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INTEGRAL EQUATION METHOD

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SHAPE IDENTIFICATION TECHNIQUE FOR A 2-D ELLIPTIC SYSTEM BY BOUNDARY INTEGRAL EQUATION METHOD

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Abstract

This paper is concerned with the identification of the geometrical structure of the boundary shape for a two-dimensional boundary value problem. The output least square identification method is considered for estimating partially unknown boundary shapes. A numerical parameter estimation technique using the spline collocation method is proposed.

INTRODUCTION

In the last decade, there has been much practical interest in domain identification for distributed parameter systems. Its application includes the optimal shape design (See Pironneau [15] and the references therein), free boundary problems arising in an oil reservoir [9], non-destructive evaluations in thermal testing of materials [3], etc. In most identification techniques, unknown domains to be identified are found by the minimization of output least square error functions (OLSI - method). Within the theoretical framework of optimal control, the existence of the optimal solution of the OLSI has been studied by Chanais [8] and Murat & Simon [14] for elliptic boundary value problems with Neumann and Dirichlet boundary conditions. In practical application of the domain identification, difficulties arise with physical domains that are often composed of non-smooth geometrical parts;hence the study of sensitivity analysis is of great importance (e.g. [10][16][18], etc). Associated with those analyses, many numerical investigations of domain identifications have been acomplished based on finite element methods (See [4][12], etc). However, the computation of the derivative of the cost function with respect to the unknown domain tends to be expensive and time-consuming. Such computations require intricate grid modification techniques for finite elements related to the decomposition of the unknown domain. The boundary integral equation method (BIE) has an advantage over the defect mentioned above. It does not require any decomposition of the unknown domain, only of its boundary. Hence the application of BIE to the domain estimation allows more sophisticated yet simpler algorithms, especially, in the case where only boundary measurement data are available to us.

Our goal in domain shape identification is to develop a feasible computational method using the BIE method. Some previous efforts are described in [19][20] and one application to a wing optimization problem was illustrated in Pironneau's book [15], Ch. 8.5. In this paper, we propose a computational method based on spline functions which approximate
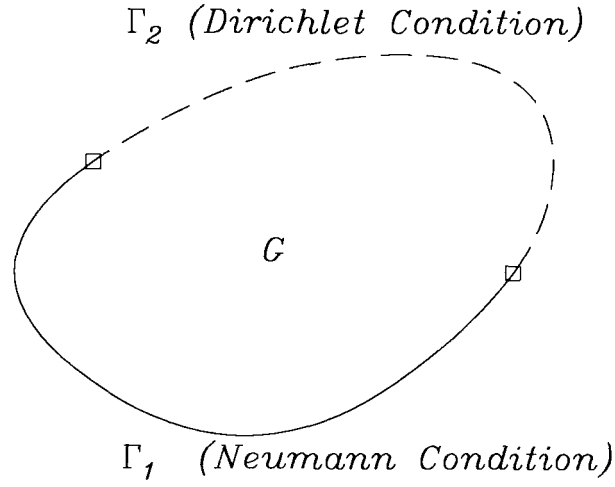
$$\Gamma_2 \ (Dirichlet \ Condition)$$

$$G$$

$$\Gamma_1 \ (Neumann \ Condition)$$

Figure 1. The Spatial Domain G and the Boundary Γ

the unknown boundary curve.

To explain our approach, we restrict our attention to a 2 - D elliptic system. Let G ($\subset R^{(2)}$) be the open domain bounded by Γ. As depicted in Fig. 1, the boundary is decomposed into two parts, i.e.,

$$\Gamma = \Gamma_1 \cup \Gamma_2.$$

The system behavior on G is governed by the following Laplace equation,

$$\Delta u = 0 \quad \text{on } G \tag{1}$$

with the mixed boundary condition

$$\frac{\partial u}{\partial n} = g_1 \quad \text{on } \Gamma_1 \tag{2}$$

$$u = g_2 \quad \text{on } \Gamma_2. \tag{3}$$

We consider the two types of domain identification problems,

Case-1: Identification of Dirichlet Boundary

Case-2: Identification of Neumann Boundary.

The problem considered in Case-1 is to estimate the geometrical shape of the Dirichlet boundary Γ_2 from the boundary observed data on Γ_1, i.e.,

$$y_1 = u \mid_{\Gamma_1} . \tag{4}$$

In Case-2, we deal with the identification on the Neumann boundary Γ_2 using the boundary measurement on Γ_2, i.e.,

$$y_2 = \frac{\partial u}{\partial n} \mid_{\Gamma_2} . \tag{5}$$

The problem treated here involves some applications to the impedance computed tomography, the identification of free boundary problems, the structure design of fluid flow, etc.

