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ITERATIVE METHODS FOR DESIGN SENSITIVITY ANALYSIS

A.D. Belegundu
Department of Mechanical Engineering
The Pennsylvania State University
University Park, PA 16802

and

B.G. Yoon
Graduate Student

ABSTRACT

A numerical approach is presented for design sensitivity analysis. The approach is based on perturbing the design variables and then using iterative schemes to obtain the response of the perturbed structure. A forward difference formula then yields the approximate sensitivity. Algorithms for displacement and stress sensitivity as well for eigenvalue and eigenvector sensitivity are developed. Results for the stress sensitivity problem are compared with the semi-analytical method. Examples are considered in structures and fluids.

INTRODUCTION

Iterative methods are presented for obtaining design sensitivity coefficients (or derivatives) of implicit functions. Design derivatives are important not only in gradient-based optimization codes, but also for examining trade-off's, system identification, and probabilistic design. Iterative methods are presented for both the algebraic and eigenvalue problems; stress, eigenvalue and eigenvector derivatives are considered. The iterative approaches provide approximate derivatives. They are very simple to implement in a program, especially for calculation of eigenvector derivatives. The idea of using iterative methods for a class of problems was suggested for one dimensional problems in (ref. 1). Here, this idea is developed to handle the matrix algebraic as well as the generalized eigenvalue problems.

The basic idea behind the approach is as follows. Let

$$g = g(\underline{b},\underline{y}) \tag{1}$$

be a continuously differentiable function of a design variable vector $\underline{b}$ of dimension (kx1), and a state variable vector $\underline{y}$ of dimension (nx1). The state variables are implicitly dependent on design through the n state equations of the form

$$\underline{\phi}(\underline{b},\underline{y}) = 0 \tag{2}$$

Let $\underline{b}^o$ be the current design and $\underline{y}^o$ be the associated state variable vector. The problem of concern is to find the sensitivity, $dg/d\underline{b}$, at the current design. The iterative method is based on perturbing each design variable, in turn, as

$$b_i^\varepsilon = b_i^o + \varepsilon \tag{3}$$

Equation (2) now becomes

$$\underline{\phi}(\underline{b}^\varepsilon,\underline{y}^\varepsilon) = \underline{0} \tag{4}$$

Now, a modified residual-correction or Newton-Raphson technique is applied to solve Eq. (4), treating $\underline{y}^\varepsilon$ as the vector of unknowns.

Then, the sensitivity vector is given approximately by

$$dg/db_i \approx [g(\underline{b}^\varepsilon, \underline{y}^\varepsilon)-g(\underline{b}^o,\underline{y}^o)]/\varepsilon \tag{5}$$

For the eigenvalue problem, as discussed later, the system in Eq. (4) is augmented by a certain orthogonality relation. Note that certain coefficient matrices involving stiffness, mass, etc. have been decomposed at the current design while solving for $\underline{y}^o$. The iterative approach presented here can be viewed as re-analysis schemes used to solve Eq. (4), which uses the already decomposed matrices. Since the perturbation ε is very small, the iterative schemes converge very rapidly.

DISPLACEMENT AND STRESS SENSITIVITY

A finite element model of the structure is assumed. The problem of obtaining design derivatives of displacements and stresses is now considered. Consider a function

$$g = g(\underline{b},\underline{z}) \tag{6}$$

which represents a stress constraint, with $\underline{b}$ = (kxl) design vector and $\underline{z}$ = (nxl) displacement vector which is obtained from the finite element equations

$$\underline{K}(\underline{b})\ \underline{z} = \underline{F}(\underline{b}) \tag{7}$$

where $\underline{K}$ is an (nxn) structural stiffness matrix, and $\underline{F}$ is an (nxl) nodal load vector. Let $\underline{b}^o$ be the current design. At this stage, the analysis has been completed. Thus, the decomposed $\underline{K}(\underline{b})^o$ and $\underline{z}^o$ are known.

