to\infty}{\to} & Q_s(t)x. \end{array}$$

It is desirable in computations to work directly with Q_s^N and hence we seek results which will guarantee the convergence $Q_s^N \to Q_s$ of this diagram. To obtain such a result, we shall make use of a uniform decay rate for the S^N defined via Q_s^N.

Theorem 3.1 *Assume that Hypothesis 3.1–3.3, 3.5 hold. Further assume that for all $s \geq t_0$, $x \in H$ and $\epsilon > 0$, we can find $\hat{t}$ such that for all $t \geq \hat{t}$, we have $\|S(t,s)x\| \leq \epsilon$ and $\|S^N(t,s)P_H^N x\| \leq \epsilon$ for all N. Then $Q_s^N(t)P_H^N x \to Q_s(t)x$ for all $t_0 \leq t < \infty$.*

Proof: Let M be the bound for Q_s and Q_s^N that are the stability solutions of Hypothesis 3.5. Let Q_k^N and Q_k be the related RIE solutions on $[t_0, t_k]$ satisfying $Q_k^N(t_k) = M \cdot I$, $Q_k(t_k) = M \cdot I$. Then for $t \leq t_k$ we have

$$\begin{aligned} \|Q_s^N(t)P_H^N x - Q_s(t)x\|^2 \;&\leq\; \|(Q_s^N(t) - Q_k^N(t))P_H^N x\|^2 + \|Q_k^N(t)P_H^N x - Q_k(t)x\|^2 \\ &\quad + \|(Q_s(t) - Q_k(t))x\|^2. \end{aligned}$$

Recalling that $Q_k^N(t) \geq Q_s^N(t)$, $Q_k(t) \geq Q_s(t)$ by construction, and using the uniform boundedness of Q_k^N and Q_s^N following from the uniform decay rate and the arguments above, we obtain for some $\hat{M}$

$$\begin{aligned} \|(Q_s^N(t) - Q_k^N(t))P_H^N x\|^2 \;&\leq\; 2\hat{M} < (Q_k^N(t) - Q_s^N(t))P_H^N x, P_H^N x >_H \\ \|(Q_s(t) - Q_k(t))x\|^2 \;&\leq\; 2\hat{M} < (Q_k(t) - Q_s(t))x, x >_H . \end{aligned}$$

Again using (2.7), we have

$$< (Q_k^N(t) - Q_s^N(t))P_H^N x, P_H^N x >_H$$
$$\leq < (Q_k^N(t_k) - Q_s^N(t_k))S^N(t_k,t)P_H^N x, S^N(t_k,t)P_H^N x >_H,$$

and

$$< (Q_k(t) - Q_s(t))x, x >_H \leq < (Q_k(t_k) - Q_s(t_k))S(t_k,t)x, S(t_k,t)x >_H .$$

Combining the above inequalities, we obtain

$$\|Q_s^N(t)P_H^N x - Q_s(t)x\|^2 \leq \|Q_k^N(t)P_H^N x - Q_k(t)x\|^2$$
$$+4M\hat{M}(\|S^N(t_k,t)x\|^2 + \|S(t_k,t)x\|^2).$$

Let k be large enough so that $\|S(t_k,t)x\| \leq \epsilon/(12M\hat{M})$, $\|S^N(t_k,t)P_H^N x\| \leq \epsilon/(12M\hat{M})$, for all N. Then let N be large enough to obtain $\|Q_k^N(t)P_H^N x - Q_k(t)x\|^2 \leq \epsilon/3$. From the previous estimates we thus find $\|Q_s^N(t)P_H^N x - Q_s(t)x\|^2 \leq \epsilon$ which yields the desired results.

We note that if Hypothesis 3.4 holds and the uniform decay assumption in Theorem 3.1 is replaced by the following: there exists $\hat{t}$ such that for any $t \geq \hat{t}$ and for any $s \in [t_0, t_f]$, $\|S(t+s,s)x\| \leq \epsilon$, $\|S^N(t+s,s)P_H^N x\| \leq \epsilon$, for all N, then the convergence of Theorem 3.1 is uniform in the bounded interval $[t_0, t_f]$.

Theorem 3.1 is not very useful in practice, since the uniform decay assumption is difficult to verify directly. However it does provide some insight and suggests more realistic conditions that might be verifiable. Recalling the definition of W-detectability and our discussions following it, we are prompted to formulate the following assumptions.

Hypothesis 3.6 (Uniform W-detectability) *The original system is detectable and there exist constants M_K, M_{KV} and $\beta > 0$ such that for each N, there exists a uniformly bounded operator valued function $K^N(\cdot) : H^N \mapsto H^N$, with $\|K^N(t)\|_{L(H^N)} \leq M_K$, for $t \in [t_0, \infty)$. If $T_{K^N}^N$ is the evolution operator corresponding to the perturbation of T^N by $K^N\sqrt{W^N}$, then $\|T_{K^N}^N(t,s)\|_{L(H^N)} \leq M_{KV}e^{-\beta(t-s)}$, for $(t,s) \in \Delta_\infty(t_0)$.*

If the Hypothesis 3.6 holds, then Q^N_{min} is the unique uniformly bounded solution of the Riccati integral equation on the infinite time interval for H^N. Under the uniform W-stabilizability Hypothesis 3.3, we have $Q^N_{min}(t) \leq M \cdot I$, for all $t \in [t_0, \infty)$. Furthermore, by application of Theorem 2.2, there exist constants $M_s, \alpha > 0$ independent of N such that the evolution operator S^N defined via Q^N_{min} satisfies

$$\|S^N(t,s)\| \leq M_s e^{-\alpha(t-s)}, \quad \text{for } (t,s) \in \Delta_\infty(t_0).$$

Thus, by Theorem 3.1, $Q^N_{min}(t)$ converges to $Q_{min}(t)$ as $N \to \infty$. We summarize the results in a major convergence theorem.

Theorem 3.2 *Assume that our system and its approximate systems satisfy Hypotheses 3.1 and 3.2 and the uniform W-stabilizability and uniform W-detectability conditions of Hypothesis 3.3, 3.6. Then the unique uniformly bounded solution Q^N of the Riccati integral equation on $[t_0, \infty)$ in H^N converges strongly to the unique uniformly bounded solution Q of the Riccati integral equation on the infinite time interval in H. Furthermore, if Hypothesis 3.2 is replaced by Hypothesis 3.4, then this convergence is uniform in t for t in any bounded interval.*

We note that in the case of a periodic system, the uniform convergence in one period implies that Q^N converges to Q uniformly in the entire interval $[t_0, \infty)$. We further remark that the convergence of the Riccati operator guaranteed by Theorem 3.2 is sufficient (using standard arguments, see [G], [BK1]) to guarantee convergence of the optimal approximate feedback system trajectories $S^N(t,s)P^N_H x$ and optimal approximate controls u^N to the optimal system trajectories $S(t,s)x$ and optimal controls u (see Theorem 3.1 of [BK1]). Moreover, one also obtains convergence of the system generated by using the approximate feedback gains with the original infinite dimensional control system (a feature that is of great practical importance), e.g., see the related remarks in section 4 of [BK1].

The hypotheses of Theorem 3.2 are much more readily verified than others guaranteeing convergence that can be found in the literature (e.g., see [G], Theorem

5.3, where one is required to show that the approximate systems are uniformly stabilized by the feedback with a uniformly bounded sequence of approximate Riccati operators). As we shall see in the later sections, there are two distinct approaches that lead to rather easy use of our Theorem 3.2 in the event one is dealing with parabolic evolution systems.

4 Parabolic evolution equations: control and approximation

In this section, we formulate the linear quadratic regulator problem for an abstract parabolic control system. We focus our attention on systems associated with a time dependent sesquilinear form. First we review the theory of parabolic evolution equations (relying heavily on [T]) and extend some related results in a form applicable to control problems. Then a control system is defined for which general assumptions of stabilizability and detectability are made. A framework for approximation schemes is presented and conditions for convergence of the operators involved are discussed under assumptions of uniform stabilizability and uniform detectability for the approximate systems. Our discussions here are in the spirit of the approaches taken in [BK1], [BI1], [BI2].

Let H, V be two complex separable Hilbert spaces with $< \cdot, \cdot >_H$, $< \cdot, \cdot >_V$ as inner products and $\| \cdot \|_H$, $\| \cdot \|_V$ as norms respectively. Let V^* be the dual space of V with $< \cdot, \cdot >_{V^*,V}$ denoting the duality pairing. The space V is assumed to be densely and continuously embedded in H, and thus there exists a constant c such that for all $\psi \in V$, $\|\psi\|_H \leq c\|\psi\|_V$. Since for each element φ of H, we can define a bounded linear functional on V by $< \varphi, \psi >_H$, for $\psi \in V$, we have the usual embedding relationship $V \subset H \subset V^*$.

For each t in the interval $[t_0, \infty)$, consider a sesquilinear form $\sigma(t; \cdot, \cdot)$ defined on $V \times V$. We assume throughout that σ has the following properties:

Hypothesis 4.1 (V-Continuity) *For each bounded interval $[t_0, t_1]$, there exists a con-*

stant c_1 such that

$$(4.1)\qquad |\sigma(t;\varphi,\psi)| \le c_1\|\varphi\|_V\|\psi\|_V, \quad \textit{for } t\in[t_0,t_1], \varphi,\psi\in V.$$

Hypothesis 4.2 (V-Ellipticity) *For each bounded interval $[t_0,t_1]$, there exist constants $c_2 > 0$, m such that*

$$(4.2)\qquad Re\ \sigma(t;\varphi,\varphi) \ge c_2\|\varphi\|_V^2 - m\|\varphi\|_H^2, \quad \textit{for } t\in[t_0,t_1], \varphi\in V.$$

Under the above assumptions, we have a well known ([FM], [K], [T], [S]) result: For each $t \in [t_0,t_1]$, there exists a unique closed operator $A(t) : V \mapsto V^*$ such that

$$(4.3)\qquad \sigma(t;\varphi,\psi) = - < A(t)\varphi,\psi >_{V^*,V}, \quad \text{for } \psi\in V.$$

Furthermore, if $\hat{A}(t)$ is defined using the same method with a sesquilinear form σ^* defined by $\sigma^*(t;\varphi,\psi) = \overline{\sigma(t;\psi,\varphi)}$, then $\hat{A}(t)$ is identical to the adjoint operator $A^*(t)$ of $A(t)$. Both operators $A(t)$, $A(t)^*$ are infinitesimal generators of analytic semigroups in V^* and an abstract parabolic evolution equation can be defined by

$$\frac{d}{dt}x(t) = A(t)x(t), \quad x(t_0) = x_0 \in V^*.$$

In order to insure the existence of an evolution operator for this equation, we must make additional assumptions on the continuity of σ with respect to t.

Hypothesis 4.3 (Smoothness in t) *For each bounded interval $[t_0,t_1]$, there exist constants K and α, $0 < \alpha \le 1$, such that for all $t,s \in [t_0,t_1]$, and for all $\varphi,\psi \in V$, we have*

$$|\sigma(t;\varphi,\psi) - \sigma(s;\varphi,\psi)| \le K|t-s|^\alpha\|\varphi\|_V\|\psi\|_V.$$

Under the above assumptions, there exists an evolution operator associated with the above evolution equation. The following theorem summarizes the properties of this evolution operator.

Theorem 4.1 ([T], pp.127, pp.145-155) *Let the Hypotheses 4.1, 4.2, 4.3 hold. Then there exists a unique evolution operator $\tilde{T}(\cdot,\cdot)$ in V^* satisfying the following conditions:*

(i) *For any* $t_0 \leq s < t \leq t_1$, *the range* $\mathcal{R}(\tilde{T}(t,s))$ *of operator* $\tilde{T}(t,s)$ *is a subset of* V.

(ii) *The operator* $\tilde{T}(t,s)A(s)$ *has a unique bounded extension in* $L(V^*)$, *for all* $t_0 \leq s < t \leq t_1$; *therefore, we can and will use the same expression for the extension.*

(iii) *For each* $\varphi \in V^*$, *the* V^**-valued function* $\tilde{T}(t,s)\varphi$ *is continuously differentiable in* t *for* $t \in (s, t_1]$, *and continuously differentiable in* s *for* $s \in [t_0, t)$. *Furthermore, for* $\varphi \in V^*$

$$\begin{aligned} \frac{d}{dt}\tilde{T}(t,s)\varphi &= A(t)\tilde{T}(t,s)\varphi, \\ \frac{d}{ds}\tilde{T}(t,s)\varphi &= -\tilde{T}(t,s)A(s)\varphi. \end{aligned}$$

(iv) *The restriction of* $\tilde{T}(t,s)$ *on* H *is strongly continuous in the* H *norm. For all* $x_0 \in H$, *the function* $x(t) = \tilde{T}(t,s)x_0$ *is in* $L^2([s,t_1];V)$ *and the derivative* $\dot{x}(t) = A(t)\tilde{T}(t,s)x_0$ *is in* $L^2([s,t_1];V^*)$. *Furthermore, there exist constants* C_1, C_2, *depending only on* c_1, c_2, m, K *and* α *such that*

$$\|\tilde{T}(t,s)x_0\|_V \leq C_1(t-s)^{-1/2}\|x_0\|_H. \tag{4.4}$$

$$\|\tilde{T}(\cdot,s)x_0\|_{L^2([s,t_1];V)} \leq C_2\|x_0\|_H. \tag{4.5}$$

All the statements in the above theorem can be found in [T]. However they are organized into several sections with somewhat different notation; we therefore give a brief argument which collects the results from the book.