EXISTENCE OF SOLUTIONS

Let θ be a constant parametrization vector among values in a given compact set $\Theta \subset R^{(n)}$. Throughout this paper, the parameter θ charactrizes the unknown boundary to be identified. We define open sets G_θ whose boundaries are given by

$$\Gamma_1 \cup \Gamma_2(\theta) \quad \text{in} \quad Case-1$$

or

$$\Gamma_1(\theta) \cup \Gamma_2 \quad \text{in} \quad Case-2.$$

From the practical point of view, actual observed data are taken from a finite number of sensors allocated on the boundary. Hence we represent the corresponding output of the system model as follows:

(Case-1)

$$y_1 = u \mid_{\Gamma_1}$$

$$= \left[\int_{\Sigma_1^1} h_1^1(\gamma_{\Gamma_1} u) d\Gamma_1, \cdots, \int_{\Sigma_1^m} h_1^m(\gamma_{\Gamma_1} u) d\Gamma_1 \right]'$$

(Case-2)

$$y_2 = \frac{\partial u}{\partial n} \mid_{\Gamma_2}$$

$$= \left[\int_{\Sigma_2^1} h_2^1(\gamma_{\Gamma_2} \frac{\partial u}{\partial n}) d\Gamma_2, \cdots, \int_{\Sigma_2^m} h_2^m(\gamma_{\Gamma_2} \frac{\partial u}{\partial n}) d\Gamma_2 \right]'$$

where γ_{Γ_i} denotes the trace operator on Γ_i and Σ_i^j are subsets of Γ_i. For economy of notation, we rewrite (4) and (5) by

$$y_i(\theta) = \begin{cases} H_1 u & \text{for the Case} - 1 \\ H_2 \frac{\partial u}{\partial n} & \text{for the Case} - 2, \end{cases} \tag{6}$$

respectively. The output least square error functions for both cases are then given by

$$J_i(\theta) = \frac{1}{2} |y_i(\theta) - y_d|^2 \qquad (i = 1, 2) \tag{7}$$

where $\{y_d\}$ denotes the corresponding actual data.

The OLSI-method is stated as follows:

(IDP) Find the optimal parameter θ^* which is the solution of

$$J_i(\theta^*) = \min_{\theta \in \Theta} J_i(\theta) \qquad (i = 1, 2). \tag{8}$$

In the sequel, we first state the existence condition of the problem for the Case-1.

Theorem 1 (*Identification of Dirichlet boundary*) *Suppose that*

(H-1) The open sets G_θ depend continuously on $\theta \in \Theta$ in the following sense:

if $\theta \to \theta^*$, then the Hausdorff distance $\delta(\overline{G_\theta}, \overline{G_\theta^*}) \longrightarrow 0$.

(H-2) There exists a bounded open set C and D such that

$$C \subset G_\theta \subset D$$

for any $\theta \in \Theta$.

(H-3) The sets G_θ have the ϵ -cone proprty.

(H-4) The external input g_1 belongs to $H^{\frac{1}{2}}(\Gamma_1)$.

(H-5) The weight functions of the measurement operator H_1 satisfy

$$h_1^i \in L^\infty(\Sigma_1^i) \qquad (i = 1, 2, \cdot, m).$$

Then there exists at least one solution.

The above results can be easily obtained from Lions' lecture notes [11] and Pironneau's book [15].

Remark 1 *The ϵ - cone property in (H-3) means that, for $\forall x \in \Gamma_2(\theta)$, there exists a direction $\eta(x)$ such that, for $\forall z \in B(x, \epsilon) \cup G_\theta$,*

$$C(\epsilon, \eta(x), z) \subset G_\theta$$

where $C(\epsilon, \eta, z)$ denotes the half-cone of angle ϵ, direction η, and vertex x intersected with the ball $B(x, \epsilon)$ of center x and radius ϵ. This property is equivalent to the corresponding boundary curve being Lipshitz continuous. (See Chanais [8]).

For the identification in Case-2, we obtain similar results.

Theorem 2 *(Identification of Neumann boundary) We suppose the hypotheses (H-1) and (H-2) in Theorem 1 and assume that*

(H-3) ' The sets G_θ have C^2 -regularity property.

(H-4) ' $g_2 \in H^{\frac{3}{2}}(\Gamma_2)$

(H-5)' $h_2^i \in L^\infty(\Sigma_2^i)$ $(i = 1, 2, \cdots, m)$

Then there exists at least one solution.

