On curve and surface stretching in turbulent flow

By N. ETEMADI[1]

Cocke (1969) proved that in incompressible, isotropic turbulence the average material line (material surface) elements increase in comparison with their initial values. We obtain rigorously, among other things, good estimates of how much they increase in terms of the eigenvalues of the Green deformation tensor.

Introduction

In the following note we will study deformation of material curves and surfaces convected in a turbulent flow. We will do this by looking at the notion of material line and surface elements as a means of generating hypotheses for the so called *flow path (Lagrangian)* of the motion. Then we will use these hypotheses to obtain upper and lower bounds for the evolution of the *ensemble average* of the arc length (surface area) of an arbitrary curve (surface) in time.

For the definition of material line and surface elements and their historical background see Monin and Yaglom (1975). Cocke (1969) is the first who generated these mathematical assumptions, see section 1.1, "implicitly" and gave a convincing proof of them for isotropic turbulence. Orszag (1970, 1977) takes these assumptions for granted, and by a variant of Cocke's arguments obtains somewhat weaker results than Cocke's, see Remark 1.2. Our work *complements* the work of Cocke. Namely, we will bring out these assumptions in section 1, and we will show, first, that once one accepts these assumptions then Cocke's (1969) results can be improved to obtain "tight" upper and lower bounds for the ensemble average of material lines and surfaces. Then, in the remaining part of section 1 and in section 2 we carry on the results to arbitrary curves and surfaces, and in turn we also obtain upper bounds for moments of the dispersion between two points moving in the flow at any given time in terms of their separation at initial time.

Throughout our work we will use $x(a,t)$ as the Lagrangian representation of the flow, i.e. the trajectory followed by the particle which is at a at initial time t_0. We will assume that x is smooth enough in a space–time region and is, for fixed t, an invertible mapping so that our manipulations are legitimate. We will also use $|v|$ as the magnitude of an arbitrary vector v, whose components, without danger of confusion, will be denoted by v_1, v_2, and v_3.

1 University of Illinois at Chicago

1. Curve Stretching

The motion of an infinitesimal material line δl is governed by the equation

$$\frac{d\delta l}{dt} = (\delta l \cdot \nabla)u \quad \text{or} \quad \frac{d\delta l}{dt} = \sum_{j=1}^{3} u_{,j}\delta l_j, \tag{1}$$

with $\delta l(t_0) = \delta^0 l$. Where $\frac{d}{dt}$ is the material derivative following the motion, see e.g. Monin and Yaglom (1975) section 24.5. It is easy to check that the above equation has the solution,

$$\delta l = \sum_{k=1}^{3} x_{,k}\delta^0 l_k \quad ; x(a,t_0) = a \ , \ (x_{i,j}(a,t_0)) = (\delta_{ij}) = I, \tag{2}$$

where I is the identity matrix.

Let $\rho(a,t)$ be the Lagrangian density. It is easy to show that

$$det(x_{i,j}(a,t)) = \frac{\rho(a,t_0)}{\rho(a,t)} = \frac{\rho_0}{\rho_t}, \tag{3}$$

where the left hand side is the determinant of the Jacobian matrix (or the Jacobian) of the transformation $y = x(a,t), t > t_0$, see Batchelor (1977) p.79. In particular, the Jacobian is one when the flow is incompressible.

