NASA
Technical
Paper
2931

AVSCOM
Technical
Report
89-B-002

August 1989

Derivation of a Tapered p-Version Beam Finite Element

Howard E. Hinnant

US ARMY
AVIATION
SYSTEMS COMMAND
AVIATION R&T ACTIVITY

NASA

NASA
Technical
Paper
2931

AVSCOM
Technical
Report
89-B-002

1989

Derivation of a Tapered p-Version Beam Finite Element

Howard E. Hinnant
Aerostructures Directorate
USAARTA-AVSCOM
Langley Research Center
Hampton, Virginia

National Aeronautics and
Space Administration
Office of Management
Scientific and Technical
Information Division

Introduction

Finite-element models are used extensively by the rotorcraft industry for determining the vibratory characteristics of helicopter airframes. These airframe models are typically large, usually containing several thousand finite elements, and their definition involves considerable effort. Results obtained by using these models usually do not correlate well with experimental data, particularly at the higher frequencies that are near the predominant rotor excitation frequencies (refs. 1 through 4). Although there are several possible reasons for the lack of correlation, one that has recently attracted attention is the accuracy of the elements themselves.

Most finite-element codes in use today employ what are commonly referred to as h-version elements. An h-version element uses fixed-order shape functions to relate the discrete nodal displacements of the element to the continuous displacements within the element. Mesh size normally controls the accuracy of the results obtained using h-version elements. As the number of elements is increased, the mesh size decreases and the results tend to converge. Tests for convergence usually involve solving a series of problems with successively refined meshes until convergence is indicated. Such convergence checks are often used in studies which employ small models. To check the finite-element model of a large structure (such as an airframe) for convergence by mesh refinement would be a formidable task not likely to be done in practice. So-called p-version elements that have recently gained attention have the potential for both improving accuracy *and* simplifying convergence checks.

The p-version element is different from the h-version element in that the user may select the order of the shape function defining the displacement behavior of the element. In general, the higher the order of the shape function, the more accurate the results. Increasing the order of the shape functions is analogous to increasing the number of elements for a comparable h-version model. This feature becomes a distinct advantage in convergence checks of large finite-element models because raising the order of the shape functions would be much easier than increasing the number of elements in the data input file.

The advantage of using p-version elements is lost if the element does not closely approximate the geometry of the structure being modeled. For example, a uniform p-version beam element would pose no advantage over a uniform h-version element in either accuracy or ease of use in modeling a tapered beam. However, a p-version beam element whose cross-sectional dimensions vary linearly with length would allow for a more accurate representation of a tapered beam than could be achieved with a uniform beam element. Because many substructures in a typical airframe consist of beams whose cross sections vary along their length, the use of p-version elements offers an advantage over h-version elements only if the p-version elements can accurately model the physical structure without resorting to excessive mesh refinement.

The Langley Research Center has underway a research structural dynamics program which is investigating finite-element modeling techniques for helicopter vibrations analyses. This activity has identified the need for a small research code containing a limited library of p-version beam and shell elements. To meet the requirements of a research tool, these elements must be well documented, easily modified, and tailored to the needs of airframe vibrations analyses. Work has been initiated at Langley to develop such a code. Initial attention is being given to the beam element. A survey of the finite-element literature revealed no commercially available codes with p-version beam elements and only a few research codes having such elements. None of the latter codes appear to be suitable for airframe vibrations analyses work. However, there has been significant work done in the area of tapered h-version beam elements which is relevant to the beam element that is the subject of this paper. Pertinent work in this area is summarized.

Tapered beam finite elements based on cubic polynomial shape functions have been under investigation for many years (refs. 5 and 6). Higher order, tapered beam elements have also been developed. Thomas and Dokumaci (ref. 7) presented two tapered beam elements using quintic polynomials. One beam element had two internal nodes, whereas the other beam element defined the second derivative at each end of the beam as a degree of freedom. Interelement continuity was then enforced for displacement and the first two derivatives, which formed a C^2-type element. A tapered beam element using seventh-degree polynomial shape functions was derived by To in reference 8. Each node had four degrees of freedom (displacement and the first three derivatives) on which interelement continuity was imposed. It should be pointed out that these higher order elements are *not* p-version elements because the order of the shape functions was fixed when the element was derived. In another paper, To (ref. 9) developed a tapered beam element incorporating shear deformation and rotary inertia but based the element formulation on cubic shape functions. More recently, a tapered p-version beam element was developed by Hodges, Hopkins, Kunz, and Hinnant (ref. 10) specifically for modeling rotor blades. To correctly model a spinning rotor blade, the element includes the nonlinear effects of large nodal displacements and rotations, aerodynamics, and inertial rotation. Such a complex element is not needed for modeling airframe structures. As a practical matter, elements used for the analysis of airframes should be relatively simple because of the large number of elements normally required to model the structure adequately.

The objective of this paper is to present the derivation of a tapered, p-version beam element for use in dynamic analyses of general structural systems. This element features hierarchical shape functions which allow higher order analyses to use element matrices established for lower order analyses. Appropriate orthogonal relations for the shape functions are employed to avoid ill-conditioned matrices and reduce the number of nonzero terms. Element matrices are explicitly formed, thus eliminating the need for numerical quadrature resulting in a simpler implementation and a reduction in roundoff error.

Symbols

A	cross-sectional area
a_i	coefficients of polynomial representing cross-sectional property
E	modulus of elasticity
f_{ui},f_{vi},$f_{\theta i}$	ith axial, lateral bending, and torsional frequency, respectively, Hz
G	shear modulus
Idof	number of internal degrees of freedom
I_{XX},I_{YY},I_{ZZ}	area moment of inertia about X-, Y-, and Z-axis, respectively
J	torsional stiffness constant
$\mathbf{K}$	stiffness matrix
$\mathbf{K}_{EE}$	external stiffness submatrix
$\mathbf{K}^u_{EI}$,$\mathbf{K}^v_{EI}$,$\mathbf{K}^w_{EI}$,$\mathbf{K}^\theta_{EI}$	external-internal stiffness coupling submatrice associated with u, v, w, and θ, respectively
$\mathbf{K}^u_{II}$,$\mathbf{K}^v_{II}$,$\mathbf{K}^w_{II}$,$\mathbf{K}^\theta_{II}$	internal stiffness submatrice associated with u, v, w, and θ, respectively
l	length of beam
$\mathbf{M}$	mass matrix
$\mathbf{M}_{EE}$	external mass submatrix
$\mathbf{M}^u_{EI}$,$\mathbf{M}^v_{EI}$,$\mathbf{M}^w_{EI}$,$\mathbf{M}^\theta_{EI}$	external-internal mass coupling submatrice associated with u, v, w, and θ, respectively
$\mathbf{M}^u_{II}$, $\mathbf{M}^v_{II}$, $\mathbf{M}^w_{II}$, $\mathbf{M}^\theta_{II}$	internal mass submatrice associated with u, v, w, and θ, respectively
Ndof	total number of degrees of freedom
Nelem	number of elements
N^u_i, N^v_i, N^w_i, N^θ_i	ith shape function associated with u, v, w, and θ, respectively