Proof: Existence. Taking $X = V^*$, we let $A(t)$ be defined as in (4.3). As indicated in [T], (pp. 144), using Theorem 5.2.1 of [T], we find there exists an evolution operator $\tilde{T}$ on V^*. The range $\mathcal{R}(\tilde{T}(t,s))$ is a subset of $\mathcal{D}(A(t)) = V$ for all $t_0 \leq s < t \leq t_1$. For any $\varphi \in V^*$, $\tilde{T}(t,s)\varphi$ is continuously differentiable in t for $t \in (s, t_1]$. Now let the sesquilinear form σ^* be defined by

$$\sigma^*(t;\varphi,\psi) = \overline{\sigma(t;\psi,\varphi)}, \quad \varphi,\psi \in V.$$

Let $A^*(t)$ be the linear operator defined via σ^*; then $A^*(t)$ is the adjoint operator of $A(t)$. As indicated by the remarks following Lemma 5.4.6 of [T], we can use the results of section 5.2 (with $\tilde{S}$ and A^* replacing U and A of [T]) to construct an operator-valued function $\tilde{S}(t,s)$, such that for all $t_0 \leq s < t \leq t_1$, $A^*(s)\tilde{S}(t,s)$ is a bounded operator in V^*, and for any $\varphi \in V^*$, $\tilde{S}(t,s)\varphi$ is continuously differentiable in s for $s \in [t_0, t)$. Furthermore, for $\varphi \in V^*$

$$\frac{d}{ds}\tilde{S}(t,s)\varphi = -A^*(s)\tilde{S}(t,s)\varphi.$$

In fact, $\tilde{S}(t,s)$ can be constructed as follows. Let $\exp\{tA^*(s)\}$ be the semi-group generated by $A^*(s)$ and we define

$$\begin{aligned}\tilde{S}(t,s) &= \exp\{(t-s)A^*(t)\} + W(t,s),\\ W(t,s) &= \int_s^t \exp\{(\tau - s)A^*(\tau)\}R(t,\tau)d\tau,\end{aligned}$$

where the function R can be computed by iterative methods using

$$R(t,s) - \int_s^t R_1(\eta,s)R(t,\eta)d\eta = R_1(t,s),$$

with $R_1(t,s) = (A^*(t) - A^*(s))\exp\{(t-s)A^*(t)\}$. Then following the same type of arguments as in [T] (pp. 149), we can conclude that $\tilde{S}(t,s) = \tilde{T}^*(t,s)$. Therefore $\tilde{T}(t,s)A(s)$ has a unique bounded extension in V^*. For all $\varphi \in V^*$, $\tilde{T}(t,s)\varphi$ is strongly differentiable in s for $s \in [t_0, t)$.

Finally statement (iv) of the above theorem can be found in the sections 5.4 and 5.5 of [T]. We note that in these sections of [T], the space X plays the role of our space H. Let $T(t,s)$ be the restriction of $\tilde{T}(t,s)$ to H; by Theorem 5.4.1 of [T], $T(t,s)$ is strongly continuous in the H norm. Furthermore the estimate (4.4) holds. For any $x_0 \in H$, let $x(t) = T(t,s)x_0$. By Lemma 5.5.2 and Proposition 5.5.1 of [T] (pp. 152 – 153) with $f \equiv 0$, the function $x(\cdot)$ is in $L^2([s,t_1];V)$ and $\dot{x}(\cdot)$ is in $L^2([s,t_1];V^*)$. In addition, the estimate (4.5) holds. We note that the constants C_1, C_2 depend only on the constants c_1, c_2, m, K and α.

Uniqueness. By Theorem 5.2.3 of [T] (pp. 128), we can conclude that the evolution operator satisfying the conditions (i) – (iv) must be unique.

We remark that the same theorem holds if we use the sesquilinear form σ^*; therefore, properties (i) – (iv) hold for the adjoint evolution operator $\tilde{T}^*(\cdot,\cdot)$. As consequence of (iv), the restriction T of $\tilde{T}$ to H is an evolution operator in H as defined in the section 2. We wish to take H as our state space; therefore we use primarily the evolution operator T in this paper. The operator $\tilde{T}$ is used in the remainder of the current section in several proofs of uniqueness theorems. The only precaution one must take is that $T(t,s)\varphi$ is continuously differentiable with respect to t in the V^* sense and the derivative of $T(t,s)\varphi$ is an element of V^*. In particular, for each $\psi \in V$, $< T(t,s)\varphi, \psi >_H$ is differentiable with respect to t, and

$$\frac{d}{dt} < T(t,s)\varphi, \psi >_H = < A(t)T(t,s)\varphi, \psi >_{V^*,V} = -\sigma(t; T(t,s)\varphi, \psi).$$

The conclusions of this theorem are very useful in defining our control system; however, the conditions of Hypothesis 4.3 are too restrictive for our use, since we may need to perturb the equation with nonsmooth but bounded (feedback) terms. We can show that if σ is perturbed with a sesquilinear form that is uniformly bounded in H, then there exists an associated evolution operator T_K which preserves most of the desirable properties of the evolution operator T. In fact, let $K(\cdot) : [t_0, \infty) \mapsto L(H)$ be a uniformly bounded operator valued measurable function. We can then define a sesquilinear form σ_K in $V \times V$ as

$$\sigma_K(t; \varphi, \psi) = \sigma(t; \varphi, \psi) - < K(t)\varphi, \psi >_H, \quad \varphi, \psi \in V.$$

It is easy to see that for each bounded interval $[t_0, t_1]$, Hypotheses 4.1 and 4.2 hold. Therefore, we can find an operator $A_K(t)$ defined on V such that (4.3) holds for σ_K and $A_K(t)$. Furthermore, we have by the definitions of $A(t)$ and $A_K(t)$ that $A_K(t)\varphi = A(t)\varphi + K(t)\varphi$ for $\varphi \in V$ and we may establish the following result.

Theorem 4.2 *Consider a sesquilinear form σ satisfying Hypotheses 4.1, 4.2, 4.3 and* *let $K(\cdot)$, σ_K be defined as above. Then there exists a unique evolution operator $T_K(\cdot,\cdot)$ in H for which the following properties hold:*

(i) *The range $\mathcal{R}(T_K(t,s))$ of the operator $T_K(t,s)$ is a subset of V, for all $t_0 \le s < t \le t_1$.*

(ii) *For $\varphi \in H$, the function $T_K(t,s)\varphi$ is differentiable with respect to t in the V^* sense, and*

$$\frac{d}{dt}T_K(t,s)\varphi = A_K(t)T_K(t,s)\varphi.$$

(iii) *For all $x_0 \in H$, the function $x(t) = T_K(t,s)x_0$ is in $L^2([s,t_1];V)$ and its derivative $\dot{x}(t)$ is in $L^2([s,t_1];V^*)$. Furthermore, there exists constants C_1, C_2 depending only on c_1, c_2, m, K and α such that*

$$\|T_K(t,s)x_0\|_V \le C_1(t-s)^{-1/2}\|x_0\|_H,$$
$$\|T_K(\cdot,s)x_0\|_{L^2([s,t_1];V)} \le C_2\|x_0\|_H.$$

Proof: Existence. Let T_K be the unique evolution operator in H corresponding to the perturbation of T by K. From the results on perturbations given in section 2, we have that T_K satisfies for all $\varphi \in H$

$$(4.6) \qquad \begin{aligned} T_K(t,s)\varphi &= T(t,s)\varphi + \int_s^t T(t,\eta)K(\eta)T_K(\eta,s)\varphi d\eta, \\ T_K(t,s)\varphi &= T(t,s)\varphi + \int_s^t T_K(t,\eta)K(\eta)T(\eta,s)\varphi d\eta. \end{aligned}$$

Since the function $K(\eta)T_K(\eta,s)\varphi$ is uniformly bounded in H norm by some constant C, using the estimate (4.4), we can find a constant $\tilde{C}$ such that for $\eta \in [s,t]$

$$\|T(t,\eta)K(\eta)T_K(\eta,s)\varphi\|_V \le \tilde{C}(t-\eta)^{-1/2}\|\varphi\|_H.$$

Therefore the integral term in equation (4.6) converges in the V sense and hence, $T_K(t,s)\varphi \in V$ for $\varphi \in H$.

Consider $x_0 \in H$, we define $x(t) = T_K(t,t_0)x_0$, and $f(t) = K(t)x(t)$. From equation (4.6), the function $x(t)$ can be written as

$$x(t) = T(t,t_0)x_0 + \int_{t_0}^t T(t,\eta)f(\eta)d\eta.$$

By the strong continuity of T_K and uniform boundedness of K, it is obvious that $f(\cdot) \in L^2([t_0, t_1]; H)$ and hence $f(\cdot) \in L^2([t_0, t_1]; V^*)$. By Theorem 5.5.1 of [T], $x(\cdot)$ is in $L^2([t_0, t_1]; V)$, is differentiable with $\dot{x}(\cdot)$ in $L^2([t_0, t_1]; V^*)$ and satisfies $\dot{x}(t) = A_K(t)x(t)$. Using the equality (4.6) and the boundedness of the perturbation, by modifying the constants C_1, C_2 in (4.4), (4.5), we can easily obtain

$$\|T_K(t,s)\varphi\|_V \leq C_1(t-s)^{-1/2}\|\varphi\|_H,$$
$$\|T_K(\cdot, t_0)\varphi\|_{L^2([t_0,t_1];V)} \leq C_2\|\varphi\|_H,$$

for all $\varphi \in H$.

Uniqueness. Let $\hat{T}_K$ satisfy the conclusions (i) – (ii) of Theorem 4.2. For all $\varphi \in H$, consider $\hat{T}_K(t,s)\varphi$ as a V^* valued function. Then we have

$$\begin{aligned} \frac{d}{d\eta}\tilde{T}(t,\eta)\hat{T}_K(\eta,s)\varphi &= -\tilde{T}(t,\eta)A(\eta)\hat{T}_K(\eta,s)\varphi \\ &\quad +\tilde{T}(t,\eta)(A(\eta)+K(\eta))\hat{T}_K(\eta,s)\varphi \\ &= \tilde{T}(t,\eta)K(\eta)\hat{T}_K(\eta,s)\varphi. \end{aligned}$$

Integrating both sides of the above equation from s to t, we obtain

$$\hat{T}_K(t,s)\varphi = \tilde{T}(t,s)\varphi + \int_s^t \tilde{T}(t,\eta)K(\eta)\hat{T}_K(\eta,s)\varphi d\eta.$$

Since T is the restriction of $\tilde{T}$ to H, $\hat{T}_K$ is a solution of (4.6). By our uniqueness results of section 2 for perturbed evolution operators, we have $\hat{T}_K = T_K$. Hence the unique solution of (4.6) is the unique evolution operator T_K generated in the theorem.

Now consider a function $f(\cdot) \in L^2([t_0, t_1]; H)$. We can then define

$$z(t) = T(t,t_0)z_0 + \int_{t_0}^t T(t,\eta)f(\eta)d\eta.$$

The function $z(\cdot)$ is the unique solution of the following initial value problem:

$$\begin{aligned} z(t) &= T(t,s)z(s) + \int_s^t T(t,\eta)f(\eta)d\eta, \quad t_0 \leq s \leq t \leq t_1. \\ z(t_0) &= z_0. \end{aligned} \tag{4.7}$$

Henceforth, we consider (4.7) as the definition of our basic evolution system. The function z corresponds to the solution of a weaker formulation of the evolution equation.

Lemma 4.1 ([T], Theorem 5.5.1) *The function $z(\cdot)$ given by (4.7) is the unique function in $L^2([t_0, t_1]; V)$ with derivative $\dot{z}(\cdot)$ in $L^2([t_0, t_1]; V^*)$ for which the following equation holds for $\psi \in V$*

$$(4.8) \qquad \begin{aligned} < z(t) - z(t_0), \psi >_H &= \int_{t_0}^{t} \{-\sigma(\eta; z(\eta), \psi) + < f(\eta), \psi >_H\} \, d\eta, \\ z(t_0) &= z_0. \end{aligned}$$

Lemma 4.2 ([T], Lemma 5.5.1) *For any two functions $z(\cdot), w(\cdot)$ in $L^2([t_0, t_1]; V)$ with derivatives $\dot{z}, \dot{w}$ in $L^2([t_0, t_1]; V^*)$, the following equality holds:*

$$\begin{aligned} < z(t), w(t) >_H &= < z(s), w(s) >_H \\ &\quad + \int_s^t \left\{< \dot{z}(\eta), w(\eta) >_{V^*,V} + \overline{< \dot{w}(\eta), z(\eta) >_{V^*,V}}\right\} d\eta, \end{aligned}$$

for all $t_0 \le s \le t \le t_1$.

