The length scale for sub-grid-scale parameterization with anisotropic resolution

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Abstract

Use of the Smagorinsky eddy-viscosity formulation and related schemes for sub-grid-scale parameterization of large eddy simulation models requires specification of a single length scale, earlier related by Lilly to the scale of filtering and/or numerical resolution. An anisotropic integration of the Kolmogoroff enstrophy spectrum allows generalization of that relationship to anisotropic resolution. It is found that the Deardorff assumption is reasonably accurate for small anisotropies and can be simply improved for larger values.

1. Introduction

Numerical integration of the time-dependent equations of fluid dynamics to simulate evolution of the largest scales of motion has been applied for about 30 years, initially in the field of weather prediction. Early workers were concerned over what to do about the scales of motion too small to be resolved by the computer, but the problem was often confused with the numerical errors introduced by finite difference algorithms. The technique that Smagorinsky (1963) introduced grew out of an empirical variable eddy viscosity method due, I believe to R. Richtmyer (no reference available), for smoothing simulated shock wave calculations. Smagorinsky and his associates recognized, however, that Richtmyer's variable viscosity was also consistent with the notion of a universal equilibrium range of turbulence. I pursued this point and used the Kolmogoroff inertial sub-range hypothesis to quantitatively relate the length scale required in the Smagorinsky formulation to the resolved scale (Lilly, 1966, 1967). I assumed isotropic resolution, although numerical simulations then and now often are carried out with anisotropic resolution. The question of how to generalize my expression for this purpose has engaged some attention. Deardorff (1970) assumed that the length scale is proportional to the cube root of the product of the resolution scales in the three directions. Alternatives have also been proposed, most of them involving the effects of a nearby boundary. Piomelli, et al (1987) review the subject and conclude that no real consensus has yet emerged, though Deardorff's formulation is widely used.

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The purpose of this note is to aid in resolving this issue. I extend my earlier calculation, oriented now toward spectral and pseudo-spectral integration techniques, to allow resolution to vary with direction. I still assume, however, that the small scale limit of the simulation is in the Kolmogoroff inertial sub-range in all directions. This is a serious limitation, as the most common reason for applying anisotropic resolution is an expectation of anisotropic and/or inhomogeneous turbulence, typically in regions close to a boundary. Thus the scale anisotropy problem is usually compounded with a boundary layer problem and with the likelihood that a classical inertial sub-range may not exist, at least in the scales resolved by the simulation. Piomelli, et al (1987) discuss and test several additional parameterizations intended to improve the boundary layer resolution, but all are modifications of an isotropic formulation related to Deardorff's. The results to be presented here are offered to test and perhaps improve the Deardorff assumption.

2. The problem and the solution

Consider the problem of integrating the Navier-Stokes equation under conditions of incompressibility and constant density, so that

$$\partial u_i/\partial t + \partial(u_i u_j)/\partial x_j + \partial p/\partial x_i = \nu\partial^2 u_i/\partial x_j{}^2 \tag{1}$$

and

$$\partial u_i/\partial x_i = 0, \tag{2}$$

where p is pressure divided by density, and other terminology is conventional. A low-pass spatial filter is applied to these equations, in recognition of the limited resolution available to any real discretization technique. In the early development era this was typically a uniform average over a box surrounding a spatial grid point. Leonard (1974) introduced the use of Gaussian filters, a technique which is widely applied in engineering fluid dynamics simulations. In many current research simulations the effective filter is the spectral cut-off of a finite Fourier transform. In any case it is assumed to be a linear operator and is here designated by angle brackets $<>$. Because it is linear it may be exchanged with the differential operators in (1) and (2) and applied to the flow variables directly. This allows the linear terms of the equations to be integrated as if they were unfiltered. The filtered momentum flux product, $< u_i u_j >$, cannot, however, be resolved directly from the filtered velocities, and must be subject to a creative parameterization.