N_i^0	ith shape function of C^0 type
N_i^1	ith shape function of C^1 type
P_u, P_v, P_w, P_θ	number of shape function associated with u, v, w, and θ, respectively
$P_2(x)$	quadratic polynomial representing generic cross-sectional property
$P_4(x)$	quartic polynomial representing generic cross-sectional property
$q_i^u, q_i^v, q_i^w, q_i^\theta$	ith discrete degree of freedom associated with u, v, w, and θ, respectively
$S_{i,j}^0$	matrix function defined by $\int_0^l P_4(x) N_i^0 N_j^0 \, dx$
$S_{i,j}^{0P}$	matrix function defined by $\int_0^l P_4(x) (N_i^0)' (N_j^0)' \, dx$
$S_{i,j}^1$	matrix function defined by $\int_0^l P_2(x) N_i^1 N_j^1 \, dx$
$S_{i,j}^{1P}$	matrix function defined by $\int_0^l P_4(x) (N_i^1)' (N_j^1)' \, dx$
$S_{i,j}^{1PP}$	matrix function defined by $\int_0^l P_4(x) (N_i^1)'' (N_j^1)'' \, dx$
T	kinetic energy
t	time, sec
V	strain energy
u, v, w	displacement along X-, Y-, and Z-axis, respectively
X, Y, Z	rectangular axis system with origin at one end of beam; X-axis along neutral axis of beam, and Y- and Z-axes oriented parallel to principal axes of beam
x, y, z	independent variable along X-, Y-, and Z-axis, respectively
α_i^0, β_i^0, γ_i^0	coefficients in generating equation for N_i^0
α_i^1, β_i^1, γ_i^1	coefficients in generating equation for N_i^1
$\delta(\ )$	first variation of ( )
$\delta_{i,j}$	$\begin{cases} 0 & (i \neq j) \\ 1 & (i = j) \end{cases}$
$\lambda(i)$	$\begin{cases} -1 & (i = 2 \text{ or } 4) \\ 1 & (\text{Otherwise}) \end{cases}$
θ	rotation about X-axis (i.e., torsion)
ρ	mass density
$\omega_{ui}, \omega_{vi}, \omega_{\theta i}$	ith axial, lateral bending, and torsional frequency, respectively, rad/sec

Primes to a symbol denote a derivative with respect to x. A dot over a symbol denotes a derivative with respect to time.

Basic Mathematical Formulation

The beam finite element is derived assuming that the beam behaves kinematically like a Bernoulli-Euler beam. Rotary inertia effects are included; however, shear flexibility is not included in the present formulation. The cross-sectional area can vary as a quadratic polynomial along the length of the beam, and the area moment of inertia can vary as a quartic polynomial. The element axis system X, Y, Z is oriented so that the X-axis is along the neutral axis of the undeformed beam, and the Y- and Z-axes are parallel to the principal axes of the cross section. (See fig. 1.) This orientation results in a zero cross product of inertia ($I_{YZ} = 0$). The continuous displacements u, v, w, and θ are assumed to be only a function of x and time.

Hamilton's principle is used in the derivation of this beam element. Assuming only conservative forces, Hamilton's principle is stated as (ref. 11)

$$\int_{t_1}^{t_2} \delta(T - V)\, dt = 0 \tag{1}$$

where t_1 and t_2 represent arbitrary times at which the state of the system is known, and T and V are the kinetic and strain energies, respectively. The kinetic energy of the beam is found in reference 12 as

$$T = \frac{1}{2} \int_0^l \rho \left[A\dot{u}^2 + A\dot{v}^2 + A\dot{w}^2 + I_{XX}\dot{\theta}^2 + I_{ZZ}(\dot{v}')^2 + I_{YY}(\dot{w}')^2\right] dx \tag{2}$$

and the strain energy follows similarly from reference 13 as

$$V = \frac{1}{2} \int_0^l \left[EA(u')^2 + EI_{ZZ}(v'')^2 + EI_{YY}(w'')^2 + GJ(\theta')^2\right] dx \tag{3}$$

The continuous problem associated with the continuous displacements u, v, w, and θ is discretized by introducing discrete degrees of freedom q_i which are related to the continuous displacements according to

$$u = \sum_{i=1}^{P_u} N_i^u q_i^u \tag{4}$$

$$v = \sum_{i=1}^{P_v} N_i^v q_i^v \tag{5}$$

$$w = \sum_{i=1}^{P_w} N_i^w q_i^w \tag{6}$$

$$\theta = \sum_{i=1}^{P_\theta} N_i^\theta q_i^\theta \tag{7}$$

Substituting equations (4) through (7) into equations (2) and (3) expresses the kinetic and strain energies in terms of discrete degrees of freedom. The application of Hamilton's principle (eq. (1)) then leads to the identification of the following nonzero terms which appear in the element mass and stiffness matrices:

$$M_{i,j}^u = \int_0^l \rho A N_i^u N_j^u\, dx \tag{8}$$

$$M_{i,j}^v = \int_0^l \rho \left[A N_i^v N_j^v + I_{ZZ}(N_i^v)'(N_j^v)'\right] dx \tag{9}$$

$$M_{i,j}^w = \int_0^l \rho \left[A N_i^w N_j^w + I_{YY}(N_i^w)'(N_j^w)'\right] dx \tag{10}$$

$$M_{i,j}^{\theta} = \int_0^l \rho I_{XX} N_i^{\theta} N_j^{\theta} \, dx \tag{11}$$

$$K_{i,j}^{u} = \int_0^l EA (N_i^{u})' (N_j^{u})' \, dx \tag{12}$$

$$K_{i,j}^{v} = \int_0^l EI_{ZZ} (N_i^{v})'' (N_j^{v})'' \, dx \tag{13}$$

$$K_{i,j}^{w} = \int_0^l EI_{YY} (N_i^{w})'' (N_j^{w})'' \, dx \tag{14}$$

$$K_{i,j}^{\theta} = \int_0^l GJ (N_i^{\theta})' (N_j^{\theta})' \, dx \tag{15}$$

Shape Functions

The discrete degrees of freedom are divided into two sets, external and internal. The 12 external degrees of freedom, which are depicted in figure 2, correspond to the usual definition of the physical nodal degrees of freedom for a beam finite element (ref. 14). The internal degrees of freedom have no physical significance but are simply the coefficients of the higher order shape functions. These internal degrees of freedom serve to increase the accuracy of the transformation from the discrete problem having a finite number of degrees of freedom to the continuous problem having an infinite number of degrees of freedom. The number of internal degrees of freedom used in the beam element is specified by the data input file and may vary from zero to theoretically infinite. Setting $P_u = P_\theta = 2$ and $P_v = P_w = 4$ in equations (4) through (7) will lead to the classical 12-degree-of-freedom beam element. Increasing any of these P's adds internal degrees of freedom to the element. The shape functions N_i are usually taken to be polynomials although in theory they can be any set of functions.