As a consequence, for any $x_0 \in H$, let $x(t) = T(t, t_0)x_0$, then

$$(4.9) \qquad \|x(t)\|_H^2 = \|x_0\|_H^2 - 2\int_{t_0}^{t} \text{Re } \sigma(\eta; x(\eta), x(\eta)) d\eta.$$

We note that if Hypotheses 4.1 - 4.3 hold, then for each bounded interval $[t_0, t_1]$, we can define $T(t, s)$ uniquely, therefore $T(t, s)$ is also uniquely defined for all $t_0 \le s \le t < \infty$. The equality (4.9) suggests a sufficient condition for the stability of T.

Hypothesis 4.4 *There exists a constant $k > 0$ such that*

$$Re \; \sigma(t; \varphi, \varphi) \ge k\|\varphi\|_H^2, \quad for \; t_0 \le t < \infty, \quad \varphi \in V.$$

Theorem 4.3 *Under Hypothesis 4.4, T is uniformly exponentially stable.*

Proof: For any $x_0 \in H$, let $x(t) = T(t,s)x_0$. Then by (4.9), we have

$$\|x(t)\|_H^2 \le \|x_0\|_H^2 - 2\int_s^t k\|x(\eta)\|^2 d\eta,$$

for all $t_0 \le s \le t < \infty$. This implies

$$\|T(t,s)\|_{L(H)} \le 1, \quad \int_s^t \|T(\eta,s)x_0\|_H^2 \le \frac{1}{2k}\|x_0\|_H^2.$$

Therefore, by Lemma 2.1, T is uniformly exponentially stable. Note, moreover that by Lemma 2.1, under Hypothesis 4.4, we can find $M, \alpha > 0$ depending only on k such that

$$\|T(t,s)\|_{L(H)} \le Me^{-\alpha(t-s)}.$$

We can now use these considerations to define an evolution equation control system of the form (4.7) via a sesquilinear form. The space H will serve as our state space, with subspace V and the sesquilinear form σ defined as above and Hypotheses 4.1–4.3 holding. Let the control space U be a Hilbert space, and let $B(\cdot) : [t_0, \infty) \mapsto L(U,H)$ be a measurable operator-valued function. We assume that there exists a constant M_B such that

$$\|B(t)\|_{L(U,H)} \le M_B, \quad \text{for } t \in [t_0, \infty).$$

For any control $u(\cdot) : [t_0, \infty) \mapsto U$, belonging to $L^2([t_0,\infty);U)$, the corresponding trajectories satisfy for $\psi \in V$

$$(4.10) \qquad < z(t) - z(s), \psi >_H = -\int_s^t \{\sigma(\eta; z(\eta), \psi) - < B(\eta)u(\eta), \psi >_H\} \, d\eta,$$

for all $(t,s) \in \Delta_\infty(t_0)$. Let $T(\cdot,\cdot)$ be the evolution operator defined via σ. By Lemma 4.1, an equivalent form of (4.10) is given by

$$(4.11) \qquad z(t) = T(t,s)z(s) + \int_s^t T(t,\eta)B(\eta)u(\eta)d\eta, \quad \text{for } (t,s) \in \Delta_\infty(t_0).$$

Let $z_0 \in H$ be the initial state of the system at t_0 and let the cost for control $u(\cdot)$ be given by

$$(4.12) \qquad J_\infty(u; z_0, t_0) = \int_{t_0}^\infty < W(t)z(t), z(t) >_H + < R(t)u(t), u(t) >_U \, dt,$$

where $W(\cdot) : [t_0, \infty) \mapsto L(H)$, $R(\cdot) : [t_0, \infty) \mapsto L(U)$. The operators $W(t), R(t)$ are assumed to be selfadjoint nonnegative definite operators, uniformly bounded in the entire interval $[t_0, \infty)$. Furthermore, there exists a constant $r > 0$ such that

$$< R(t)v, v >_U \quad \geq \quad r\|v\|_U^2, \quad \text{for } t \geq t_0, \quad v \in U.$$

Recalling the discussions of section 2, we note that the standing assumptions of that section hold. Therefore, for a given nonnegative definite self-adjoint operator G on H, the Riccati integral equation in each finite time interval $[t_0, t_k]$,

$$\begin{aligned} Q_k(s)x &= T^*(t_k, s)GT(t_k, s)x \\ &\quad + \int_s^{t_k} T^*(\eta, s)\left[W(\eta) - Q_k(\eta)B(\eta)R^{-1}(\eta)B^*(\eta)Q_k(\eta)\right] T(\eta, s)x d\eta \end{aligned}$$

has a unique self-adjoint solution Q_k.

For the control problem in the infinite time interval $[t_0, \infty)$, we need stabilizability and detectability conditions to assure existence and uniqueness of a uniformly bounded solution of the Riccati integral equation.

Hypothesis 4.5 (Detectability) *There exists a uniformly bounded operator valued function $\Psi(\cdot) : [t_0, \infty) \mapsto L(H)$, such that if we denote by S_Ψ the evolution operator corresponding to the perturbation of T by $\Psi(\cdot)W^{1/2}(\cdot)$, the following estimate holds for $x \in H$:*

$$\|S_\Psi(t, s)x\|_H \leq Me^{-\omega(t-s)}\|x\|_H,$$

for some constants $M, \omega > 0$.

Hypothesis 4.6 (Stabilizability) *There exists a uniformly bounded operator valued function $K(\cdot) : [t_0, \infty) \mapsto L(H, U)$, such that if we denote by S_K the evolution operator corresponding to the perturbation of T by $B(\cdot)K(\cdot)$, the following estimate holds for $x \in H$:*

$$\|S_K(t, s)x\|_H \leq Me^{-\omega(t-s)}\|x\|_H,$$

for some constants $M, \omega > 0$.

We remark that Hypothesis 4.6 is stronger than "W-Stabilizability"; however under the Hypothesis 4.5 by Theorem 2.2, these two types of stabilizability assumption are equivalent.

To this point we have defined a control system using an abstract parabolic evolution equation that fits into the general framework of section 3. Under Hypotheses 4.5 and 4.6, we may apply the theory of the previous sections to establish the following results for our control problem:

(i) The Riccati integral equation in the infinite time interval $[t_0, \infty)$ has a unique uniformly bounded solution $Q(\cdot)$.

(ii) Let S_Q be the evolution operator corresponding to the perturbation of $T(\cdot,\cdot)$ by $-BR^{-1}B^*Q(\cdot)$. For each initial state z_0, the unique optimal trajectory is given by $S_Q(t,t_0)z_0$.

We turn next to give results for finite dimensional approximations of our control system. As in section 3, let $\{H^N\}_{N=1}^{\infty}$ be a sequence of finite dimensional subspaces of $V \subset H$. Let P_H^N be the orthogonal projection operator from H onto H^N. Since H^N is an approximation of H, we assume that for every $\varphi \in H$, $\|P_H^N\varphi - \varphi\|_H \to 0$, as $N \to \infty$. In addition, we require that H^N is an approximation of V as well, so that for all $\varphi \in V$, $\|P_H^N\varphi - \varphi\|_V \to 0$, as $N \to \infty$. We note that in fact this latter convergence implies the convergence in H for $\varphi \in H$ since V is continuously and densely embedded in H.

Let $\{U^N\}_{N=1}^{\infty}$ be a sequence of finite dimensional subspaces of U. Let P_U^N be the orthogonal projection operator from U onto U^N. We assume U^N approximates U in the following sense: for $v \in U$, $\|P_U^N v - v\|_U \to 0$, as $N \to \infty$.

For each N, we define a sesquilinear form σ^N as the restriction of σ to $H^N \times H^N$ and define a linear operator $A^N(t) : H^N \mapsto H^N$ by

$$- < A^N(t)\varphi^N, \psi^N >_H = \sigma^N(t; \varphi^N, \psi^N), \quad \text{for } \varphi^N, \psi^N \in H^N.$$

By continuity of σ with respect to t, the operator valued function $A^N(t)$ is continuous in time. As a consequence, there exists a unique differentiable evolution operator $T^N(\cdot,\cdot)$ in H^N generated by $A^N(t)$; that is for $\varphi^N \in H^N$ we have

$$\frac{d}{dt}T^N(t,s)\varphi^N = A^N(t)T^N(t,s)\varphi^N.$$

Note immediately that the σ^N's satisfy the Hypotheses 4.1–4.3 with the same constants c_1, c_2, m, α and K, therefore for each fixed interval $[t_0, t_1]$, there exist constants C_1, C_2 independent of N such that for all $\varphi^N \in H^N$,

$$\|T^N(t,s)\varphi^N\|_V \le C_1(t-s)^{-\frac{1}{2}}\|\varphi^N\|_H; \tag{4.13}$$

$$\|T^N(\cdot,t_0)\varphi^N\|_{L^2([t_0,t_1];V)} \le C_2\|\varphi^N\|_H. \tag{4.14}$$

The approximation properties of the evolution operator T^N are summarized by the the following convergence theorem.

Theorem 4.4 *Let Hypotheses 4.1– 4.3 hold and let $T(\cdot,\cdot)$ and $T^N(\cdot,\cdot)$ be defined as above where $\|P_H^N\varphi - \varphi\|_V \to 0$ as $N \to \infty$ for $\varphi \in V$; then the following properties hold:*

(i) *There exist constants M_T and ω such that for all N,*

$$\|T^N(t,s)\|_{L(H^N)} \le M_T e^{\omega(t-s)}, \quad \|T(t,s)\|_{L(H)} \le M_T e^{\omega(t-s)}, \quad t_0 \le s \le t < \infty.$$

(ii) *For any finite interval $[a,b] \subset [t_0,\infty)$ and any $\varphi \in H$, we have*

$$\|T^N(t,s)P_H^N\varphi - T(t,s)\varphi\|_H \to 0, \quad asN \to \infty, \quad a \le s \le t \le b.$$

Furthermore the convergence is uniform for all $a \le s \le t \le b$.

Proof: (i) By Lemma 4.2, and (4.9) for every $\varphi \in H^N$ we have

$$\|T^N(t,s)\varphi\|_H^2 = \|\varphi\|_H^2 - 2\int_s^t \mathrm{Re}\,\sigma^N(\eta; T^N(\eta,s)\varphi, T^N(\eta,s)\varphi)d\eta.$$

Under Hypothesis 4.2,

$$\|T^N(t,s)\varphi\|_H^2 \le \|\varphi\|_H^2 + 2\int_s^t m\|T^N(\eta,s)\varphi\|_H^2 d\eta.$$

Using Gronwall's inequality, we obtain

$$\|T^N(t,s)\varphi\|_H^2 \leq \|\varphi\|_H^2 e^{2|m|(t-s)}.$$

Noting that the same estimates hold for $T(t,s)\varphi$, we obtain (i).

(ii) Let $\varphi \in H$, define $w(t) = T(t,s)\varphi$ and $w^N(t) = T^N(t,s)P_H^N\varphi$ and let $z^N(t) = w(t) - w^N(t)$. We note that $z^N(t)$ is not an element of H^N, in fact

$$z^N(t) - P_H^N z^N(t) = w(t) - P_H^N w(t). \tag{4.15}$$

Since $w(t)$ is differentiable in the V^* sense, $w^N(t)$ is differentiable, and both functions are in $L^2([a,b];V)$ with derivatives in $L^2([a,b];V^*)$. By (4.9), Lemma 4.2 and definitions of the operators $A(t), A^N(t)$, we obtain

$$\begin{aligned}
&\|z^N(t)\|_H^2 \\
&= \|z^N(0)\|_H^2 + 2\int_s^t \text{Re } < A(\eta)w(\eta) - A^N(\eta)w^N(\eta), z^N(\eta) >_{V^*,V} d\eta \\
&= \|z^N(0)\|_H^2 - 2\int_s^t \text{Re } \left\{\sigma(\eta; w(\eta), z^N(\eta)) - \sigma^N(\eta; w^N(\eta), P_H^N z^N(\eta))\right\} d\eta \\
&-2\int_s^t < A^N(\eta)w^N(\eta), z^N(\eta) - P_H^N z^N(\eta) >_{V^*,V} d\eta.
\end{aligned}$$

Since the duality pairing reduces to the H-inner product on $H \times H$, we have

$$\begin{aligned}
&< A^N(\eta)w^N(\eta), z^N(\eta) - P_H^N z^N(\eta) >_{V^*,V} \\
&=< A^N(\eta)w^N(\eta), z^N(\eta) - P_H^N z^N(\eta) >_H .
\end{aligned}$$