The proof of *Theorem 2* can be obtained by extending the results of Chanais [8]. For the precise definition of C^m - regularity property, we refer to Adams [1].

INTEGRAL EQUATION MODEL AND ITS NUMERICAL SCHEMES

In this section, we replace the elliptic boundary value problem given in Section 1 by an equivalent integral equation on the boundary curve. A numerical method is given based on the spline collocation method.

Boundary Integral Equation Model

The Green's representation formula yields the relation (See e. g. [6])

$$
\begin{aligned}
c(x_0)u(x_0) = & \\
& \int_{\Gamma_1} \left\{ u(\frac{\partial}{\partial n_x} \log |x - x_0|) - g_1 \log |x - x_0| \right\} d\Gamma_1 \\
& + \int_{\Gamma_2} \left\{ g_2(\frac{\partial}{\partial n_x} \log |x - x_0|) - \frac{\partial u}{\partial n_x} \log |x - x_0| \right\} d\Gamma_2
\end{aligned}
$$

where
$$
c(x_0) = \begin{cases} 2\pi & \text{for } x_0 \in G \\ \pi & \text{for } x_0 \in \Gamma \quad \text{(smooth boundary)} \\ 0 & \text{otherwise} \end{cases} .
$$

In the sequel, we consider the case where the boundary curve Γ is represented by a parametric representation,

$$
\Gamma = \left\{ \xi(t) = (\xi_1(t), \xi_2(t)) \,\middle|\, \begin{array}{l} t \in [0,1], \xi \text{ is a Jordan} \\ \text{curve and } \left|\frac{d\xi}{dt}\right| \neq 0 \end{array} \right\} .
$$

Moreover, the decomposition of the boundary Γ into the Dirichlet and Neumann boundary is assumed to be given by

$$
\Gamma_1 = \{ \xi(t) \mid t \in [0, \bar{t}] \}
$$

6

$$\Gamma_2 = \{\xi(t) \mid t \in [\bar{t}, 1]\}.$$

Thus, the boundary state and its flux on the boundary can be rewritten by using the arc length t along the curve Γ. Let us define

$$\phi(t) \stackrel{\text{def}}{=} u(\xi(t)) \qquad for \quad t \in [0, \bar{t}]$$

$$\phi_n(t) \stackrel{\text{def}}{=} \frac{\partial u}{\partial n}(\xi(t)) \qquad for \quad t \in [\bar{t}, 1].$$

Then the elliptic boundary value problem can be reduced to the Fredholm integral equation for ϕ and ϕ_n,

$$(\pi I - K_1)\phi(s) + L_1\phi_n(s) = -L_2 g_1(s) + K_2 g_2(s)$$

$$for \quad 0 < s < \bar{t} \tag{9}$$

$$-K_3\phi(s) + L_3\phi_n(s) = -L_4 g_1(s) + (K_4 - \pi I)g_2(s)$$

$$for \quad \bar{t} < s < 1, \tag{10}$$

where

$$K_1 = K_3 \stackrel{\text{def}}{=} \int_0^{\bar{t}} \left\{ \frac{\partial}{\partial n_t} \log |\xi(s) - \xi(t)| \right\} (\cdot) \left| \frac{d\xi}{dt} \right| dt$$

$$K_2 = K_4 \stackrel{\text{def}}{=} \int_{\bar{t}}^1 \left\{ \frac{\partial}{\partial n_t} \log |\xi(s) - \xi(t)| \right\} (\cdot) \left| \frac{d\xi}{dt} \right| dt$$

$$L_1 = L_3 \stackrel{\text{def}}{=} \int_{\bar{t}}^1 \{\log |\xi(s) - \xi(t)|\}(\cdot) \left| \frac{d\xi}{dt} \right| dt$$

and

$$L_2 = L_4 \stackrel{\text{def}}{=} \int_0^{\bar{t}} \{\log |\xi(s) - \xi(t)|\}(\cdot) \left| \frac{d\xi}{dt} \right| dt,$$

respectively.

Numerical Scheme by Spline Collocation Method

Many numerical methods have been proposed for the solution of integral equations, the most commonly used of these being Galerkin, product integration and collocation methods (See *e. g.*, [2]). Although an asymptotic error analysis is well established for the Galerkin

7

method, its numerical implementation becomes complicated since element matrices of discretized equations are computed through a time-consuming double integration. Owing to the simple numerical treatment, the collocation method for solving integral equations has been studied by many authors. In this paper, we use the spline collocation method for which the asymptotic convergence analysis is available (See [17]). In the sequel, we briefly mention this method for the models (9) and (10).

