N89 - 22821

Renormalization group analysis of turbulence

By L. M. SMITH

1. Objective

The objective is to understand and extend a recent theory of turbulence based on dynamic renormalization group (RNG) techniques. The application of RNG methods to hydrodynamic turbulence has been explored most extensively by Yakhot and Orszag (1986). They calculate an eddy viscosity consistent with the Kolmogorov inertial range by systematic elimination of the small scales in the flow. Further, assumed smallness of the nonlinear terms in the redefined equations for the large scales results in predictions for important flow constants such as the Kolmogorov constant. The authors emphasize that no adjustable parameters are needed. The parameterization of the small scales in a self-consistent manner has important implications for sub-grid modeling.

2. The RNG Transformation

Renormalization group methods were first developed for quantum field theories. They were later applied to the theory of critical points in materials that undergo phase transitions (Ma, 1976). Predictions for the universal exponents characterizing the behavior of thermodynamic quantities near critical points are quite accurate. The common feature of the physical phenomena amenable to RNG analysis is a lack of characteristic length and time scales.

The lack of characteristic length and time scales in turbulence makes RNG methods attractive. The universality of the inertial range spectrum in widely varying turbulent flows is also suggestive.

The RNG transformation consists of two steps. First, small scales are eliminated by an averaging procedure. Second, space is rescaled. New independent variables are defined on the original intervals by the rescaling. In most cases, the dependent variables must also be rescaled.

A set of equations is renormalizable if it is unchanged by the RNG transformation. Renormalizability implies scale invariance. Usually a set of equations is renormalizable only for specific values of its coefficients and the scaling parameters. These points are called fixed points. However, the physics of more general cases is often well described by the physics at a fixed point.

The method of attack is to iterate the RNG transformation of the equations. With each transformation the coefficients in the equations change. One looks for a situation in which this iteration procedure converges.

In addition to redefining coefficients of existing terms, the scale elimination often generates terms of different form than those in the original equations.

These new terms can be classified as irrelevant, marginal or relevant according to whether they decay, are constant or grow when rescaled. One can ask if a fixed point exists in the absence of new terms. If so, all new terms must be irrelevant for the system to be truly renormalizable at that point.

3. The Basic Premise of the RNG Analysis of Turbulence

The theory is based on the postulated equivalence between inertial range solutions of the Navier Stokes equations subject to initial and boundary conditions, and homogeneous isotropic flow driven by a Gaussian random force (Forster et. al., 1977, Yakhot and Orszag, 1986). The model equations are then

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla P + \nu_o \nabla^2 \mathbf{v} \tag{1}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{2}$$

where $\mathbf{v}(\mathbf{x}, t)$ is the velocity, P the pressure, ρ the density, ν_o the kinematic viscosity and $\mathbf{f}$ the forcing. The domain of equations (1) and (2) is unbounded.

The white noise force is given by its correlation function in wavevector, frequency space. The correlation is assumed to obey a power law spectrum,

$$\langle f_i(\mathbf{k}, \omega) f_j(-\mathbf{k}, -\omega) \rangle \propto |\mathbf{k}|^{-y} \tag{3}$$

where the brackets indicate an ensemble average. The exponent y is chosen to give the inertial range energy spectrum. Once y is fixed, there are no adjustable parameters in the problem.

4. A Revised RNG Analysis

Yakhot and Orszag show that analysis of (1)-(3) using the full RNG transformation yields the scaling laws of velocity correlations, and thus the energy spectrum. If y is set equal to the number of dimensions, 3, the Kolmogorov spectrum is recovered: $E(k) \propto k^{-5/3}$ where $k = |\mathbf{k}|$. Amplitudes, however, are left undetermined.

By performing only the scale elimination, and abandoning the rescaling, they are able to find both scaling laws and amplitudes. Then $E(k) = Ko_{RNG}\epsilon^{2/3}k^{-5/3}$, where ϵ is the dissipation rate and Ko_{RNG} is the RNG prediction for the Kolmogorov constant.

Rescaling is used only to justify neglect of new terms generated by the elimination procedure. The terms of concern are cubic in the velocity vector and are marginal with respect to the fixed point found in their absence. One wonders how the results would change if the cubic terms are retained. A goal of the present research is to assess the effect of these terms on the system.

5. The Effect of the Small Scales

The theory developed by Yakhot and Orszag is an attempt to calculate the effect of the small scales on the large scales in turbulence. Their method determines that the large scales 'feel' the small scales as an eddy viscosity.

Equations (1) and (2) are written in wavevector, frequency space and the pressure is eliminated by taking the curl of the curl. The equations for the Fourier coefficients of the velocity field are then expanded in a power series via the introduction of an ordering parameter which multiplies the nonlinear term.

A narrow band of wavenumbers is removed by averaging over their force field. The averaging procedure replaces the contribution of the nonlinear interaction of those wavenumbers with a term linear in the velocity vector for the remaining wavenumbers. The nonlinear interaction is only approximately represented in this term. The approximation is due to truncation of the power series at second order and neglect of terms cubic in the velocity vector for the remaining wavenumbers.

