

IMPROVEMENTS TO THE CONTINUE FEATURE IN TRANSIENT ANALYSIS

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SUMMARY

The CONTINUE feature in transient analysis as implemented in the standard release of COSMIC/NASTRAN has inherent errors associated with it. As a consequence, the results obtained by a CONTINUED restart run do not, in general, match the results that would have been obtained in a single run without the CONTINUE feature. These inherent errors have been eliminated by improvements to the restart logic that have been developed by RPK Corporation and that are available on all RPK-supported versions of COSMIC/NASTRAN. These improvements ensure that the results of a CONTINUED transient analysis run are the same as those of a non-CONTINUED run. In addition, the CONTINUE feature has been extended to transient analysis involving uncoupled modal equations. The improvements and enhancement have been illustrated by examples.

INTRODUCTION

In transient analysis, it is frequently necessary to continue the integration of equations beyond the last (or from any earlier intermediate) output time for which the results were obtained in a previous run. Thus, the initial time for the new run is to be a specified output time of the previous run and the initial conditions for the new run are to be the same as the conditions existing at the specified output time of the previous run. The CONTINUE feature in NASTRAN makes it possible to do this without re-executing the entire problem.

The CONTINUE feature involves performing a checkpoint run of a transient analysis problem followed by a restart run with the aim of continuing the integration from the last (or from any earlier intermediate) output time for which the results were obtained in the checkpoint run. The details of the usage of this feature are given in Vol. 2, Section 2.9.6 of the NASTRAN User's Manual (Reference 1). The theoretical aspects of this feature are discussed in Section 11.4.2 of the NASTRAN Theoretical Manual (Reference 2) and the programming aspects are detailed in Section 4.65.7.3 of the NASTRAN Programmer's Manual (Reference 3).

EQUATIONS FOR TRANSIENT ANALYSIS

The equations used in transient analysis are discussed in detail in References 2 and 3. For the sake of convenience, they are summarized in the Appendix.

The most important equation employed in transient analysis is Equation (1) given in the appendix. After the integration is started, this equation is used to continue the integration by computing the displacements at successive times until there is a change in the time step. In order to compute the displacements at each time, this equation requires, in addition to the loads at that time, the displacements and the loads at the two previous times.

When there is a change in the time step employed for the integration, Equation (2) in the appendix is used to compute the displacements for the first time after the time step change. It is important to note that this equation is employed only once for each time step change. Equation (1) is then employed to resume the integration until there is another time step change. Every time Equation (2) is employed, it requires, in addition to the loads at that time, the displacements at the three previous times and the loads at the previous time.

CURRENT IMPLEMENTATION OF THE CONTINUE FEATURE

When the CONTINUE feature is employed in the standard release of COSMIC/NASTRAN to restart a transient analysis problem, Equation (3) in the appendix is used to start the integration from the specified time. Equation (1) or (2), as appropriate, is then used to continue the process. In order to start the integration, Equation (3) uses the displacements, velocities and accelerations at the specified output time of the checkpoint run. Unlike Equations (1) and (2), Equation (3) uses neither the loads at the specified output time nor any displacements at solution times prior to that output time.

The CONTINUE feature, as described above, is based on the inherent assumption that the output times of the checkpoint run do not include all of the solution times of the integration. In other words, it is assumed that the TSTEP bulk data card (Reference 1) in the checkpoint run involves one or more non-unity skip factors for output. As a result, the CONTINUE feature initiates the integration in the restart run by considering the conditions (namely, the displacements, velocities and accelerations) only at the specified output time of the previous run; it does not take into account the conditions at solution times just before that output time because the latter conditions are assumed to be not available to the restart run.

The above approach to the CONTINUE feature introduces inherent errors into the procedure (Reference 2). As a consequence, the results of a CONTINUED restart run do not, in general, match the results that would have been obtained in a single run without using the CONTINUE feature.

It can also be seen from Equation (3) and the associated Equations (3a) through (3d) that the initial time step used in a CONTINUED run can affect its results. As a result, the errors mentioned above may be magnified further if the initial time step in the CONTINUED restart run is not the same as that used in the checkpoint run just before the restart (Reference 2).