The derivative of the function g with respect to the ith design variable is given by

$$dg/db_i = \partial g/\partial b_i + \partial g/\partial \underline{z} \cdot d\underline{z}/db_i \tag{8}$$

The partial derivatives $\partial g/\partial \underline{b}$ and $\partial g/\partial \underline{z}$ are readily available using the finite element relations. The problem, therefore, is to compute the displacement sensitivity, $d\underline{z}/d\underline{b}$. An iterative approach for computing this is now given.

Corresponding to the ith design variable, let the perturbed design vector, $\underline{b}^\varepsilon$, be defined as

$$\underline{b}^\varepsilon = (b_1^0, b_2^0, \ldots, b_i^0 + \varepsilon, \ldots, b_k^o)^T \tag{9}$$

The perturbation ε is relatively small, and a value of 1% of b_i has found to work well in practice. The choice of ε is based on balancing the truncation and cancellation errors. The problem is to find $\underline{z}^\varepsilon$, the solution of

$$\underline{K}(\underline{b}^\varepsilon)\ \underline{z}^\varepsilon = \underline{F}(\underline{b}^\varepsilon) \tag{10}$$

using the decomposed $\underline{K}(\underline{b}^o)$ and $\underline{z}^o$. A modified version of the residual-correction scheme given in (ref. 2) is given below.

Algorithm 1 (Displacement and Stress Sensitivity)

Step (0). Set j=0. Choose the perturbation ε and the error tolerance Δ.

Define $\underline{b}^{\varepsilon}$ as in Eq. (9).

Step (i). Calculate the residual $\underline{r}^j$ from

$$\underline{r}^j = \underline{K}(\underline{b}^{\varepsilon})\ \underline{z}^j - \underline{F}(\underline{b}^{\varepsilon}) \tag{11}$$

Step (ii). Solve for the correction $\underline{e}^j$ from

$$\underline{K}(\underline{b}^o)\ \underline{e}^j = -\underline{r}^j \tag{12}$$

Step (iii). Update

$$\underline{z}^{j+1} = \underline{z}^j + \underline{e}^j \tag{13}$$

Step (iv). Check the convergence criterion

$$||\ \underline{z}^{j+1} - \underline{z}^j||\ \leq \Delta \tag{14}$$

If (14) is satisfied, then set $\underline{z}^{\varepsilon} = \underline{z}^{j+1}$ and compute the displacement sensitivity as

$$d\underline{z}\ /\ db_i = (\underline{z}^{\varepsilon} - \underline{z}^{o})\ /\ \varepsilon \tag{15}$$

The stress sensitivity can be recovered from Eq. (8). If Eq. (14) is not satisfied, set j = j+1 and re-execute steps (i)-(iv) above.

Numerical results and comparison with the exact and semi-analytical methods discussed in the literature are presented subsequently. Theoretically, it can be shown that the above scheme will converge provided [2]:

$$r_{\sigma}\ [\underline{I} - \underline{K}^{-1}(\underline{b}^{o})\underline{K}(\underline{b}^{\varepsilon})] < 1 \tag{16}$$

where $r_{\sigma}(\underline{A})$ = spectral radius of the matrix $\underline{A}$, which is the maximum size of the eigenvalues of $\underline{A}$. In the problem considered here, $\underline{K}(\underline{b}^o)$ and $\underline{K}(\underline{b}^{\varepsilon})$ are roughly equal owing to ε being small, and (16) can generally be expected to hold.

EIGENVALUE AND EIGENVECTOR SENSITIVITY

Eigenvalue sensitivity is useful when resonant frequencies or critical buckling loads need to be restricted. Exact analytical expressions for eigenvalue sensitivity can be readily derived for the case of non-repeated roots [3]. The problem of obtaining eigenvector sensitivity, on the other hand, is more complicated and is an area of current interest [4-7]. Eigenvector sensitivity is useful in obtaining the design derivatives of forced dynamic response. Here, an iterative approach is presented for approximate derivatives of eigenvalues and eigenvectors. The approach is particularly easy to implement in a program and provides both eigenvalue and eigenvector derivatives simultaneously. Further, the derivative of a particular eigenvector does not require knowledge of all eigenvectors of the problem, as with certain analytical methods.

