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EXPLICIT, IMPLICIT, AND HYBRID METHODS

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Time integration methods can be separated into two groups: explicit and implicit. Roughly speaking, methods which do not involve the solution of any algebraic equations are called explicit, whereas those that require the solution of equations are called implicit.

The relative advantages and disadvantages of explicit and implicit methods are summarized in Fig. 1. It is interesting to observe that the positive attributes of these two methods form complementary sets, so that if the positive attributes of the two methods can be combined into a single method, a truly powerful method would be achieved.

An important point which is brought out in Fig. 1 is that whereas implicit methods are unconditionally stable for linear problems, stability does not imply accuracy and in fact the stability of implicit methods has often misled structural analysts into using time steps which yield very poor accuracy. Furthermore, no current time integration will undoubtedly be an important topic for future research.

Relative merits of explicit and implicit integration methods

Explicit

+ very simple and trouble free algorithm, complex phenomena easily included

+ accuracy is assured if Δt stable for large systems

+ no stiffness matrix necessary - saves storage

- conditionally stable, small Δt

Implicit

+ unconditionally stable, large Δt

- complex algorithm with low reliability in nonlinear situations

- accuracy can deteriorate

- Newton form has large core storage requirements

Figure 1

The major trend of the past decade of research on time integration procedures has been hybridization methods so as to take advantage of the complementary nature of the positive attributes of explicit and implicit integration. The types of hybridization are indicated in Fig. 2. References for these methods are as follows: partitioning [1-7], operating splitting methods [8-11], semi-implicit methods [9-12]. It should be noted that the distinction between semi-implicit methods and operator splitting methods is rather fuzzy; both groups of methods try to achieve unconditional stability through some modification of the evolution operator which either completely obviates the need for solving any equations or reduces the size of the system to be solved.

Objective of current research in time integration:

to exploit the advantages of implicit and explicit methods through hybridization (advantages of the two methods are complementary!)

directions:

partitioning: different operators on different parts of the mesh

semi-implicit methods: unconditionally stable methods that require no solution of equations or smaller systems

operator splitting methods: split $\underset{\sim}{A}$ to simplify solution - similar to semi-implicit

Figure 2

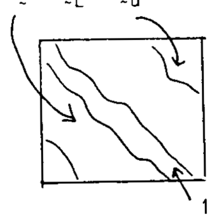
The major shortcoming of operator splitting methods has been the rapid deterioration of their accuracy with increasing time step. For example, if we consider the Trujillo semi-implicit method, which is illustrated in Fig. 3, we find that as the Courant number increases the accuracy diminishes dramatically. In Reference [10] it is shown that the phase velocity in a one-dimensional mesh in the Trujillo method is such that the shorter waves essentially only advance one mesh length during a time step; thus, the effect of the semi-implicit integrator, as shown in Fig. 4, is to retard wave velocity so severely that regardless of the size of the time step a quasi-Courant condition applies in that the numerical waves only traverse a single element in a time step. This distortional characteristic of semi-implicit methods has also been noted in the rigid-body modes by Park and Housner [12]. In Reference [12] several techniques for improving the accuracy of semi-implicit methods were developed, but we have not had time to check their effects independently.

TRUJILLO SEMI-IMPLICIT

(ref. 9)

$$(\underset{\sim}{M} + \Delta t\ \underset{\sim}{K})\ \underset{\sim}{u}^{n+1} = \underset{\sim}{M}\ \underset{\sim}{u}^{n} + \Delta t\ \underset{\sim}{s}^{n+1}$$

let $\quad \underset{\sim}{K} = \underset{\sim}{K}_L + \underset{\sim}{K}_u$



$\frac{1}{2}$ to $\underset{\sim}{K}_L$ and $\underset{\sim}{K}_u$

$$(\underset{\sim}{M} + \Delta t\ \underset{\sim}{K}_L)\ \underset{\sim}{u}^{n+1/2} = (\underset{\sim}{M} - \Delta t\ \underset{\sim}{K}_u)\ \underset{\sim}{u}^{n} + \Delta t\ \underset{\sim}{s}^{n+1}$$