In addition to the above deficiency, the current implementation of the CONTINUE feature has the further limitation that it is restricted to coupled equations and is not applicable to transient analysis involving uncoupled modal equations.

IMPROVED IMPLEMENTATION OF THE CONTINUE FEATURE

RPK has developed improvements to the CONTINUE feature that have removed the deficiencies described above. This involved extensive changes to the code in the TRD (Transient Analysis - Displacement approach) module, including, in particular, the addition of an extra output data block to this module, and minor related changes elsewhere. These changes also required minor modifications to the DMAP of Displacement Approach Rigid Formats 9 (Direct Transient Response) and 12 (Modal Transient Response). These improvements are available on all RPK-supported versions of COSMIC/NASTRAN, beginning with the 1988 release.

The highlights of RPK's improvements are described below.

1. The loads and displacements for all solution times of the checkpoint run, as well as the time steps employed to obtain each of those solutions, are made available to the restart run. With this information, the restart run using the CONTINUE feature now initiates the continuation of the integration using the same procedure as that used in a non-CONTINUED run. This applies even in those cases where the time step used in the CONTINUED restart run is different from that used in the earlier checkpoint run just before restart. As a result, Equation (3) is no longer used and the restart run uses Equation (1) or (2), as appropriate, to initiate the integration.
2. The key to providing the information required for the above procedure is the new data block output from the TRD module. There are two possible cases that need to be considered with regard to the generation of this data block.

Case 1. The TSTEP bulk data card in the checkpoint run has no non-unity skip factors

In this case, the output times, by definition, include all solution times. As a result, the displacements for all solution times are available from the standard UDVT (displacement-velocity-acceleration) matrix data block resulting from the TRD module in the checkpoint run. Accordingly, the new data block in this case is designed to contain only the loads for all solution times as well as the time steps used to obtain each of the solutions. This case is identified in the new data block (for subsequent use in the restart run) by setting the fourth word of its six-word trailer to 1.

Case 2. The TSTEP bulk data card in the checkpoint run consists of one or more non-unity skip factors

Since the output times in this case are only a subset of the solution times, the new data block is designed to contain not only the loads but also the displacements for all solution times, as well as the time steps used to obtain each of those solutions. Further, the new data block in this case has an additional special record that has information relating each output time to its corresponding solution time. This case is identified in the new data block by setting the fourth word of its six-word trailer to 0.

3. Using the information provided by the new data block described above, the restart run using the improved CONTINUE feature determines whether all of the loads and displacements

required for continuing the integration are available just from this data block alone (Case 2 above) or from both this data block and the UDV_T data block (Case 1 above).

It should be emphasized that the new data block is generated only in checkpoint runs since its sole use is in a subsequent restart run using the CONTINUE feature. Also, it should be noted that, even when it is generated, this data block does not require any new computations since all of the information it needs is already available.

4. The restart run then determines if the time step to be used for continuing the integration in the restart run is the same as, or different from, the time step that was used for obtaining the solution for the specified output time of the checkpoint run. If the former condition is true, Equation (1) in the appendix is used to initiate the integration. If the latter condition is true, Equation (2) in the appendix is used to initiate the integration. Depending upon which of these two equations is used, the required loads and displacements for starting the integration are retrieved from the appropriate data block or data blocks mentioned earlier.
5. The procedure outlined above for the CONTINUE feature ensures that the CONTINUED run starts the integration in the same manner as the integration would have continued had it proceeded further in the original checkpoint run. This therefore ensures that the results obtained by a CONTINUED run will be the same as those of a non-CONTINUED run.
6. Separately, RPK has extended the CONTINUE feature to transient analysis involving uncoupled modal equations. Since the equations in this case have closed-form solutions (References 2 and 3), this development introduces no errors.

EXAMPLES

In order to illustrate the improvements to the CONTINUE feature mentioned above, NAS-TRAN Demonstration Problem No. D09-01-1A (Transient Analysis with Direct Matrix Input) was selected. Two variations of this problem with the same standard input data, but with different sets of TSTEP bulk data input, were analyzed. These two cases are identified as Examples 1 and 2.