Shape Functions for u and θ

The shape functions N_i^u and N_i^θ are identical and have C^0-type continuity. That is, continuity is enforced across element boundaries, but continuity of the derivatives across element boundaries is not enforced. Shape functions satisfying C^0 continuity will be denoted by N_i^0 (i.e., $N_i^u = N_i^\theta = N_i^0$). The first two shape functions in this set are the well-known linear polynomials (ref. 15)

$$N_1^0 = -\frac{x}{l} + 1 \tag{16}$$

$$N_2^0 = \frac{x}{l} \tag{17}$$

The higher order C^0-type shape functions used herein were derived subject to two requirements: First, the C^0 continuity is enforced by restricting the higher order shape functions to be zero at the element boundaries, and second, the set of higher order polynomials must be orthogonal with respect to their first derivative. Orthogonality of the first derivative was chosen over polynomial orthogonality because the element mass and stiffness matrices obtained by requiring first-derivative orthogonality contain fewer nonzero terms; thus explicit integration is facilitated. This also results in matrices which are better conditioned than those obtained from orthogonal shape functions. These requirements are expressed mathematically by the following equations:

$$N_i^0(0) = 0 \qquad (i \geq 3) \tag{18}$$

$$N_i^0(l) = 0 \qquad (i \geq 3) \tag{19}$$

$$\int_0^l (N_i^0)'(N_j^0)' \, dx = \frac{1}{l}\delta_{i,j} \qquad (i \geq 3, j \geq 3) \tag{20}$$

The first three higher order C^0-type shape functions in the set defined by equations (18) through (20) are

$$N_3^0 = \sqrt{3}\left(\frac{x^2}{l^2} - \frac{x}{l}\right) \tag{21}$$

$$N_4^0 = \sqrt{5}\left(-2\frac{x^3}{l^3} + 3\frac{x^2}{l^2} - \frac{x}{l}\right) \tag{22}$$

$$N_5^0 = \sqrt{7}\left(5\frac{x^4}{l^4} - 10\frac{x^3}{l^3} + 6\frac{x^2}{l^2} - \frac{x}{l}\right) \tag{23}$$

In general, once N_3^0 is known, the $i+1$ shape function can be found from the recursive formula

$$N_{i+1}^0 = (\alpha_i^0 + x\beta_i^0)N_i^0 - \gamma_i^0 N_{i-1}^0 \qquad (i \geq 3) \tag{24}$$

where

$$\alpha_i^0 = \frac{(2i-3)\sqrt{2i-1}}{i\sqrt{2i-3}} \qquad (i \geq 3) \tag{25}$$

$$\beta_i^0 = -\frac{(4i-6)\sqrt{2i-1}}{il\sqrt{2i-3}} \qquad (i \geq 3) \tag{26}$$

$$\gamma_i^0 = \frac{(i-3)\sqrt{2i-1}}{i\sqrt{2i-5}} \qquad (i \geq 3) \tag{27}$$

Shape Functions for v and w

The shape functions N_i^v and N_i^w require C^1-type continuity, meaning that both the functions and their first derivatives must be continuous across element boundaries. The shape functions N_i^v and N_i^w are identical except that $N_2^w = -N_2^v$ and $N_4^w = -N_4^v$ to ensure that the discrete rotational degrees of freedom q_5 and q_{11} have the sense indicated in figure 2. Shape functions satisfying C^1 continuity will be denoted by N_i^1 (i.e., $N_i^v = \lambda(i)N_i^w = N_i^1$). The first four C^1-type functions for the beam element are (ref. 15)

$$N_1^1 = 2\frac{x^3}{l^3} - 3\frac{x^2}{l^2} + 1 \tag{28}$$

$$N_2^1 = \frac{x^3}{l^2} - 2\frac{x^2}{l} + x \tag{29}$$

$$N_3^1 = -2\frac{x^3}{l^3} + 3\frac{x^2}{l^2} \tag{30}$$

$$N_4^1 = \frac{x^3}{l^2} - \frac{x^2}{l} \tag{31}$$

The derivation of the higher order C^1-type shape functions employed in this paper is based on the same philosophy as the derivation of the C^0-type shape functions. To ensure C^1 continuity the higher order shape functions must have zero slope and displacement at the element boundaries. Consistent with the previous discussion on the orthogonality properties of the C^0-type shape functions, the C^1-type shape functions are required to be orthogonal in their second derivative. The specific requirements are

$$N_i^1(0) = 0 \qquad (i \geq 5) \tag{32}$$

$$(N_i^1)'(0) = 0 \qquad (i \geq 5) \tag{33}$$

$$N_i^1(l) = 0 \qquad (i \geq 5) \tag{34}$$

$$(N_i^1)'(l) = 0 \qquad (i \geq 5) \tag{35}$$

$$\int_0^l (N_i^1)''(N_j^1)''\,dx = \frac{1}{l^3}\delta_{i,j} \qquad (i \geq 5, j \geq 5) \tag{36}$$

The first three higher order C^1-type shape functions in the set defined by equations (32) through (36) are

$$N_5^1 = \sqrt{5}\left(\frac{x^4}{2l^4} - \frac{x^3}{l^3} + \frac{x^2}{2l^2}\right) \tag{37}$$

$$N_6^1 = \sqrt{7}\left(-\frac{x^5}{l^5} + \frac{5x^4}{2l^4} - 2\frac{x^3}{l^3} + \frac{x^2}{2l^2}\right) \tag{38}$$

$$N_7^1 = \sqrt{9}\left(\frac{7x^6}{3l^6} - 7\frac{x^5}{l^5} + \frac{15x^4}{2l^4} - \frac{10x^3}{3l^3} + \frac{x^2}{2l^2}\right) \tag{39}$$

If N_5^1 is given, N_i^1 can be found from the recursive relation

$$N_{i+1}^1 = (\alpha_i^1 + x\beta_i^1)N_i^1 - \gamma_i^1 N_{i-1}^1 \qquad (i \geq 5) \tag{40}$$

where

$$\alpha_i^1 = \frac{(2i-5)\sqrt{2i-3}}{i\sqrt{2i-5}} \qquad (i \geq 5) \tag{41}$$

$$\beta_i^1 = -\frac{(4i-10)\sqrt{2i-3}}{il\sqrt{2i-5}} \qquad (i \geq 5) \tag{42}$$

$$\gamma_i^1 = \frac{(i-5)\sqrt{2i-3}}{i\sqrt{2i-7}} \qquad (i \geq 5) \tag{43}$$

Matrix Functions

Inspection of equations (2) and (3) indicates that there are eight cross-sectional properties associated with a beam element. Two of them, ρA and EA, are represented by the quadratic polynomial

$$P_2(x) = a_2x^2 + a_1x + a_0 \tag{44}$$

The remaining six properties, ρI_{XX}, ρI_{YY}, ρI_{ZZ}, GJ, EI_{YY}, and EI_{ZZ}, are represented by the quartic polynomial

$$P_4(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{45}$$

Because these polynomials can represent any of the aforementioned cross-sectional properties by substituting in the appropriate values for the coefficients a_i, equations (44) and (45) are referred to herein as "generic cross-sectional property polynomials." Note that $P_2(x)$ is a subset of $P_4(x)$, and, therefore, the coefficients a_0 through a_2 refer to both polynomials. The context in which the coefficients are used will clearly specify which generic cross-sectional property polynomial is being referenced.