Consider the generalized eigenvalue problem

$$\underline{K}(\underline{b})\underline{y} = \lambda\ \underline{M}(\underline{b})\underline{y} \tag{17}$$

where λ is a particular non-repeated eigenvalue and $\underline{y}$ is the associated eigenvector. For the frequency problem, $\underline{K}$ and $\underline{M}$ represent the structural stiffness and mass matrices, respectively. For the buckling problem, $\underline{K}$ and $\underline{M}$ represent the structural stiffness and geometric stiffness matrices, respectively. It is desired to find the sensitivities $d\lambda/d\underline{b}$ and $d\underline{y}/d\underline{b}$. Let $\underline{b}^o$ be the current design vector and $(\lambda_o, \underline{y}^o)$ be a given eigenvalue-eigenvector pair at the current design. Let $\underline{b}^\varepsilon$ be a perturbed design vector as given in Eq. (9). The residual is given by

$$\underline{R} = \underline{K}(\underline{b}^\varepsilon)\underline{y}^\varepsilon - \lambda_\varepsilon\ \underline{M}(\underline{b}^\varepsilon)\underline{y}^\varepsilon \tag{18}$$

The object is to solve the nonlinear equations $\underline{R} = \underline{0}$ for the unknowns λ_ε and y^ε; the Newton-Raphson technique is used for this purpose. The Jacobian $\underline{J}$ of the system in Eq. (18) is $[\partial\underline{R}/\partial\underline{y}^\varepsilon,\ \partial\underline{R}/\partial\lambda_\varepsilon]$. The Newton-Raphson equations are consequently:

$$[\underline{K}(\underline{b}^\varepsilon)-\lambda_\varepsilon\underline{M}(\underline{b}^\varepsilon) \qquad -\underline{M}(\underline{b}^\varepsilon)\underline{y}\ ] \begin{pmatrix}\delta\underline{y}\\ \delta\lambda\end{pmatrix} = -\underline{R} \tag{19}$$

Note, however, that Eq. (19) represents a system with n equations and (n+1) unknowns; an additional equation is needed. This additional equation is obtained by introducing the normalization condition

$$-\underline{y}^T\ \underline{M}\ \delta\underline{y} = 0 \tag{20}$$

which states that the change in the eigenvector is orthogonal to the original eigenvector with respect to the mass matrix. In fact, the above scheme has been used as a re-analysis approach in (ref. 8). Here, an additional modification is made: the Jacobian matrix in Eq. (19) is modified by replacing $\underline{K}(\underline{b}^\varepsilon)$ by $\underline{K}(\underline{b}^o)$, $\underline{y}^\varepsilon$ by $\underline{y}^o$ and λ_ε by λ_o. The motivation for this, as in the previous section, is tc preserve a constant coefficient matrix in the iterative scheme. The resulting efficiency has not been found to affect the convergence of the procedure owing to the relatively small size of ε. The above modifications lead to an iterative scheme based on solving the system.

$$[\underline{C}] \begin{pmatrix}\delta\underline{y}\\ \delta\lambda\end{pmatrix} = \begin{pmatrix}-\underline{R}\\ 0\end{pmatrix} \tag{21}$$

where

$$\underline{C} = \begin{Bmatrix} \underline{K}(\underline{b}^o)-\lambda_o\underline{M}(\underline{b}^o) & -\underline{M}(\underline{b}^o)\underline{y}^o \\ -\underline{y}^{o^T}\underline{M}(\underline{b}^o) & 0 \end{Bmatrix} \quad (22)$$

The coefficient matrix is symmetric and nonsingular for the case of non-repeated roots [8]. Gaussian elimination can be used to solve Eq. (21). The algorithm for eigenvalue-eigenvector sensitivity is now given.

Algorithm 2 (Eigenvalue-Eigenvector Sensitivity)

Step (0). Set $j=0$. Choose the perturbation ε and the error tolerances Δ_1 and Δ_2.

Define $\underline{b}^\varepsilon$ as in Eq. (9). Decompose the matrix $\underline{C}$ given in Eq. (22).