We select an increasing sequence of mesh points

$$\Delta^{N,M} = \Delta^N \cup \Delta^M$$

such that

$$\Delta^N = \left\{ t_i^N = i\bar{t}/N \qquad i = 0, 1, \cdots, N \right\}$$
$$\Delta^M = \left\{ t_i^M = \bar{t} + i(1-\bar{t})/M \qquad i = 0, 1, \cdots, M \right\}.$$

In addition, we introduce the nodal points

$$\tilde{t}_j^{N,M} = \begin{cases} t_j^N - \frac{\bar{t}}{2N} & \text{with} \quad j = 1, 2, \cdots, N \\ \\ t_j^M - \frac{1-\bar{t}}{2M} & \text{with} \quad j = N+1, \cdots, N+M. \end{cases}$$

By $S^k(\Delta^N)$ (resp. $S^k(\Delta^M)$) we denote the space of all $(k-1)$ -times continuously differentiable splines of degree k subordinate to the partitions Δ^N (resp. Δ^M), if $k \geq 1$. By $S_0(\Delta^N)$ (resp. $S_0(\Delta^M)$) we denote the corresponding step functions. We approximate the solutions of (9) and (10) as

$$\begin{cases} \phi(s) \approx \phi_\Delta^N(s) \in S^k(\Delta^N) & \text{for } 0 < s < \bar{t} \\ \\ \phi_n(s) \approx \phi_{n,\Delta}^M(s) \in S^k(\Delta^M) & \text{for } \bar{t} < s < 1 \end{cases}$$

Then the spline collocation method is to find

$$(\phi_\Delta^N, \phi_{n,\Delta}^M) \in S^k(\Delta^N) \times S^k(\Delta^M)$$

so as to satisfy the collocation equations,

$$(\pi I - K_1)\phi_\Delta^N(\tilde{t}_j^{N,M}) \quad + \quad L_1\phi_{n,\Delta}^M(\tilde{t}_j^{N,M})$$

$$= \quad -L_2 g_1(\tilde{t}_j^{N,M}) + K_2 g_2(\tilde{t}_j^{N,M})$$

$$\text{for} \quad j = 1, 2, \cdots, N \tag{11}$$

$$-K_3\phi_\Delta^N(\tilde{t}_j^{N,M}) \quad + \quad L_3\phi_{n,\Delta}^M(\tilde{t}_j^{N,M})$$

$$= \quad -L_4 g_1(\tilde{t}_j^{N,M}) + (K_4 - \pi I) g_2(\tilde{t}_j^{N,M})$$

$$\text{for} \quad j = N+1, \cdots, N+M. \tag{12}$$

COMPUTATIONAL METHOD FOR DOMAIN IDENTIFICATION

In this section, using the integral equation approach as stated in the previous section, we consider the finite approximation for the problem (IDP).

Discretized Optimization Problem

Let $\eta(t, \theta)$ and $\varsigma(t, \theta)$ be smooth curves which specify the unknown Dirichlet and Neumann boundaries, *i. e.*,

(Case-1)

$$\Gamma_2(\theta) \quad = \quad \{\eta(t, \theta) \mid \bar{t} \le t \le 1, \ \theta \in \Theta$$

$$\eta \in C^2(\bar{t}, 1) \quad \text{with}$$

$$\eta(\bar{t}, \theta) = \xi(\bar{t}) \quad \eta(1, \theta) = \xi(0)\}, \tag{13}$$

(Case-2)

$$\Gamma_1(\theta) \quad = \quad \{\xi(t) = \varsigma(t, \theta) \mid 0 \le t \le \bar{t}, \ \theta \in \Theta$$

$$\eta \in C^2(\bar{t}, 1) \quad \text{with}$$

$$\varsigma(0, \theta) = \xi(1) \quad \varsigma(\bar{t}, \theta) = \xi(\bar{t})\}. \tag{14}$$

We rewrite the system models (9) and (10) as

$$A_i(\theta, s) \begin{bmatrix} \phi \\ \phi_n \end{bmatrix} = f_i(\theta, s) \tag{15}$$

$$(i = 1, 2)$$

where

(Case-1)

$$A_1(\theta) = \begin{bmatrix} \pi I - K_1, & L_1(\theta) \\ -K_3(\theta), & L_3(\theta) \end{bmatrix}$$

$$f_1(\theta) = \begin{bmatrix} -L_2 g_1 + K_2(\theta) g_2 \\ -L_4(\theta) g_1 + (K_4(\theta) - \pi I) g_2 \end{bmatrix}$$

