DYNAMICS OF ARTICULATED STRUCTURES

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For purposes of this presentation, an articulated structure is defined as an assembly of flexible bodies that may be coupled by kinematic connections and force elements that permit large relative displacement and rotation. Kinematics of such systems is defined using one reference frame for each body in the system and deformation modal coordinates that define displacement fields within flexible bodies. Deformation kinematics are defined by both elastic vibration and static correction deformation modes. Linear elastic deformation is presumed; i.e., a linear stress-strain relation is valid and relative displacements within each elastic component are small enough so that the theory of linear elasticity applies. Coupling of reference and modal coordinates leads to a system of nonlinear equations of motion. Methods of automatically generating and solving these equations of motion are outlined.

- Large Displacement and Rotations of Body Reference Frames (reference coordinates)

- Elastic Vibration and Static Correction Deformation Modes (modal coordinates)

- Coupled Nonlinear Equations in Reference and Modal Coordinates

- Automated Equation Generation and Solution

EULER'S THEOREM

Euler's theorem guarantees existence of a unit vector u about which an x'-y'-z' reference frame may be rotated by an angle χ to bring it from a reference x-y-z frame to a general orientation. Components of the unit vector u and angle χ of rotation are used to define orientation of a reference frame in space.
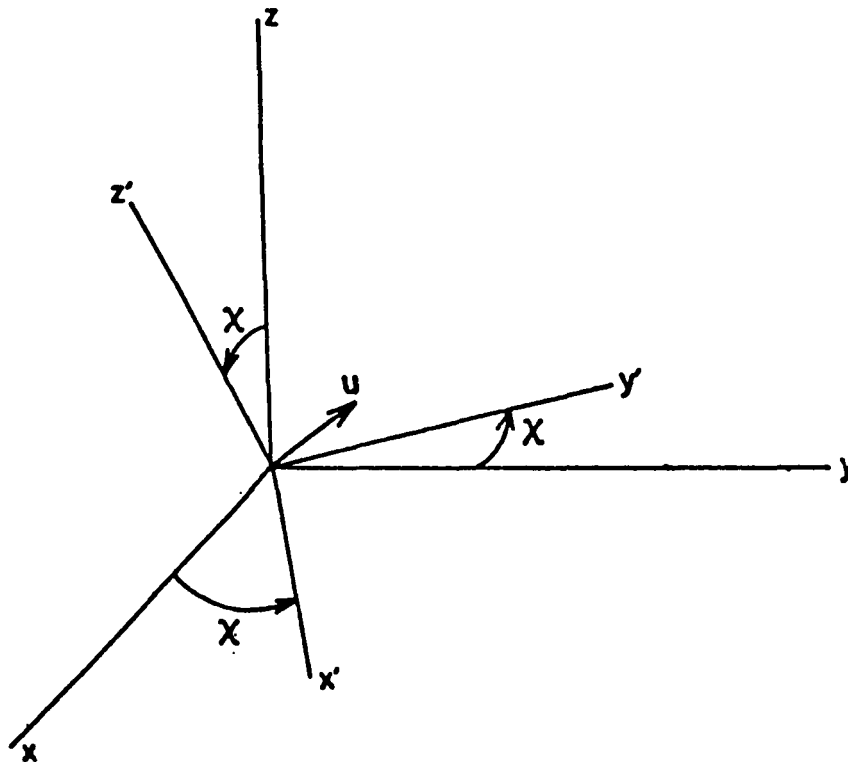
EULER PARAMETERS

A set of four Euler parameters is defined as **p**, as shown on the chart. These four parameters are not independent, since the vector **p** must be a unit vector in R^4. The direction cosine transformation matrix from the x'-y'-z' reference frame to the x-y-z frame is defined as shown. The quadratic nature of terms in the transformation matrix, as functions of Euler parameters, leads to attractive properties when writing velocity and acceleration equations that are needed in the equations of motion. Furthermore, use of Euler parameters avoids singular orientation difficulties that are associated with a set of three rotation parameters, such as Euler angles or Bryant angles [1,2].

- $e_0 = \cos\chi/2$, $e = u\sin\chi/2$

- $p = [e_0, e^T]^T = [e_0, e_1, e_2, e_3]^T$

- $p^T p = \cos^2\chi/2 + u^T u \sin^2\chi/2 = 1$

- $s = As'$, $A = 2\begin{bmatrix} e_0^2 + e_1^2 - \frac{1}{2} & e_1 e_2 - e_0 e_3 & e_1 e_3 + e_0 e_2 \\ e_1 e_2 + e_0 e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2 e_3 - e_0 e_1 \\ e_1 e_3 - e_0 e_2 & e_2 e_3 + e_0 e_1 & e_0^2 + e_3^2 - \frac{1}{2} \end{bmatrix}$