For both of the above examples, checkpoint and restart runs were made on the DEC VAX version using the standard release of COSMIC/NASTRAN and on RPK's CRAY version of COSMIC/NASTRAN using the improvements to the CONTINUE feature described in the paper. The results of these analyses are presented in Table 1 (for Example 1) and Table 2 (for Example 2). To facilitate comparison of the results, the restart runs were made from an intermediate output time of the checkpoint runs. It is quite clear from these tables that the results of the CONTINUED restart runs on RPK's CRAY version match those of the non-CONTINUED checkpoint runs perfectly, thereby validating the improvements. The results of the DEC VAX version do not exhibit the same correlation.

In order to illustrate the extension of the CONTINUE feature to transient analysis involving uncoupled modal equations, NASTRAN Demonstration Problem No. D12-01-1A (Transient Analysis of a Free One Hundred Cell Beam) was selected. This case is identified as Example 3.

Checkpoint and restart runs for the above problem were made on RPK's CRAY version mentioned above. (No runs were made on the standard release of COSMIC/NASTRAN as it does not support the CONTINUE feature for uncoupled modal equations.) Again, to facilitate comparison of the results, the restart run was made from an intermediate output time of the checkpoint run. The results are presented in Table 3. It is again quite clear from this table that the results of the CONTINUED restart run on RPK's CRAY version match those of the non-CONTINUED checkpoint run perfectly, thereby validating the new development.

CONCLUDING REMARKS

The CONTINUE feature in transient analysis as implemented in the standard release of COSMIC/NASTRAN has inherent errors associated with it. As a consequence, the results obtained by a CONTINUED run do not, in general, match the results that would have been obtained in a single run without the CONTINUE feature. These inherent errors have been eliminated by improvements to the restart logic that have been developed by RPK Corporation and that are available on all RPK-supported versions of COSMIC/NASTRAN. These improvements ensure that the results of a CONTINUED transient analysis run are the same as those of a non-CONTINUED run. In addition, the CONTINUE feature has been extended to transient analysis involving uncoupled modal equations. The improvements and enhancement have been illustrated by examples.

REFERENCES

1. The NASTRAN User's Manual, NASA SP-222(08), June 1986.
2. The NASTRAN Theoretical Manual, NASA SP-221(06), January 1981.
3. The NASTRAN Programmer's Manual, NASA SP-223(05), December 1978.

TABLE 1. RESULTS FOR EXAMPLE 1

(Transient Analysis Involving Coupled Equations)

Displacements for Extra Point 10 of NASTRAN Demonstration Problem No. D09-01-1A

TSTEP Bulk Data Card Input

Checkpoint Run: Number of time steps = 200
Time increment = 0.010 sec.
Skip factor for output = 10

Number of time steps = 50
Time increment = 0.015 sec.
Skip factor for output = 5

Restart Run: Number of time steps = 100
Time increment = 0.010 sec.
Skip factor for output = 5

NOTES:

1. The restart run on the DEC VAX version uses the CONTINUE feature available in the standard release of COSMIC/NASTRAN.
2. The restart run on RPK's CRAY version uses the improved CONTINUE feature described in the paper. In this case, since the initial time step used in the restart run (0.010 sec.) is the same as that used to obtain the output of the checkpoint run from where the restart is initiated, the CONTINUE features uses Equation (1) in the appendix to initiate the integration.
3. The restart run was initiated by setting the parameter NCOL in the rigid format DMAP to 11 just before the TRLG module (see Volume 2, Section 2.9.6 of Reference 1 for details), thereby triggering the continuation of the integration from the 11th output time of the checkpoint run.
4. Note that the time step changes from 0.010 sec. to 0.015 sec. at time = 2.000 sec.
5. The % Error in the table is calculated by the following relationship:

$$\frac{(\text{Displacement from Restart Run}) - (\text{Displacement from Checkpoint Run})}{(\text{Displacement from Checkpoint Run})} \times 100$$