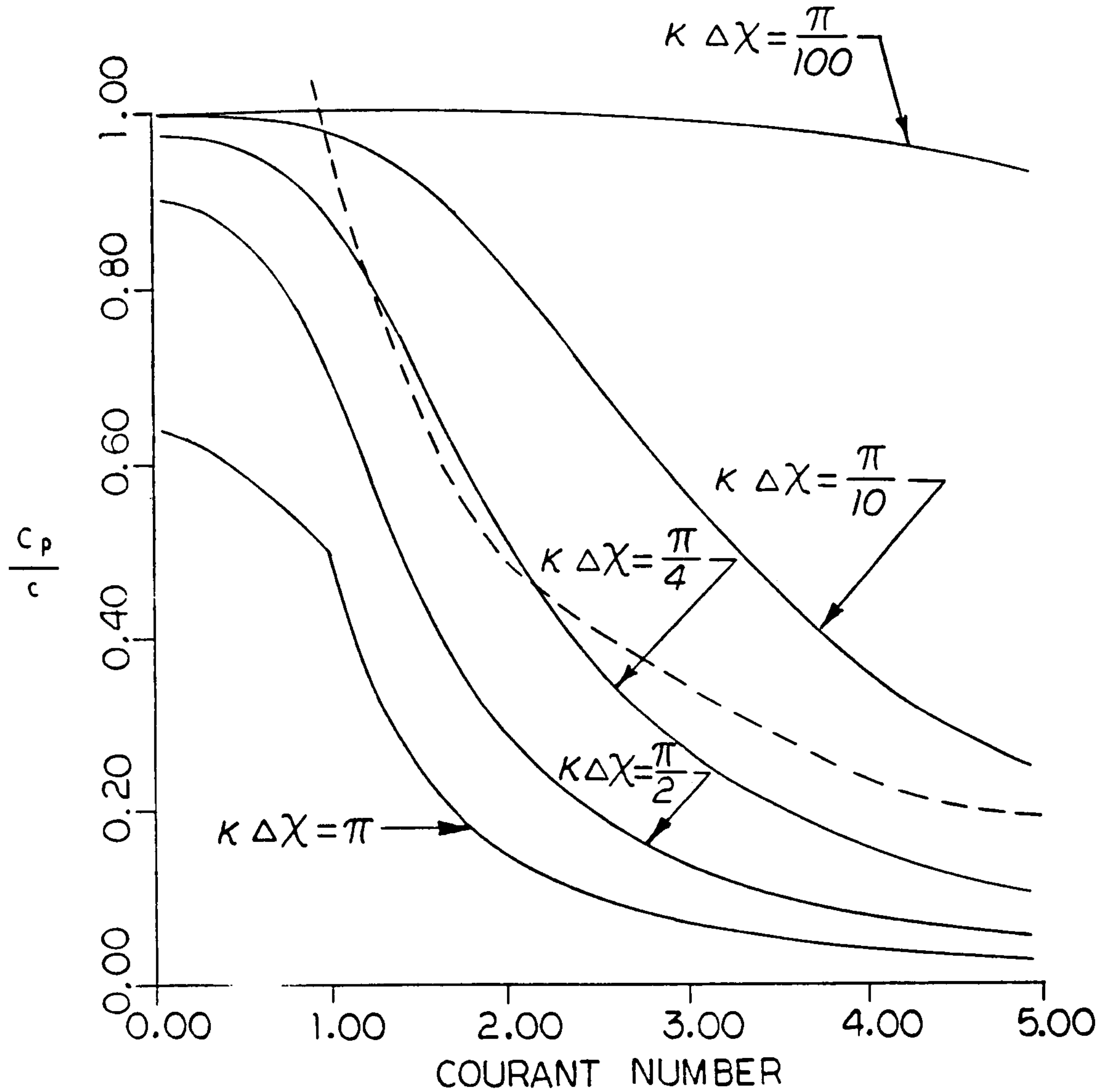
$$(\underset{\sim}{M} + \Delta t\ \underset{\sim}{K}_u)\ \underset{\sim}{u}^{n+1} = (\underset{\sim}{M} - \Delta t\ \underset{\sim}{K}_u)\ \underset{\sim}{u}^{n+1/2} + \Delta t\ \underset{\sim}{s}^{n+1}$$

similar to 2 passes of Gauss - Seidel

unconditionally stable

accuracy?

Figure 3



Phase error as a function of Courant number, r, for various wave numbers in the Trujillo method; the Courant line is dashed and corresponds to $c_p \Delta t = \Delta x$.