Next consider

$$|\delta l|^2 = \delta^0 l^T (x_{i,j})^T (x_{i,j}) \delta^0 l = \delta^0 l^T W \delta^0 l. \tag{4}$$

Clearly W is a symmetric non-negative definite matrix. Since its determinant is $(\rho_0/\rho_t)^2$, consequently its eigenvalues, say w_1, w_2, w_3, are strictly positive and we have

$$det(W) = det(X_{i,j})^2 = w_1 w_2 w_3 = \frac{\rho_0^2}{\rho_t^2}. \tag{5}$$

Let $A = (a_{ij})$ be the unitary (rotation) matrix corresponding to diagonalization of W. Thus $|A\delta^0 l| = |\delta^0 l|$. We can rewrite,

$$\begin{aligned} \frac{|\delta l|^2}{|\delta^0 l|^2} &= w_1 \frac{(A\delta^0 l)_1^2}{|A\delta^0 l|^2} + w_2 \frac{(A\delta^0 l)_2^2}{|A\delta^0 l|^2} + w_3 \frac{(A\delta^0 l)_3^2}{|A\delta^0 l|^2} \\ &= (sin^2\theta cos^2\psi) w_1 + (sin^2\theta sin^2\psi) w_2 + (cos^2\theta) w_3, \end{aligned} \tag{6}$$

where θ, ψ are the usual spherical coordinates of the unit vector $\frac{A\delta^0 l}{|A\delta^0 l|}$.

Now we are in a position to take ensemble averages of both sides of this equation. To do so, we need to make some assumptions about the joint probability distribution of the variables involved; namely w_i's, θ and ψ. This is where the physics of the problem come in. The following three assumptions have been extracted from the work of Cocke (1969) who proved them to be true for incompressible, isotropic turbulence.

Assumptions;

(1) w_i's are independent of θ and ψ,

(2) θ and ψ are uniformly distributed over the unit sphere,

(3) w_i's are identically distributed.

Assumption one implies that the random orientation of the element δl is uncorrelated with its random deformations along the principal axis. Assumption two simply says that δl is equally likely to be oriented in any direction. Finally, assumption three means that the random deformations of δl along its principal axis have the same probability law.

Proposition 1.1. Under Assumptions (1) and (2) we have,

$$\frac{1}{3}\langle\sqrt{w_1}+\sqrt{w_2}+\sqrt{w_3}\,\rangle \le \langle\frac{|\delta l|}{|\delta^0 l|}\rangle \le \frac{1}{2}\langle\sqrt{w_1}+\sqrt{w_2}+\sqrt{w_3}\,\rangle, \tag{i}$$

$$\frac{1}{2}\langle(w_1+w_2+w_3)^{\frac{1}{2}}\rangle \le \langle\frac{|\delta l|}{|\delta^0 l|}\rangle \le \langle(w_1+w_2+w_3)^{\frac{1}{2}}\rangle. \tag{ii}$$

Proof. Since $\sqrt{x}$ is concave and the coefficients of w_i's add up to one, from (6) and (19) we obtain,

$$\frac{|\delta l|}{|\delta^0 l|} \ge (sin^2\theta cos^2\psi)\sqrt{w_1}+(sin^2\theta sin^2\psi)\sqrt{w_2}+(cos^2\theta)\sqrt{w_3}. \tag{7}$$

Also it is clear from (6) that,

$$\frac{|\delta l|}{|\delta^0 l|} \le |sin\theta cos\psi|\sqrt{w_1}+|sin\theta sin\psi|\sqrt{w_2}+|cos\theta|\sqrt{w_3}. \tag{8}$$

Next, use the joint density function of θ and ψ, i.e. $\frac{1}{4\pi}sin\theta$ $0 \le \theta \le \pi$, $0 \le \psi \le 2\pi$, to conclude that the average of the coefficients of w_i's and their square roots, in (6), are $\frac{1}{3}$ and $\frac{1}{2}$, respectively, and this will in turn give us (i). Since the coefficient of w_i's in (6) are bounded by one we are only left to show the left hand side of (ii). To see this note that again by the fact that $\sqrt{x}$ is concave, from (6) and (19) we have,

$$\begin{aligned}\frac{|\delta l|}{|\delta^0 l|} &= (w_1+w_2+w_3)^{\frac{1}{2}}[\frac{(sin^2\theta cos^2\psi)w_1+(sin^2\theta sin^2\psi)w_2+(cos^2\theta)w_3}{w_1+w_2+w_3}]^{\frac{1}{2}}\\ &\ge (w_1+w_2+w_3)^{\frac{1}{2}}[\frac{|sin\theta cos\psi|w_1+|sin\theta sin\psi|w_2+|cos\theta|w_3}{w_1+w_2+w_3}].\end{aligned} \tag{9}$$