A lumped mass finite-element formulation is used to carry out vibration and static correction mode analysis of each deformable body in an articulated structure. A typical point P^i is defined in the undeformed state of the body by a constant vector $r_0{}^i$. During the process of deformation, this point undergoes displacement u^i in the body reference frame, as shown. Lumped masses m_i at each note in the finite-element model are used in defining kinetic properties of the flexible body [3,4].
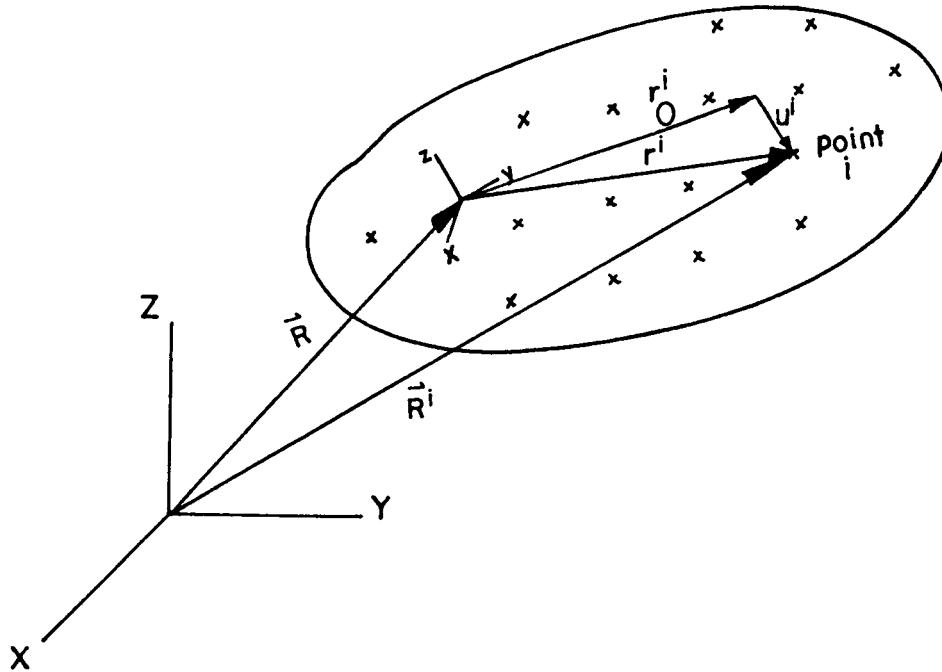
A vector u of modal displacements is defined as a linear combination of a set of deformation modes ϕ^j, $j = 1, \cdots, m$. The displacement vector u^i of point i in the body is provided by a projection matrix P^i. As noted earlier, the direction cosine transformation matrix for the reference frame associated with the body is a function of the Euler parameters of that reference frame. Finally, the global position vector R^i of point i on the body is given as shown.

$u = a_j \phi^j$ – nodal displacement relative to reference frame

$\phi^j = j=1, \cdots, m$ – deformation modes

$u^i = P^i u$ – elastic displacment of mode i

$A(p)$ – direction cosine matrix of reference frame

p – Euler parameters of reference frame

$R^i = R + A\left(r_0^i + P^i a_j \phi^j\right)$

VIBRATION AND STATIC CORRECTION MODES

Boundary conditions must be selected for characterizing deformation of flexible components. Since kinematic constraints on bodies in an articulated structure often lead to statically indeterminant sets of boundary conditions, a statically determinant or underdetermined set of boundary conditions is selected for use in vibration analysis. Unit loads associated with deleted kinematic constraints are used to define static correction modes [3]. These calculations are carried out with any standard finite-element code that is capable of generating lumped mass information. Constants that will appear subsequently in the equations of motion are calculated using information generated within the finite-element code.

- Select Boundary Conditions for Flexible Components

- Calculate Natural Vibration Modes

- Calculate Static correction Modes for Deleted Constraints

- Calculate Constants for Equations of Motion

The position relationship derived earlier is differentiated to obtain the global velocity vector of node i in the body, as shown. The time derivative of the direction cosine orientation matrix yields an expression in the time derivative of Euler parameters as shown [2,3]. The velocity vector may thus be written in matrix form for use in derivation of the equations of motion of the system.

$$\dot{R}^i = \dot{R} + \dot{A}r^i + AP^i \phi^j \dot{a}_j$$

$$\dot{A}r^i = -2E(p)\overset{+}{r}\dot{p}$$

$$\dot{R}^i = \begin{bmatrix} I & -2E\overset{+i}{r} & A\psi^i_t \end{bmatrix} \begin{bmatrix} \dot{R} \\ \dot{p} \\ \dot{a} \end{bmatrix}$$

$$\psi^i_t = P^i \begin{bmatrix} \phi^1, \cdots, \phi^m \end{bmatrix}$$

KINEMATIC CONSTRAINTS

A variety of kinematic couplings between flexible bodies is derived in Refs. 2 and 3. Joint definition ξ_j η_j ζ_j reference frames are fixed to the deformable body to define information required to write kinematic constraint equations associated with each joint in the system. Shown on the chart below are spherical, revolute, and universal joints, for which constraint equations may be found in Ref. 3.
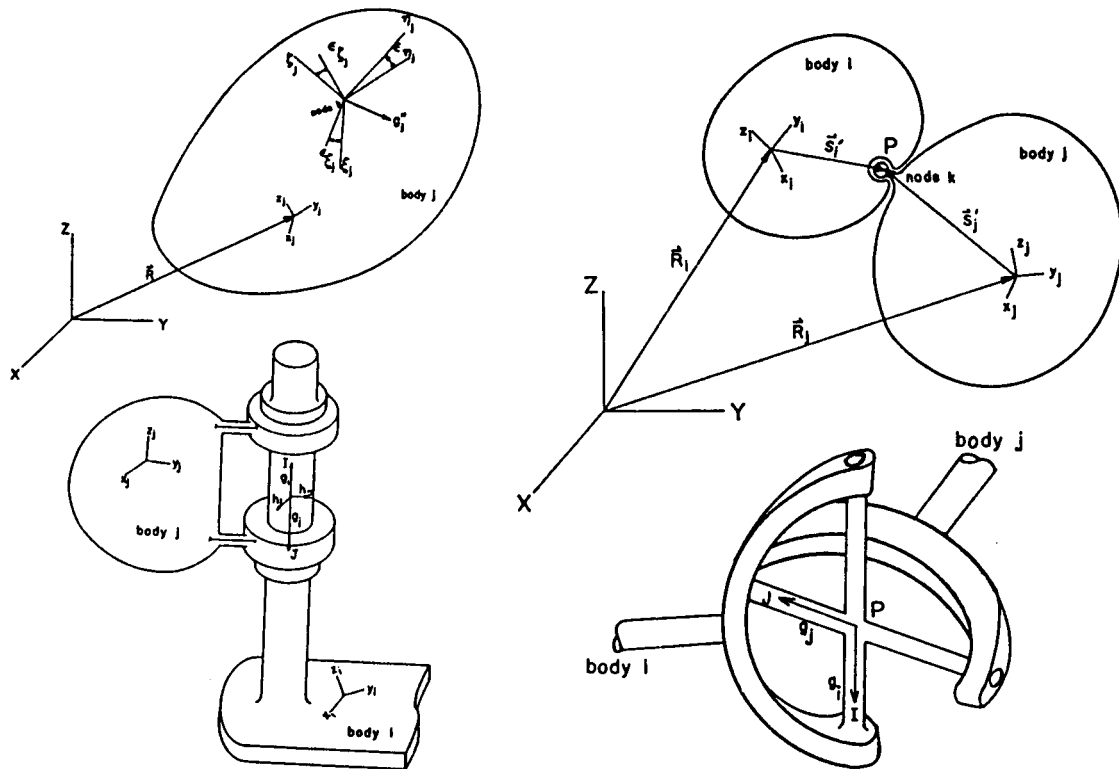
The kinetic energy of a flexible body may be written in terms of time derivatives of reference frame generalized coordinates and modal coordinates, as shown. Since vectors r^i and matrices E, G, and A are nonlinear functions of Euler parameters, the mass matrix of the flexible body is a nonlinear function of generalized coordinates, as a result of geometric nonlinearities in the system kinematics.