Figure 4

Avoidance of equation solution and unconditional stability can be achieved by rational Runge-Kutta methods [13], see Fig. 5. Again, the accuracy of these methods deteriorates rather quickly when the time step is much larger than the stability limit for the explicit methods. These methods seem to be most suited to parabolic systems. For structural mechanics, which involves a combination of hyperbolic and parabolic behavior, their lack of accuracy is generally unacceptable.

Rational Runge Kutta

$$\underset{\sim}{M}\,\underset{\sim}{v}_1 + \underset{\sim}{K}\,\underset{\sim}{\theta}_n = \underset{\sim}{f}_{n+1}$$

$$\underset{\sim}{M}\,\underset{\sim}{v}_2 + \underset{\sim}{K}(\underset{\sim}{\theta}_n + c_2\,\Delta t\,\underset{\sim}{v}_1) = \underset{\sim}{f}_{n+1}$$

$$\underset{\sim}{b} = b_1\,\underset{\sim}{v}_1 + b_2\,\underset{\sim}{v}_2$$

$$\underset{\sim}{\theta}_{n+1} = \underset{\sim}{\theta}_n + \Delta t\;\underset{\sim}{e}/(\underset{\sim}{b}^T\,\underset{\sim}{b})$$

where $\underset{\sim}{e} = 2(\underset{\sim}{v}_1^T\,\underset{\sim}{b})\,\underset{\sim}{v}_1 - (\underset{\sim}{v}_1^T\,\underset{\sim}{v}_1)\,\underset{\sim}{b}$

unconditionally stable and second order accurate

if $c = \frac{1}{2}$, $b_1 = 2$, $b_2 = -1$ Hairer 1980 (ref. 13)

no solution of equations if $\underset{\sim}{M}$ diagonal

$\lambda < 0$ if Δt is too large

partitioned Rational Runge Kutta methods, Liu et al.

IJNME, 1581-1597, (1984), 1984 (ref. 14)

Figure 5

A very novel operating splitting method, which exploits the unique features of the finite-element assembly operation, has recently been developed by Hughes and coworkers [15]. This method only required conversion of the element matrices, so while the method does not completely avoid the solution of equations as in semi-implicit methods, the size of equations to be solved is reduced substantially, see Fig. 6. Hughes and coworkers make a very compelling argument that this type of method will prove particularly beneficial in three-dimensional applications.

We have tested an early version of the method in both parabolic systems and elastic-plastic structural mechanics problems. In comparing the method with a conjugate gradient method, we found that the element-by-element and conjugate gradient methods were of comparable speed. When used with large time steps in structural dynamics problems, we were not able to achieve reasonable accuracy unless we made a large number of sweeps during each time step. On the other hand, we found the method to be very useful in crash-type problems in conjunction with explicit techniques. As a deforming structure becomes mostly plastic, it becomes possible to increase the explicit time step quite a bit if the element-by-element procedure is used to stabilize elements which unload into the elastic regime. This would detract somewhat from the accuracy if it occurs in many element. However, in general, phase accuracy is not an overriding concern in crash-type problems, so that the potential of these methods for stabilizing explicit methods is worth investigating. We have not yet tried the later versions of the element-by-element technique which are reported to be substantially more accurate. Reference [16] reports a procedure which reduces the computational effort required in solving the element equations by as much as an order of magnitude.

ELEMENT-BY-ELEMENT OPERATOR SPLIT

Hughes, Levit, Winget ASCE J. Eng. Mech. Div. April 1983 (ref. 8)

Comp. Meth. Appl. Mech. Eng. 1983 (ref. 15)

Ortiz, Pinsky and Taylor (ref. 17)

Recall implicit Eqns.

$$(\underset{\sim}{M} + \Delta t\ \underset{\sim}{K})\ \underset{\sim}{u}^{n+1} = \bar{\underset{\sim}{f}} \qquad (1)$$

$$(\underset{\sim}{I} + \Delta t\ \underset{\sim}{M}^{-1}\ \underset{\sim}{K})\ \underset{\sim}{u} = \underset{\sim}{f} = \underset{\sim}{M}^{-1}\ \bar{\underset{\sim}{f}} \qquad \text{(superscript dropped)}$$

$$(2)$$

Approximation

$$(\underset{\sim}{I} + \Delta t\ \underset{\sim}{M}^{-1} \sum_e \underset{\sim}{K}_e) = \prod_e \underbrace{(\underset{\sim}{I} + \Delta t\ \underset{\sim}{M}^{-1}\ \underset{\sim}{K}_e)}_{\underset{\sim}{G}_e}$$