**TABLE 1. RESULTS FOR EXAMPLE 1
(Continued)**

Time (sec.) (See Note 4 above)	Results from DEC VAX Version of 1988 COSMIC/NASTRAN (See Note 1 above)			Results from RPK's CRAY Version of 1988 COSMIC/NASTRAN (See Note 2 above)		
	Displacements from Checkpoint Run	Displacements from Restart Run	% Error (See Note 5 above)	Displacements from Checkpoint Run	Displacements from Restart Run	% Error (See Note 5 above)
.000	0.000000E+00			0.000000E+00		
.100	8.404462E-01			8.404462E-01		
.200	9.099538E-01			9.099538E-01		
.300	1.447636E-01	See		1.447636E-01	See	
.400	-7.532177E-01	Note 3		-7.532178E-01	Note 3	
.500	-9.602748E-01	above		-9.602749E-01	above	
.600	-2.864749E-01			-2.864749E-01		
.700	6.501075E-01			6.501076E-01		
.800	9.903484E-01			9.903484E-01		
.900	4.221459E-01			4.221458E-01		
1.000	-5.332897E-01	-5.332897E-01	.00	-5.332899E-01	-5.332899E-01	.00
1.100	-9.995404E-01	-9.972498E-01	-.23	-9.995404E-01	-9.995404E-01	.00
1.200	-5.489158E-01	-5.464416E-01	-.45	-5.489157E-01	-5.489157E-01	.00
1.300	4.052275E-01	4.056157E-01	.10	4.052276E-01	4.052276E-01	.00
1.400	9.876569E-01	9.856030E-01	-.21	9.876569E-01	9.876569E-01	.00
1.500	6.641117E-01	6.614999E-01	-.39	6.641117E-01	6.641117E-01	.00
1.600	-2.686209E-01	-2.693950E-01	.29	-2.686212E-01	-2.686212E-01	.00
1.700	-9.549485E-01	-9.531747E-01	-.19	-9.549486E-01	-9.549486E-01	.00
1.800	-7.653049E-01	-7.626103E-01	-.35	-7.653047E-01	-7.653047E-01	.00
1.900	1.263505E-01	1.274941E-01	.91	1.263508E-01	1.263508E-01	.00
2.000	9.021050E-01	9.006485E-01	-.16	9.021051E-01	9.021051E-01	.00
2.015	9.624534E-01	9.606622E-01	-.19	9.624535E-01	9.624535E-01	.00
2.075	9.827350E-01	9.800652E-01	-.27	9.827349E-01	9.827349E-01	.00
2.150	5.388174E-01	5.363594E-01	-.46	5.388172E-01	5.388172E-01	.00
2.225	-1.927009E-01	-1.936352E-01	.48	-1.927012E-01	-1.927012E-01	.00
2.300	-8.213626E-01	-8.202745E-01	-.13	-8.213628E-01	-8.213628E-01	.00
2.375	-1.011611E+00	-1.009082E+00	-.25	-1.011611E+00	-1.011611E+00	.00
2.450	-6.618994E-01	-6.592784E-01	-.40	-6.618993E-01	-6.618993E-01	.00
2.525	4.110985E-02	4.242327E-02	3.19	4.111014E-02	4.111014E-02	.00
2.600	7.221763E-01	7.214810E-01	-.10	7.221765E-01	7.221765E-01	.00
2.675	1.017772E+00	1.015439E+00	-.23	1.017772E+01	1.017772E+00	.00
2.750	7.701184E-01	7.673931E-01	-.35	7.701182E-01	7.701182E-01	.00

TABLE 2. RESULTS FOR EXAMPLE 2

(Transient Analysis Involving Coupled Equations)

Displacements for Extra Point 10 of NASTRAN Demonstration Problem No. D09-01-1A

TSTEP Bulk Data Card Input

Checkpoint Run: Number of time steps = 200
Time increment = 0.005 sec.
Skip factor for output = 10