Now the average of the coefficient of w_i's in the above line is $\frac{1}{2}$ and we are through by Assumption (1).///

Proposition 1.2. Under Assumptions (1) and (2) and/or (1) and (3) we have,

$$\frac{1}{3}\langle w_1^{\frac{p}{2}} + w_2^{\frac{p}{2}} + w_3^{\frac{p}{2}}\rangle \le \langle(\frac{|\delta l|}{|\delta^0 l|})^p\rangle \le \langle w_1^{\frac{p}{2}} + w_2^{\frac{p}{2}} + w_3^{\frac{p}{2}}\rangle \text{ fo } 0 < p < 2, \qquad (i)$$

$$\langle\frac{|\delta l|^2}{|\delta^0 l|^2}\rangle = \frac{1}{3}\langle w_1 + w_2 + w_3\rangle \text{ for } p = 2, \qquad (ii)$$

$$\langle\frac{w_1^{\frac{1}{2}} + w_2^{\frac{1}{2}} + w_3^{\frac{1}{2}}}{3}\rangle^p \le \langle(\frac{|\delta l|}{|\delta^0 l|})^p\rangle \le \langle\frac{w_1^{\frac{p}{2}} + w_2^{\frac{p}{2}} + w_3^{\frac{p}{2}}}{3}\rangle \text{ for } 2 < p < \infty, \qquad (iii)$$

$$\frac{1}{6}\langle log(w_1 w_2 w_3)\rangle \le \langle log(\frac{|\delta l|}{|\delta^0 l|})\rangle. \qquad (iv)$$

Proof. In light of what we have said in the proof of the above proposition, (ii) is immediate under Assumptions (1) and (2). Now we can conclude (ii) by utilizing Assumptions (1) and (3) in (6) to obtain,

$$\begin{aligned}\langle\frac{|\delta l|^2}{|\delta^0 l|^2}\rangle &= \langle sin^2\theta cos^2\psi\rangle\langle w_1\rangle + \langle sin^2\theta sin^2\psi\rangle\langle w_2\rangle + \langle cos^2\theta\rangle\langle w_3\rangle. \\ &= \langle sin^2\theta cos^2\psi + sin^2\theta sin^2\psi + cos^2\theta\rangle\langle w_1\rangle = \langle w_1\rangle \\ &= \frac{1}{3}\langle w_1 + w_2 + w_3\rangle.\end{aligned} \qquad (10)$$

The left hand side of (i) and the right hand side of (iii) follow from an inequality like the one given in (7). $\sqrt{x}$ should be replaced by $x^{\frac{p}{2}}$, and the direction of the inequality should be reversed due to convexity when $p > 2$. The right hand side of (i) is trivially true and the left hand side of (iii) is a consequence of $\langle\frac{|\delta l|}{|\delta^0 l|}\rangle^p \le \langle(\frac{|\delta l|}{|\delta^0 l|})^p\rangle$, $p \ge 1$, see (18), and the right hand side of (i) for $p = 1$.

Finally to obtain (iv) use the concave function $log(x)$ rather than $\sqrt{x}$ in (7) under Assumption (1) and (2), and an argument similar to the one given at (10) under Assumption (1) and (3).///

Remark 1.1. Note that the left hand side inequalities in the above propositions are strict unless $w_1 = w_2 = w_3$. For all the concave functions involved are strictly increasing. Clearly this happens only when W is the identity matrix, meaning pure rotation. Furthermore, this has to be the case with probability one in order to have equality in the above propositions, which is certainly not an interesting case.