Step (i). Define the residual

$$\underline{R}^j = \underline{K}(\underline{b}^\varepsilon)\underline{y}^j - \lambda_j\underline{M}(\underline{b}^\varepsilon)\underline{y}^j \quad (23)$$

Step (ii). Solve the algebraic equations

$$[\underline{C}] \begin{pmatrix} \delta\underline{y} \\ \delta\lambda \end{pmatrix} = \begin{pmatrix} -\underline{R}^j \\ 0 \end{pmatrix} \quad (24)$$

for $\delta\underline{y}$ and $\delta\lambda$.

Step (iii). Update

$$\underline{y}^{j+1} = \underline{y}^j + \delta\underline{y}$$

$$\lambda_{j+1} = \lambda_j + \delta\lambda \quad (25)$$

Step (iv). Check the convergence criterion

$$|| \delta y || \leq \Delta_1 , | \delta\lambda| \leq \Delta_2 \quad (26)$$

If (26) is satisfied, then set $\underline{y}^\varepsilon = \underline{y}^{j+1}$, $\lambda_\varepsilon = \lambda_{j+1}$ and compute the sensitivity as

$$d\underline{y}/db_i = (\underline{y}^\varepsilon - \underline{y}^o)/\varepsilon$$

$$d\lambda/db_i = (\lambda_\varepsilon - \lambda_o)/\varepsilon \quad (27)$$

If (26) is not satisfied, set $j = j+1$ and re-execute steps (i)-(iv) above.

Numerical results are presented in the following section.

NUMERICAL RESULTS

Thin plate problem

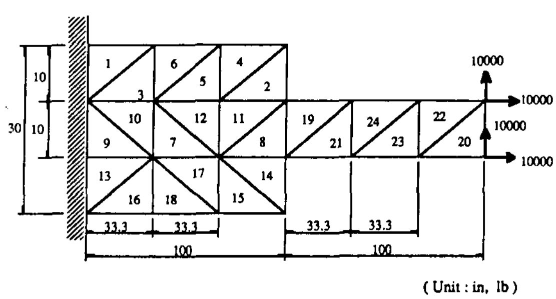
Consider the plane stress problem in Fig. 1, where inverse thicknesses are the design variables. That is, the reciprocal of the plate thickness is chosen as a design variable. Inverse design variables are used in optimal design literature because they linearize the stress function and lead to improved convergence. The stress constraint function is the von Mises failure criterion in element j, given by

$$g_j = \sigma_{VM}/\sigma_a - 1 \tag{28}$$

where $\sigma_{VM}^2 = \sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2$ and σ_a = constant allowable stress limit. Constant strain triangular elements are used. For brevity, only the design sensitivity coefficients, dg_{19}/db_{19} and dg_{24}/db_{24}, are presented in Table 1. The sensitivity vectors have been obtained using Algorithm 1 discussed earlier. In Table 1, the results obtained by the iterative method are compared with the semi-analytical method used widely in the literature, based on the formula in Eq. (8) with $d\underline{z}/db_i$ obtained from

$$\underline{K}\, d\underline{z}/db_i = - \frac{\underline{K}(\underline{b}^{\varepsilon})-\underline{K}(\underline{b}^{o})}{\varepsilon}\, \overset{o}{\underline{z}} + \frac{[\underline{F}(\underline{b}^{\varepsilon})-\underline{F}(\underline{b}^{o})]}{\varepsilon} \tag{29}$$

The results are also compared with the exact sensitivity obtained using analytical derivatives. It is interesting to note that the semi-analytical method yields the same result as the first iteration of the iterative method. However, the iterative method further improves upon this and approaches the exact sensitivity (Table 1). While all methods yield values of acceptable accuracy, the comparison serves to illustrate the nature of the iterative process. This aspect is shown graphically in Fig. 2. It is noted that when using direct variables (as opposed to reciprocal variables), the semi-analytical method yields essentially exact sensitivity owing to the fact that stiffness is a linear function of design variables.



E (Young's modulus) = 30.E6 psi
t (Thickness of element) = 1.4 in
μ (Poisson ratio) = 0.2
σ (Allowable stress) = 20000. psi

Figure 1.