Number of time steps = 50
Time increment = 0.015 sec.
Skip factor for output = 5

Restart Run: Number of time steps = 50
Time increment = 0.015 sec.
Skip factor for output = 5

NOTES:

1. The restart run on the DEC VAX version uses the CONTINUE feature available in the standard release of COSMIC/NASTRAN.
2. The restart run on RPK's CRAY version uses the improved CONTINUE feature described in the paper. In this case, since the initial time step used in the restart run (0.015 sec.) is not the same as that used to obtain the output of the checkpoint run from where the restart is initiated (0.005 sec.), the CONTINUE feature uses Equation (2) in the appendix to initiate the integration.
3. The restart run was initiated by setting the parameter NCOL in the rigid format DMAP to 21 just before the TRLG module (see Volume 2, Section 2.9.6 of Reference 1 for details), thereby triggering the continuation of the integration from the 21st output time of the checkpoint run.
4. Note that the time step changes from 0.005 sec. to 0.015 sec. at time = 1.000 sec.
5. The displacement shown in the parentheses below is obtained only in the checkpoint run because this run involves a change in time step. There is no corresponding displacement from the restart run as the latter run does not involve any change in time step. In general, each change in time step results in such an extra displacement in the output.
6. The % Error in the table is calculated by the following relationship:

$$\frac{(\text{Displacement from Restart Run}) - (\text{Displacement from Checkpoint Run})}{(\text{Displacement from Checkpoint Run})} \times 100$$

TABLE 2. RESULTS FOR EXAMPLE 2
(Continued)

Results from DEC VAX Version of 1988 COSMIC/NASTRAN (See Note 1 above)				Results from RPK's CRAY Version of 1988 COSMIC/NASTRAN (See Note 2 above)		
Time (sec.) (See Note 4 above)	Displacements from Checkpoint Run	Displacements from Restart Run	% Error (See Note 6 above)	Displacements from Checkpoint Run	Displacements from Restart Run	% Error (See Note 6 above)
.000	0.000000E+00			0.000000E+00		
.050	4.792385E-01			4.792386E-01		
.100	8.412145E-01			8.412146E-01		
.150	9.973578E-01			9.973578E-01		
.200	9.094625E-01			9.094625E-01		
.250	5.990351E-01			5.990351E-01		
.300	1.420328E-01			1.420328E-01		
.350	-3.497228E-01			-3.497229E-01		
.400	-7.559065E-01	See Note 3 above		-7.559065E-01	See Note 3 above	
.450	-9.771311E-01			-9.771311E-01		
.500	-9.592662E-01			-9.592662E-01		
.550	-7.066831E-01			-7.066831E-01		
.600	-2.811852E-01			-2.811851E-01		
.650	2.131146E-01			2.131147E-01		
.700	6.552684E-01			6.552685E-01		
.750	9.370877E-01			9.370878E-01		
.800	9.896156E-01			9.896156E-01		
.850	7.999993E-01			7.999992E-01		
.900	4.146350E-01			4.146349E-01		
.950	-7.218431E-02			-7.218443E-02		
1.000	-5.413412E-01	-5.413412E-01	.00	-5.413413E-01	-5.413413E-01	.00
1.015	(-6.622212E-01)	See Note 5		(-6.622213E-01)	See Note 5	
1.075	-9.761183E-01	-9.721630E-01	-.41	-9.761184E-01	-9.761184E-01	.00
1.150	-8.898796E-01	-8.838871E-01	-.67	-8.898796E-01	-8.898796E-01	.00
1.225	-3.286562E-01	-3.238251E-01	-1.47	-3.286560E-01	-3.286560E-01	.00
1.300	4.079918E-01	4.090828E-01	.27	4.079919E-01	4.079919E-01	.00
1.375	9.268688E-01	9.236374E-01	-.35	9.268689E-01	9.268689E-01	.00
1.450	9.510176E-01	9.451886E-01	-.61	9.510176E-01	9.510176E-01	.00
1.525	4.675484E-01	4.622332E-01	-1.14	4.675483E-01	4.675483E-01	.00
1.600	-2.654808E-01	-2.674452E-01	.74	-2.654810E-01	-2.654810E-01	.00
1.675	-8.568062E-01	-8.543713E-01	-.28	-8.568063E-01	-8.568063E-01	.00
1.750	-9.908002E-01	-9.852656E-01	-.56	-9.908002E-01	-9.908002E-01	.00