Corollary 1.1. (Cocke). For an isotropic incompressible turbulence,

$$\langle log\frac{|\delta l|}{|\delta^0 l|}\rangle > 0 \ , \ \langle\frac{|\delta l|^p}{|\delta^0 l|^p}\rangle > 1,$$

for any $p > 0$.

Proof. Cocke (1969) gives an a priori proof to the effect that the Assumption (1) to (3) are true for isotropic incompressible flow. This invokes the above proposition and together with Remark 1.1 and the fact that $det(x_{i,j}) = 1$, see (3), we obtain the first inequality. The second one follows from (18), and the one we have just established as follows;

$$\langle\frac{|\delta l|^p}{|\delta^0 l|^p}\rangle = exp[log\langle\frac{|\delta l|^p}{|\delta^0 l|^p}\rangle] \geq exp[\langle log\frac{|\delta l|^p}{|\delta^0 l|^p}\rangle] = exp[p\langle log\frac{|\delta l|}{|\delta^0 l|}\rangle] > 1. \qquad (11)$$

We could have also used the arithmetic geometric mean inequality, $(w_1^r + w_2^r + w_3^r)/3 \geq \sqrt[3]{(w_1 w_2 w_3)^r}$, to achieve the same end.///

Remark 1.2. The proof of the Assumptions (1) to (3) is the thrust of the work in Cocke (1969) in which he has also shown Proposition 1.2 (ii) and (iv), the above remark and corollary by using the same argument. Orszag (1970) takes these assumptions for granted, follows Cocke's argument and obtains a weaker result, $\langle\frac{|\delta l|^2}{|\delta^0 l|^2}\rangle > 1$, see the above corollary for $p = 2$. Note that this result *does not imply* that the average material line stretches. Now if we agree on all three assumptions, then it is trivial to show that $\langle W\rangle = \gamma(\delta_{ij})$; where γ is the right hand side of (ii) in Proposition 1.2. In this connection see also Orszag (1977), p.240–241.

Next we extend the above results to an arc length following the flow. The statement of the inequalities needed to carry this on can be found in the appendix. Let $C(s;t_0) : [a,b] \longrightarrow \mathbf{R}^3$ be a parametric representation of a *non-random* curve at time t_0. Then $C(s;t) = x(C(s;t_0),t)$ is the corresponding *random curve* , following the flow at time t. Let $\mathbf{C}_{t_0}$ and $\mathbf{C}_t$, $t \geq t_0$ be their arc lengths, respectively. For homogenous turbulence define,

$$\alpha(t) = \langle\frac{\sqrt{w_1}+\sqrt{w_2}+\sqrt{w_3}}{3}\rangle \begin{array}{l} \leq \ \beta(t) = \langle\sqrt{\frac{w_1+w_2+w_3}{3}}\rangle \\ \leq \ \gamma(t) = \sqrt{\langle\frac{w_1+w_2+w_3}{3}\rangle} \end{array},$$

where the first and second inequalities are the consequence of (19) and (18) respectively.

Theorem 1.1. For an isotropic, incompressible flow,

$$\mathbf{C}_{t_0} \leq exp\{\langle log(\mathbf{C}_t)\rangle\} \leq \langle\mathbf{C}_t\rangle \leq \gamma(t)\mathbf{C}_{t_0}, \qquad (i)$$

$$\mathbf{C}_{t_0} \le \alpha(t)\mathbf{C}_{t_0} \le \langle \mathbf{C}_t \rangle \le \frac{3}{2}\alpha(t)\mathbf{C}_{t_0}, \qquad (ii)$$

$$\frac{\sqrt{3}}{2}\beta(t)\mathbf{C}_{t_0} \le \langle \mathbf{C}_t \rangle \le \sqrt{3}\beta(t)\mathbf{C}_{t_0}. \qquad (iii)$$