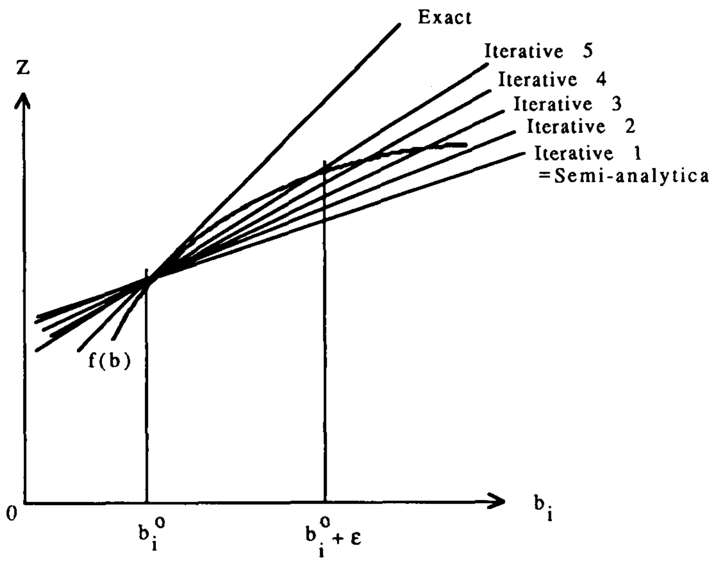


Figure 2.

Table 1.

Method		dg_{19}/db_{19}	dg_{24}/db_{24}
	1	8.7098	5.6437
Iterative	2	8.7949	5.6980
	3	8.7957	5.6986
Semi-analytical		8.7098	5.6437
Exact		8.7969	5.7002

Plane Frame Problem

Consider the frame structure in Fig. 3. The design variables associated with the I-section are $\underline{b} = (h, w, t_w, t_f)$ as shown in Fig. 3. The current design is $\underline{b} = (3.0, 3.0, 0.3, 0.5)$ for each element. The sensitivity of the lowest eigenvalues and corresponding eigenvector obtained using Algorithm 2 given earlier is presented in Tables 2 and 3, respectively. For the eigenvector, only selected sensitivity coefficients are presented for brevity. The maximum number of iterations required for an error tolerance of 10^{-7} is five. Thus, we see that convergence of the algorithm is very rapid and simple to implement. Also, the algorithm does not require computation of all eigenvectors to find the sensitivity of a few specific eigenvectors. However, if the sensitivity of all eigenvectors is required, then alternative approaches may be preferable.

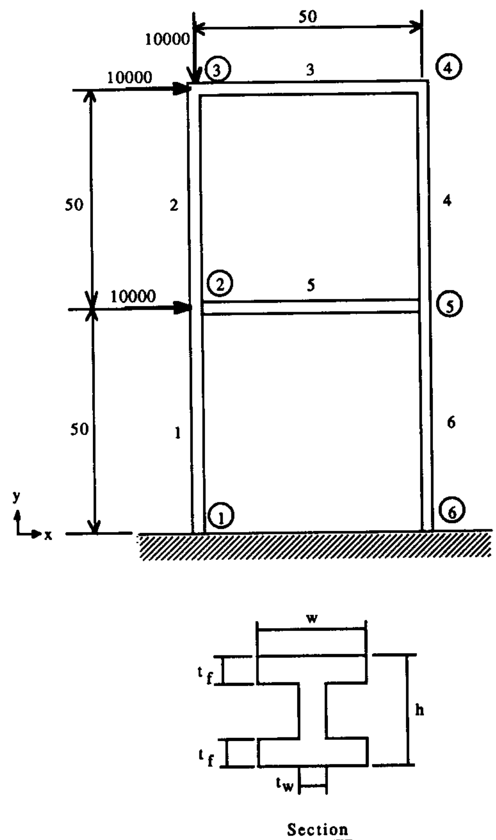


Figure 3.

Table 2.