Substituting the generic cross-sectional property polynomials for the actual cross-sectional properties in equations (8) through (15) and substituting generic N^0 or N^1 shape functions for the displacement-specific shape functions, the 10 terms in equations (8) through (15) are reduced to 5 unique terms. These five unique terms are referred to herein as "matrix functions," and have the following definitions:

$$S^0_{i,j} = \int_0^l P_4(x) N^0_i N^0_j \, dx \tag{46}$$

$$S^{0P}_{i,j} = \int_0^l P_4(x) (N^0_i)' (N^0_j)' \, dx \tag{47}$$

$$S^1_{i,j} = \int_0^l P_2(x) N^1_i N^1_j \, dx \tag{48}$$

$$S^{1P}_{i,j} = \int_0^l P_4(x) (N^1_i)' (N^1_j)' \, dx \tag{49}$$

$$S^{1PP}_{i,j} = \int_0^l P_4(x) (N^1_i)'' (N^1_j)'' \, dx \tag{50}$$

The matrix functions are functions of a cross-sectional property which is shown as an argument in parentheses when appropriate, for example, $S^0_{i,j}(\rho A)$. The expressions for the nonzero terms in the mass and stiffness matrices given by equations (8) through (15) can now be expressed in terms of the matrix functions as follows:

$$M^u_{i,j} = S^0_{i,j}(\rho A) \tag{51}$$

$$M^v_{i,j} = S^1_{i,j}(\rho A) + S^{1P}_{i,j}(\rho I_{ZZ}) \tag{52}$$

$$M^w_{i,j} = \lambda(i)\lambda(j) \left[S^1_{i,j}(\rho A) + S^{1P}_{i,j}(\rho I_{YY}) \right] \tag{53}$$

$$M^\theta_{i,j} = S^0_{i,j}(\rho I_{XX}) \tag{54}$$

$$K^u_{i,j} = S^{0P}_{i,j}(EA) \tag{55}$$

$$K^v_{i,j} = S^{1PP}_{i,j}(EI_{ZZ}) \tag{56}$$

$$K^w_{i,j} = \lambda(i)\lambda(j) S^{1PP}_{i,j}(EI_{YY}) \tag{57}$$

$$K^\theta_{i,j} = S^{0P}_{i,j}(GJ) \tag{58}$$

where

$$\lambda(i) = \left\{ \begin{array}{ll} -1 & (i = 2 \text{ or } 4) \\ 1 & (\text{Otherwise}) \end{array} \right\} \tag{59}$$

Explicit expressions for the five sets of matrix functions for $i = 1, \infty$ and $j = i, \infty$ are given in appendix A. Expressions for $j = 1, i-1$ are not given because the matrix functions are symmetric (i.e., $S_{i,j} = S_{j,i}$).

Element Mass and Stiffness Matrices

The nonzero mass and stiffness terms given by equations (51) through (54) and equations (55) through (58), respectively, must be appropriately assembled to form the element mass and stiffness matrices. This procedure depends on the arrangement of the discrete degrees of freedom in the element vector of unknowns. The first 12 degrees of freedom in the vector of unknowns are the external degrees of freedom associated with the classical beam element. The higher order (internal) degrees of freedom are positioned after the external degrees of freedom in the vector of unknowns. Specifically, all the internal q^u's are grouped together, then all the internal q^v's are grouped together, and so on. The final arrangement of unknowns is given as follows:

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \\ q_{10} \\ q_{11} \\ q_{12} \\ q_{13} \\ \vdots \\ q_{12+P_u-2} \\ q_{13+P_u-2} \\ \vdots \\ q_{12+P_u-2+P_v-4} \\ q_{13+P_u-2+P_v-4} \\ \vdots \\ q_{12+P_u-2+P_v-4+P_w-4} \\ q_{13+P_u-2+P_v-4+P_w-4} \\ \vdots \\ q_{12+P_u-2+P_v-4+P_w-4+P_\theta-2} \end{Bmatrix} = \begin{Bmatrix} u(0) \\ v(0) \\ w(0) \\ \theta(0) \\ -w'(0) \\ v'(0) \\ u(l) \\ v(l) \\ w(l) \\ \theta(l) \\ -w'(l) \\ v'(l) \\ q_3^u \\ \vdots \\ q_{P_u}^u \\ q_5^v \\ \vdots \\ q_{P_v}^v \\ q_5^w \\ \vdots \\ q_{P_w}^w \\ q_3^\theta \\ \vdots \\ q_{P_\theta}^\theta \end{Bmatrix} \tag{60}$$

Mass Matrix

The element mass matrix is partitioned into several submatrices consistent with the arrangement of the degrees of freedom in the vector of unknowns given in equation (60). This leads to a matrix of the form

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{EE} & \mathbf{M}_{EI}^u & \mathbf{M}_{EI}^v & \mathbf{M}_{EI}^w & \mathbf{M}_{EI}^\theta \\ & \mathbf{M}_{II}^u & 0 & 0 & 0 \\ & & \mathbf{M}_{II}^v & 0 & 0 \\ & sym & & \mathbf{M}_{II}^w & 0 \\ & & & & \mathbf{M}_{II}^\theta \end{bmatrix} \tag{61}$$

The submatrix $\mathbf{M}_{EE}$ is termed the external submatrix because it is associated only with the external degrees of freedom. This submatrix, which is given by the following equation, is a 12 by 12 matrix and reduces to the classical consistent mass matrix for constant cross-sectional properties:

$\mathbf{M}_{EE} =$

$$
\begin{bmatrix}
S^0_{1,1}(\rho A) & 0 & 0 & 0 & 0 & 0 & S^0_{1,2}(\rho A) & 0 & 0 & 0 & 0 & 0 \\
 & \substack{S^1_{1,1}(\rho A)\\+S^{1P}_{1,1}(\rho I_{ZZ})} & 0 & 0 & 0 & \substack{S^1_{1,2}(\rho A)\\+S^{1P}_{1,2}(\rho I_{ZZ})} & 0 & \substack{S^1_{1,3}(\rho A)\\+S^{1P}_{1,3}(\rho I_{ZZ})} & 0 & 0 & 0 & \substack{S^1_{1,4}(\rho A)\\+S^{1P}_{1,4}(\rho I_{ZZ})} \\
 & & \substack{S^1_{1,1}(\rho A)\\+S^{1P}_{1,1}(\rho I_{YY})} & 0 & \substack{-S^1_{1,2}(\rho A)\\-S^{1P}_{1,2}(\rho I_{YY})} & 0 & 0 & 0 & \substack{S^1_{1,3}(\rho A)\\+S^{1P}_{1,3}(\rho I_{YY})} & 0 & \substack{-S^1_{1,4}(\rho A)\\-S^{1P}_{1,4}(\rho I_{YY})} & 0 \\
 & & & S^0_{1,1}(\rho I_{XX}) & 0 & 0 & 0 & 0 & 0 & S^0_{1,2}(\rho I_{XX}) & 0 & 0 \\
 & & & & \substack{S^1_{2,2}(\rho A)\\+S^{1P}_{2,2}(\rho I_{YY})} & 0 & 0 & 0 & \substack{-S^1_{2,3}(\rho A)\\-S^{1P}_{2,3}(\rho I_{YY})} & 0 & \substack{S^1_{2,4}(\rho A)\\+S^{1P}_{2,4}(\rho I_{YY})} & 0 \\
 & & & & & \substack{S^1_{2,2}(\rho A)\\+S^{1P}_{2,2}(\rho I_{ZZ})} & 0 & \substack{S^1_{2,3}(\rho A)\\+S^{1P}_{2,3}(\rho I_{ZZ})} & 0 & 0 & 0 & \substack{S^1_{2,4}(\rho A)\\+S^{1P}_{2,4}(\rho I_{ZZ})} \\
 & & & & & & S^0_{2,2}(\rho A) & 0 & 0 & 0 & 0 & 0 \\
 & & \mathit{sym} & & & & & \substack{S^1_{3,3}(\rho A)\\+S^{1P}_{3,3}(\rho I_{ZZ})} & 0 & 0 & 0 & \substack{S^1_{3,4}(\rho A)\\+S^{1P}_{3,4}(\rho I_{ZZ})} \\
 & & & & & & & & \substack{S^1_{3,3}(\rho A)\\+S^{1P}_{3,3}(\rho I_{YY})} & 0 & \substack{-S^1_{3,4}(\rho A)\\-S^{1P}_{3,4}(\rho I_{YY})} & 0 \\
 & & & & & & & & & S^0_{2,2}(\rho I_{XX}) & 0 & 0 \\
 & & & & & & & & & & \substack{S^1_{4,4}(\rho A)\\+S^{1P}_{4,4}(\rho I_{YY})} & 0 \\
 & & & & & & & & & & & \substack{S^1_{4,4}(\rho A)\\+S^{1P}_{4,4}(\rho I_{ZZ})}
\end{bmatrix}
\tag{62}
$$

The submatrices $\mathbf{M}^u_{EI}$, $\mathbf{M}^v_{EI}$, $\mathbf{M}^w_{EI}$, and $\mathbf{M}^\theta_{EI}$ couple the external and internal degrees of freedom in the element mass matrix. These submatrices, which are shown in equations (63) through (66), are variable sized arrays of an order equal to 12 by the number of internal degrees of freedom corresponding to the particular continuous variable. For example, let P_u represent the number of discrete degrees of freedom associated with u. Then, because there are always two external discrete degrees of freedom associated with u, the dimensions of $\mathbf{M}^u_{EI}$ are 12 by $P_u - 2$. It is of interest to note that the coupling terms associated with the higher order internal degrees of freedom become zero if a sufficient number of internal degrees of freedom are included.

$$
\mathbf{M}^{u}_{EI} = \begin{bmatrix}
S^0_{1,3}(\rho A) & S^0_{1,4}(\rho A) & S^0_{1,5}(\rho A) & S^0_{1,6}(\rho A) & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & \dots \\
S^0_{2,3}(\rho A) & S^0_{2,4}(\rho A) & S^0_{2,5}(\rho A) & S^0_{2,6}(\rho A) & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & \dots
\end{bmatrix} \tag{63}
$$

$$
\mathbf{M}^{v}_{EI} = \begin{bmatrix}
0 & 0 & \dots & 0 & 0 & \dots \\
S^1_{1,5}(\rho A) + S^{1P}_{1,5}(\rho I_{ZZ}) & S^1_{1,6}(\rho A) + S^{1P}_{1,6}(\rho I_{ZZ}) & \dots & S^1_{1,10}(\rho A) + S^{1P}_{1,10}(\rho I_{ZZ}) & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
S^1_{2,5}(\rho A) + S^{1P}_{2,5}(\rho I_{ZZ}) & S^1_{2,6}(\rho A) + S^{1P}_{2,6}(\rho I_{ZZ}) & \dots & S^1_{2,10}(\rho A) + S^{1P}_{2,10}(\rho I_{ZZ}) & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
S^1_{3,5}(\rho A) + S^{1P}_{3,5}(\rho I_{ZZ}) & S^1_{3,6}(\rho A) + S^{1P}_{3,6}(\rho I_{ZZ}) & \dots & S^1_{3,10}(\rho A) + S^{1P}_{3,10}(\rho I_{ZZ}) & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
S^1_{4,5}(\rho A) + S^{1P}_{4,5}(\rho I_{ZZ}) & S^1_{4,6}(\rho A) + S^{1P}_{4,6}(\rho I_{ZZ}) & \dots & S^1_{4,10}(\rho A) + S^{1P}_{4,10}(\rho I_{ZZ}) & 0 & \dots
\end{bmatrix} \tag{64}
$$

$$
\mathbf{M}^{w}_{EI} = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots \\
S^1_{1,5}(\rho A) + S^{1P}_{1,5}(\rho I_{YY}) & S^1_{1,6}(\rho A) + S^{1P}_{1,6}(\rho I_{YY}) & \cdots & S^1_{1,10}(\rho A) + S^{1P}_{1,10}(\rho I_{YY}) & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots \\
-S^1_{2,5}(\rho A) - S^{1P}_{2,5}(\rho I_{YY}) & -S^1_{2,6}(\rho A) - S^{1P}_{2,6}(\rho I_{YY}) & \cdots & -S^1_{2,10}(\rho A) - S^{1P}_{2,10}(\rho I_{YY}) & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots \\
S^1_{3,5}(\rho A) + S^{1P}_{3,5}(\rho I_{YY}) & S^1_{3,6}(\rho A) + S^{1P}_{3,6}(\rho I_{YY}) & \cdots & S^1_{3,10}(\rho A) + S^{1P}_{3,10}(\rho I_{YY}) & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots \\
-S^1_{4,5}(\rho A) - S^{1P}_{4,5}(\rho I_{YY}) & -S^1_{4,6}(\rho A) - S^{1P}_{4,6}(\rho I_{YY}) & \cdots & -S^1_{4,10}(\rho A) - S^{1P}_{4,10}(\rho I_{YY}) & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots
\end{bmatrix} \tag{65}
$$

$$
\mathbf{M}^{\theta}_{EI} = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots \\
S^0_{1,3}(\rho I_{XX}) & S^0_{1,4}(\rho I_{XX}) & \cdots & S^0_{1,8}(\rho I_{XX}) & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots \\
S^0_{2,3}(\rho I_{XX}) & S^0_{2,4}(\rho I_{XX}) & \cdots & S^0_{2,8}(\rho I_{XX}) & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots
\end{bmatrix} \tag{66}
$$