So (2) becomes

$$\prod_e \underset{\sim}{G}_e\ \underset{\sim}{u} \equiv (\underset{\sim}{G}_1\ \underset{\sim}{G}_2 \cdots\cdots \underset{\sim}{G}_{ne})\ \underset{\sim}{u} = \underset{\sim}{f}$$

Procedure

$$\underset{\sim}{u}_{[0]} = \underset{\sim}{f} \qquad\qquad \underset{\sim}{G}_e\ \underset{\sim}{u}_{[e]} = u_{[e-1]}$$

ONLY ELEMENT MATRICES NEED TO BE INVERTED!

Figure 6

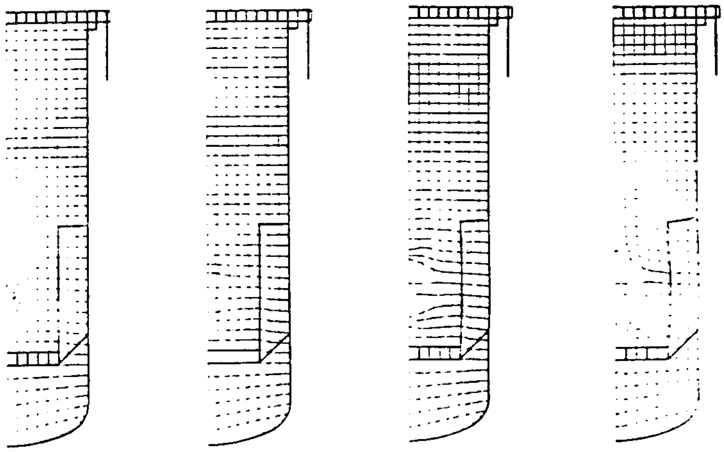
For problems with different time scales such as the space-structure deployment problem, where high-frequency impacts occur in conjunction with low-frequency rigid-body motions, the <u>partitioned</u> methods are quite promising. Partitioned methods are defined as those which employ different time steps or different integrators in different parts of the mesh. During an input, it would be desirable to use different time steps in the vicinity of the impact in solving a large-scale structure problem. By doing this, accuracy could be retained in all parts of the mesh without engendering large expense. The potential of these methods is indicated in Fig. 7.

Δt_{FL} = 0.14 MSEC $\qquad$ Δt_{ST} = 0.04 MSEC

RUNNING TIME FOR 60 MSEC SIMULATION

E^1 420 SEC

E^4 - E, E - I 140 SEC IBM 370/195



$\ell_{min} \sim 2$

c ~ 1500

FOR 60 SEC SEISMIC SIMULATION

RUNNING TIME E - I: 4.2×10^5 SEC

Figure 7

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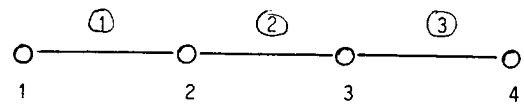
OF POOR QUALITY

Considerable progress has recently been achieved in mesh partitions with different time steps, see References 5 to 12; 14 to 16, and 18 to 20. Basically, two types of mixed time partitions have been involved: element partitions and nodal partitions. The algorithm for nodal partition is shown in Fig. 8. Nodal partitions appear to provide the best accuracy, but their analysis has been impeded by the fact that the amplification matrix is not symmetric.

Subcycling with Nodal Partition

$$\underset{\sim}{u}^{n+1} = \underset{\sim}{u}^{n} + \Delta t \, \underset{\sim}{M}^{-1} \underbrace{(\underset{\sim}{s}^{n} - \underset{\sim}{K}\,\underset{\sim}{u}^{n})}_{\underset{\sim}{f}^{n}}$$



nodes 1 and 2 with Δt

nodes 3 and 4 with $2\Delta t$

computations in cycle

update u_1, u_2

update f_1, f_2

update u_1, u_2

update f_i, $i = 1$ to 3

update u_1, u_2, u_3, u_4

update f_1, f_2

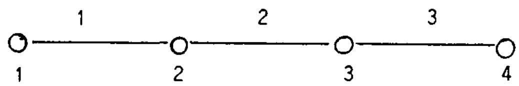
- amplification matrix is not symmetric

Figure 8

An elemental partition is shown in Fig. 9. Element partitions are associated with symmetric amplification matrices and in Ref. [20] a proof of sufficient conditions or stability has been given for a first-order, linear system with different time steps. The proof applies to both explicit and implicit integrators.