I assume the decomposition of the momentum flux product applied by Smagorinsky, Lilly, and Deardorff, i.e.

$$< u_i u_j > = < u_i >< u_j > - \tau_{ij} + \delta_{ij} E,$$
$$\tau_{ij} = - < u_i u_j > + < u_i >< u_j > - \delta_{ij} E$$
$$E = (< {u_i}^2 > - < u_i >^2)/2 \tag{3}$$

with τ_{ij} designated as the subgrid scale (SGS) stress, E the SGS kinetic energy, and δ_{ij} is the Kronecker delta. The filtered equations of motion and continuity are now written as

$$\partial < u_i > /\partial t + \partial(< u_i >< u_j >)/\partial x_j + \partial \pi/\partial x_i$$
$$= \partial \tau_{ij}/\partial x_j + \nu \partial^2 < u_i > /\partial {x_j}^2 \tag{4}$$
$$\partial < u_i > /\partial x_i = 0, \tag{5}$$

where $\pi = < p > + E$.

The Smagorinsky parameterization for the SGS stress is of the eddy viscosity type, given by

$$\tau_{ij} = K(S_{ij} + S_{ji}), \tag{6}$$

where $S_{ij} = \partial < u_i > /\partial x_j$, and

$$K = \lambda^2 S, \tag{7}$$

where $S^2 = S_{ij}(S_{ij} + S_{ji})$. The length scale λ was assumed by Smagorinsky to be proportional to the grid resolution. Some investigators have replaced S by ω, the square root of the enstrophy, i.e. $\omega^2 = (\partial u_i/\partial x_j - \partial u_j/\partial x_i)^2/2$. Not much difference is found in practice, corresponding to the fact that the volume integrals of S^2 and ω^2 are identical and they have the same Fourier spectral amplitudes.

I evaluated the length scale λ by assuming the validity of inertial cascade theory and neglecting intermittency effects. If the resolved kinetic energy equation is derived by multiplying (4) by $< u_i >$, the contribution from the eddy stress term is

$$< u_i > \partial \tau_{ij}/\partial x_j = \partial(< u_i > \tau_{ij})/\partial x_j - KS^2$$
$$= \partial(< u_i > \tau_{ij})/\partial x_j - \lambda^2 S^3 \tag{8}$$

The first term on the rhs is variable in sign and vanishes or is small in the volume average, while the second term is always negative. It is assumed that the loss of energy from the resolved scales, $\lambda^2 S^3$, is the same as the viscous dissipation of energy, ϵ. These are equated after evaluating S^2 from spectral integration, i. e.

$$S^2 = \int_o^{k_m} k^2 E(k) dk \tag{9}$$

where $E(k)$ is the spectral amplitude of kinetic energy, defined as an integral along the surfaces of spheres in spectral space. The wavenumber k_m must correspond to the resolution limit, which is the cut-off wavenumber for spectral discretization. It is assumed that at and near k_m the energy spectrum is given by the inertial range form, i.e.

$$E(k) = \alpha\epsilon^{2/3}k^{-5/3}, \tag{10}$$

where α is the Kolmogoroff constant, about 1.5, and ϵ is the viscous dissipation of kinetic energy. Substitution of (10) into (9) leads to the result of integration

$$S^2 = (3/4)\alpha\epsilon^{2/3}{k_m}^{4/3} \tag{11}$$

The rate of energy removal from the resolved flow is evaluated by substituting (11) into the last term of (8) so that

$$\lambda^2 S^3 = (3\alpha/4)^{3/2}\lambda^2\epsilon {k_m}^2 \tag{12}$$

Equating this expression to dissipation allows evaluation of the length scale as

$$\lambda = S^{-3/2}\epsilon^{1/2} = (4/3\alpha)^{3/4}{k_m}^{-1} \tag{13}$$

One may now set k_m = constant/Δ, where Δ is the grid interval, assumed to be isotropic. The minimum value for the constant, assuming the spectral wavelengths are bounded by the Nyquist limit, is two. When pseudo-spectral integration techniques are applied, in order to avoid aliasing the constant must be no less than three, which increases λ by 50%.

I now wish to drop the assumption of resolution isotropy. For simplicity I assume, however, axisymmetry, i.e. that the limits of resolution in two of the three dimensions are the same. The typical situation involves *increased* resolution close to a boundary, either through a variable grid length or use of Chebyshev polynomials. Within the axisymmetric framework one also may consider the effects of *decreased* resolution in one direction. In the conclusion section I argue that the results for different resolution in all three directions can be determined adequately from the axisymmetric case.