• Lumped masses m_i, $i=1,\cdots,n$

$$T = \tfrac{1}{2} \begin{bmatrix} \dot{R} \\ \dot{p} \\ \dot{a} \end{bmatrix}^T M \begin{bmatrix} \dot{R} \\ \dot{p} \\ \dot{a} \end{bmatrix}$$

$$M = \begin{bmatrix} \displaystyle\sum_{i=1}^{N} m_i & -2E \displaystyle\sum_{i=1}^{N} m_i r^{+i} & A \displaystyle\sum_{i=1}^{N} m_i \psi_t^i \\[2ex] & 4G^T \left(\displaystyle\sum_{i=1}^{N} m_i \tilde{r}^{iT} \tilde{r}^i \right) G & -2 \displaystyle\sum_{i=1}^{N} m_i G^T \tilde{r}^{iT} \psi_t^i \\ & & + 2p \displaystyle\sum_{i=1}^{N} m_i r^{iT} \psi_t^i \\[2ex] \text{symmetric} & & \displaystyle\sum_{i=1}^{N} m_i \psi_t^{iT} \psi_t^i \end{bmatrix}$$

The strain energy U of the flexible body may be written explicitly in terms of a modal stiffness matrix $\mathbf{K_{aa}}$, as shown. Generalized forces defined through direct application of the definition of virtual work lead to nonlinear algebraic expressions in generalized coordinates, as shown. These forces include both externally applied forces and forces of interaction due to compliant couplings between bodies and feedback control actuators.

$$U = \frac{1}{2} \begin{bmatrix} R \\ p \\ a \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ \hline 0 & K_{aa} \end{bmatrix} \begin{bmatrix} R \\ p \\ a \end{bmatrix}$$

$$\delta W = \sum_{i=1}^{N} F^{i^T} \begin{bmatrix} I & -2A\tilde{r}^iG & A\Psi_t^i \end{bmatrix} \begin{bmatrix} \delta R \\ \delta p \\ \delta a \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{N} F^{i^T} & \sum_{i=1}^{N} -2F^{i^T}A\tilde{r}^iG & \sum_{i=1}^{N} F^{i^T}A\Psi_t^i \end{bmatrix} \begin{bmatrix} \delta R \\ \delta p \\ \delta a \end{bmatrix}$$

$$= \begin{bmatrix} Q_R^T & Q_p^T & Q_a^T \end{bmatrix} \begin{bmatrix} \delta R \\ \delta p \\ \delta a \end{bmatrix}$$

EQUATIONS OF MOTION OF A SINGLE BODY

The equations of motion of an unconstrained individual flexible body are shown below [3]. The system of equations for an articulated structure that is made up of multiple bodies connected by kinematic constraints is developed [3] using the Lagrange multiplier form of multi-body system dynamics [1,2]. Evaluation of individual terms appearing in the coefficient matrix of accelerations and on the right side of the equations of motion is derived by expanding the expressions shown and calculating constant coefficients associated with deformation modes and mass distribution.

$$
\begin{bmatrix}
\sum\limits_{i=1}^{N} m_i & -2\sum\limits_{i=1}^{N} m_i \tilde{r}^i & A\sum\limits_{i=1}^{N} m_i \psi_t^i & 0 \\
& 4G^T\left(\sum\limits_{i=1}^{N} m_i \tilde{r}^{i^T}\tilde{r}^i\right)G & \begin{matrix}-2G^T\sum\limits_{i=1}^{N} m_i \tilde{r}^{i^T}\psi_t^i \\ +2p\sum\limits_{i=1}^{N} m_i \tilde{r}^{i^T}\psi_t^i\end{matrix} & 2p \\
& \text{symmetric} & \sum\limits_{i=1}^{N} m_i \psi_t^{i^T}\psi_t^i & 0 \\
& & & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{R} \\ \ddot{p} \\ \ddot{a} \\ \lambda
\end{bmatrix}
=
\begin{bmatrix}
\sum\limits_{i=1}^{N} F^i \\
\sum\limits_{i=1}^{N} 2G^T \tilde{r}^i A^T F^i \\
\sum\limits_{i=1}^{N} (A\psi_t^i)^T F^i \\
0
\end{bmatrix}
$$