The mass submatrices corresponding to the internal degrees of freedom are given by the following equations (eqs. (67) through (70)). Note that these submatrices are symmetric and banded. Their size depends on the number of internal degrees of freedom employed for a particular continuous variable. Given that P_v is the number of discrete degrees of freedom associated with v and that there are always four external degrees of freedom associated with v, the size of $\mathbf{M}^{v}_{II}$ is $P_v - 4$ by $P_v - 4$.

$$
\mathbf{M}^{u}_{II} = \begin{bmatrix}
S^0_{3,3}(\rho A) & S^0_{3,4}(\rho A) & \cdots & S^0_{3,7}(\rho A) & 0 & 0 & 0 & \cdots \\
 & S^0_{4,4}(\rho A) & S^0_{4,5}(\rho A) & \cdots & S^0_{4,8}(\rho A) & 0 & 0 & \cdots \\
sym & & S^0_{5,5}(\rho A) & S^0_{5,6}(\rho A) & \cdots & S^0_{5,9}(\rho A) & 0 & \cdots \\
 & & & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix} \tag{67}
$$

$$\mathbf{M}^v_{II} = \begin{bmatrix} S^1_{5,5}(\rho A) + S^{1P}_{5,5}(\rho I_{ZZ}) & S^1_{5,6}(\rho A) + S^{1P}_{5,6}(\rho I_{ZZ}) & \cdots & S^1_{5,11}(\rho A) + S^{1P}_{5,11}(\rho I_{ZZ}) & 0 & 0 & 0 & \cdots \\ & S^1_{6,6}(\rho A) + S^{1P}_{6,6}(\rho I_{ZZ}) & S^1_{6,7}(\rho A) + S^{1P}_{6,7}(\rho I_{ZZ}) & \cdots & S^1_{6,12}(\rho A) + S^{1P}_{6,12}(\rho I_{ZZ}) & 0 & 0 & \cdots \\ sym & & S^1_{7,7}(\rho A) + S^{1P}_{7,7}(\rho I_{ZZ}) & S^1_{7,8}(\rho A) + S^{1P}_{7,8}(\rho I_{ZZ}) & \cdots & S^1_{7,13}(\rho A) + S^{1P}_{7,13}(\rho I_{ZZ}) & 0 & \cdots \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{68}$$

$$\mathbf{M}^w_{II} = \begin{bmatrix} S^1_{5,5}(\rho A) + S^{1P}_{5,5}(\rho I_{YY}) & S^1_{5,6}(\rho A) + S^{1P}_{5,6}(\rho I_{YY}) & \cdots & S^1_{5,11}(\rho A) + S^{1P}_{5,11}(\rho I_{YY}) & 0 & 0 & 0 & \cdots \\ & S^1_{6,6}(\rho A) + S^{1P}_{6,6}(\rho I_{YY}) & S^1_{6,7}(\rho A) + S^{1P}_{6,7}(\rho I_{YY}) & \cdots & S^1_{6,12}(\rho A) + S^{1P}_{6,12}(\rho I_{YY}) & 0 & 0 & \cdots \\ sym & & S^1_{7,7}(\rho A) + S^{1P}_{7,7}(\rho I_{YY}) & S^1_{7,8}(\rho A) + S^{1P}_{7,8}(\rho I_{YY}) & \cdots & S^1_{7,13}(\rho A) + S^{1P}_{7,13}(\rho I_{YY}) & 0 & \cdots \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{69}$$

$$\mathbf{M}^\theta_{II} = \begin{bmatrix} S^0_{3,3}(\rho I_{XX}) & S^0_{3,4}(\rho I_{XX}) & \cdots & S^0_{3,9}(\rho I_{XX}) & 0 & 0 & 0 & \cdots \\ & S^0_{4,4}(\rho I_{XX}) & S^0_{4,5}(\rho I_{XX}) & \cdots & S^0_{4,10}(\rho I_{XX}) & 0 & 0 & \cdots \\ sym & & S^0_{5,5}(\rho I_{XX}) & S^0_{5,6}(\rho I_{XX}) & \cdots & S^0_{5,11}(\rho I_{XX}) & 0 & \cdots \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{70}$$

Stiffness Matrix

The stiffness matrix is partitioned similar to the mass matrix and has the form

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{EE} & \mathbf{K}^u_{EI} & \mathbf{K}^v_{EI} & \mathbf{K}^w_{EI} & \mathbf{K}^\theta_{EI} \\ & \mathbf{K}^u_{II} & 0 & 0 & 0 \\ & & \mathbf{K}^v_{II} & 0 & 0 \\ & sym & & \mathbf{K}^w_{II} & 0 \\ & & & & \mathbf{K}^\theta_{II} \end{bmatrix} \tag{71}$$

The external stiffness submatrix $\mathbf{K}_{EE}$ is given by the 12 by 12 matrix in the following equation and reduces to the classical beam element stiffness matrix for constant cross-sectional properties.

$$\mathbf{K}_{EE} =$$

$$\begin{bmatrix}
S^{0P}_{1,1}(EA) & 0 & 0 & 0 & 0 & 0 & S^{0P}_{1,2}(EA) & 0 & 0 & 0 & 0 & 0 \\
 & S^{1PP}_{1,1}(EI_{ZZ}) & 0 & 0 & 0 & S^{1PP}_{1,2}(EI_{ZZ}) & 0 & S^{1PP}_{1,3}(EI_{ZZ}) & 0 & 0 & 0 & S^{1PP}_{1,4}(EI_{ZZ}) \\
 & & S^{1PP}_{1,1}(EI_{YY}) & 0 & -S^{1PP}_{1,2}(EI_{YY}) & 0 & 0 & 0 & S^{1PP}_{1,3}(EI_{YY}) & 0 & -S^{1PP}_{1,4}(EI_{YY}) & 0 \\
 & & & S^{0P}_{1,1}(GJ) & 0 & 0 & 0 & 0 & 0 & S^{0P}_{1,2}(GJ) & 0 & 0 \\
 & & & & S^{1PP}_{2,2}(EI_{YY}) & 0 & 0 & 0 & -S^{1PP}_{2,3}(EI_{YY}) & 0 & S^{1PP}_{2,4}(EI_{YY}) & 0 \\
 & & & & & S^{1PP}_{2,2}(EI_{ZZ}) & 0 & S^{1PP}_{2,3}(EI_{ZZ}) & 0 & 0 & 0 & S^{1PP}_{2,4}(EI_{ZZ}) \\
 & & & & & & S^{0P}_{2,2}(EA) & 0 & 0 & 0 & 0 & 0 \\
 & & sym & & & & & S^{1PP}_{3,3}(EI_{ZZ}) & 0 & 0 & 0 & S^{1PP}_{3,4}(EI_{ZZ}) \\
 & & & & & & & & S^{1PP}_{3,3}(EI_{YY}) & 0 & -S^{1PP}_{3,4}(EI_{YY}) & 0 \\
 & & & & & & & & & S^{0P}_{2,2}(GJ) & 0 & 0 \\
 & & & & & & & & & & S^{1PP}_{4,4}(EI_{YY}) & 0 \\
 & & & & & & & & & & & S^{1PP}_{4,4}(EI_{ZZ})
\end{bmatrix} \tag{72}$$