Moreover, P_H^N is the orthogonal projection operator and hence the last term in the above equation equals to zero. Using the definition of σ^N, the sesquilinearity of σ and (4.15), we find

$$\begin{aligned}
\|z^N(t)\|_H^2 &= \|\varphi - P_H^N\varphi\|_H^2 - 2\int_s^t \text{Re } \sigma(\eta; w(\eta), z^N(\eta) - P_H^N z^N(\eta))d\eta \\
&-2\int_s^t \text{Re } \sigma(\eta; w(\eta), P_H^N z^N(\eta))d\eta \\
&+2\int_s^t \text{Re } \sigma^N(\eta; w^N(\eta), P_H^N z^N(\eta))d\eta
\end{aligned}$$

$$
\begin{aligned}
&= \|\varphi - P_H^N\varphi\|_H^2 - 2\int_s^t \text{Re } \sigma(\eta; w(\eta), w(\eta) - P_H^N w(\eta))d\eta \\
&\quad -2\int_s^t \text{Re } \sigma(\eta; w(\eta) - P_H^N w(\eta), P_H^N z^N(\eta))d\eta \\
&\quad -2\int_s^t \text{Re } \sigma^N(\eta; P_H^N z^N(\eta), P_H^N z^N(\eta))d\eta.
\end{aligned}
$$

Since P_H^N is the orthogonal projection, we find $< P_H^N z^N, w - P_H^N w >_H = 0$, so that from (4.15)

$$
\|P_H^N z^N(t)\|_H^2 = \|z^N(t)\|_H^2 - \|w(t) - P_H^N w(t)\|_H^2.
$$

Combining this with the previous equation, we have

$$
\|P_H^N z^N(t)\|_H^2 = \Theta^N(t,s) - 2\int_s^t \text{Re } \sigma^N(\eta; P_H^N z^N(\eta), P_H^N z^N(\eta))d\eta,
$$

where $\Theta^N(t,s)$ is given by

$$
\begin{aligned}
\Theta^N(t,s) &= \|\varphi - P_H^N\varphi\|_H^2 - \|w(t) - P_H^N w(t)\|_H^2 \\
&\quad -2\int_s^t \text{Re } \sigma(\eta; w(\eta), w(\eta) - P_H^N w(\eta))d\eta \\
&\quad -2\int_s^t \text{Re } \sigma(\eta; w(\eta) - P_H^N w(\eta), P_H^N z^N(\eta))d\eta.
\end{aligned}
$$

Using the V-ellipticity of σ^N, we find

$$
\|P_H^N z^N(t)\|_H^2 \le |\Theta^N(t,s)| + 2|m|\int_s^t \|P_H^N z^N(\eta)\|_H^2 d\eta. \tag{4.16}
$$

To use Gronwall's inequality to conclude convergence of $P_H^N z^N$, it suffices to show that $|\Theta^N(t,s)|$ goes to zero uniformly for all $a \le s \le t \le b$. By the continuity and uniform boundedness of T, the term $\|T(t,s)\varphi - P_H^N T(t,s)\varphi\|_H^2$ goes to zero uniformly for all $a \le s \le t \le b$. Using the V-continuity of σ, the two integrals in Θ can be bounded by

$$
\begin{aligned}
&2c_1\int_s^t \left\{\|P_H^N w(\eta) - w(\eta)\|_V \|P_H^N z^N(\eta)\|_V + \|w(\eta)\|_V \|P_H^N w(\eta) - w(\eta)\|_V\right\} d\eta \\
&\le 2c_1 \left[\|w(\cdot)\|_{L^2([a,b];V)} + \|P_H^N z^N(\cdot)\|_{L^2([a,b];V)}\right] \left[\int_s^t \|w(\eta) - P_H^N w(\eta)\|_V^2 d\eta\right]^{\frac{1}{2}}.
\end{aligned}
$$

Using the inequalities (4.5) and (4.14), we observe that the functions w, w^N are in a bounded subset of $L^2([a,b];V)$. By dominated convergence arguments, the above integral converges to zero. Furthermore by taking $t=b$, $s=a$, we obtain that this convergence is uniform for all $a \le s \le t \le b$. Therefore $|\Theta^N(t,s)|$ converges to zero uniformly for all $a \le s \le t \le b$. Finally, from (4.15), we have

$$\|T(t,s)\varphi - T^N(t,s)P_H^N\varphi\|_H^2 = \|P_H^N z^N(t)\|_H^2 + \|w(t) - P_H^N w(t)\|_H^2,$$

and the uniform convergence of $P_H^N z^N(t)$ in t implies $T^N(t,s)P^N\varphi$ converges to $T(t,s)\varphi$ uniformly for all $a \le s \le t \le b$.

Since we can define operators $A^*(t), A^{*N}, T^*(t,s)$ and $T^*(t,s)$ by using the sesquilinear form σ^* as we indicated after (4.3), the convergence of $T^{*N}(t,s)$ to $T^*(t,s)$ can be shown using the same arguments as in the proof of the above theorem.

Having defined our approximate (uncontrolled) system and established the convergence of Theorem 4.4, we return to the control problem for (4.10)–(4.12). Approximations of functions B, W, R are defined as follows:

$$\begin{array}{lll} B^N(\cdot) : [t_0,\infty) \mapsto L(U^N, H^N), & B^N(t)v^N = P_H^N B(t)v^N, & v^N \in U^N; \\ W^N(\cdot) : [t_0,\infty) \mapsto L(H^N), & W^N(t)\varphi^N = P_H^N W(t)\varphi^N, & \varphi^N \in H^N; \\ R^N(\cdot) : [t_0,\infty) \mapsto L(U^N), & R^N(t)v^N = P_U^N R(t)v^N, & v^N \in U^N. \end{array}$$

Let G be the nonnegative selfadjoint operator in the finite interval cost functional associated in the usual manner with (4.12) for our control system in H. Let $G^N = P_H^N G$ and $z_0^N = P_H^N z_0$.

In each subspace H^N, a finite dimensional control system is thus defined by

$$z^N(t) = T^N(t,s)z^N(s) + \int_s^t T^N(t,\eta)B^N(\eta)u^N(\eta)d\eta, \tag{4.17}$$

with $u^N(\cdot) \in L^2([t_0,\infty);U^N)$ and $z^N(t_0) = z_0^N$. The cost functionals for the associated finite time interval problems are given by

$$\begin{aligned} J^N(u^N; z_0^N, t_0, t_k) &= < G^N z^N(t_k), z^N(t_k) >_H \\ &+ \int_{t_0}^{t_k} \left\{ < W^N(t)z^N(t), z^N(t) >_H + < R^N(t)u^N(t), u^N(t) >_U \right\} dt, \end{aligned} \tag{4.18}$$

while the cost functional for the infinite time interval problem is given by

$$(4.19) \qquad J_\infty^N(u^N; z_0^N; t_0) = \int_{t_0}^{\infty} \left\{ < W^N(t)z^N(t), z^N(t) >_H + < R^N(t)u^N(t), u^N(t) >_U \right\} dt.$$

To obtain the uniform convergence of the operator valued functions, we make additional assumptions on the continuity of B, W, R.

Hypothesis 4.7 (Parameter smoothness) *The operator valued functions B, W, R are piecewise strongly continuous functions on $[t_0, \infty)$.*

Lemma 4.3 *Under Hypothesis 4.7, the following convergence is uniform in t for t in any bounded interval:*

$$\begin{aligned} &\|B^N(t)P_U^N v - B(t)v\|_H \to 0, && v \in U; \\ &\|B^{*N}(t)P_H^N \varphi - B^*(t)\varphi\|_U \to 0, && \varphi \in H; \\ &\|W^N(t)P_H^N \varphi - W(t)\varphi\|_H \to 0, && \varphi \in H; \\ &\|R^N(t)P_U^N v - R(t)v\|_U \to 0, && v \in U, \end{aligned}$$

as $N \to \infty$. The operator $G^N P_H^N$ also converges strongly to G as $N \to \infty$.

Proof: We only prove the uniform convergence of B^N, the remainder of the arguments being similar. For simplicity, we without loss of generality assume that the function $t \mapsto B(t)$ is strongly continuous. For a given $v \in U$, the pointwise convergence of the functions $B^N(t)P_U^N v$ to $B(t)v$ is given by our assumptions on the approximation properties of the spaces H^N, U^N. To conclude uniform convergence in t, it is enough to show that the functions $B^N(t)P_U^N v$ are equi-continuous. That is, for any $\epsilon > 0$, there exists $\delta > 0$ such that for all N, if $|t-s| \leq \delta$, we have $\|B^N(t)P_U^N v - B^N(s)P_U^N v\|_H \leq \epsilon$. By definition of B^N, we have

$$\begin{aligned} \|B^N(t)P_U^N v - B^N(s)P_U^N v\|_H &\leq \|B(t)P_U^N v - B(s)P_U^N v\|_H \\ &\leq \|B(t)(P_U^N v - v)\|_H + \|B(t)v - B(s)v\|_H \\ &\quad + \|B(s)(P_U^N v - v)\|_H \\ &\leq 2M_B \|P_U^N v - v\|_U + \|B(t)v - B(s)v\|_H. \end{aligned}$$

By continuity of B, we conclude that $B^N(t)P_U^N v$ are equi-continuous functions of t in a bounded interval. Hence the convergence is uniform in any bounded interval.

It is easy to verify that B^N, W^N, R^N are uniformly bounded and W^N, R^N are nonnegative self-adjoint operators. In addition there exists a constant $r > 0$ such that for all N,

$$< R^N(t)v^N, v^N >_U \geq r\|v^N\|_U^2, \quad \text{for } t \in [t_0, \infty), v^N \in U^N.$$

Consider any finite time interval $[t_0, t_k]$, and let Q_k, Q_k^N be the unique self-adjoint solutions of the Riccati integral equations in H and H^N associated with the control systems (4.11), (4.17) respectively. Then it follows from Theorem 4.4 and the discussions of section 3 (in particular see Hypotheses 3.1, 3.2 and the remarks just prior to Hypothesis 3.4) that for each k, $Q_k^N(t)$ converges to $Q_k(t)$ strongly and the convergence is uniform in t for $t \in [t_0, t_k]$.

We assumed above (Hypotheses 4.5, 4.6) that the control system (4.10)–(4.12) in H is detectable and stabilizable. Therefore there exists a unique uniformly bounded solution Q of the Riccati integral equation on the infinite time interval $[t_0, \infty)$. In order to approximate Q by a uniformly bounded solution of the Riccati integral equation in H^N, we have to show that the approximate control systems defined here are also detectable and stabilizable. More importantly, recalling the results (e.g., see Theorem 3.2) of section 3, we need uniform detectability and uniform stabilizability for the approximate systems (4.17), (4.19).

Based on the stabilizability and detectability properties of the original system, for a given approximation scheme, we would like to show that the following conditions hold:

Condition US (Uniform stabilizability) *There exist constants $M_K, M, \omega > 0$ independent of N such that for each of the approximate systems, we can find a uniformly bounded operator valued function $K^N(\cdot) : [t_0, \infty) \mapsto L(H^N, U^N)$ such that*

$$\|K^N(t)\|_{L(H^N, U^N)} \leq M_K,$$

and if T_K^N *is the evolution operator corresponding to the perturbation of* T^N *by* $B^N K^N(\cdot)$, *then*

$$\|T_K^N(t,s)\|_{L(H^N)} \le Me^{-\omega(t-s)}, \quad \textit{for } (t,s) \in \Delta_\infty(t_0).$$

Condition UD (Uniform detectability) *There exist constants* $M_\Psi, M, \omega > 0$ *independent of* N *such that for each of the approximation systems, we can find a uniformly bounded operator valued function* $\Psi^N(\cdot) : [t_0, \infty) \mapsto L(H^N)$ *such that*

$$\|\Psi^N(t)\|_{L(H^N)} \le M_\Psi,$$

and if T_Ψ^N *is the evolution operator corresponding to the perturbation of* T^N *by* $\Psi^N(W^N)^{1/2}(\cdot)$, *then*

$$\|T_\Psi^N(t,s)\|_{L(H^N)} \le Me^{-\omega(t-s)}, \quad \textit{for } (t,s) \in \Delta_\infty(t_0).$$

We may summarize our findings as follows.

Theorem 4.5 *Under Hypotheses 4.1-4.3, 4.5-4.7, the conditions* $\|P_H^N \varphi - \varphi\|_V \to 0$ *for* $\varphi \in V$, $\|P_U^N v - v\|_U \to 0$ *for* $v \in U$ *and the Conditions US and UD, there exists a unique uniformly bounded solution* Q^N *of the Riccati integral equation on the infinite time interval* $[t_0, \infty)$ *for each approximate system in* H^N. *Furthermore, the sequence* $Q^N(t)P_H^N$ *converges strongly to* $Q(t)$ *and the convergence is uniform in* t *for* t *in any bounded interval.*

These results follow from Theorem 3.2 and our discussions above. We have thus reduced our problem of ensuring convergence of the Riccati variables to one of guaranteeing uniform stabilizability and detectability of the approximate systems. In the following two sections, two different approaches to obtaining uniform detectability (Condition UD) and stabilizability (Condition US) are presented.

5 Dissipativity and uniform stabilizability / detectability

The original control system (4.10) defined in section 4 was assumed to be stabilizable and detectable (i.e., we assumed Hypotheses 4.5, 4.6 held). For a given evolution system, often an easy way to ascertain stability is using the dissipativity of the system. In particular, if a system satisfies Hypothesis 4.4, by Theorem 4.3 the associated evolution operator is uniformly exponentially stable. This naturally suggests a sufficient condition for stabilizability of a control system.