The coefficient of the linear term is an integral. The integral is evaluated in the limit $0 \sim k << k_c$, where k_c is the low wavenumber cutoff of the eliminated band. In this limit the integral is proportional to k^2. Thus the large scales see the small scales as (approximately) a viscous term.

Iteration of the elimination procedure produces an equation for the large scales and long times identical in form to equation (1). The molecular viscosity ν_o is replaced by an eddy viscosity,

$$\frac{\partial \mathbf{v}}{\partial t} + \bar{\lambda}(\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f} - \frac{\bar{\lambda}}{\rho}\nabla P + \nu_T \nabla^2 \mathbf{v} \tag{4}$$

where ν_T is the eddy viscosity and $\bar{\lambda}$ is the nondimensionalized ordering parameter. The eddy viscosity depends on k_c,

$$\nu_T = \nu_o[1 + \beta^3 \epsilon (k_c^{-4} - k_d^{-4})]^{1/3} \tag{5}$$

where β is a function of $\bar{\lambda}$, k_c is the last eliminated wavenumber and k_d is the viscous cutoff. If one now takes the limit $k_c \sim k \sim 0$,

$$\nu_T \sim \beta \epsilon^{1/3} k^{-4/3}. \tag{6}$$

In this limit the 'renormalized' equation (4) has a fixed point, at which all subsequent analysis takes place.

6. Evaluation of Universal Flow Constants

The nondimensionalized ordering parameter $\bar{\lambda}$ is proportional to $(y+1)^{1/2}$, where $-y$ is the power of k in the force correlation function (see (3)). Recall $y = 3$ for a Kolmogorov inertial range. Using $y = 3$, the fixed point value of $\bar{\lambda}$

is 11.5. However, the RNG results depend on the simultaneous extrapolation of $k_c \sim k \sim 0$ and y approaching -1.

The assumption of small $(y+1)$ allows a power series expansion of the renormalized equations. The equation for the zeroth order velocity coefficient is the Langevin equation,

$$(-i\omega + \nu_T k^2)\bar{\mathbf{v}}^{(0)}(\mathbf{k},\omega) = \bar{\mathbf{f}}, \tag{7}$$

where $\bar{\mathbf{v}}(\mathbf{k},\omega)$ is the Fourier amplitude, $\bar{\mathbf{v}} = \bar{\mathbf{v}}^{(0)} + \bar{\lambda}\bar{\mathbf{v}}^{(1)} + \bar{\lambda}^2\bar{\mathbf{v}}^{(2)} + \ldots$, k and ω are small, and ν_T is given by equation (6). The value of β at the fixed point, evaluated to zeroth order in $\bar{\lambda}$, is found to be $\beta = .493$.

Evaluation of the Kolmogorov constant follows a power series expansion at the fixed point of the energy equation (the equation for the two-point velocity correlations). The analysis is again for $k_c \sim k \sim 0$ with ν_T given by (6). When terms to lowest nontrivial order are retained, its value is found to be $Ko_{RNG} = 1.617$ (Dannevik et. al., 1987).

The same scale elimination procedure can be performed on the equations for the advection of a passive scalar. The renormalized equations for the large scales are characterized by an eddy diffusivity. A power series expansion with $k_c \sim k \sim 0$ leads to a prediction for the Batchelor constant, Ba.

7. Goals

a. Further steps in the analysis remain to be understood. For example, the RNG $\kappa - \epsilon$ model leads to a prediction for the von Karman constant.

b. One would like to assess the importance of the neglected new terms generated by the scale elimination procedure. These terms are cubic in the transform coefficients of the velocity vector for the large scales.

c. An exploration of other forcing functions will be revealing.

i. We have already investigated the possibility of a non-white noise, power law spectrum. The force correlation is allowed to fall off with frequency and considered proportional to $k^{-y}\omega^{-b}$. We find that there is no positive value of b consistent with both dimensional analysis and a Kolmogorov energy spectrum.

ii. A completely satisfactory theory of turbulence should account for spatial intermittency. In the present RNG formulation, intermittency might be included by changing the statistics assumed for the forcing function.

d. It should be possible to extend the theory to flows other than homogeneous, isotropic turbulence. Perhaps one can do away with the artificial forcing for realizable kinds of turbulence.

e. Finally, the RNG sub-grid and $\kappa - \epsilon$ models should be tested.

REFERENCES

DANNEVIK, P., YAKHOT, V. & ORSZAG, S. A. 1987 *Phys. Fluids* **30**, 2021.

FORSTER, D., NELSON, D. R. & STEPHEN, M. J. 1977 *Physical Review A* **16**, 732.

KRAICHNAN, R. H. 1987 *Phys. Fluids* **30**, 2400.

MA, S. K. 1976 Modern Theory of Critical Phenomena - Frontiers in Physics, **46**.

YAKHOT, V. & ORSZAG, S. A. 1986 *Journal of Scientific Computing* **1**, 3.