TABLE 3. RESULTS FOR EXAMPLE 3

(Transient Analysis Involving Uncoupled Modal Equations)

Displacements for Grid Point 26, Component T3, of
NASTRAN Demonstration Problem No. D12-01-1A

TSTEP Bulk Data Card Input

Checkpoint Run: Number of time steps = 30
Time increment = 0.001388 sec.
Skip factor for output = 1

Restart Run: Number of time steps = 14
Time increment = 0.001388 sec.
Skip factor for output = 1

NOTES:

1. For the restart run of this problem, RPK's CRAY version uses the CONTINUE feature that has been extended to uncoupled modal equations and that is mentioned in the paper. No runs were made on the standard release of COSMIC/NASTRAN as it does not have this capability.
2. The restart run was initiated by setting the parameter NCOL in the rigid format DMAP to 17 just before the TRLG module (see Volume 2, Section 2.9.6 of Reference 1 for details), thereby triggering the continuation of the integration from the 17th output time of the checkpoint run.
3. The % Error in the table is calculated by the following relationship:

$$\frac{(\text{Displacement from Restart Run}) - (\text{Displacement from Checkpoint Run})}{(\text{Displacement from Checkpoint Run})} \times 100$$

TABLE 3. RESULTS FOR EXAMPLE 3
(Continued)

Results from CRAY Version of 1988 COSMIC/NASTRAN (See Note 1 above)			
Time (sec.)	Displacements from Checkpoint Run	Displacements from Restart Run	% Error (See Note 3 above)
0.000000E+00	0.000000E+00		
1.388000E-03	3.346152E-03		
2.776000E-03	1.799760E-02		
4.164000E-03	3.245619E-02		
5.552000E-03	2.809689E-02		
6.940000E-03	3.562258E-03		
8.328000E-03	-3.010927E-02		
9.716000E-03	-6.037387E-02		
1.110400E-02	-7.236326E-02		
1.249200E-02	-5.503825E-02		
1.388000E-02	-1.209147E-02		
1.526800E-02	3.983471E-02		
1.665600E-02	8.073034E-02		
1.804400E-02	9.407332E-02		
1.943200E-02	7.293699E-02		
2.082000E-02	2.460154E-02		
2.220800E-02	-3.235601E-02	-3.235601E-02	.00
2.359600E-02	-7.679893E-02	-7.679893E-02	.00
2.498400E-02	-9.321987E-02	-9.321987E-02	.00
2.637200E-02	-7.703266E-02	-7.703266E-02	.00
2.776000E-02	-3.604619E-02	-3.604619E-02	.00
2.914800E-02	1.333591E-02	1.333591E-02	.00
3.053600E-02	5.347509E-02	5.347509E-02	.00
3.192400E-02	7.224338E-02	7.224338E-02	.00
3.331200E-02	6.658045E-02	6.658045E-02	.00
3.470000E-02	4.226934E-02	4.226934E-02	.00
3.608800E-02	1.031458E-02	1.031458E-02	.00
3.747600E-02	-1.823067E-02	-1.823067E-02	.00
3.886400E-02	-3.631594E-02	-3.631594E-02	.00
4.025200E-02	-4.226599E-02	-4.226599E-02	.00
4.164000E-02	-3.859226E-02	-3.859226E-02	.00

APPENDIX

The equations in this appendix are all taken from Section 4.65.7.3 of the NASTRAN Programmer's Manual (Reference 3). Readers are referred to that section for the complete definition and description of the various terms involved in the equations.