No. of design variable	Eigenvalue sensitivity
1	4359.6
2	1746.9
3	1418.1
4	10481.0
5	807.7
6	-1077.6
7	-4369.5
8	-6465.5
9	503.2
10	-2058.8
11	-7315.0
12	-12353.0
13	807.7
14	-1077.6
15	-4369.5
16	-6465.5
17	5957.6
18	1964.0
19	540.4
20	11784.0
21	4359.6
22	1746.9
23	1418.1
24	10481.0

Table 3.

No. of degree of freedom	dy/db_5	dy/db_{17}
4	0.036758	0.021438
5	0.000370	0.002413
6	-0.000868	0.003890
7	-0.028564	-0.007628
8	0.000378	0.002547
9	-0.002667	-0.000389
10	-0.028327	-0.007628
11	-0.000653	-0.002547
12	0.001614	-0.000389
13	0.036339	0.021438
14	-0.000407	-0.002413
15	0.000147	0.003890

Fluid Mechanics Problem

The objective of this problem in Fig. 4 is to obtain the sensitivities of the maximum absolute eigenvalue and eigenvector of the amplification matrix G of the incompressible Euler equations in fluid mechanics (ref. 9). This problem is motivated from a study of the stability of the computational algorithm. The Euler equations are

$$[\underline{I} - \Delta t\, \underline{D} + \frac{\varepsilon_i}{4} (1-\cos\theta_x)\, \underline{I} + i \frac{\Delta t}{\Delta x} \underline{A} \sin\theta_x] (\underline{I} - \Delta t\, D)^{-1}$$

$$+[\underline{I} - \Delta t\, \underline{D} + \frac{\varepsilon_i}{4} (1-\cos\theta_y)\, \underline{I} + i \frac{\Delta t}{\Delta y} \underline{B} \sin\theta_y] (\underline{I} - \Delta t\, \underline{D})^{-1}$$

$$+[\underline{I} - \Delta t\, \underline{D} + \frac{\varepsilon_i}{4} (1-\cos\theta_z)\, \underline{I} + i \frac{\Delta t}{\Delta z} \underline{C} \sin\theta_z] (\underline{G} - \underline{I})$$

$$= \Delta t\, \underline{D} - \frac{\varepsilon_e}{2} [(1 - \cos\theta_x)^2 + (1 - \cos\theta_y)^2 + (1 - \cos\theta_z)^2]\, \underline{I}$$

$$- i\, [\frac{\Delta t}{\Delta x} \underline{A} \sin\theta_x + \frac{\Delta t}{\Delta y} \underline{B} \sin\theta_y + \frac{\Delta t}{\Delta z} \underline{C} \sin\theta_z]$$

where, $\underline{I}$ is a (4x4) identity matrix.
The source Jacobian Matrix is

$$[D] = \begin{Bmatrix} 0 & 0 & 0 & 0 \\ 1/r & 0 & 2\omega+2u_y/r & 0 \\ 0 & -2\omega-u_y/r & -u_x/r & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix}$$

and the flux Jacobian matrices are

$$[A] = \begin{Bmatrix} 0 & \beta & 0 & 0 \\ 1 & 2u_x & 0 & 0 \\ 0 & u_y & u_x & 0 \\ 0 & u_z & 0 & u_x \end{Bmatrix}$$

$$[B] = \begin{Bmatrix} 0 & 0 & \beta & 0 \\ 0 & u_y & u_x & 0 \\ 1 & 0 & 2u_y & 0 \\ 0 & 0 & u_z & u_y \end{Bmatrix}$$

$$[C] = \begin{Bmatrix} 0 & 0 & 0 & \beta \\ 0 & u_z & 0 & u_x \\ 0 & 0 & u_z & u_y \\ 1 & 0 & 0 & 2u_z \end{Bmatrix}$$

The time step is

$$\Delta t = \frac{CFL}{\sqrt{\lambda_x^2 + \lambda_y^2 + \lambda_z^2}}$$

The maximum eigenvalue of matrices [A], [B], [C] are

$$\lambda_x = \frac{1}{\Delta x} \left[\left| u_x \right| + \sqrt{u_x^2 + \beta} \right]$$

$$\lambda_y = \frac{1}{\Delta y} \left[\left| u_y \right| + \sqrt{u_y^2 + \beta} \right]$$

$$\lambda_z = \frac{1}{\Delta z} \left[\left| u_z \right| + \sqrt{u_z^2 + \beta} \right]$$