Hypothesis 5.1 *There exists a uniformly bounded function $K(\cdot) : [t_0, \infty) \mapsto L(H, U)$ and a constant $k > 0$ such that for all $\varphi \in V$,*

$$\sigma(t; \varphi, \varphi) + < B(t)K(t)\varphi, \varphi >_H \geq k\|\varphi\|_H^2, \quad for\ t \in [t_0, \infty).$$

Lemma 5.1 *Under Hypothesis 5.1, the control system defined by (4.10)-(4.12) is stabilizable. In fact if T_K is the evolution operator corresponding to the perturbation of the evolution operator T by $-BK$, we can find constants $M, \alpha > 0$ such that*

$$\|T_K(t, s)\|_{L(H)} \leq Me^{-\alpha(t-s)}, \quad for\ (t, s) \in \Delta_\infty(t_0).$$

As a consequence there exists a constant C such that for all $x_0 \in H$, we can find a control $u(\cdot) \in L^2([s, \infty); U)$ with a cost

$$J_\infty(u; x_0, s) \leq C\|x_0\|^2, \quad for\ s \in [t_0, \infty).$$

Proof: Let $K(\cdot)$ be the operator valued function in Hypothesis 5.1. Define the perturbed sesquilinear form $\sigma_K(t; \varphi, \psi) = \sigma(t; \varphi, \psi) + < B(t)K(t)\varphi, \psi >_H$. Then T_K is associated with σ_K as in Theorem 4.2 with $-B(t)K(t)$ as the perturbation term. Under our assumptions, by Theorem 4.3, there exist $M, \alpha > 0$ such that

$$\|T_K(t, s)\|_{L(H)} \leq Me^{-\alpha(t-s)}, \quad \text{for } (t, s) \in \Delta_\infty(t_0).$$

For any $x_0 \in H$, let $v(t) = -K(t)T_K(t, s)x_0$; it is easy to see that the corresponding trajectory is $x(t) = T_K(t, s)x_0$.

By our standing assumptions, the operator valued functions $W(\cdot), R(\cdot), B(\cdot)$ are uniformly bounded in the entire interval $[t_0, \infty)$. Take

$$C \geq (\|W(t)\|_{L(H)} + \|K^*(t)R(t)K(t)\|_{L(H)})M^2/2\alpha, \quad \text{for } t \geq t_0.$$

Then

$$\begin{aligned} J_\infty(v; x_0, s) &= \int_s^\infty \{< W(t)x(t), x(t) >_H + < R(t)K(t)x(t), K(t)x(t) >_U\} \, dt \\ &\leq \int_s^\infty \left\{ \|W(t)\|_{L(H)} + \|K^*(t)R(t)K(t)\|_{L(H)} \right\} M^2 e^{-2\alpha(t-s)} \|x_0\|_H^2 dt \\ &\leq C\|x_0\|_H^2. \end{aligned}$$

Similarly, a sufficient condition for detectability can be stated as following.

Hypothesis 5.2 *There exists a uniformly bounded operator valued function $\Psi(\cdot)$: $[t_0, \infty) \mapsto L(H)$, and constant $\lambda > 0$ such that*

$$\sigma(t; \varphi, \varphi) + < \Psi(t) W^{\frac{1}{2}}(t)\varphi, \varphi >_H \geq \lambda \|\varphi\|_H^2, \quad \varphi \in V.$$

Lemma 5.2 *Under Hypothesis 5.2, the control system defined by (4.10)- (4.12) is detectable.*

In the remainder of the current section, we assume that Hypotheses 5.1, 5.2 hold for our control system in H. The strict H-dissipativity Hypotheses 5.1, 5.2 on the evolution systems are stronger than the usual stabilizability and detectability hypotheses; however, they are in general easy to verify for a wide class of problems. Moreover, the constants M and the decay rates α depend only on the values of k and λ. Thus, this type of approach suggests that approximate systems which preserve the H-dissipativity might be uniformly stabilizable and uniformly detectable. Pursuing this type of argument, we shall try to show that the following conditions are implied by Hypothesis 5.1, 5.2. (As in the discussions of section 4 surrounding (4.17)- (4.19), we assume that $B^N(t) = P_H^N B(t)$ and $W^N(t) = P_H^N W(t)$.)

Condition 5.1 *There exists a constant $\tilde{k} > 0$ such that for every N, there exists a uniformly bounded operator valued function $K^N(\cdot) : [t_0, \infty) \mapsto L(H^N, U^N)$ so that*

$$\sigma^N(t; \varphi^N, \varphi^N) + < B^N(t) K^N(t) \varphi^N, \varphi^N >_H \geq \tilde{k} \|\varphi^N\|_H^2 \tag{5.1}$$

holds for all $\varphi^N \in H^N$.

Condition 5.2 *There exists a constant* $\tilde{\lambda} > 0$ *such that for every* N*, there exists a uniformly bounded operator valued function* $\Psi^N(\cdot) : [t_0, \infty) \mapsto L(H^N)$ *so that*

$$\sigma^N(t; \varphi^N, \varphi^N) + < \Psi^N(t)(W^N(t))^{\frac{1}{2}}\varphi^N, \varphi^N >_H \geq \tilde{\lambda}\|\varphi^N\|_H^2 \tag{5.2}$$

holds for all $\varphi^N \in H^N$.

Note that if the original system satisfies Hypotheses 5.1 and 5.2, by the definition of σ^N we have

$$\sigma^N(t; \varphi^N, \varphi^N) + < P_H^N B(t)K(t)\varphi^N, \varphi^N >_H \;\geq\; k\|\varphi^N\|_H^2 \tag{5.3}$$

$$\sigma^N(t; \varphi^N, \varphi^N) + < P_H^N \Psi(t)W^{\frac{1}{2}}(t)\varphi^N, \varphi^N >_H \;\geq\; \lambda\|\varphi^N\|_H^2. \tag{5.4}$$

Let us compare inequality (5.3) to (5.1); if we could take $K^N(t) = K(t)$ in (5.1), then Condition 5.1 holds trivially. However, a careful examination of inequalities (5.3) and (5.4) reveals that they do not provide stabilizability and detectability of the approximate system. In the case of (5.3) vs. (5.1) we observe that the range of the operator $K(\cdot)$ hypothesized in Hypothesis 5.1 is not necessarily in U^N and $K(\cdot)$ cannot be used as a stability operator for the control system in H^N, U^N as required in Condition 5.1. Comparing (5.4) and (5.2) and recalling that $W^N(t) = P_H^N W(t)$, we see that the the choice $\Psi^N = P_H^N \Psi P_H^N$ would suffice only in the case where $P_H^N(P_H^N W(t))^{1/2} = W^{1/2}(t)$.

Before we state additional conditions for the approximate systems, let us consider several interesting cases for which stabilizability and detectability are preserved.

I) Dissipative systems. Suppose that Hypotheses 5.1, 5.2 hold for $K(t) \equiv 0$ and $\Psi(t) \equiv 0$; then by definition of the sesquilinear form σ^N, the Conditions 5.1, 5.2 hold with $K^N(t) \equiv 0$ and $\Psi^N(t) \equiv 0$. This is the case when the homogeneous system is itself dissipative.

II) Finite dimensional control. Suppose the control space U is finite dimensional. Taking $U^N = U$, we can use $K^N(t) = K(t)$ and the approximate systems are uniformly stabilizable.

III) Special stability operators. Consider the inequalities in Hypotheses 5.1, 5.2 again. We can assume that these inequalities hold where $B(t)K(t)$ and $\Psi(t)W^{1/2}(t)$ are nonnegative definite self-adjoint operators. Suppose that there exist scalar functions $\kappa(t) \geq 0$ and $\mu(t) \geq 0$ such that

$$B(t)K(t) \leq \kappa(t)B(t)B^*(t), \quad \Psi(t)W^{1/2}(t) \leq \mu(t)W(t).$$

Then taking $K(t) = \kappa(t)B^*(t)$ and $\Psi(t) = \mu(t)W^{1/2}(t)$, we find Hypotheses 5.1 and 5.2 also hold. If we modify slightly the definition of the sesquilinear form σ^N by

$$\begin{aligned} \hat{\sigma}^N(t;\varphi^N,\psi^N) &= \sigma^N(t;\varphi^N,\psi^N) + < B(t)[I - P_U^N]K(t)\varphi^N, \psi^N > \\ &\quad + < \Psi(t)[I - P_H^N]W^{\frac{1}{2}}(t)\varphi^N, \psi^N >, \end{aligned}$$

the sesquilinear form $\hat{\sigma}^N$ satisfies the Conditions 5.1 and 5.2. Indeed we note that the perturbation terms satisfy

$$\begin{aligned} &< B(t)[I - P_U^N]K(t)\varphi^N, \varphi^N > \\ &\quad = \kappa(t) < [I - P_U^N]B^*(t)\varphi^N, [I - P_U^N]B^*(t)\varphi^N >_U \geq 0, \\ &< \Psi(t)[I - P_H^N]W^{1/2}\varphi^N, \varphi^N > \\ &\quad = \mu(t) < [I - P_H^N]W^{1/2}(t)\varphi^N, [I - P_H^N]W^{1/2}(t)\varphi^N >_H \geq 0. \end{aligned}$$

Thus by taking $K^N(t) = P_U^N K(t)$, $\Psi^N(t) = P_H^N \Psi(t)$, the Conditions 5.1, 5.2 hold for $\hat{\sigma}^N$. On the other hand, the perturbation terms go to zero as N goes to ∞. Therefore if we use $\hat{\sigma}^N$ as the sesquilinear form for the approximate control system in H^N, the corresponding evolution operator $\hat{T}^N$ should also converge to T.

The three cases above motivate us to consider the following modifications of the sesquilinear form in H^N. Let the operator valued functions $K(\cdot), \Psi(\cdot)$ be as in Hypotheses 5.1 and 5.2; define $\hat{\sigma}^N$ as:

$$\begin{aligned} \hat{\sigma}^N(t;\varphi^N,\psi^N) &= \sigma^N(t;\varphi^N,\psi^N) + < B(t)[I - P_U^N]K(t)\varphi^N, \psi^N > \\ &\quad + < \Psi(t)[I - P_H^N]W^{\frac{1}{2}}(t)\varphi^N, \psi^N >, \end{aligned}$$

for all $\varphi^N, \psi^N \in H^N$. Let $\hat{A}^N(t) : H^N \mapsto H^N$ be defined by

$$-< \hat{A}^N(t)\varphi^N, \psi^N >_H = \hat{\sigma}^N(t; \varphi^N, \psi^N), \quad \varphi^N, \psi^N \in H^N.$$

Let $\hat{T}^N(\cdot,\cdot)$ be the evolution operator generated by $\hat{A}^N(t)$.

We can repeat the arguments in the proof of Theorem 4.4 using $\hat{T}^N$ in place of T^N. In the arguments, there is an extra term

$$2\int_s^t \text{Re } < \Lambda^N(\eta)w^N(\eta), z^N(\eta) >_H d\eta$$

on the right side of the inequalities, where $\Lambda^N(\eta) \in L(H^N)$ is given by

$$\Lambda^N(t) = P_H^N B(t)[I - P_U^N]K(t) + P_H^N \Psi(t)[I - P_H^N]W^{1/2}(t).$$

There exists a constant C independent of N such that

$$\|\Lambda^N(\eta)\|_{L(H^N)} \leq C, \quad \eta \in [a, b],$$

and, furthermore,

$$\|\Lambda^N(\eta)P_H^N\varphi\|_H \to 0, \quad \text{for } \varphi \in H.$$

Recalling $z^N = w - w^N$ and using (4.15), we have $w^N = w - z^N = P_H^N w - P_H^N z^N$. We thus find (suppressing the argument η throughout)

$$\begin{aligned}
\text{Re } < \Lambda^N w^N, z^N >_H &= \left|< \Lambda^N(P_H^N w - P_H^N z^N), P_H^N z^N >_H\right| \\
&\leq \|\Lambda^N P_H^N w\|_H \|P_H^N z^N\|_H + C\|P_H^N z^N\|_H^2 \\
&\leq \frac{1}{2}\|\Lambda^N P_H^N w\|_H^2 + (C + \frac{1}{2})\|P_H^N z^N\|_H^2.
\end{aligned}$$

The integral (with respect to η) of the first term in this last expression $\to 0$ uniformly in t, s and can be added to the term $\Theta^N(t, s)$ in (4.16), while the integral of the second term can be included with the integral term in the right side of (4.16). We thus can argue:

Theorem 5.1 *Under Hypotheses 5.1, 5.2, the conclusions (i), (ii) of Theorem 4.4 hold for $\hat{T}^N$.*

Now consider $\hat{T}^N$ as the evolution operator for our approximate control systems in H^N; the convergence of the solutions of the Riccati integral equation in any finite time interval still holds. To generalize the arguments in the three special cases I), II), III) above, we make the following additional assumptions.