Proof. The above corollary implies that for the non-random vector $\delta^0 l = C'(s;t_0)$,

$$\langle log(|\sum_{k=1}^{3} x_{,k} C_k'(s;t_0)|)\rangle = \langle log(|\delta l|)\rangle \ge log(|\delta^0 l|) = log(|C'(s;t_0)|). \qquad (12)$$

Consequently,

$$\begin{aligned} exp\{\langle log(\mathbf{C}_t)\rangle\} &= exp\{\langle log[\int_a^b |\frac{\partial C(s;t)}{\partial s}|\, ds]\rangle\} \\ (by\ (20),\ S_1 = [a,b]) &\ge \int_a^b exp\{\langle log(|\frac{\partial C(s;t)}{\partial s}|)\rangle\}\, ds \\ &= \int_a^b exp\{\langle log(|\sum_{k=1}^{3} x_{,k}(C(s;t_0),t) C_k'(s;t_0)|)\rangle\}\, ds \qquad (13) \\ (by\ (12)) &\ge \int_a^b exp\{log(|C'(s;t_0)|)\} ds \\ &= \int_a^b |C'(s;t_0)|\, ds = \mathbf{C}_{t_0}. \end{aligned}$$

Next, inequality in (i) is an immediate consequence of (18) with $\phi = exp(x)$. The following one is true by virtue of (18) with $\phi = \sqrt{x}$, and Proposition 1.2 (ii). For the rest of the inequalities all we need is Proposition 1.1.///

Remark 1.3. Since in an inviscid flow vortex lines remain vortex lines, the above theorem is an statement about their evolution in time when the flow is isotropic and incompressible. The same is also true for vortex sheets which will follow from the discussion in section 2.

Remark 1.4. The reason for presenting various upper and lower bounds is for their potential in applications. For instance, with regard to the above remark and under the same conditions, one can compute $\gamma^2(t)$ as the ratio of the enstrophy at time t to time t_0.

Proposition 1.2 will give us "tight" upper and lower bounds for $\langle \int_a^b |\frac{\partial C(s;t)}{\partial s}|^p ds\rangle$ in an obvious way. This can be used partially to get information about the moments of $\mathbf{C}_t$.

Theorem 1.2. For an isotropic, incompressible flow, let $\alpha_p(t) = \langle w_1^{\frac{p}{2}} + w_2^{\frac{p}{2}} + w_3^{\frac{p}{2}} \rangle / 3$. Then,for $p > 0$,

$$(\mathbf{C}_{t_0})^p \leq \langle (\mathbf{C}_t)^p \rangle \tag{i}$$

for $0 < p \leq 1$,

$$\alpha_p(t)[min_{s\epsilon[0,1]}\{|C'(s;t_0)|^{p-1}\}]\ \mathbf{C}_{t_0} \leq \langle (\mathbf{C}_t)^p \rangle \leq \langle \mathbf{C}_t \rangle^p \leq c_2^p\ (\mathbf{C}_{t_0})^p \tag{ii}$$

for $p \geq 1$,

$$(\mathbf{C}_{t_0})^p \leq c_1^p\ (\mathbf{C}_{t_0})^p \leq \langle \mathbf{C}_t \rangle^p \leq \langle (\mathbf{C}_t)^p \rangle \leq 3\alpha_p(t)[max_{s\epsilon[0,1]}\{|C'(s;t_0)|^{p-1}\}]\ \mathbf{C}_{t_0} \tag{iii}$$

where c_2 (c_1) is the minimum (maximum) of the coefficients of $\mathbf{C}_{t_0}$ in the upper (lower) bounds for $\langle \mathbf{C}_t \rangle$ in Theorem 1.1, and without loss of generality, we have assumed $[a, b] = [0, 1]$.