Data

Grid sizes in x, y and z directions are $\Delta x = 1/16$, $\Delta y = \pi/32$, $\Delta z = 1/32$.
Parameter of time-derivative term is $\beta = 1$.
Radius is $r=2$, angular velocity of propeller is $\omega=2$.
Parameter CFL (Courant-Friedrichs-Lewy Number) is = 5.
Fluid velocities in x, y, z directions are $u_x = 0.5$, $u_y = 1$, $u_z = 1$.
The implicit second-order artificial viscosity is $\varepsilon_i = 0$.
The explicit fourth-order artificial viscosity is $\varepsilon_e = 0$.
The lower boundaries of wavenumbers θ_x, θ_y, θ_z are = 0, and the upper boundaries are = π.

Results

The optimum values of wavenumbers θ_x, θ_y, θ_z at which the absolute value of maximum eigenvalue are maximum, are obtained by using the optimization program LINRM [10]. The results are $\theta_x = \theta_y = \theta_z = \pi/2$. All sensitivity calculations are now done at these values of θ_x, θ_y and θ_z. The sensitivity of the maximum eigenvalue, absolute maximum eigenvalue and corresponding eigenvector are shown in Tables 4 and 5.

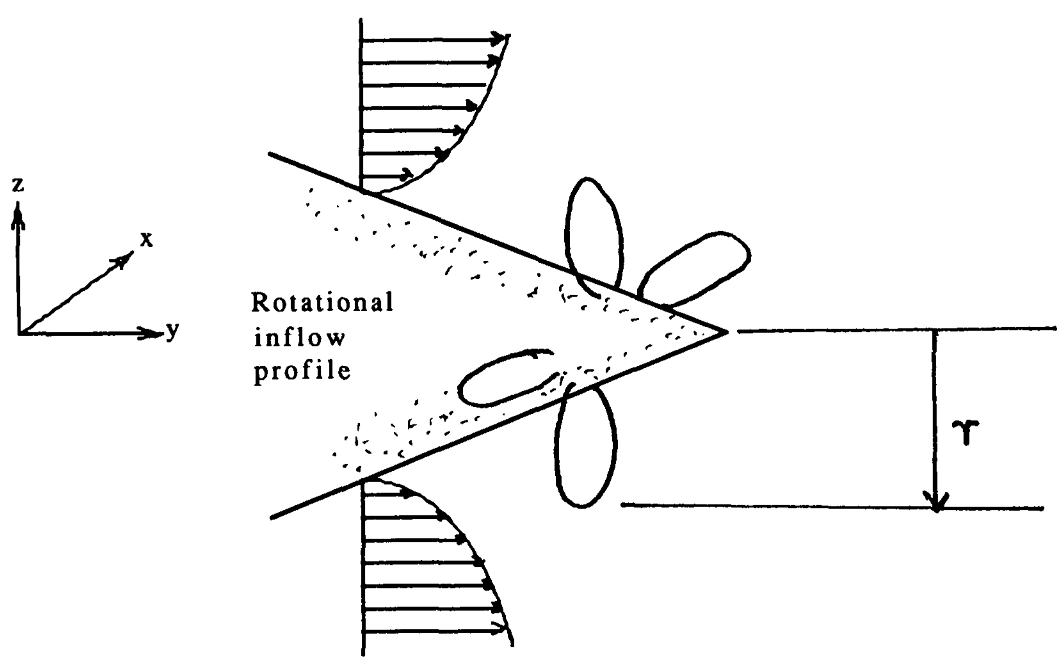


Figure 4.