Subcycling with Elemental Partition

$$\underset{\sim}{u}^{n+1} = \underset{\sim}{u}^{n} + \Delta t(\underset{\sim}{M}^{-1})\underbrace{(\underset{\sim}{s}^{n} - \underset{\sim}{K}\,\underset{\sim}{u}^{n})}_{\underset{\sim}{f}^{n}}$$ 1st order system i.e. heat conduction diffusion



elements 1 - with Δt

elements 2 & 3 - with $2\Delta t$

computations in cycle

update u_1, u_2

update f_1

update u_1, u_2 sometimes deleted

update f_i i = 1 to 3

update u_1, u_2, u_3, u_4

update f_1

amplification matrix eqns are symmetric

Figure 9

Partitioned implicit methods are particularly well-suited to iterative solvers. Whereas for Newton-type solvers, several different triangulations have to be stored for mixed time integration, this is not necessary for iterative solvers. To illustrate the nature of the solutions which can be obtained from these methods the results of the thermal transient problem in Fig. 10 are shown in Figs. 11 and 12. An interesting observation from Fig. 12 is that when the time step ratio is extremely large (32:1 in case 2), stability is maintained but large errors develop. It has become clear that methods of this type must use a smooth transition of time steps from the smallest time step to the largest time step. Thus, an important ingredient in any mixed time integration procedure is a strategy which automatically selects the time steps within the different regimes according to accuracy requirements and provides smooth transitions of time steps between regions where very large time steps can be used and those where very small time steps can be used.