(Case-2)

$$A_2(\theta) = \begin{bmatrix} \pi I - K_1(\theta), & L_1(\theta) \\ -K_3(\theta), & L_3 \end{bmatrix}$$

$$f_2(\theta) = \begin{bmatrix} -L_2(\theta) g_1 + K_2(\theta) g_2 \\ -L_4(\theta) g_1 + (K_4 - \pi I) g_2 \end{bmatrix}.$$

Without loss of generality, we may set observation regions as

$$\Sigma_i = \left\{ \xi(t) \mid t \in]t_p^i - \sigma_1^i, t_p^i + \sigma_2^i[\subset [0, \bar{t}] \right\}$$

$$\text{for} \quad i = 1, 2, \cdots, m, \tag{16}$$

where σ_1^i and σ_2^i are given constants. Then observation model (6) is replaced by

$$y_i(\theta) = \begin{cases} \mathcal{H}_1 \phi(\theta), \\ \mathcal{H}_2 \phi_n(\theta) \end{cases}$$

where

$$[\mathcal{H}_k]_i \overset{\text{def}}{=} \int_{t_p^i - \sigma_1^i}^{t_p^i + \sigma_2^i} h_k^i(\xi(t))(\cdot) \left| \frac{d\xi}{dt} \right| dt,$$

$$\text{for} \quad k = 1, 2 \quad i = 1, 2, \cdots, m.$$

In the above equation, $h_k^i(\xi(t))$ denote the weighting functions of measurement operator H_k in *Theorem-1* or *Theorem-2*. For the state approximation, we choose the set of B-splines (See [5]) as basis elements in $H^r(0, 1)$ where r is a positive integer. Let

10

$\{B_{j,r}^{N(M)}(t)\}_{j=-[(r-1)/2]}^{N(M)+[r/2]}$ be a sequence of r-th order B-splines with the knot $\Delta^{N(M)}$. Then the solution of (15) is approximated by

$$\phi(s) \approx \phi_\Delta^N(s) = \sum_{j=-[\frac{(r-1)}{2}]}^{N+[\frac{r}{2}]} \alpha_{j+[\frac{(r-1)}{2}]+1}^N B_{j,r}^N(s)$$

$$\phi_n(s) \approx \phi_{n,\Delta}^M(s) = \sum_{j=-[\frac{(r-1)}{2}]}^{M+[\frac{r}{2}]} \beta_{j+[\frac{(r-1)}{2}]+1}^M B_{j,r}^M(s).$$

The collocation equations for (15) yield the set of linear systems for the unknown coefficient vector,

$$\begin{cases} w^{N,M} = (\alpha^N, \ \beta^M) \\ A_i^{N,M}(\theta)w^{N,M} = f_i^{N,M}(\theta) \\ \qquad (i=1,2) \end{cases} \tag{17}$$

where $A_i^{N,M}$ and $f_i^{N,M}$ denote the corresponding element matrix and vector for the Case-1 and Case-2. Associated with the observation operator $\mathcal{H}_i$, the matrices $\mathcal{H}_i^{N,M}$ can be constructed as

$$\mathcal{H}_i^{N,M}w^{N,M} = \begin{cases} \mathcal{H}_1\phi_\Delta^N \\ \mathcal{H}_2\phi_{n,\Delta}^M \end{cases}.$$

Using this notation, the numerical data for the parameter θ can be obtained by computing

$$y_i^{N,M}(\theta) = \mathcal{H}_i^{N,M}w^{N,M}$$

$$\text{for} \quad i=1,2.$$

Thus we describe the computational method for the problem (IDP):

(IDP)^{M,N} Given the data $\{y_d\}$, find the solution $\theta_i^* \in \Theta$ which minimizes

$$J_i^{N,M}(\theta) = \frac{1}{2}\left|y_i^{N,M}(\theta) - y_d\right|^2 \tag{18}$$

subject to (17).

In the sequel, we discuss the computor implementation for solving the problem (IDP)^{M,N}.