The submatrices $\mathbf{K}^{u}_{EI}$, $\mathbf{K}^{v}_{EI}$, $\mathbf{K}^{w}_{EI}$, and $\mathbf{K}^{\theta}_{EI}$ couple the external and internal degrees of freedom in the element stiffness matrix and are given by the following equations. These submatrices have the same dimensions and properties as the corresponding external-internal coupling mass submatrices (eqs. (63) through (66)) discussed previously.

$$\mathbf{K}^{u}_{EI} = \begin{bmatrix}
S^{0P}_{1,3}(EA) & S^{0P}_{1,4}(EA) & 0 & \dots \\
0 & 0 & 0 & \dots \\
0 & 0 & 0 & \dots \\
0 & 0 & 0 & \dots \\
0 & 0 & 0 & \dots \\
0 & 0 & 0 & \dots \\
S^{0P}_{2,3}(EA) & S^{0P}_{2,4}(EA) & 0 & \dots \\
0 & 0 & 0 & \dots \\
0 & 0 & 0 & \dots \\
0 & 0 & 0 & \dots \\
0 & 0 & 0 & \dots \\
0 & 0 & 0 & \dots
\end{bmatrix} \tag{73}$$

$$
\mathbf{K}_{EI}^{v} = \begin{bmatrix}
0 & 0 & \dots & 0 & 0 & \dots \\
S_{1,5}^{1PP}(EI_{ZZ}) & S_{1,6}^{1PP}(EI_{ZZ}) & \dots & S_{1,8}^{1PP}(EI_{ZZ}) & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
S_{2,5}^{1PP}(EI_{ZZ}) & S_{2,6}^{1PP}(EI_{ZZ}) & \dots & S_{2,8}^{1PP}(EI_{ZZ}) & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
S_{3,5}^{1PP}(EI_{ZZ}) & S_{3,6}^{1PP}(EI_{ZZ}) & \dots & S_{3,8}^{1PP}(EI_{ZZ}) & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
S_{4,5}^{1PP}(EI_{ZZ}) & S_{4,6}^{1PP}(EI_{ZZ}) & \dots & S_{4,8}^{1PP}(EI_{ZZ}) & 0 & \dots
\end{bmatrix} \tag{74}
$$

$$
\mathbf{K}_{EI}^{w} = \begin{bmatrix}
0 & 0 & \dots & 0 & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
S_{1,5}^{1PP}(EI_{YY}) & S_{1,6}^{1PP}(EI_{YY}) & \dots & S_{1,8}^{1PP}(EI_{YY}) & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
-S_{2,5}^{1PP}(EI_{YY}) & -S_{2,6}^{1PP}(EI_{YY}) & \dots & -S_{2,8}^{1PP}(EI_{YY}) & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
S_{3,5}^{1PP}(EI_{YY}) & S_{3,6}^{1PP}(EI_{YY}) & \dots & S_{3,8}^{1PP}(EI_{YY}) & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
-S_{4,5}^{1PP}(EI_{YY}) & -S_{4,6}^{1PP}(EI_{YY}) & \dots & -S_{4,8}^{1PP}(EI_{YY}) & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots
\end{bmatrix} \tag{75}
$$

$$
\mathbf{K}_{EI}^{\theta} = \begin{bmatrix}
0 & 0 & \dots & 0 & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
S_{1,3}^{0P}(GJ) & S_{1,4}^{0P}(GJ) & \dots & S_{1,6}^{0P}(GJ) & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
S_{2,3}^{0P}(GJ) & S_{2,4}^{0P}(GJ) & \dots & S_{2,6}^{0P}(GJ) & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots
\end{bmatrix} \tag{76}
$$

The stiffness submatrices associated with the internal degrees of freedom have similar characteristics as their mass counterparts and are defined as follows:

$$\mathbf{K}^{u}_{II} = \begin{bmatrix} S^{0P}_{3,3}(EA) & S^{0P}_{3,4}(EA) & S^{0P}_{3,5}(EA) & 0 & 0 & 0 & \cdots \\ & S^{0P}_{4,4}(EA) & S^{0P}_{4,5}(EA) & S^{0P}_{4,6}(EA) & 0 & 0 & \cdots \\ sym & & S^{0P}_{5,5}(EA) & S^{0P}_{5,6}(EA) & S^{0P}_{5,7}(EA) & 0 & \cdots \\ & & & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{77}$$

$$\mathbf{K}^{v}_{II} = \begin{bmatrix} S^{1PP}_{5,5}(EI_{ZZ}) & S^{1PP}_{5,6}(EI_{ZZ}) & \cdots & S^{1PP}_{5,9}(EI_{ZZ}) & 0 & 0 & 0 & \cdots \\ & S^{1PP}_{6,6}(EI_{ZZ}) & S^{1PP}_{6,7}(EI_{ZZ}) & \cdots & S^{1PP}_{6,10}(EI_{ZZ}) & 0 & 0 & \cdots \\ sym & & S^{1PP}_{7,7}(EI_{ZZ}) & S^{1PP}_{7,8}(EI_{ZZ}) & \cdots & S^{1PP}_{7,11}(EI_{ZZ}) & 0 & \cdots \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{78}$$

$$\mathbf{K}^{w}_{II} = \begin{bmatrix} S^{1PP}_{5,5}(EI_{YY}) & S^{1PP}_{5,6}(EI_{YY}) & \cdots & S^{1PP}_{5,9}(EI_{YY}) & 0 & 0 & 0 & \cdots \\ & S^{1PP}_{6,6}(EI_{YY}) & S^{1PP}_{6,7}(EI_{YY}) & \cdots & S^{1PP}_{6,10}(EI_{YY}) & 0 & 0 & \cdots \\ sym & & S^{1PP}_{7,7}(EI_{YY}) & S^{1PP}_{7,8}(EI_{YY}) & \cdots & S^{1PP}_{7,11}(EI_{YY}) & 0 & \cdots \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{79}$$

$$\mathbf{K}^{\theta}_{II} = \begin{bmatrix} S^{0P}_{3,3}(GJ) & S^{0P}_{3,4}(GJ) & \cdots & S^{0P}_{3,7}(GJ) & 0 & 0 & 0 & \cdots \\ & S^{0P}_{4,4}(GJ) & S^{0P}_{4,5}(GJ) & \cdots & S^{0P}_{4,8}(GJ) & 0 & 0 & \cdots \\ sym & & S^{0P}_{5,5}(GJ) & S^{0P}_{5,6}(GJ) & \cdots & S^{0P}_{5,9}(GJ) & 0 & \cdots \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{80}$$

All the relations needed to completely define the tapered p-version beam element for an arbitrary number of internal degrees of freedom are now in hand. These include the matrix definitions given in equations (61) through (80), the definition of the cross-sectional property polynomials given by equations (44) and (45), and the explicit expressions for the five sets of matrix functions given in appendix A.