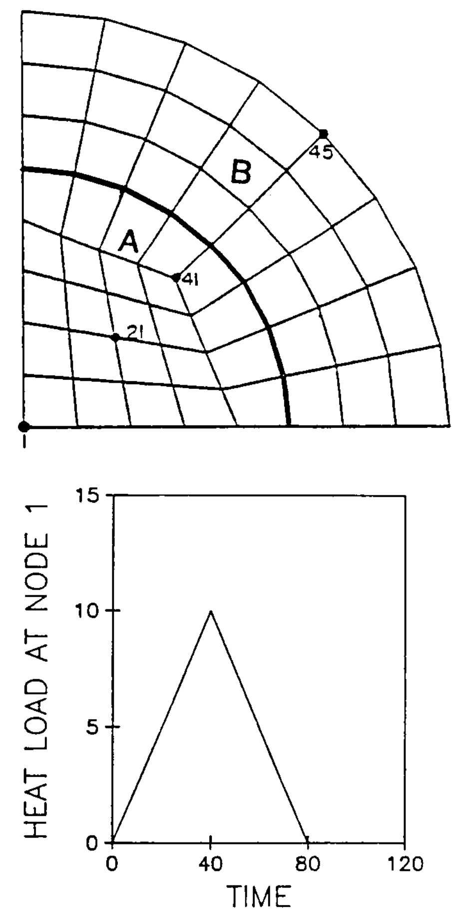


Figure 10

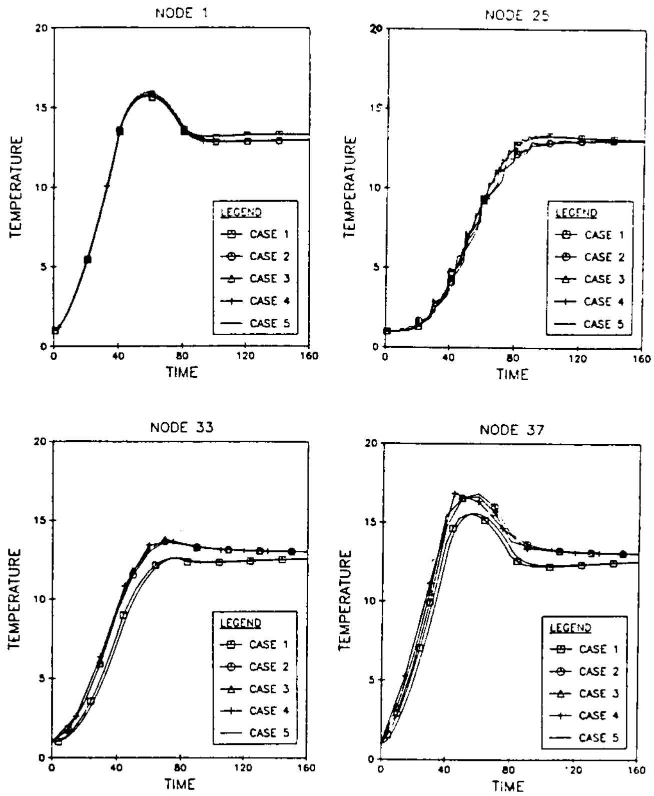


Figure 11

Case 1: E - 4E
2: E - 10I (α = 1)
3: E - 10I (α = ½)
4: 3I - 15I
5: E - E

(From ref. 20)

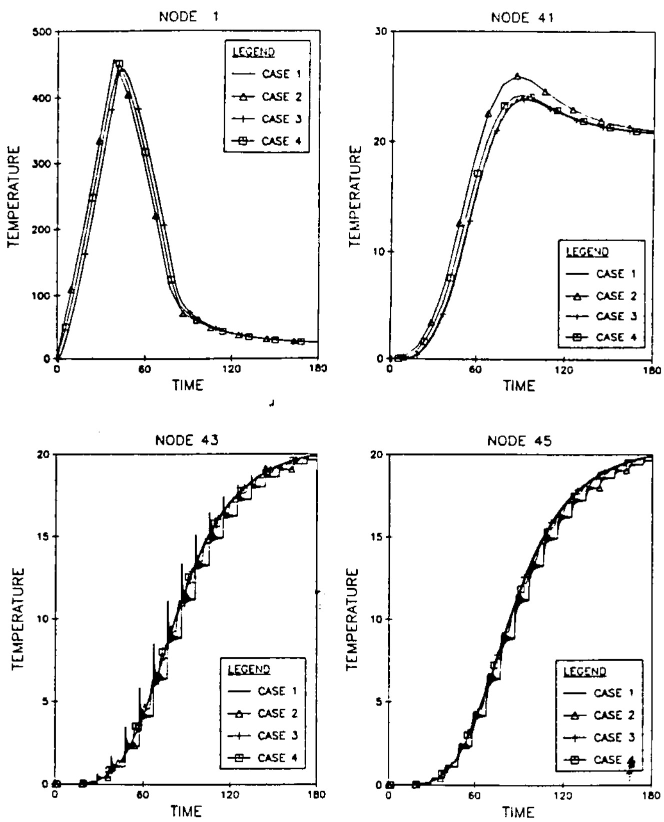


Figure 12

Case 1: E - E, Δt = 0.3
2: 32I - E, Δt = 0.3
3: E - 3I, Δt = 1.2
4: 5I - I, Δt = 1.2

(From ref. 20)

The potential of these methods even in two-dimensional problems is quite tremendous, as evidenced by the comparisons shown in Fig. 13. This is a two-dimensional heat conduction problem with a large range of thermal conductivities. As can be seen from the comparison, savings of a factor of 2 to 5 can be achieved even in moderately sized two-dimensional problems. These types of savings have important implications in a computer-aided engineering environment, where the analysis of a new concept must be achieved in a reasonable amount of time if the design process is to be interactive.

These mixed-time integration procedures are in many ways still in their infancy. The applications to nonlinear problems and contact-impact problems will probably require special strategies in order to exploit these methods to their fullest advantage. It would also be desirable to develop stability analyses in the linear regime for second-order systems, such as the equations of motion, and for nodal partitions.

This class of methods, when combined with iterative solvers, would be uniquely suited for parallel architecture computers. In principle, each subdomain with a particular time step could be treated by a different CPU. Data transfer between subdomains would only be necessary for interface data.

Storage and Running Time Comparisons

Storage

Method	I-4E-8E-8I $\Delta t = 1$	I $\Delta t = 1$	I-E-2E-2I $\Delta t = 4$	I $\Delta t = 4$
nonzero terms in $\underset{\sim}{K}$	3288	33771	6089	33771
average semibandwidth	6	69	12	69
solution time CPU-s	90.	516.	70.	139.

Note: • problem is linear;

• 8 x 50 mesh is numbered for large bandwidth to simulate 3D problems.

Figure 13

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