Admissible Class of Parameters

Solving the optimization problem $(IDP)^{M,N}$, we need to specify the unknown boundary curve so as to satisfy the hypotheses as stated in *Theorems-1* and *-2*. To this end, we approximate the unknown boundary curve by cubic B-spline functions $\{B_{i,3}(t)\}$. In the sequel, we consider one simple curve as follows:

(Case-1)

$$\eta(t,\theta) = (\eta_1(t), \eta_2^M(t,\theta))$$

$$\text{for} \quad \bar{t} \leq t \leq 1$$

where

$$\eta_1(t) = \frac{\xi_1(0) - \xi_1(\bar{t})}{1 - \bar{t}}(t - \bar{t}) + \xi_1(\bar{t})$$

$$\eta_2(t,\theta) = \sum_{i=-1}^{M+1} \alpha_i^M(\theta) B_{i,3}^M(t)$$

(Case-2)

$$\varsigma(t,\theta) = (\varsigma_1(t), \varsigma_2^N(t,\theta))$$

$$\text{for} \quad 0 \leq t \leq \bar{t}$$

where

$$\varsigma_1(t) = \frac{\xi_1(\bar{t}) - \xi_1(0)}{\bar{t}}t + \xi_1(0)$$

$$\varsigma_2(t,\theta) = \sum_{i=-1}^{N+1} \beta_i^N(\theta) B_{i,3}^N(t)$$

In the above equations, the Fourier coefficients $\{\alpha_i^M(\theta)\}_{i=-1}^{M+1}$ and $\{\beta_i^N(\theta)\}_{i=-1}^{N+1}$ are obtained through the following linear systems:

$$\begin{cases} \Lambda_1 \alpha^M = \underline{\theta}_1 \\ \Lambda_2 \beta^N = \underline{\theta}_2 \end{cases}$$

where Λ_1 and $\underline{\theta}_1$ are given by

$$\Lambda_1 = \begin{bmatrix} -\frac{M}{2(1-\bar{t})}, & 0, & \frac{M}{2(1-\bar{t})} & & & \\ 1/6, & 2/3, & 1/6 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & 1/6, & 2/3, & 1/6 \\ & & & -\frac{M}{2(1-\bar{t})}, & 0, & \frac{M}{2(1-\bar{t})} \end{bmatrix},$$

12

$$\underline{\theta}_1 = [\theta_1, \xi_2(\bar{t}), \theta_2, \cdots, \theta_M \xi_2(0), \theta_{M+1}]'.$$

The matrix Λ_2 and vector θ_2 can be described in the same way. The number of dimensions in θ are thus set as

$$dim(\theta) = n = \begin{cases} M + 1 & \text{for the Case} - 1 \\ N + 1 & \text{for the Case} - 2 \end{cases}.$$

We note that the unknown boundaries constructed by these curves satisfy the hypotheses (H-1) and (H-3) in *Theorem-1*, and (H-1) and (H-3)' in *Theorem-2*. Furthermore, in order to assure the hypothesis (H-2) in both theorems, we impose the following constraints:

$$\bar{\theta}_L^{M(N)} \le \underline{\theta}_i^{M(N)} \le \bar{\theta}_U^{M(N)} \quad (i = 1, 2)$$

where the lower and upper bounds are given constant vectors. The precise form of the unknown boundary will be shown in the numerical experiments.

Optimization Algorithm

Under the admissible parameter class stated above, we can easily evaluate the gradient of the cost functional (18) with respect to $\underline{\theta}_i$ (i=1,2). Hence many optimization techniques are readily applicable to our problem. Our approach for this optimization problem is to use the trust region method. The trust region scheme is briefly stated as follows: Let $\{x_k\}_{k=1,2,\cdots}$ be a sequence generated by this algorithm. At the current point x_k, we build a model of the cost functional (We usually choose a quadratic model). Then we define a region around x_k where we believe this model to be an adequate approximation of the functional. Using this model, we seek a feasible direction so as to guarantee a sufficient decrease in the model of cost. Once we obtain the feasible direction, the exact cost functional is evaluated at the new point. If its value has decreased enough, this new point is acceptable and updated as the next iterate. The trust region is then expanded. Otherwise the new point is rejected and the current trust region is reduced. The effectiveness of this algorithm is its global

13

convergence properties;namely, this algorithm makes it possible to ensure convergence to a critical point (optimal solution), even from starting points (initial guesses) that are far away from the optimal solution. For detailed discussions, we refer to [13] etc.

NUMERICAL EXPERIMENTS

In this section, we tested our method for two examples. Although these are quite simple, problems treated here can be easily extended to the more interesting topics, such as the identification of free boundary and optimal shape design problems. For the implementation of the trust region algorithm, we used a Fortran software package created by Dr. R. G. Carter, ICASE (See [7] for more details). Test computations were carried out on the Gould NP1 at the NASA Langley Research Center.