Hypothesis 5.3 *Consider $K(\cdot), \Psi(\cdot)$ as in Hypotheses 5.1, 5.2 and assume there exist constants $\hat{k} < k$, $\hat{\lambda} < \lambda$ and $\hat{N}$ such that for all $N \geq \hat{N}$*

$$
\begin{aligned}
< (I - P_U^N)K(t)\varphi^N, (I - P_U^N)B^*(t)\varphi^N > &\geq -\hat{k}\|\varphi^N\|_H^2, \\
< (I - P_H^N)W^{\frac{1}{2}}(t)\varphi^N, (I - P_H^N)\Psi^*(t)\varphi^N > &\geq -\hat{\lambda}\|\varphi^N\|_H^2,
\end{aligned}
$$

for all $\varphi^N \in H^N$.

Lemma 5.3 *Under Hypotheses 5.1, 5.2 and 5.3, the approximate control systems are uniformly stabilizable and detectable.*

Proof: We assume without loss of generality that $\hat{N} = 1$. Let $\tilde{k} = k - \hat{k}$ and $\tilde{\lambda} = \lambda - \hat{\lambda}$. By Hypothesis 5.3, $\tilde{k} > 0$ and $\tilde{\lambda} > 0$. Take $K^N(t) = P_U^N K(t)$ and $\Psi^N(t) = P_H^N \Psi(t)$; then with this choice of K^N, Ψ^N, the Conditions 5.1, 5.2 hold for $\hat{\sigma}^N$. Let $\hat{T}_K^N$, $\hat{T}_\Psi^N$ be evolution operators corresponding to the perturbations of $\hat{T}^N$ by $B^N K^N$ and $\Psi^N (W^N)^{1/2}$ respectively. By Theorem 4.3, there exist constants $M, \alpha > 0$ depending on $\tilde{k}$, $\tilde{\lambda}$ only such that

$$
\|\hat{T}_K^N(t,s)\|_{L(H^N)} \leq Me^{-\alpha(t-s)}, \quad \|\hat{T}_\Psi^N(t,s)\|_{L(H^N)} \leq Me^{-\alpha(t-s)}.
$$

By the general framework of section 3, there exists a unique solution $\hat{Q}^N$ of the Riccati equation on the infinite time interval for each control system in H^N. The operator $\hat{Q}^N(t)P_H^N$ converges strongly to the unique solution $Q(t)$ of the Riccati integral equation for the original system in H as $N \to \infty$. The convergence is uniform in t for t in any bounded interval.

We summarize the results for our dissipativity approach to uniform stabilizability and detectability in the following theorem.

Theorem 5.2 *Consider the parabolic control system defined by (4.10)- (4.12) under Hypothesis 4.1–4.3, 4.5–4.7, and the corresponding approximate systems as defined via* $\hat{\sigma}^N$, $\hat{T}^N$ *as in this section where* $P_H^N \to I$ *strongly in* V *and* $P_U^N \to I$ *strongly in* U. *Under the "H-dissipativity" Hypotheses 5.1, 5.2 and the consistency Hypothesis 5.3, the following conclusions hold:*

(i) *There exist unique uniformly bounded solutions* $\hat{Q}^N, Q$ *of the Riccati integral equations in the infinite time interval* $[t_0, \infty)$ *for each of the approximate control systems and the original system, respectively. There exists a constant* M *such that for all* $t \in [t_0, \infty)$

$$\|\hat{Q}^N(t)\|_{L(H^N)} \leq M; \quad \|Q(t)\|_{L(H)} \leq M.$$

(ii) *Let* $\hat{S}^N$, S *be the perturbed evolution operator corresponding to the perturbations of* $\hat{T}^N, T$ *by* $-B^N(R^N)^{-1}B^{*N}\hat{Q}^N$ *and* $-BR^{-1}BQ$ *respectively; then there exist constants* $M, \alpha > 0$ *such that*

$$\|\hat{S}^N(t,s)\|_{L(H^N)} \leq Me^{-\alpha(t-s)}; \quad \|S(t,s)\|_{L(H)} \leq Me^{-\alpha(t-s)},$$

for all $(t,s) \in \Delta_\infty(t_0)$.

(iii) *As* $N \to \infty$, $\hat{Q}^N, \hat{S}^N$ *converge to* Q, S *in the following sense: for all* $\varphi \in H$

$$\|\hat{Q}^N(t)P_H^N\varphi - Q(t)\varphi\|_H \to 0;$$
$$\|\hat{S}^N(t,s)P_H^N\varphi - S(t,s)\varphi\|_H \to 0.$$

The convergence is uniform in (t,s) *in any bounded interval.*

The advantage of using the dissipativity approach outlined above is that the hypotheses are readily checked. The H-dissipativity can sometimes be replaced by even weaker dissipativity conditions for which one can obtain an exponential decay rate (e.g., see [Ch], [La]). For parabolic systems with strict V-ellipticity, we can avoid use of this type of argument as we shall see in the next section. However these results might be useful for systems without strict V-ellipticity or possibly even some hyperbolic systems (e.g., see [BKS], [BKW]).

6 Periodic systems: compactness and uniform stabilizability / detectability

One of the special features of parabolic evolution systems as defined in section 4 is that the evolution operator $T(t,s)$ is also a bounded linear operator from H to V. Since often the space V is compactly embedded in H, $T(t,s)$ is thus a compact operator. Using this fact, we can show that the convergence of the sequence of operators $T^N(t,s)$ to $T(t,s)$ is in a stronger sense. In this section, by combining periodicity and compactness of the evolution operators T^N and T, we can show that the approximation schemes discussed in section 4 preserve detectability and stabilizability.

Some ideas for the stability of periodic evolution operators can be found in [H1], [H2]. The use of compact embedding ideas for the proof of operator norm convergence can be found in [BI2] (The authors gratefully acknowledge K. Ito for fruitful discussions regarding this approach).

In this section we make a periodicity assumption for our control system:

Hypothesis 6.1 *There exists a constant $\theta > 0$ such that*

(i) *The sesquilinear form σ is θ-periodic in time;*

(ii) *The operator valued functions B, R, W are periodic in time with period θ.*

Lemma 6.1 *Under the above assumption, the evolution operator $T(\cdot,\cdot)$ of the corresponding homogeneous evolution equation is θ-periodic.*

Proof: For any $s \leq t$, and all $\varphi, \psi \in V$, we have

$$\begin{aligned} < T(t+\theta, s+\theta)\varphi, \psi >_H &= < \varphi, \psi >_H - \int_{s+\theta}^{t+\theta} \sigma(\tau; T(\tau, s+\theta)\varphi, \psi) d\tau \\ &= < \varphi, \psi >_H - \int_{s}^{t} \sigma(\tau; T(\tau+\theta, s+\theta)\varphi, \psi) d\tau. \end{aligned}$$

By the uniqueness of the solution of the weak form of our evolution equation, we have $T(t+\theta, s+\theta)\varphi = T(t,s)\varphi$, for $\varphi \in H$.

Under the periodicity Hypothesis 6.1, the continuity assumptions and the uniform boundedness assumptions of the control system need only to be verified in the bounded interval $[0, \theta]$. The above lemma shows that the periodicity of the evolution operator is given by the periodicity of the corresponding sesquilinear form σ. The following theorem plays a very important role in the study of periodic systems. We give its proof in order to remind the reader of the dependency of certain bounds involved.

Theorem 6.1 ([H1], [H2]) *Let $T(\cdot,\cdot)$ be a θ-periodic evolution operator. Then $T(\cdot,\cdot)$ is uniformly exponentially stable if and only if there exist an integer n and a constant $\beta < 1$ such that*

$$\|T(n\theta, 0)\|_{L(H)} \leq \beta. \tag{6.1}$$

Proof: a) Let $T(\cdot,\cdot)$ be uniformly exponentially stable; that is, there exist constants $M, \omega > 0$ such that

$$\|T(t,s)\|_{L(H)} \leq Me^{-\omega(t-s)}, \quad (t,s) \in \Delta_\infty(0).$$

Therefore, if we take n large enough such that $M\exp\{-\omega n\theta\} < 1$, and let $\beta = M\exp\{-\omega n\theta\}$, we have that (6.1) holds.

b) Suppose (6.1) holds. Let C be a constant such that for all $0 \leq s \leq t \leq n\theta$, $\|T(t,s)\| \leq C$. Now for any $0 \leq s \leq t < \infty$ and $t - s > \theta$, we can find integers k and m such that

$$k \leq \frac{t-s}{n\theta} \leq k+1, \quad (m-1)\theta \leq s \leq m\theta.$$

Therefore, by the semi-group property of $T(\cdot,\cdot)$, we get

$$T(t,s) = T(t,(nk+m)\theta)T((nk+m)\theta, m\theta)T(m\theta, s).$$

By definition of k and m, we have $m\theta - s \leq n\theta$ and $t - (nk+m)\theta \leq n\theta$; then by (6.1), we have

$$\|T(t,s)\|_{L(H)} \leq C^2\beta^k \leq C^2 e^{k\log\beta}$$

Since $\beta < 1$, we have $\log\beta < 0$; therefore we find

$$\|T(t,s)\|_{L(H)} \leq C^2 e^{-\log\beta} \cdot e^{(k+1)\log\beta} \leq Me^{-\omega(t-s)}.$$

with $M = C^2/\beta$ and $\omega = -\log\beta/n\theta$.

Since the evolution operator T is also $n\theta$-periodic, therefore we can assume without loss of generality that $\|T(\theta,0)\|_{L(H)} \leq \beta < 1$. The following lemma is an interesting consequence of the above theorem.

Lemma 6.2 *For a periodic system, if the stabilizability and the detectability assumptions are satisfied, then we can find θ-periodic operator valued functions $\hat{K}(\cdot)$ and $\hat{\Psi}(\cdot)$ such that the evolution operator $\hat{T}_K$ and $\hat{T}_\Psi$ corresponding to the perturbation of the evolution operator T by $B\hat{K}$, $\hat{\Psi}W^{1/2}$ are also uniformly exponentially stable.*

Proof: Suppose K, Ψ are operator valued functions in the stabilizability and detectability assumptions. Let T_K, T_Ψ be the evolution operators corresponding to perturbation of T by BK and $\Psi W^{1/2}$ respectively. Without lost of generality, we can assume that there exists a constant $\beta < 1$ such that

$$\|T_K(\theta,0)\| \leq \beta; \tag{6.2}$$

$$\|T_\Psi(\theta,0)\| \leq \beta. \tag{6.3}$$

Now define θ-periodic operator valued functions $\hat{K}, \hat{\Psi}$ as $\hat{K}(t) = K(t)$, $\hat{\Psi}(t) = \Psi(t)$, for $t \in [0,\theta)$, and extend periodically for $t \geq \theta$. Then we have $\hat{T}_K(\theta,0) = T_K(\theta,0)$, $\hat{T}_\Psi(\theta,0) = T_\Psi(\theta,0)$; therefore (6.2), (6.3) still hold for the new evolution systems. By Theorem 6.1, we conclude that $\hat{T}_K$, $\hat{T}_\Psi$ are also uniformly exponentially stable.

As a consequence of this lemma, we can assume without loss of generality that the operator valued functions K, Ψ in Hypotheses 4.5 and 4.6 are θ-periodic. In fact we can make the following equivalent assumptions:

Hypothesis 6.2 *There exist a constant $\beta < 1$ and θ-periodic operator valued functions K, Ψ, such that if T_K, T_Ψ are the evolution operators corresponding to the perturbation of the evolution operator T by $BK, \Psi W^{1/2}$ respectively, then*

$$\|T_K(\theta,0)\| \leq \beta, \quad \|T_\Psi(\theta,0)\| \leq \beta.$$

For the remainder of this section, we shall assume Hypotheses 6.1 and 6.2 hold and we focus on the uniform stabilizability and the uniform detectability of the approximate control systems. Let $K^N(t) = P_U^N K(t)$, $\Psi^N(t) = P_H^N \Psi$ and T_K^N, T_Ψ^N be the evolution operators corresponding to the perturbations of T^N by $B^N K^N$ and $\Psi^N (W^N)^{1/2}$ respectively. As we have seen in the proof of the Theorem 6.1, the constant M and decay rate ω depend only on the uniform bound of T and constant β. We already know (use the arguments of Theorem 4.4 and uniform boundedness of the perturbations) that $\|T_K^N(t,s)\|_{L(H^N)}$ and $\|T_\Psi^N(t,s)\|_{L(H^N)}$ can be uniformly bounded by a constant C independent of N for all $0 \leq s \leq t \leq \theta$; therefore to obtain uniform stabilizability and uniform detectability, we only have to show (see b) of the proof of Theorem 6.1) that we can find $\hat{\beta} < 1$ and N_0 such that for all $N \geq N_0$, we have

$$\|T_K^N(\theta,0)\|_{L(H^N)} \leq \hat{\beta}; \tag{6.4}$$

$$\|T_\Psi^N(\theta,0)\|_{L(H^N)} \leq \hat{\beta}. \tag{6.5}$$

If, on the other hand, the convergence of $T_K^N(t,s)P_H^N$ to $T_K(t,s)$ ($T_\Psi^N(t,s)P_H^N$ to $T_\Psi(t,s)$) is in the operator norm, then we can readily establish that (6.4), (6.5) hold for N large enough. The following theorem is very useful in the proof of this desired convergence.