Results are obtained by writing the enstrophy spectrum function in three-dimensional spectral space and integrating it over prolate or oblate spheroids, that is figures obtained by rotating an ellipse about its long or short axis. This may not correspond accurately to the resolution limits in a model using Fourier modes in two dimensions and Chebyshev polynomials or finite differences in the third, but it is more readily compared with the isotropic case. Similar results can be obtained by assuming a cylindrical volume in wavenumber space, but they differ only slightly and not in any qualitatively important manner.

If the three-dimensional spectrum function is isotropic, as assumed, it is $E(k)/4\pi k^2$, and the 3-d enstrophy or S^2 spectrum is $E(k)/4\pi$. The wavenumber spheroid is assumed to be bounded by the equation for an ellipse, i.e.

$$(k_h/k_{hm})^2 + (k_z/k_{zm})^2 = 1$$

where k_h is the wavenumber in the axisymmetric direction, notationally assumed to be horizontal, and k_z is that in the vertical direction, with k_{hm} and k_{zm} the horizontal and vertical spectral limits. In place of (9), the evaluation of S^2 is accomplished by integration over the spheroid as follows:

$$S^2 = \int_0^{k_{zm}} \int_0^{k_{h1}} E(k) k_h dk_h dk_z \tag{14}$$

The limit on the first integral is $k_{h1} = k_{hm}(1 - {k_z}^2/{k_{zm}}^2)^{1/2}$. Note that if $E(k)$ were replaced by 4π in (14), the integral would give the volume of the wavenumber spheroid, $4\pi {k_{hm}}^2 k_{zm}$. With the energy spectrum given by (10), with $k^2 = {k_h}^2 + {k_z}^2$, integration of (14) yields

$$S^2 = (3/4)\alpha\epsilon^{2/3} {k_{hm}}^{8/9} {k_{zm}}^{4/9} y(r) \tag{15}$$

$$\text{where } r = k_{hm}/k_{zm} \text{and } y = r^{10/9} \int_0^1 [r^2 + (1-r^2)x^2]^{-5/6} dx.$$

The particular form for S is chosen so that if it is substituted into the first equality of (13), with $y = 1$ and the wavenumbers assumed to be inversely proportional to the grid spacings Δx, Δy, Δz, one obtains the expression assumed by Deardorff, i.e. $\lambda \sim (\Delta x \Delta y \Delta z)^{1/3}$. More generally, upon substituting S into (13) one obtains

$$\lambda = \lambda_D y^{-3/4}, \lambda_D = (4/3\alpha)^{3/4} {k_{hm}}^{-2/3} {k_{zm}}^{-1/3}, \tag{16}$$

with λ_D equivalent to Deardorff's expression.

The integral for y in (15) can be reduced to the forms

$$y = r^{4/9}(l - r^2)^{-1/2} \int_0^{x_m} (\cos x)^{-1/3} dx, \tan x_m = (1-r^2)^{1/2}/r, \text{for } r < 1 \tag{17a}$$

$$y = r^{4/9}(r^2 - 1)^{-1/2} \int_0^{x_m} (\cos x)^{-2/3} dx, \sin x_m = (r^2-1)^{1/2}/r, \text{for } r > 1 \tag{17b}$$

For the extreme cases $r = 0, \infty$, the integration limits x_m become $\pi/2$ in both expressions, and the integrals are evaluated in terms of tabulated Gamma functions. Thus for small r, $S^2 \sim {k_{hm}}^{4/3}$ and $\lambda \sim {k_{hm}}^{-1}$, while for large r, $S^2 \sim {k_{hm}}^{1/3} k_{zm}$ and $\lambda \sim {k_{hm}}^{-1/4} {k_{zm}}^{-3/4}$. A somewhat more revealing interpretation can be obtained by differentiating λ with respect to r. This yields the following:

$$r\lambda^{-1}\partial\lambda/\partial r = -(3r/4y)\partial y/\partial r = \begin{cases} -1/3 \text{ for } r \ll 1 & (18a) \\ 0 \text{ for } r = 1 & (18b) \\ 5/12 \text{ for } r \gg 1 & (18c) \end{cases}$$

$$\partial^2\lambda/\partial(lnr)^2 = (4/27)\lambda_D \text{ at } r = 1. \qquad (18d)$$

These show that the Deardorff expression is valid in the vicinity of $r = 1$, but that with increasing anisotropy in either direction λ becomes larger than λ_D. Fig. 1 is a plot of λ/λ_D. The data for the curve labelled "Length scale" were calculated to two-three digit accuracy by using the (18d) for $.05 < r < 20$ and matching that with an expansion of the expressions in (17a,b) around their limiting values. The straight lines are plots of the asymptotic forms, that is $y^{-3/4}$ for the y's given by (17a) for $r \ll 1$ and (17b) for $r \gg 1$.