Example-1 : Identification of Dirichlet Boundary

The boundary Γ was decomposed into Γ_1 and Γ_2 by $\bar{t} = 0.75$. The boundary inputs were set as

$$g_1(\xi(t)) = \begin{cases} 1 & \text{for} \quad t \in [0.25, 0.5] \\ 0 & \text{for} \quad t \in [0., 0.5) \cup (0.5, 0.75] \end{cases},$$

$$g_2(\eta(t)) = 0 \quad \text{for} \quad t \in (0.75, 1).$$

The number of knot sequence in *Example-1* was set as $N = 24$ and $M = 8$. To discretize the system model by the spline collocation method, we use parabolic B-splines (i. e. $r = 2$). The number of sensors was taken as $m = 24$. The initial guesses for the parameters were given by

$$\theta_i^{(0)} = \begin{cases} 1 & \text{for} \quad i = 2, \cdots, 8 \\ 0 & \text{for} \quad i = 1, 9 \end{cases}.$$

Table 1 shows the estimated parameter values using the artificially generated data. Figure 2 represents the estimated parameter function $\eta_2^M(t, \theta)$ and true boundary shape.

Example-2 : Identification of Neumann Boundary

In this example, the boundary Γ was devided by $\bar{t} = 0.25$. The boundary inputs were preassigned as

$$g_1(\varsigma(t)) = 0 \qquad \text{for} \quad t \in (0., 0.25),$$

$$g_2(\xi(t)) = \begin{cases} 1 & \text{for} \quad t \in [0.5, 0.75] \\ 0 & \text{for} \quad t \in [0.25, 0.5) \cup (0.75, 1.] \end{cases}$$

The number of knot sequence in *Example-2* was set as $N = 8$ and $M = 24$. We also use parabolic B-spline functions for the discretized system model (17). The number of sensors and the initial guesses for the parameters were taken the same as in *Example - 1*. Table 2 shows the estimated parameter values and Figure 3 represents the estimated parameter curve $\varsigma^N(t, \theta)$ and true boundary shape.

ACKNOWLEDGEMENT

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	Initial Guess	Iteration 10	Iteration 20	Iteration 90
$\hat{\theta}_1$	0.0000	0.2260	0.2598	0.4982
$\hat{\theta}_2$	1.0000	1.1571	1.0794	1.0800
$\hat{\theta}_3$	1.0000	1.1756	1.1982	1.2000
$\hat{\theta}_4$	1.0000	1.2833	1.2947	1.3002
$\hat{\theta}_5$	1.0000	1.3051	1.3569	1.3499
$\hat{\theta}_6$	1.0000	1.2857	1.2978	1.2999
$\hat{\theta}_7$	1.0000	1.1755	1.1981	1.2000
$\hat{\theta}_8$	1.0000	1.1565	1.0786	1.0800
$\hat{\theta}_9$	0.0000	-0.2261	-0.2599	-0.5021
Value of Cost	1.1796×10^{-3}	1.7248×10^{-6}	5.3033×10^{-11}	4.1711×10^{-16}

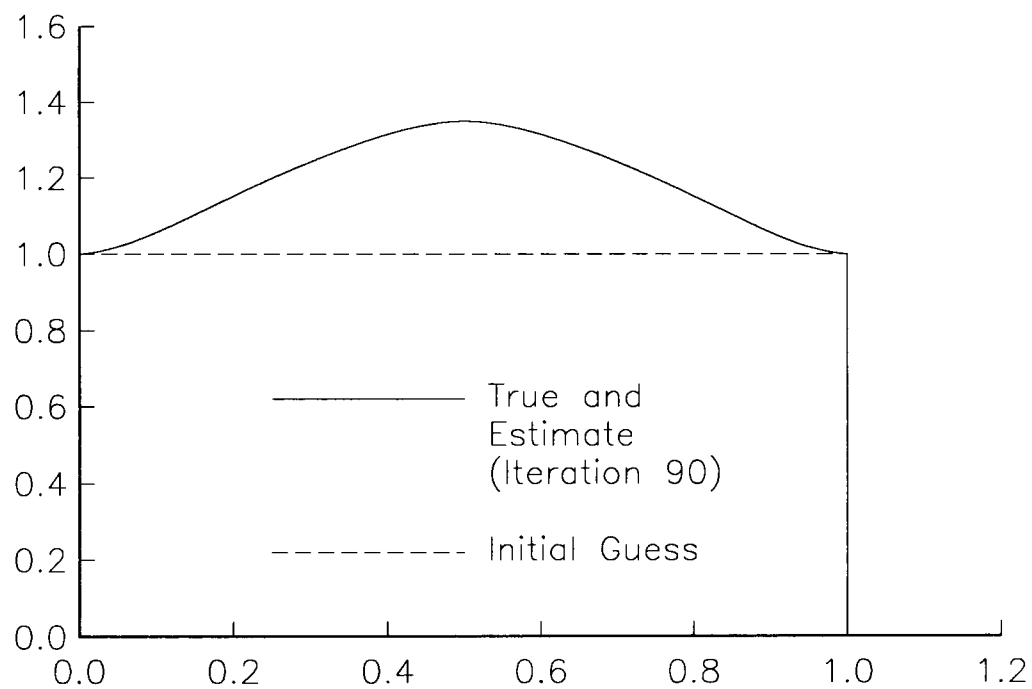Table 1. Estimated Values in *Example-1*.