Theorem 6.2 *Let H, V be Hilbert spaces as defined in section 4 with V compactly embedded in H. Consider a sequence of bounded linear operators $\mathcal{T}^N$ defined on H and bounded linear operator $\mathcal{T}$ defined on H. Suppose the range of $\mathcal{T}^N$ and $\mathcal{T}$ are in V, and the following conditions hold:*

(i) *There exists a constant C such that*

$$\|\mathcal{T}^N\|_{L(H,V)} \leq C, \quad \|\mathcal{T}\|_{L(H,V)} \leq C;$$

(ii) *For any $\varphi \in H$, we have*

$$\|\mathcal{T}^N\varphi - \mathcal{T}\varphi\|_H \to 0, \quad \|\mathcal{T}^{*N}\varphi - \mathcal{T}^*\varphi\|_H \to 0,$$

as $N \to \infty$.

Then the convergence of the sequence of operators $\mathcal{T}^N$ to $\mathcal{T}$ is in the operator norm, that is $\|\mathcal{T}^N - \mathcal{T}\|_{L(H)} \to 0$, as $N \to \infty$.

Before we give the proof of this theorem, let us state a useful lemma.

Lemma 6.3 *Consider a nonincreasing sequence of compact sets E_k, $E_k \supseteq E_{k+1}$, $k = 1, 2, 3, \ldots$. If we have $\bigcap_{k=1}^{\infty} E_k = \{0\}$, then for each $\epsilon > 0$, we can find k_0 large enough such that for all $k \geq k_0$, E_k is a subset of a ball $B(0, \epsilon)$ in H defined by $B(0, \epsilon) = \{\varphi \in H \mid \|\varphi\|_H \leq \epsilon\}$.*

Proof: Suppose there exists $\epsilon > 0$ such that for every k, we can find $\varphi_k \in E_k$, such that $\|\varphi_k\|_H > \epsilon$. Since the sequence $\{\varphi_k\}$ is in E_1 which is compact, we can assume that φ_k converges to an element φ in E_1. Obviously $\|\varphi\|_H > \epsilon/2$. On the other hand, φ must be in E_k for all k, therefore φ is also in the set $E = \bigcap_{k=1}^{\infty} E_k$. But since $E = \{0\}$, this is a contradiction.

Proof of Theorem 6.2: i) By definition of the operator norm, we have

$$\|\mathcal{T}^N - \mathcal{T}\|_{L(H)} = \sup\left\{\|\mathcal{T}^N\varphi - \mathcal{T}\varphi\|_H \mid \varphi \in H, \|\varphi\|_H \leq 1\right\}.$$

Now let us define the set F_k as

$$F_k = \bigcup_{N=k}^{\infty} \{\mathcal{T}^N\varphi - \mathcal{T}\varphi \mid \varphi \in H, \|\varphi\|_H \leq 1\}.$$

Let E_k be the H closure of F_k. The sequence of operators $\mathcal{T}^N$ converges to $\mathcal{T}$ in operator norm if and only if for all $\epsilon > 0$, we can find k_0 such that for all $k \geq k_0$, we have $E_k \subseteq B(0, \epsilon)$.

ii) We observe that E_k is a closed set in H and hence in V, and by our assumption $E_k \subset V$ is bounded in V norm, in fact

$$E_k \subset \{\varphi \mid \varphi \in V, \|\varphi\|_V \leq 2C\}.$$

Therefore, E_k is a compact set. By definition of E_k, we know that $\{E_k\}$ is a nonincreasing sequence of compact sets.

iii) Let us define $E = \bigcap_{k=1}^{\infty} E_k$. Suppose $\varphi \in E$; since φ is in the closure of F_k for all k, we can find a sequence γ^N with $\|\gamma^N\|_H \leq 1$, such that $\varphi^N = (\mathcal{T}^N - \mathcal{T})\gamma^N$ converges to φ. Therefore, for any $\psi \in H$, we have

$$\begin{aligned} < \varphi, \psi >_H &= \lim_{N \to \infty} < \varphi^N, \psi >_H \\ &= \lim_{N \to \infty} < \gamma^N, (\mathcal{T}^{*N} - \mathcal{T}^*)\psi >_H . \end{aligned}$$

Since γ^N is uniformly bounded in N and $(\mathcal{T}^{*N} - \mathcal{T}^*)\psi$ goes to zero as $N \to \infty$, we have $< \varphi, \psi >_H = 0$ for all $\psi \in H$, and therefore $\varphi = 0$.

Using the previous lemma and i), we obtain $\|\mathcal{T}^N - \mathcal{T}\|_{L(H)} \to 0$, as $N \to \infty$.

Now recall the definition of the approximate control systems defined in section 4, and consider $\mathcal{T}^N = T_K^N(\theta, 0)P_H^N$, $\mathcal{T} = T_K(\theta, 0)$. By the results of the section 4, we can easily verify that the assumptions of Theorem 6.2 are satisfied. Therefore, we have

$$\|T_K^N(\theta, 0)P_H^N - T(\theta, 0)\|_{L(H)} \to 0,$$

as $N \to \infty$. Now let β be the constant in Hypothesis 6.2; letting $\epsilon = 1 - \beta$, we can find N_0 large enough such that for all $N \geq N_0$ we have

$$\|T_K^N(\theta, 0)P_H^N - T(\theta, 0)\|_{L(H)} \leq \frac{\epsilon}{2}.$$

Therefore, for all $N \geq N_0$, we have

$$\begin{aligned} \|T_K^N(\theta, 0)\|_{L(H^N)} &\leq \|T_K(\theta, 0)\|_{L(H)} + \|T_K^N(\theta, 0)P_H^N - T_K(\theta, 0)\|_{L(H)} \\ &\leq 1 - \frac{\epsilon}{2}. \end{aligned}$$

We summarize our discussion in the following theorem:

Theorem 6.3 *Let H, V be the Hilbert spaces used in the section 4 and assume that V is compactly embedded in H. Suppose that Hypotheses 4.1–4.3, 4.7 and 6.1, 6.2 hold. Let $H^N \subset V$ be the finite dimensional approximation spaces and let the approximate control system be defined as in section 4. Then we can find N_0 large enough such* *that for all $N \geq N_0$, the approximate control systems are uniformly stabilizable and uniformly detectable. As a consequence, if Q^N, Q are the unique solutions of the Riccati integral equations on the infinite time interval in H^N, H respectively, and S^N, S are the evolution operators corresponding to the perturbations of T^N, T by $-B^N(R^N)^{-1}B^{*N}Q^N$, $-BR^{-1}B^*Q$ respectively, then we have:*

$$\|Q^N(t)P_H^N\varphi - Q(t)\varphi\|_H \to 0, \quad \|S^N(t,s)P_H^N - S(t,s)\|_{L(H)} \to 0,$$

as $N \to \infty$. The convergences are uniform in $[0,\theta]$ and for $(t,s) \in \triangle(0,\theta)$ respectively.

We remark that an autonomous system is a particular case of a periodic system; therefore the approach used here can also be applied to time invariant systems as considered in [BK1]. In the case of parabolic systems with strict V-ellipticity where V is compactly embedded in H, the arguments in this section offer an alternative (and more succinct) approach to uniform stabilizability/detectability from the dissipativity approach of section 5.

7 Parabolic partial differential equation control systems: An example

In this section we consider control systems governed by second order parabolic partial differential equations with distributed scalar control. We indicate briefly how one formulates the associated control and approximate problems in the framework of section 4. For this class of systems we show that one can, under standard assumptions, readily verify the conditions for continuity, ellipticity, stabilizability and detectability required in section 4. For Galerkin type approximation schemes based on spline subspaces, Conditions US and UD are readily established.

Let Ω be a bounded open subset of $\Re^n$ with an infinitely differentiable boundary Γ given by a variety of dimension $n-1$ and consider the following homogeneous second order parabolic partial differential equation ([L1], [L2, pp. 100]):

$$\frac{\partial}{\partial t}z(t,\xi) = \sum_{i,j=1}^{n}\frac{\partial}{\partial \xi_i}\left(a_{i,j}(t,\xi)\frac{\partial}{\partial \xi_j}z(t,\xi)\right)+\sum_{i=1}^{n}b_i(t,\xi)\frac{\partial}{\partial \xi_i}z(t,\xi) + c(t,\xi)z(t,\xi), \quad t>0, \xi\in\Omega, \tag{7.1}$$

where $\xi = (\xi_1,\cdots,\xi_n) \in \Re^n$. Generic boundary conditions are given by

$$\gamma(\xi)\sum_{i,j=1}^{n}a_{i,j}(t,\xi)\frac{\partial}{\partial \xi_j}z(t,\xi)\eta_i(\xi)+\beta(t,\xi)z(t,\xi)=0, \quad t>0,\xi\in\Gamma, \tag{7.2}$$

where $\eta(\xi) = (\eta_1(\xi),\cdots,\eta_n(\xi))$ is the outward unit normal vector at a point ξ on the boundary Γ of Ω. Note that for all $\xi\in\Gamma$, if $\gamma(\xi)\neq 0$, we can divide (7.2) by $\gamma(\xi)$; therefore, we can assume without loss of generality that γ takes only values 0 or 1. We choose as our state space $H = L^2(\Omega)$; the appropriate choice for V depends on the boundary conditions and we consider several special cases.

(i) Consider the case $\gamma(\xi)\equiv 0$, $\beta(t,\xi)\equiv 1$, and thus equation (7.2) specifies the usual Dirichlet boundary condition. We then define $V = H_0^1(\Omega)$, and a sesquilinear form σ_1 by

$$\begin{aligned}\sigma_1(t;\varphi,\psi) &= \int_\Omega \sum_{i,j=1}^{n} a_{i,j}(t,\xi)\frac{\partial}{\partial \xi_j}\varphi(\xi)\frac{\partial}{\partial \xi_i}\bar{\psi}(\xi)d\xi \\ &\quad -\int_\Omega\sum_{i=1}^{n}\left\{b_i(t,\xi)\frac{\partial}{\partial \xi_i}\varphi(\xi)\bar{\psi}(\xi)+c(t,\xi)\varphi(\xi)\bar{\psi}(\xi)\right\}d\xi.\end{aligned}$$

(ii) If we consider the case for $\gamma(\xi)\equiv 1$, $\beta(t,\xi)\equiv 0$, we obtain a Neumann boundary condition. We then choose $V = H^1(\Omega)$ and note that the integrals in the definition of σ_1 above are also defined for any functions φ,ψ in $H^1(\Omega)$. Therefore the sesquilinear form σ_2 for Neumann boundary conditions can be taken as the same as for Dirichlet conditions,$\sigma_2=\sigma_1$, and thus only the spaces V are changed.

(iii) Consider the case $\gamma(\xi)\equiv 1$ and $\beta(t,\xi)\not\equiv 0$ which results in Robin or mixed boundary conditions. We again choose $V = H^1(\Omega)$, and define a sesquilinear form σ_3 by

$$\sigma_3(t;\varphi,\psi)=\sigma_2(t;\varphi,\psi)+\int_\Gamma \beta(t,\xi)\varphi(\xi)\bar{\psi}(\xi)d\xi.$$

By taking $\gamma(\xi) = 0$ on a part Γ_1 of the boundary and $\gamma(\xi) = 1$ on $\Gamma - \Gamma_1$, we can obtain other mixed boundary conditions. The choice of space V should also be modified accordingly. In all the cases above, let V be the appropriate choice of Sobolev space (either H_0^1 or H^1), and let σ denote the corresponding sesquilinear form defined on $V \times V$. Then a solution $z(t)$ of equations (7.1) and (7.2) satisfies

$$\frac{d}{dt} < z(t), \psi >_H = -\sigma(t; z(t), \psi), \quad \text{for all } \psi \in V. \tag{7.3}$$

As usual (7.3) is called the weak form of equations (7.1) and (7.2); see equations (4.7), (4.8) and (4.10), (4.11) of section 4.

The continuity and the ellipticity conditions for the sesquilinear forms σ_i can be characterized by properties of the coefficients $a_{i,j}, b_i, c$ and β. The standard assumptions ([L2], pp. 100) for V-continuity and Hölder continuity of the sesquilinear forms σ_i are as follows: For each fixed t, the functions $a_{i,j}(t,\cdot)$, $b_i(t,\cdot)$, $c(t,\cdot)$ are elements of $L^\infty(\Omega)$, while the function $\beta(t,\cdot)$ is an element of $L^\infty(\Gamma)$. Furthermore, for each bounded interval $[a,b]$, there exist constants $C > 0$ and $0 < \gamma \leq 1$, such that each of the coefficients $a_{i,j}, b_i, c, \beta$ satisfy the bounds $\|f(t,\cdot)\|_{L^\infty} \leq C$ and $\|f(t,\cdot) - f(s,\cdot)\|_{L^\infty} \leq C|t-s|^\alpha$ for t, s in $[a,b]$, where L^∞ is $L^\infty(\Omega)$ or $L^\infty(\Gamma)$ as appropriate.