An accurate approximation for $.02 < r < 50$ is

$$\lambda/\lambda_D = r^{(2/27)^{1/2}} + r^{-(2/27)^{1/2}} - 1 \qquad (19)$$

This is obtained as a solution to (18d), but with the rhs replaced by $(2/27)(\lambda + \lambda_D)$. This relation is also plotted in Fig. 1, and is imperceptibly different from the "Length scale" except at the most extreme anisotropies.

3. Conclusions

The results of the above calculation indicate that the Deardorff formulation is accurate to within 20% for $.2 < r < 5$. For anisotropies greater than that the length scale should be increased, with (19) sufficiently accurate for anisotropies less than a factor of 50 either way. Such large anisotropies are unlikely to produce accurate simulations in any case, and probably need to be enhanced by more sophisticated parameterizations than the Smagorinsky formulation.

Upon consideration of the nature of these solutions, it seems evident that they can be extended to the case where the resolution is different in all three directions. Since equation (18) or (19) is symmetric in lnr, it doesn't seem to matter much whether two wavenumber limits are larger and one smaller, or one larger and two smaller. Apparently the results are only sensitive to the largest ratio of the length scales, so presumably the existence of a third dimension with an intermediate resolution would have little effect.

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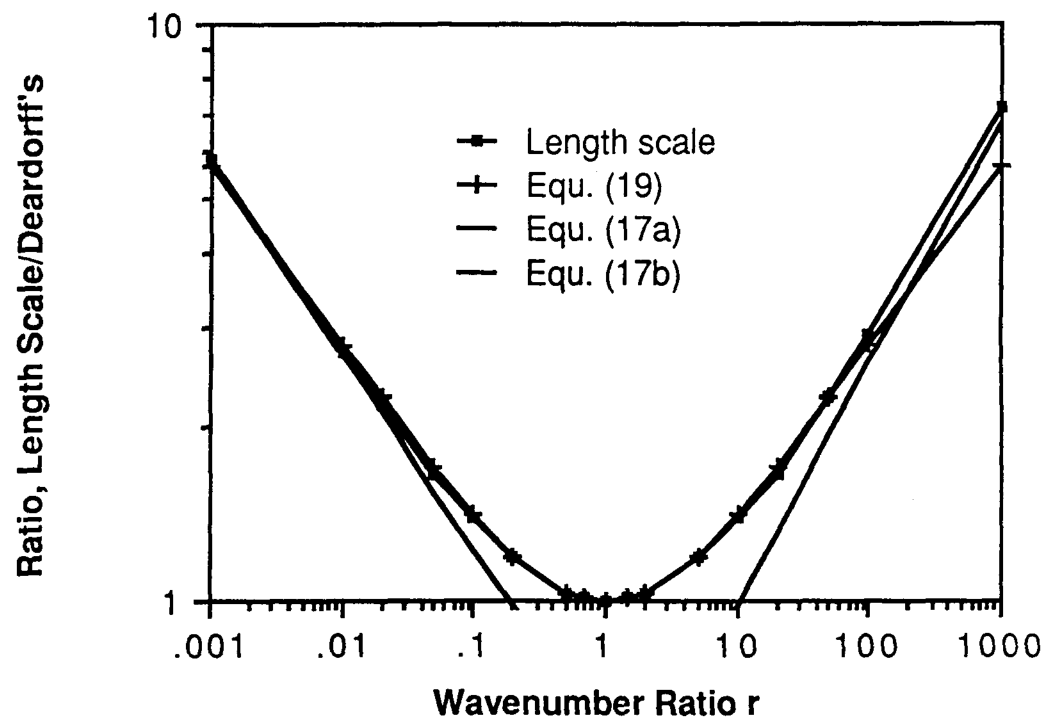

FIGURE 1. The turbulence length scale, normalized by Deardorff's (1970) assumption, and determined by solution of (13) and (15), for various ratios of the limiting wavenumbers, k_{hm}/k_{zm}. The straight lines area symptotic forms from (17a,b) and the curve marked with + signs is an approximation given by (19).