Figure 2. True Curve and Estimated Boundary in *Example-1*.

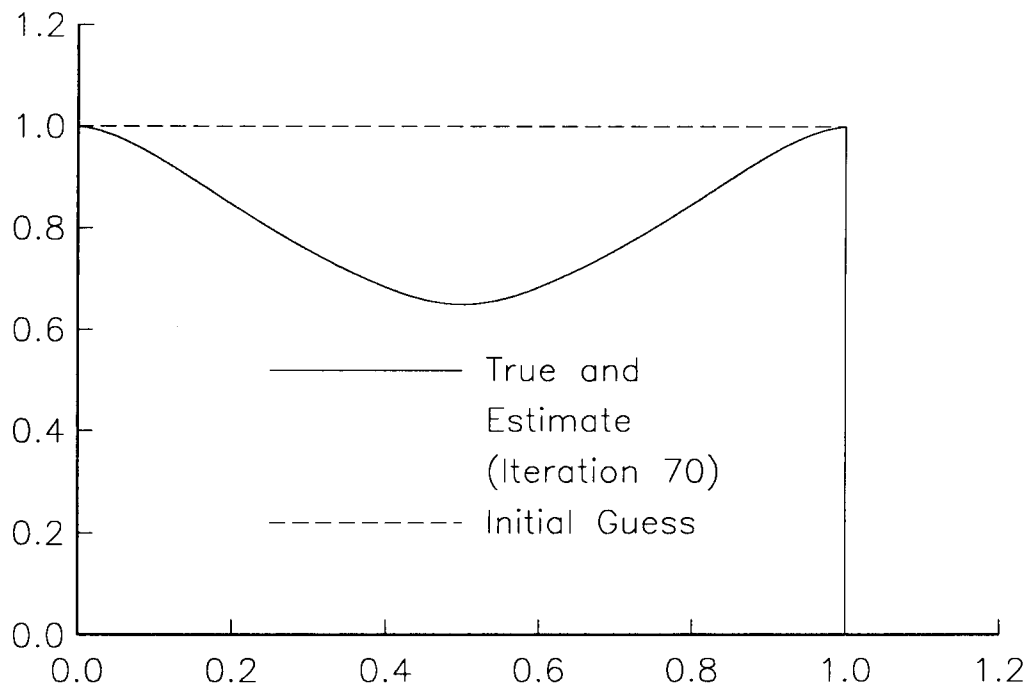|	Initial Guess	Iteration 10	Iteration 40	Iteration 70
$\hat{\theta}_1$	0.0000	−0.6170	−0.5162	−0.5000
$\hat{\theta}_2$	1.0000	0.9264	0.9198	0.9200
$\hat{\theta}_3$	1.0000	0.8876	0.7953	0.8000
$\hat{\theta}_4$	1.0000	0.8860	0.7332	0.7000
$\hat{\theta}_5$	1.0000	1.0128	0.8162	0.6500
$\hat{\theta}_6$	1.0000	0.8860	0.7347	0.7000
$\hat{\theta}_7$	1.0000	0.8876	0.7962	0.8000
$\hat{\theta}_8$	1.0000	0.9264	0.9199	0.9200
$\hat{\theta}_9$	0.0000	0.6170	0.5155	0.5000
Value of Cost	1.8867×10^{-3}	2.3881×10^{-5}	5.5471×10^{-8}	1.3553×10^{-18}

Table 2. Estimated Values in *Example-2*.

Figure 3. True Curve and Estimated Boundary in *Example-2*.

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16. Abstract

This paper is concerned with the identification of the geometrical structure of the boundary shape for a two-dimensional boundary value problem. The output least square identification method is considered for estimating partially unknown boundary shapes. A numerical parameter estimation technique using the spline collocation method is proposed.

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