$$
+
\begin{bmatrix}
2\dot{E}\left(\sum\limits_{i=1}^{N} m_i \tilde{r}^i\right)\dot{p} - 4E\left(\sum\limits_{i=1}^{N} m_i \dot{\tilde{r}}^i\right)\dot{p} \\[1em]
\hline
-8\dot{G}^T\left(\sum\limits_{i=1}^{N} m_i \tilde{r}^{i^T}\tilde{r}^i\right)G\dot{p} + 4\dot{G}^T\left(\sum\limits_{i=1}^{N} m_i \tilde{r}^{i^T}\psi_t^i\right)\dot{a} + 2G^T\left(\sum\limits_{i=1}^{N} m_i \dot{\tilde{r}}^{i^T}\psi_t^i\right)\dot{a} \\[0.8em]
-2p\left(\sum\limits_{i=1}^{N} m_i \dot{\tilde{r}}^{i^T}\psi_t^i\right)\dot{a} - 4G^T\dfrac{d}{dt}\left(\sum\limits_{i=1}^{N} m_i \tilde{r}^{i^T}\tilde{r}^i\right)G\dot{p} \\[1em]
\hline
-K_{aa}a + 2\left(\sum\limits_{i=1}^{N} m_i \dot{\tilde{r}}^{i^T}\psi_t^i\right)^T G\dot{p} - 2\left(\sum\limits_{i=1}^{N} m_i \tilde{r}^{i^T}\psi_t^i\right)^T \dot{p}^T\dot{p} \\[0.8em]
+ 4\left(\sum\limits_{i=1}^{N} m_i \psi_t^i\right)^T (G\dot{p})^T (G\dot{p}) r^i + 2\left(\sum\limits_{i=1}^{N} m_i \psi_t^i\right)^T (G\dot{p})^T \psi_t^i \dot{a} \\[1em]
\hline
-2\dot{p}^T\dot{p}
\end{bmatrix}
$$

CONSTANTS FROM FINITE-ELEMENT MODEL

Nine sets of constant vectors and matrices shown are calculated, using data generated in the finite-element deformation analysis of each flexible body. These constants are computed using an intermediate processing program [4].

$$C_1 = \sum_{i=1}^{N} m_i r_0^i$$

$$C_6(j) = \sum_{i=1}^{N} m_i \tilde{\phi}_t^{ij} r_0^i \quad , \qquad j=1,2,\ldots,m$$

$$C_2(j) = \sum_{i=1}^{N} m_i \phi_t^{ij} \quad , \qquad j=1,2,\ldots,m$$

$$C_7(j,k) = \sum_{i=1}^{N} m_i \tilde{\phi}_t^{ij} \phi_t^{ik} \quad , \quad \begin{array}{l} j=1,2,\ldots,m \\ k=1,2,\ldots,m \end{array}$$

$$C_3 = \sum_{i=1}^{N} m_i \tilde{r}_0^i \tilde{r}_0^i$$

$$C_8(j) = \sum_{i=1}^{N} m_i {r_0^i}^T \phi_t^{ij} \quad , \qquad j=1,2,\ldots,m$$

$$C_4(j) = \sum_{i=1}^{N} m_i \tilde{r}_0^i \tilde{\phi}_t^{ij} \quad , \qquad j=1,2,\ldots,m$$

$$C_9(j,k) = \sum_{i=1}^{N} m_i {\phi_t^{ij}}^T \phi_t^{ik} \quad , \quad \begin{array}{l} j=1,2,\ldots,m \\ k=1,2,\ldots,m \end{array}$$

$$C_5(j,k) = \sum_{i=1}^{N} m_i \tilde{\phi}_t^{ij} \tilde{\phi}_t^{ik} \quad , \quad \begin{array}{l} j=1,2,\ldots,m \\ k=1,2,\ldots,m \end{array}$$

NONLINEAR TERMS IN EQUATIONS OF MOTION

Three typical nonlinear terms appearing in the equations of motion presented earlier are shown here, evaluated as linear and quadratic expressions in generalized coordinates. All such terms are coded in a flexible-body module of the Dynamic Analysis and Design System (DADS) computer code. These terms are evaluated at every time step in numerical integration of the coupled system of nonlinear equations of motion.

$$\sum_{i=1}^{N} m_i \ddot{r}^i = \ddot{C}_1 + [\ddot{C}_2(j)a_j]$$

$$\sum_{i=1}^{N} m_i \tilde{r}^{i^T} \tilde{r}^i = -C_3 - \sum_{i=1}^{m} \left[C_4(j) + [C_4(j)]^T \right] a_j - \sum_{j=1}^{m} \sum_{k=1}^{m} C_5(j,k)a_j a_k$$

$$\sum_{i=1}^{N} m_i \tilde{r}^{i^T} \psi_t^i = \left[C_6(1) + \sum_{k=1}^{m} C_7(1,k)a_k, \ C_6(2) + \sum_{k=1}^{m} C_7(2,k)a_k, \cdots, \right.$$

$$\left. C_6(m) + \sum_{k=1}^{m} C_7(m,k)a_k \right]$$

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NUMERICAL SOLUTION

A variable order, variable step size numerical integration algorithm is used to compute the solution of differential-algebraic equations of motion for articulated structures. Since step size and order selected by the algorithm reflect the error tolerance required and the frequency of oscillation that develops, integration cost is influenced by selection of deformation modes in the model. Numerical results accumulated to date [4,5] show that prudent selection of a combination of vibration and static correction modes gives reasonable results. Substantial work remains to be done in rational selection of these deformation modes.