Proof. The easiest way to handle (i) is to raise the left hand side of (i) in Theorem 1.1 to power p, take p inside the log, and then use (18) with $\phi = exp(x)$. The left hand side of (ii) and the right hand side of (iii) are also the cosequences of (18) with $\phi = x^p$ in an obvious way, and the remaining parts are easily followed by Theorem 1.1.///

The next corollary will give us information about the moments of evolution of a straight line segment in turbulent flow and also the moments of dispersion of two points as time goes on.

Corollary 1.2. Let $d_1 = min[c_1^p, \alpha_p(t)]$, $d_2 = max[c_2^p, 3\alpha_p(t)]$ with c_1, c_2 and $\alpha_p(t)$ as in the above theorem.Let $C(s;t_0) = a_1 + s(a_2 - a_1)$, $s\epsilon[0, 1]$. Then for an isotropic, incompressible turbulence,

$$d_1|a_2 - a_1|^p \leq \langle (\mathbf{C}_t)^p \rangle \leq d_2|a_2 - a_1|^p \quad for \quad p > 0, \tag{i}$$

$$\langle |x(a_2, t) - x(a_1, t)|^p \rangle \leq d_2|a_2 - a_1|^p \quad for \quad p > 0. \tag{ii}$$

Proof. Note that $|\frac{\partial C(s;t)}{\partial s}| = |a_2 - a_1|$ and use the above theorem.///

2. Surface Stretching

In this section we will extend the above results to the evolution of a surface area in turbulent flow. It turns out, just as in the work of Cocke (1969), that only minor modifications are needed to do so.

Let $\delta^0 l$ and $\delta^0 k$ be two infinitesimal material line at $t = t_0$. We can form an infinitesimal material surface by taking the vector product of these vectors, i.e. $\delta^0 S = \delta^0 l \times \delta^0 k$. This at time t becomes,

$$\delta S = \delta l \times \delta k = (\sum_{j=1}^{3} X_{,j}\delta^0 l_j) \times (\sum_{j=1}^{3} X_{,j}\delta^0 k_j). \tag{14}$$

Let the matrices W, A and the eigenvalues, w_i's, of W, be as before. From the identity,

$$|v_1 \times v_2|^2 = |v_1|^2|v_2|^2 - (v_1 \cdot v_2)^2, \tag{15}$$

for any given vectors v_1 and v_2 and the fact that $|\delta^0 S|^2 = |A\delta^0 l \times A\delta^0 k|^2$, A being unitary, we can easily obtain,

$$\begin{aligned}\frac{|\delta S|^2}{|\delta^0 S|^2} &= w_2 w_3 \frac{(A\delta^0 l \times A\delta^0 k)_1^2}{|A\delta^0 l \times A\delta^0 k|^2} + w_1 w_3 \frac{(A\delta^0 l \times A\delta^0 k)_2^2}{|A\delta^0 l \times A\delta^0 k|^2} + w_1 w_2 \frac{(A\delta^0 l \times A\delta^0 k)_3^2}{|A\delta^0 l \times A\delta^0 k|^2} \\ &= (sin^2\theta cos^2\psi) w_2 w_3 + (sin^2\theta sin^2\psi) w_1 w_3 + (cos^2\theta) w_1 w_2.\end{aligned} \tag{16}$$

Now (16) plays the role of (6), and we only need to modify Assumption (3) as follows;

<u>*Assumption*;</u>

(4) w_1w_2, w_1w_3 and w_2w_3 are identically distributed. (*Note that Assumptions (3) and (4) are equivalent for incompressible flows.*)

Now all we need to do is to replace w_1, w_2, w_3 by w_2w_3, w_3w_1, w_1w_2 ; $\delta l, \delta^0 l$ by $\delta S, \delta^0 S$, and $\mathbf{C_t}, \mathbf{C_{t_0}}$ by $\mathbf{S_t}, \mathbf{S_{t_0}}$, and $[a, b]$ by $\mathbf{D}$, respectively, in the above results, including the remarks, to obtain the new ones corresponding to surfaces. Where we let $S(u, v; t_0)$ be a parametric representation of a *nonrandom* surface on a region $\mathbf{D}$ on the plain, $\mathbf{S_{t_0}}$ be its area, and,

$$\mathbf{S_t} = \int\int_{\mathbf{D}} \left|\frac{\partial x(S(u,v;t_0),t)}{\partial u} \times \frac{\partial x(S(u,v;t_0),t)}{\partial v}\right| dudv, \tag{17}$$

the area at time t.