Table 4. Eigenvalue Sensitivity for Fluid Mechanics Problem

$\Delta = 1 \times 10^{-8}$, $\varepsilon = 0.01$

Design Variable	Eigenvalue Sensitivity	
	$d\lambda/db$	$d\lvert\lambda\rvert/db$
u_x	-0.03928 -0.11272 i	-0.08543
u_y	0.25361 -0.02037 i	0.21790
u_z	-0.23827 -0.36236 i	-0.37489
CFL	0.13254 +0.06004 i	0.14541
ε_i	-0.12951 -0.29317 i	-0.24668
ε_e	-0.84451 -0.46701 i	-0.54753
ω	-0.01807 +0.02823 i	-0.00358
θ_x	-0.00138 -0.00002 i	-0.00125
θ_y	-0.00182 -0.00010 i	-0.00167
θ_z	-0.00204 -0.00189 i	-0.00267

$b^o = (0.5, 1, 1, 5, 0, 0, 2, \pi/2, \pi/2, \pi/2)$

$\underline{\lambda}^o = 0.94882 + 0.47287\, i$

$|\underline{\lambda}^o| = 1.0601$

Table 5. Eigenvector Sensitivity for Fluid Mechanics Problem

$\Delta = 1 \times 10^{-8}$, $\varepsilon = 0.01$

Degree of Freedom	dy/d CFL	$dy/d\omega$
1	-0.22740 -0.27610 i	0.03483 +0.01911 i
2	-0.34796 -0.03808 i	0.00969 -0.00969 i
3	-0.20260 -0.10825 i	0.07636 +0.00170 i
4	-0.19075 -0.10196 i	0.03520 -0.01693 i

SUMMARY AND CONCLUSIONS

A numerical method has been presented for design sensitivity analysis. The idea is based on using iterative methods for re-analysis of the structure due to a small perturbation in the design variable. A forward difference scheme then yields the approximate sensitivity. Algorithms for displacement and stress sensitivity as well as for eigenvalues and eigenvector sensitivity are developed. The iterative schemes have been modified so that the coefficient matrices are constant and hence decomposed only once. The convergence is found to be very rapid. Further, implementation of the algorithms is simple.

REFERENCES

1. Adelman, H.M., Haftka, R.T., Camarda, C.J. and Walsh, J.L.: Structural Sensitivity Analysis: Methods, Applications and Needs. Proc. Symposium on Recent Experiences in Multidisciplinary Analysis and Optimization, NASA Langley Research Center, Hampton, VA, April 1984, pp. 367-384.

2. Atkinson, K.E.: An Introduction to Numerical Analysis. John Wiley, New York, 1978.

3. Haug, E.J. and Arora, J.S.: Applied Optimal Design. John Wiley, New York, 1979.

4. Reiss, R.: Design Derivatives of Eigenvalues and Eigenfunctions for Self-Adjoint Distributed Parameter Systems. AIAA Journal, Vol. 24, No. 7, July 1986, pp. 1169-1172.

5. Hou, J.W. and Yuan, J.Z.: Calculation of Eigenvalue and Eigenvector Derivatives for Nonlinear Beam Vibrations. Submitted to AIAA Journal, Feb. 1986.

6. Ojalvo, I.U.: Efficient Computation of Mode-Shape Derivatives for Large Dynamic Systems. AIAA Journal, Vol. 25, No. 10, 1987, pp. 1386-1390.

7. Cardani, C. and Mantegazza, P., "Calculation of Eigenvalue and Eigenvector Derivatives for Algebraic Flutter and Divergence Eigenproblems. AIAA Journal, Vol. 17, No. 4, April 1979, pp. 408-412.

8. Robinson, A.R. and Harris, J.F.: Improving Approximate Eigenvalues and Eigenvectors. Journal of the Engineering Mechanics Division, Vol. 97, No. EM2, April 1971, pp. 457-475.

9. Merkle, C.L., Tsai, Y.P. and Huang, T.T.: Euler Equation Analysis of Rotational Flow Through Propellers. Presented at the 17th Symposium of Naval Hydrodynamics, Belgium, 1988.

10. Belegundu, A.D. and Arora, J.S.: A Recursive Quadratic Programming Method with Active Set Strategy for Optimal Design. Int'l. Journal for Numerical Methods in Engineering, Vol. 20, 1984, pp. 803-816.