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- Variable Order, Variable Step Size Numerical Integration

- Integration Cost is a Function of Frequency Content

- Mixed Vibration and Static Correction Modes Give Best Results

FLEXIBLE HINGE DOOR EXAMPLE

The flexible door structure shown is kinematically coupled to a body that is taken to be rigid ground. The revolute joints shown are misaligned so that there is no deformation when the door structure lies in the Y-Z plane. Any rotation of the door structure leads to deformation of the beam and plate structure of the door, which tends to bring it back to the undeformed state.
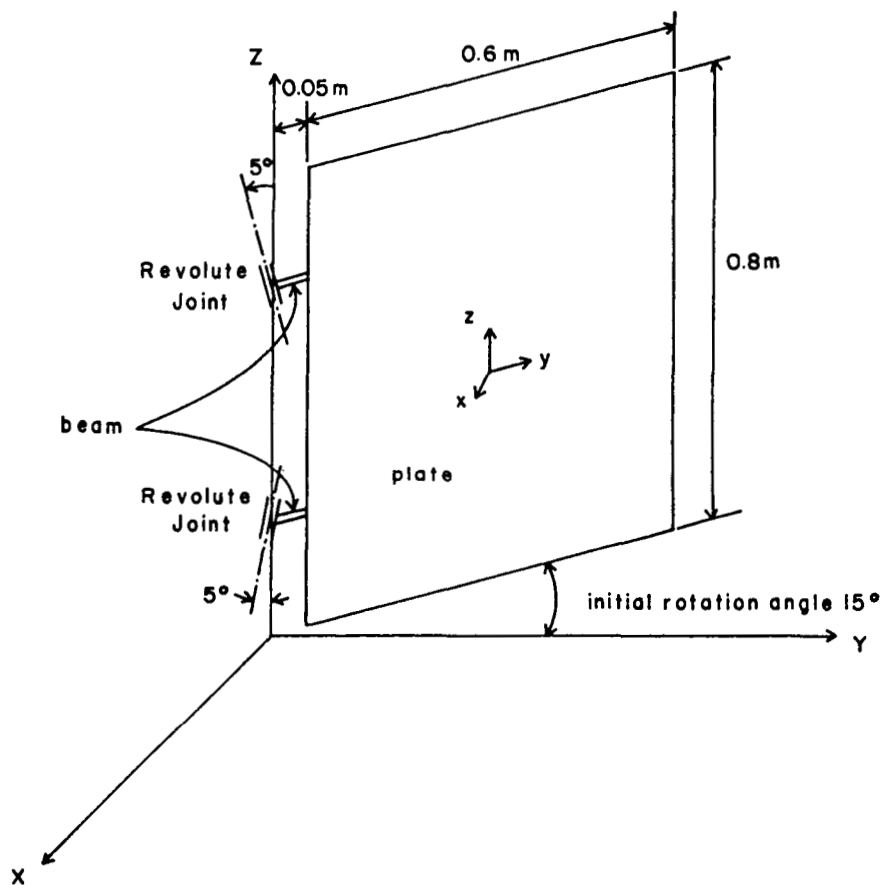
FINITE-ELEMENT MODEL OF DOOR STRUCTURE

A modest finite-element model of the door structure, using plate and beam elements, is shown.
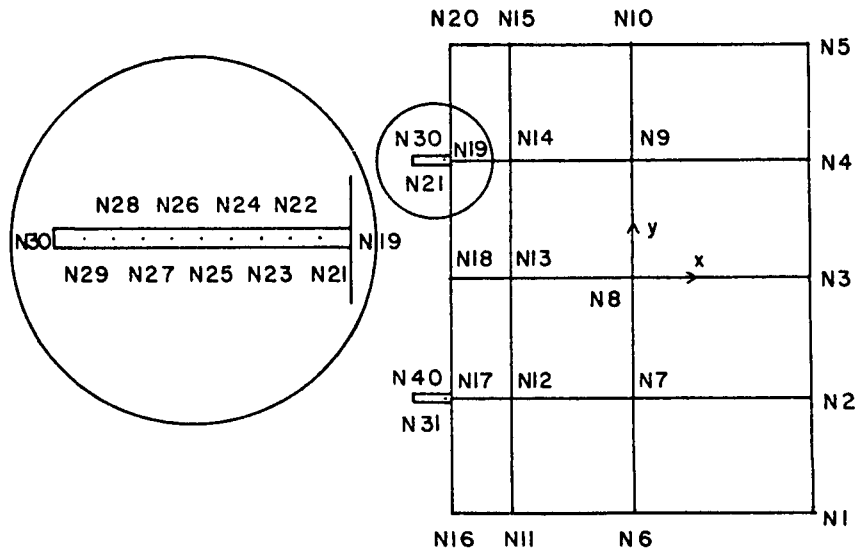
N** : Node Number