The only *nonsymbolic* modification of the proofs are: (a) In Theorem 1.2 the area of $\mathbf{D}$ has to be one or otherwise the right (left) hand side of (*iii*) ((*ii*)) has to be multiplied by that area to the power $p - 1$ due to the correct usage of Jensen's inequality; (b) In Corollary 1.2 the notion of a distance between two points has to be replaced by an area of a region on a *plane* and its left hand side of (ii) to be interpreted correctly.

<u>*Remark* 2.1</u> We could have always used Proposition 1.2 and its counterpart for surfaces to obtain upper and lower bounds for moments of material lines, material surfaces, arc lengths, and surface areas at the expense of having different *constants*. Compare Proposition 1.1 with Proposition 1.2 when $p = 1$. The difference between these two propositions becomes significant if one can realize physically non-isotropic incompressible flows that satisfy only one pair of the assumptions, involved in these propositions, rather than all of them. Finally, for analogous results concerning homogeneous turbulence we invite the reader to consult Corrsin (1972).

Appendix

(Jensen's Inequality) Let X be a random variable and ϕ a convex (concave) function containing the range of X. Assume both X and $\phi(X)$ have ensemble averages, then

$$\phi(\langle X\rangle) \leq \ (\geq)\ \langle\phi(X)\rangle. \tag{18}$$

Proof. See any standard graduate textbook in probability theory or measure theory e.g. Billingsley (1986) p.283.

The following special case of Jensen's inequality has been used frequently; let p_1, p_2, and p_3 be three positive numbers whose total sum is one, let a_1, a_2, and a_3 be any real numbers. Then with ϕ as above we have,

$$\phi(\sum_{i=1}^{3} p_i a_i) \ \leq\ (\geq)\ \sum_{i=1}^{3} p_i \phi(a_i). \tag{19}$$

Proof. Let X in (18) be the random variable which takes the value a_i with probability p_i, $i = 1, 2, 3.///$

(Dunford and Schwartz[5], p.535). Let (S, Σ, μ) and (S_1, Σ_1, μ_1) be positive measure spaces. Assume $\mu(S) = 1$. Then if K is a $\mu \times \mu_1$ – measurable function defined on $S \times S_1$,

$$\int_{S_1} exp\{\int_S log|K(s,s_1)|\mu(ds)\}\mu_1(ds_1) \leq exp\{\int_S [log(\int_{S_1} |K(s,s_1)|\mu_1(ds_1))]\mu(ds)\}. \tag{20}$$

REFERENCES

BATCHELOR, G. K. 1977 *An Introduction to Fluid Dynamics.* Cambridge University Press.

BATCHELOR, G. K. 1952 Proc. Roy. Soc. (London) A213, 349.

BILLINGSLEY, P. 1986 Probability and measure. Wiley- -interscience pub., New York.

COCKE, W. J. 1969 *The Physics of Fluids* **12**, 2488.

CORRSIN, S. 1972 *The Physics of Fluids* **15**, 1370.

DUNFORD, N. & SCHWARTZ, J. T. 1958 *Linear Operators. Part I.* Interscience pub., New York.

MONIN, A. S. & YAGLOM A. M. 1975 *Statistical fluid mechanics. Vol. II.* The MIT Press.

ORSZAG, S. A. 1970 *The Physics of Fluids* **13**, 2203.

ORSZAG, S. A. 1977 Fluid Dynamics. Gordon and Beach Sci.Pub.