It is easily seen that under these assumptions, the V-continuity and V-Hölder continuity Hypotheses 4.1 and 4.3 hold for σ_1 and σ_2 defined above. In the case of σ_3, a boundary integral is involved; but under our assumptions on the smoothness of the boundary, the following estimates hold (see ([L2], pp. 17, Theorem 3.2, pp. 23): For any $\varphi, \psi \in H^1(\Omega)$, the restrictions of φ, ψ to the boundary Γ belong to $L^2(\Gamma)$. Furthermore, there exists a constant C such that

$$\left| \int_\Gamma \varphi \bar{\psi} d\xi \right| \leq \|\varphi\|_{L^2(\Gamma)} \|\psi\|_{L^2(\Gamma)} \leq C \|\varphi\|_{H^1(\Omega)} \|\psi\|_{H^1(\Omega)}.$$

Furthermore, for all $\epsilon > 0$, there exists constant $C(\epsilon)$ such that

$$\|\varphi\|^2_{L^2(\Gamma)} \leq \epsilon \|\varphi\|^2_{H^1(\Omega)} + C(\epsilon) \|\varphi\|^2_{L^2(\Omega)}.$$

With these estimates, it is readily argued that σ_3 also satisfies the Hypotheses 4.1 and 4.3 of section 4.

To assure V-ellipticity we again make standard assumptions: for any bounded interval $[a, b]$, there exists a constant $\nu > 0$ such that for all $t \in [a, b]$ and $\xi \in \Omega$,

$$\nu \sum_{i=1}^{n} \zeta_i^2 \leq \sum_{i,j=1}^{n} a_{i,j}(t, \xi)\zeta_i\zeta_j,$$

for all $\zeta = (\zeta_1, \cdots, \zeta_n) \in \Re^n$. Under this assumption, it is readily seen that for each bounded time interval $[a, b]$, there exist constants $c_2 > 0$ and m, such that

$$\text{Re } \sigma_i(t; \varphi, \varphi) \ \geq \ c_2\|\varphi\|_V^2 - m\|\varphi\|_H^2,$$

for all $\varphi \in V$ and $t \in [a, b]$. That is, each of the sesquilinear forms σ_i, $i = 1, 2, 3$, defined above satisfies Hypothesis 4.2.

In the remaining part of the this section, let H, V be the spaces of functions appropriate for a specific problem, and let σ be the sesquilinear form defined for that problem as above. For the control space U, we choose $U = L^2(\Omega)$, with the control system being defined by (see (4.10)-(4.12))

$$\frac{d}{dt} < z(t), \psi >_H = -\sigma(t; z(t), \psi) + < B(t)u(t), \psi >_H, \tag{7.4}$$

for all ψ in V. We choose the cost functional given by (4.12). Here we define the operators $B(t), W(t), R(t)$ by the following

$$[B(t)v](\xi) = b(t, \xi)v(\xi), \quad \text{for } v \in L^2(\Omega),$$
$$[W(t)\varphi](\xi) = w(t, \xi)\varphi(\xi), \quad \text{for } \varphi \in L^2(\Omega),$$
$$[R(t)v](\xi) = r(t, \xi)v(\xi), \quad \text{for } v \in L^2(\Omega),$$

where b, w, r are scalar valued functions on $[t_0, \infty) \times \Omega$. In this case, the uniform boundedness and the positivity of the corresponding operators can be readily characterized by conditions on the functions b, w and r. We assume that for each fixed t, $b(t, \cdot), w(t, \cdot)$ and $r(t, \cdot)$ are elements of $L^\infty(\Omega)$. As $L^\infty(\Omega)$ valued functions of t, these functions are assumed continuous. Furthermore, assume there exists a constant C such that for all $t \in [t_0, \infty)$, the functions $b(t,\cdot), w(t,\cdot), r(t,\cdot)$ satisfy the bound $\|f(t,\cdot)\|_{L^\infty(\Omega)} \leq C$. The functions w, r are assumed to be nonnegative and, indeed we assume there exists a constant $r_0 > 0$ such that $r(t,\xi) \geq r_0$, a.e. in Ω, for $t \geq t_0$.

Under these assumptions, the operator valued functions B, W, R satisfy the standing assumptions of sections 2 and 4. It remains to consider Hypotheses 4.5 and 4.6 (stabilizability and detectability of the original system) as well as Conditions US and UD once we have introduced approximations. For the problems considered here, we can use the definition of the sesquilinear forms to give sufficient conditions for Hypotheses 5.1 and 5.2 (and hence Hypotheses 4.5 and 4.6) to hold. To this end, we assume that there exist constants $\mu > 0$ and $\rho > 0$ such that for $t \in [t_0, \infty)$, $|b(t,\xi)| \geq \mu$, $|w(t,\xi)| \geq \rho$, a.e. in Ω.

Under this assumption, it is readily seen that there exists constants $l \geq 0$ and $k > 0$, such that each of the sesquilinear forms σ_i satisfies for $\varphi \in V$

$$\sigma(t;\varphi,\varphi) + l < B(t)B^*(t)\varphi, \varphi >_H \geq k\|\varphi\|_H^2,$$

and

$$\sigma(t;\varphi,\varphi) + l < W(t)\varphi, \varphi >_H \geq k\|\varphi\|_H^2.$$

Thus, Hypotheses 5.1 and 5.2 hold with $K(t) = lB^*(t)$ and $\Psi(t) = lW^{1/2}(t)$.

For our approximate systems, we choose approximation spaces H^N and U^N as in sections 3 and 4 generated by finite element or spline basis elements chosen so that $H^N \subset V$ and $U^N \subset U$ yield the desired convergence properties for P_H^N and P_U^N respectively (see [C, Chaps. 2, 3], [B], [Sc]). The approximating systems are then defined as in section 5. It follows immediately that Hypothesis 5.3 holds and hence the conclusions of Theorem 5.2 are valid for the class of examples considered in this section.

We note that under periodicity assumptions we could have applied the alternative approach of section 6 to these examples since (see [A]) both $V = H^1(\Omega)$ and

$V = H_0^1(\Omega)$ embed compactly in $H = L^2(\Omega)$.

In some of our related efforts, we have numerically tested the ideas presented in this paper on one dimensional versions of the example of this section. In these examples $\Omega = (0,1)$ and we have to date used either linear or cubic B-splines to generate the approximation spaces H^N and U^N. (In fact, when Ω is a parallelepiped, the above theory still is applicable and tensor products of one dimensional elements are a good choice for approximation elements.) We have considered several examples with time dependent periodic coefficients; for these examples we could use eigendirection analysis (see [W]) to give an analytic analysis for the feedback control problems. The resulting analytical solutions were used for comparison with the numerical solutions obtained using software implementations based on the theory developed in this paper. Quite satisfactory results were obtained and, as noted in the Introduction, these are being detailed in a separate manuscript under preparation.

In concluding we note that the theory in this paper is also applicable to higher order parabolic systems (as well as to some boundary damped hyperbolic systems [BKS], [BKW]). In particular one dimensional Euler-Bernoulli beam models with Kelvin-Voigt damping satisfy (see [BI1]) the strong ellipticity assumptions needed in the theory developed above. While boundary control (as treated in [BI2]) for such models constitute an obvious class of problems, distributed control as treated in this paper is essential in cases where nonuniform piezoelectric layers along the beam are used to implement the feedback controls.

References

[A] Robert A. Adams, *Sobolev Spaces,* Academic Press, 1975.

[B] Carl de Boor, *A Practical Guide to Splines,* Springer-Verlag, 1978.

[BK1] H. T. Banks and K. Kunisch, *The linear regulator problem for parabolic systems,* SIAM J. Control and Optimization, 22 (1984), 684 - 698.

[BK2] H.T. Banks and K. Kunisch, *Estimation Techniques for Distributed Parameter Systems,* Birkhäuser Boston, 1989.

[BKS] H.T. Banks, S.L. Keeling and R.J. Silcox, *Optimal control techniques for active noise suppression*, Proc. 27th IEEE Conf. on Dec. and Control, (Austin, TX, Dec. 7–9,1988), 2006–2011.

[BKW] H.T. Banks, S.L. Keeling and C. Wang, *Linear quadratic tracking problems in infinite dimensional Hilbert spaces and a finite dimensional approximation framework*, LCDS/CCS Report #88-28, 1988.

[BI1] H. T. Banks and K. Ito, *A unified framework for approximation in inverse problems for distributed parameter systems,* LCDS/CCS #87-42; Control-Theory and Adv. Tech., 4 (1988), 73–90.

[BI2] H. T. Banks and K. Ito, *On a variational approach to a class of boundary control problems: Numerical approximations,* To appear.

[C] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems,* North-Holland, 1978.

[Ch] G. Chen, *Energy decay estimates and exact boundary value controllability for wave equation in a bounded domain,* J. Math. Pures Appl., 58 (1979), 249–725.

[CP] Ruth Curtain and A. J. Pritchard, *The infinite-dimensional Riccati equation for systems defined by evolution operators,* SIAM J. Control and Optimization, 14 (1976), 951 - 983.

[Da] G. Da Prato, *Synthesis of optimal control for an infinite dimensional periodic problem,* SIAM J. Control and Optimization, 25 (1987), 706 –714.

[DI1] G. Da Prato and A. Ichikawa, *Quadratic control for linear time varying systems,* SIAM J. Control and Optimization, to appear.

[DI2] G. Da Prato and A. Ichikawa, *Quadratic control for linear periodic systems,* Appl. Math. Optim., 18 (1988), 39–66.

[Dt] Richard Datko, *Uniform asymptotic stability of evolution processes in a Banach space,* SIAM J. Math. Anal., 3 (1972), 428 – 445.

[FM] H. Fujita and A. Mizutani, *On the finite element method for parabolic equations, I; Approximation of holomorphic semi-groups,* J. Math. Soc. Japan, 28 (1976), 749 – 771.

[G] J. S. Gibson, *The Riccati integral equations for optimal control problems on Hilbert spaces,* SIAM J. Control and Optimization, 17 (1979), 537 – 565.

[H1] Jack Hale, *Ordinary Differential Equations,* Second Edition, Robert E. Krieger Publishing Company, 1980.

[H2] Jack Hale, *Functional Differential Equations,* Springer-Verlag, 1977.

[K] S. G. Krein, *Linear Differential Equations in Banach Space,* Translations of Mathematical Monographs, Volume 29, American Mathematical Society, 1971.

[L1] J.L. Lions, *Contrôle Optimal de Systèmes Gouvernés par des Equations aux Dériées Partielles,* Dunod Gauther-Villars, 1968.

[L2] J.L. Lions, *Equations Différentelles Opérationelles,* Springer-Verlag, 1961.

[La] J. Lagnese, *Decay of solution of wave equations in a bounded region with boundary dissipation,* J. Diff. Equations, 50 (1983), 163–182.

[LT] I. Lasiecka and R. Triggiani, *The regulator problem for parabolic equations with Dirichlet boundary control,* Parts I, II, Appl. Math. Optim. 16 (1987), 147–168, 187–216.

[S] R. E. Showalter, *Hilbert Space Methods for Partial Differential Equations,* Pitman, 1977.

[Sc] Larry Schumaker, *Spline Functions: Basic Theory,* John Wiley & Sons, 1981.

[T] H. Tanabe, *Equations of Evolution,* Pitman, 1979.

[W] Chunming Wang, *Approximation methods for linear quadratic regulator problems with nonautonomous periodic parabolic systems,* Ph.D Thesis, Brown University, May 1988.

Report Documentation Page

NASA — National Aeronautics and Space Administration

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA CR-181772 ICASE Report No. 89-4		

4. Title and Subtitle	5. Report Date
OPTIMAL FEEDBACK CONTROL INFINITE DIMENSIONAL PARABOLIC EVOLUTION SYSTEMS: APPROXIMATION TECHNIQUES	January 1989
	6. Performing Organization Code
7. Author(s) H. T. Banks C. Wang	8. Performing Organization Report No. 89-4
	10. Work Unit No. 505-90-21-01
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665-5225	11. Contract or Grant No. NAS1-18107 NAS1-18605
	13. Type of Report and Period Covered Contractor Report
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Langley Research Center Hampton, VA 23665-5225	14. Sponsoring Agency Code

15. Supplementary Notes

Langley Technical Monitor:
Richard W. Barnwell

Final Report

16. Abstract

We present a general approximation framework for computation of optimal feedback controls in linear quadratic regular problems for nonautonomous parabolic distributed parameter systems. This is done in the context of a theoretical framework using general evolution systems in infinite dimensional Hilbert spaces. We discuss conditions for preservation under approximation of stabilizability and detectability hypotheses on the infinite dimensional system. The special case of periodic systems is also treated.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
parallel programming, distributed memory machines, data structures, sparse linear systems	61 - Computer Programming & Software 64 - Numerical Analysis Unclassified - unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of pages	22. Price
Unclassified	Unclassified	59	A04

NASA FORM 1626 OCT 86