Plate ; 12 (Membrane + Bending) Elements

$$E = 2.0 \times 10^{12} \ N/m^2$$

Beam ; 10 Beam Elements at each beam

$$E = 2.0 \times 10^{11} \ N/m^2$$

STATIC CORRECTION MODES

Boundary conditions for finite-element analysis are selected so that the center point of the bottom hinge is fixed in space and x- and y-coordinates of the top hinge point are likewise fixed. Five kinematic constraints are thus suppressed, two rotations at the bottom hinge and two rotations and one translation at the top hinge. Unit torques and a unit force are applied to calculate five static correction modes to represent deformation of the structure. Vibration modes are likewise calculated [4].
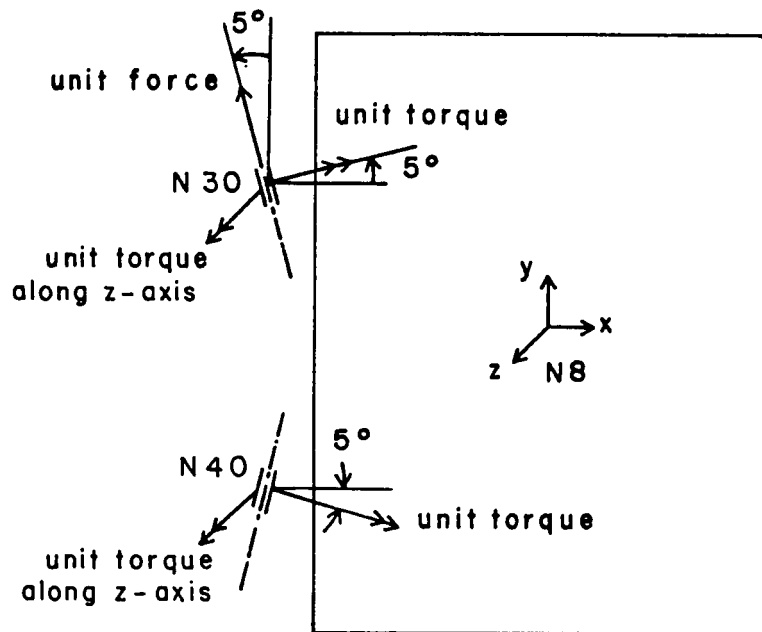
FIRST APPROXIMATE SOLUTION

Two models were used in a preliminary analysis. The first consisted of only five normal vibration modes, with numerical results for the X-coordinate of the center of the door shown as a solid line. A five static correction mode approximate solution is shown with a dotted line, reflecting much lower frequency of vibration of the door structure. To evaluate reasonables of predictions, additional modes and combinations of modes are selected.

—: 5N SOLUTION
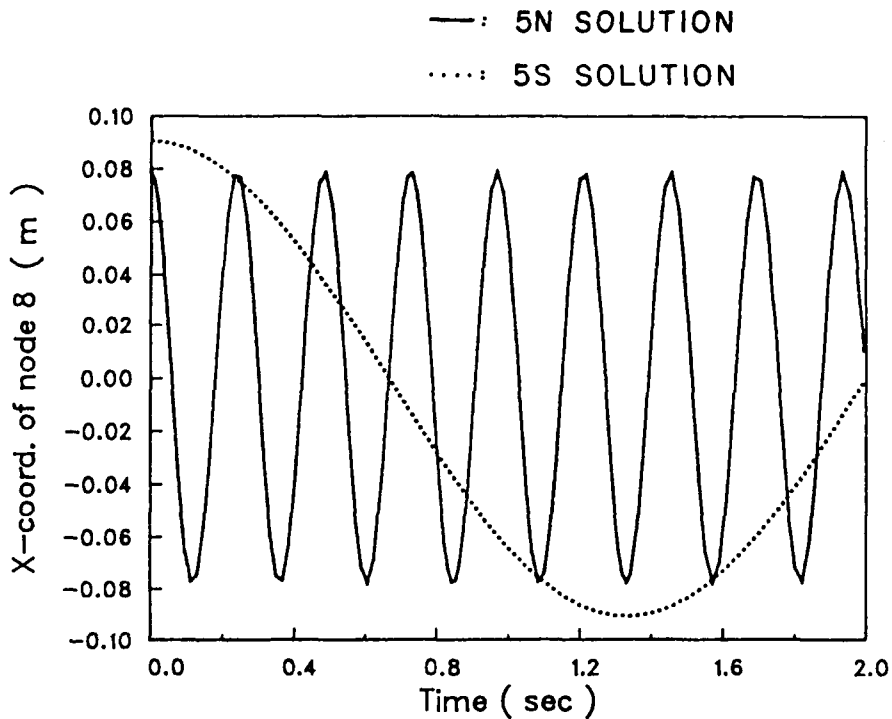
····: 5S SOLUTION

SECOND APPROXIMATE SOLUTION

The nine normal mode solution shown reflects a somewhat lower overall vibration frequency, but is still unreasonable. As shown by the dotted and dashed curves, adding four normal vibration modes to the five static correction modes yields only a slight change in the prediction obtained from purely five static correction modes. This suggests that static correction modes dominate the dynamics of this example.