The following general terminology is used in all of the equations in this appendix:

- Δt --- time step
- u_i --- displacement at time t_i
- \dot{u}_i --- velocity at time t_i
- \ddot{u}_i --- acceleration at time t_i
- P_i --- load at time t_i
- N_i --- non-linear load at time t_i
- [K] --- stiffness matrix
- [B] --- damping matrix
- [M] --- mass matrix

The matrices [C], [D] and [E] used in the equations in this appendix are functions of the [K], [B] and [M] matrices and the time step Δt and are defined as follows:

$$[C] = (2/\Delta t^2) [M] - (1/3) [K] \quad (a)$$

$$[D] = (1/\Delta t^2) [M] + (1/2\Delta t) [B] + (1/3) [K] \quad (b)$$

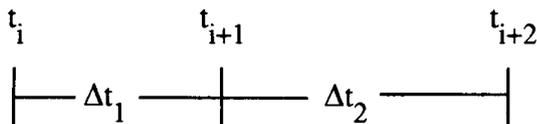
$$[E] = -(1/\Delta t^2)[M] + (1/2\Delta t) [B] - (1/3) [K] \quad (c)$$

Equation Used to Compute Successive Displacements After the Integration is Started

$$[D] \{u_{i+2}\} = (1/3) \{P_i + P_{i+1} + P_{i+2}\} + \{N_{i+1}\} + [C] \{u_{i+1}\} + [E] \{u_i\} \quad (1)$$

Equation Used to Compute the First Displacement After a Time Step Change

It is assumed here that the time step changes from Δt_1 to Δt_2 at time t_{i+1} . This is shown below.



The [C], [D] and [E] matrices are formed with using $\Delta t = \Delta t_2$.

$$[D] \{u_{i+2}\} = (1/3) \{P_i^1 + P_{i+1} + P_{i+2}\} + \{N_{i+1}\} + [C]\{u_{i+1}\} + [E]\{u_i^1\} \quad (2)$$

where:

$$\{\dot{u}_{i+1}\} = (1/\Delta t_1) (\{u_{i+1}\} - \{u_i\}) \quad (2a)$$

$$\{\ddot{u}_{i+1}\} = (1/\Delta t_1^2) (\{u_{i+1}\} - 2\{u_i\} + \{u_{i-1}\}) \quad (2b)$$

$$\{\dot{u}_i^1\} = \{\dot{u}_{i+1}\} - \{\ddot{u}_{i+1}\} \Delta t_2 \quad (2c)$$

$$\{u_i^1\} = \{u_{i+1}\} - \Delta t_2 \{\dot{u}_{i+1}\} + (\Delta t_2^2/2) \{\ddot{u}_{i+1}\} \quad (2d)$$

and

$$\{P_i^1\} = [M] \{\ddot{u}_{i+1}\} + [B] \{\dot{u}_i^1\} + [K] \{u_i^1\} \quad (2e)$$

Equation Used by the CONTINUE Feature to Compute the First Displacement of the CONTINUED Run

$$[D]\{u_1\} = (1/3) \{P_{-1} + P_0 + P_1\} + \{N_0\} + [C]\{u_0\} + [E]\{u_{-1}\} \quad (3)$$

where

$$\{P_0\} = [K] \{u_0\} + [B] \{\dot{u}_0\} + [M] \{\ddot{u}_0\} \quad (3a)$$

$$\{u_{-1}\} = \{u_0\} - \Delta t \{\dot{u}_0\} + (\Delta t^2/2) \{\ddot{u}_0\} \quad (3b)$$

$$\{\dot{u}_{-1}\} = \{\dot{u}_0\} - \{\ddot{u}_0\} \Delta t \quad (3c)$$

and

$$\{P_{-1}\} = [M] \{\ddot{u}_0\} + [B] \{\dot{u}_{-1}\} + [K] \{u_{-1}\} \quad (3d)$$

where $\{u_0\}$, $\{\dot{u}_0\}$ and $\{\ddot{u}_0\}$ are the displacements, velocities and accelerations, respectively, at the specified output time t_0 of the checkpoint run and Δt is the initial time step for the restart. $\{P_1\}$ is the load at time $t_0 = t_0 + \Delta t$ and $\{N_0\}$ is the initial non-linear load.