─── :   9N SOLUTION

······ :   5S SOLUTION

─── : 4N5S SOLUTION
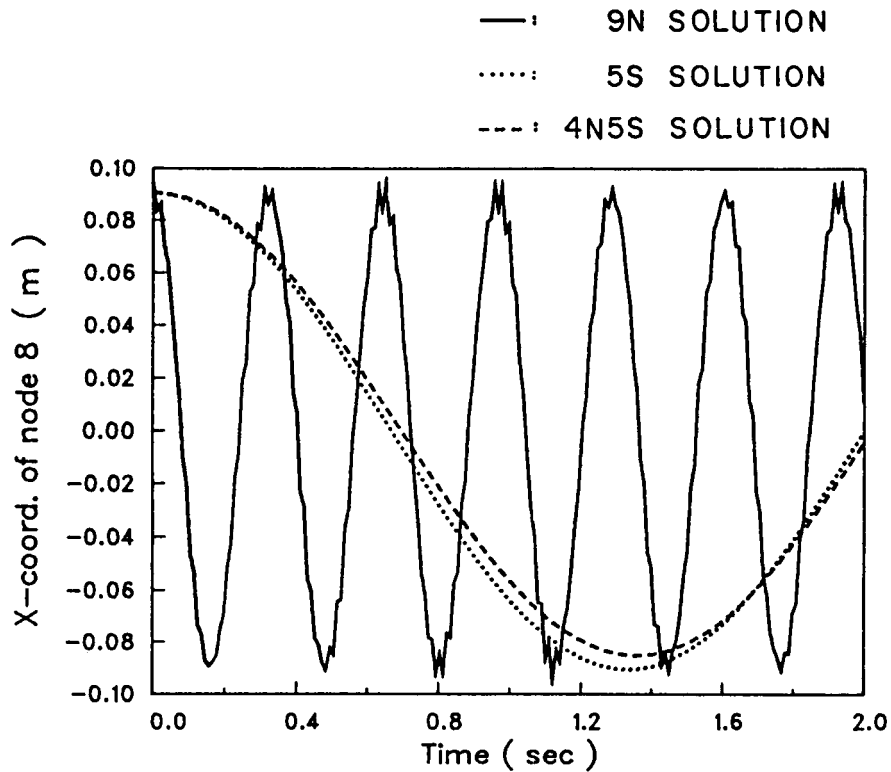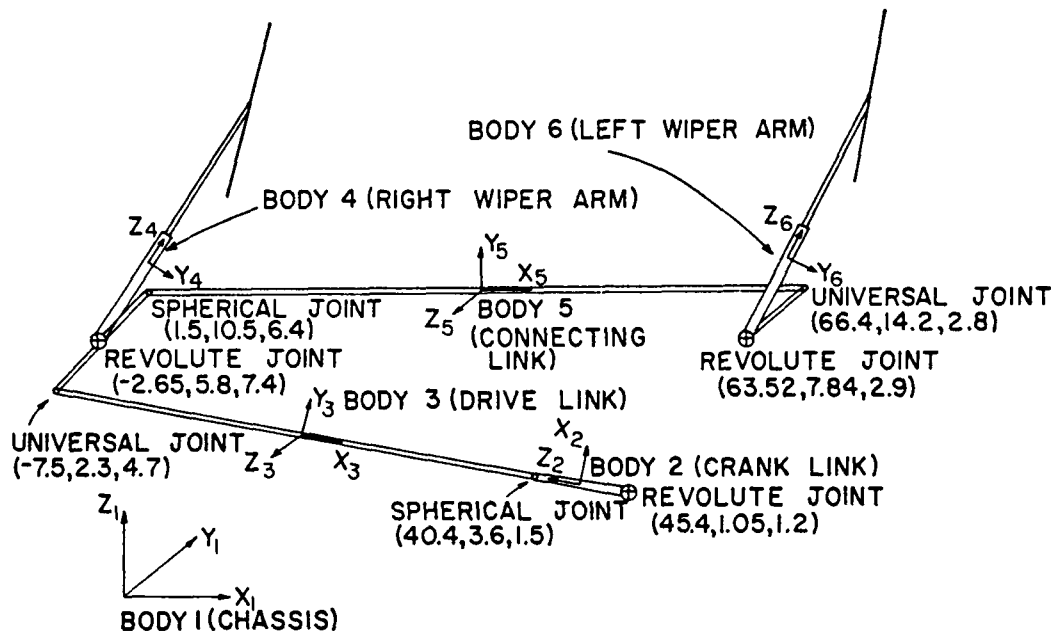
COMPARISON OF COMPUTER SIMULATION TIMES

As shown in the table below, retention of a substantial number of high-frequency normal vibration modes leads to very small step size and ultimately exceptionally large computer times. The computer times indicated are on a heavily loaded Prime 750 supermini computer.

Comparison of Simulation Times

| Type	T end [sec]	CPU [sec]	RMS integration stepsize [sec]
5S solution	2.0	106	0.47812E-01
5N solution	2.0	401	0.90934E-02
9N solution	2.0	7471	0.65035E-03
4N5S solution	2.0	7281	0.75491E-03

WINDSHIELD WIPER APPLICATION

The schematic shown is a model of an automotive windshield wiper
assembly, in which the crank-link and two connecting links are taken as
rigid. The left and right wiper arms are modeled as flexible bodies. The
mechanism is driven by applying a torque to the crank link that is a function
of motor speed.

BODY 6 (LEFT WIPER ARM)

BODY 4 (RIGHT WIPER ARM)

Z_4

Y_5

Z_6

Y_4

X_5

Y_6

SPHERICAL JOINT
(1.5, 10.5, 6.4)

Z_5 BODY 5
(CONNECTING
LINK)

UNIVERSAL JOINT
(66.4, 14.2, 2.8)

REVOLUTE JOINT
(-2.65, 5.8, 7.4)

REVOLUTE JOINT
(63.52, 7.84, 2.9)

UNIVERSAL JOINT
(-7.5, 2.3, 4.7)

Y_3 BODY 3 (DRIVE LINK)

X_2

Z_3 X_3

Z_2 BODY 2 (CRANK LINK)

Z_1

SPHERICAL JOINT
(40.4, 3.6, 1.5)

REVOLUTE JOINT
(45.4, 1.05, 1.2)

Y_1

X_1

BODY 1 (CHASSIS)

FINITE-ELEMENT MODEL OF WIPER ARMS

A modest Beam finite-element model of each wiper arm is constructed as shown. Friction torque, as a function of wiper tip velocity, is introduced as a force acting in the system, as shown.

N**: NODE NUMBER
BEAM ELEMENTS
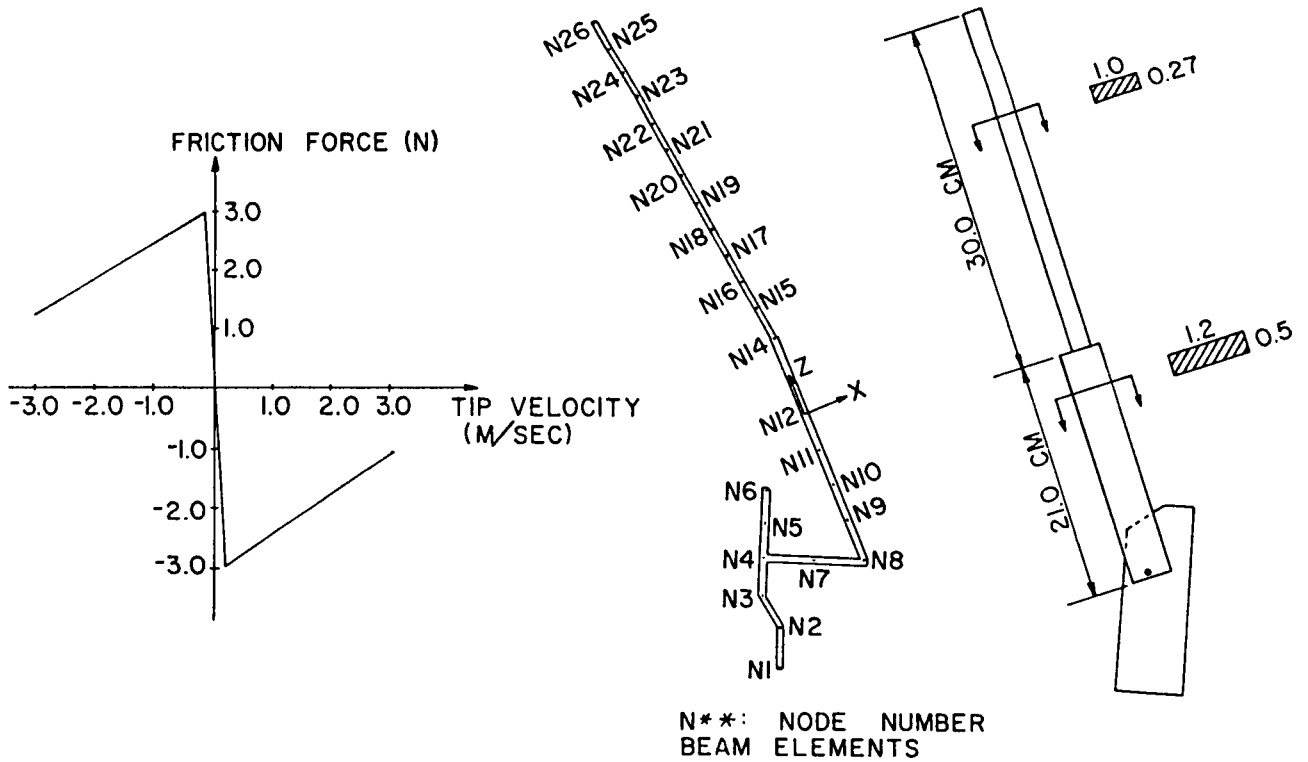
NUMERICAL RESULTS FOR WIPER ARM TIP VELOCITY

A flexible body solution shown in the solid line predicts vibration at a frequency of approximately fourteen cycles per second, relative to essentially the same gross motion predicted by a rigid body model of the windshield wiper mechanism. Experimental results with the actual system indicate an oscillation frequency of approximately thirteen cycles per second, very close to that predicted by the articulated structure model.
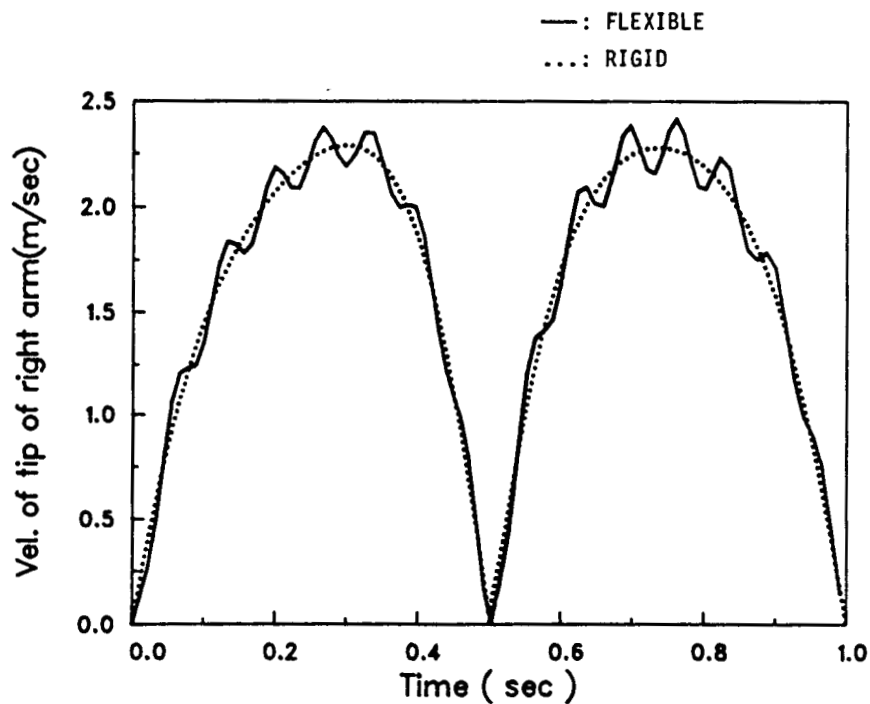
STATUS AND DEVELOPMENTS

The DADS flexible system dynamics code is now functioning and has been used to analyze a number of small and intermediate scale applications. A commercial version of the software is expected to be available from Computer Aided Design Software Incorporated, of Oakdale, Iowa. Extensions are currently under way to enhance capability of the code to represent selected aspects of space structure dynamics.

- DADS Flexible Code Is Now Functioning

- A Commercial Version of The Code Will Be Available Late In 1985

- Extensions Are Under way To Enhance Capabilities For Space Structure Dynamics

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