Numerical Validation and Preliminary Performance Analysis

The beam element developed herein is capable of emulating four different types of beam elements: uniform h-version, uniform p-version, tapered h-version, and tapered p-version. A uniform (h- or p-version) element is created by restricting a_1 through a_4 to be zero in the generic cross-sectional property polynomials. An h-version (uniform or tapered) element is created by restricting $P_u = P_\theta = 2$ and $P_v = P_w = 4$. Once the uniform versions of the element are validated, they can be used to approximate a tapered geometry for the purpose of validating the tapered versions of the element. Similarly, the p-versions of the element should converge to the same results as the h-versions of the element.

Computational Approach

Numerical results for most of the cases analyzed were computed on an IBM Model 80 computer operating at 16 MHz and equipped with an 80387 math coprocessor and running PC-DOS 3.3. Because of computer core limitations, those uniform h-version cases containing 15 or more elements were run on a DEC 3200 VAX workstation running VMS. However, all computations used eight-byte floating-point representation of real numbers conforming to the IEEE standard (default on the IBM computer but requires a /G compiler option on the VAX computer). Because the computer code used to generate these results was not written to take advantage of the symmetry and bandedness of the global mass and stiffness matrices, a fair comparison of compute times cannot be given. The only measure of computational efficiency presented is the number of degrees of freedom versus the accuracy of the models. The eigenvalue extraction technique used is a combination of the determinant search method and inverse iteration (ref. 16). The determinant search phase employs both the bisection method and the secant method to converge to an eigenvalue. Inverse iteration is then used to obtain the associated eigenvector.

Validation of Uniform h- and p-Versions

A uniform beam element can be validated rather easily because closed-form solutions to the axial and torsional vibrations and "nearly closed-form" solutions to the bending vibration problem exist. (See ref. 12.) The derivation of the closed-form expressions for these frequencies is presented in appendix B because the derivations commonly found in the literature do not include the rotary inertia terms.

The uniform, cantilevered steel beam with a circular cross section shown in figure 3 was used to validate the uniform versions of the element developed herein. Representative h- and p-version finite-element models of the beam are also shown in figure 3. The material properties of the beam used in the numerical studies were as follows:

$$E = 30.0 \times 10^6 \text{ lb/in}^2$$

$$G = 11.6 \times 10^6 \text{ lb/in}^2$$

$$\rho = \frac{0.284}{386.4} \text{ lb-sec}^2/\text{in}^4$$

For validation of the uniform versions of the finite element developed herein, the lowest frequencies of this beam were calculated "exactly" using the closed-form solutions in appendix B and numerically using both the h- and p-version elements. The first four bending frequencies, the first torsional frequency, and the first axial frequency, from appendix B, are

$$f_{v1} = 27.4651 \text{ Hz}$$

$$f_{v2} = 171.6155 \text{ Hz}$$

$$f_{v3} = 478.2670 \text{ Hz}$$

$$f_{v4} = 930.8451 \text{ Hz}$$

$$f_{\theta 1} = 523.4749 \text{ Hz}$$

$$f_{u1} = 841.7997 \text{ Hz}$$

These six frequencies will be regarded as "exact" for the purpose of validating the uniform h- and p-versions of the beam element. Table I shows the frequencies obtained using the uniform h-version of the finite element as the number of elements used was varied from 1 to 14. The exact frequencies are given at the bottom of the table for reference. Although only the first bending frequency truly converged with 14 elements, the other frequencies are nearly converged (within 0.06 percent).

Table II shows the frequencies obtained using one p-version element as the number of internal degrees of freedom is increased from 0 to 13 for each of the four continuous displacements. That is, in table II, 13 internal degrees of freedom refers to 13 internal degrees of freedom each for u, v, w, and θ, for an aggregate total of 52 internal degrees of freedom. Again, the exact frequencies are noted at the bottom of the table for reference. It is clear that all the listed frequencies have converged with 13 internal degrees of freedom, and that the converged values are essentially equal to the exact values. In figures 4 through 9 the information in tables I and II is plotted in terms of the percent error versus the number of degrees of freedom. The plots clearly show that both the p-version element and the h-version element converge to the exact answer, and that the p-version element has a faster convergence rate than the h-version element.

Validation of Tapered h- and p-Versions

The tapered steel beam with circular cross section shown in figure 10 was used to validate the tapered versions of the finite element developed herein. Also shown in figure 10 are representative h- and p-version models. Because the results shown in tables I and II have validated the uniform versions of the beam element, the exact frequencies of the tapered beam can be determined with a uniform h-version model if a sufficiently large number of elements are employed.

The frequencies obtained for the tapered beam using uniform h-version finite elements are presented in table III. It should be noted that the frequencies in table III approach the converged values from below (whereas in tables I and II, the frequencies converge from above). Also, the number of elements required to reach convergence for the tapered beam is over seven times that required for a comparable uniform beam. Both these behaviors can be explained by the fact that, as uniform elements are added, the geometry of the model is changing. The geometry approaches a smoothly tapered beam as the number of elements in the model approaches infinity. Here, the model is assumed to be converged with 100 elements (600 degrees of freedom).

The frequencies predicted by using a tapered h-version finite-element model are shown in table IV. In this case the geometry of the problem is represented exactly using one element, and therefore the geometry of the model is not changed as the number of elements is increased. It should be noted that now convergence is from above and is achieved using approximately one seventh as many elements as in table III.

Table V shows the frequencies which were obtained for the tapered beam using one tapered p-version element and varying the number of internal degrees of freedom from 0 to 19. With 19 internal degrees of freedom, 22nd order polynomials are used for lateral bending and 20th order polynomials for the axial and torsional displacements. The reason for showing the results of such a high order element is to demonstrate the numerical stability of the element. It is seen that the p-version results in table V converge very close to the values predicted by the h-version beams in tables III and IV but with the use of far fewer degrees of freedom.

The data in tables III through V are graphically depicted in figures 11 through 16 in terms of percent error versus the number of degrees of freedom. These figures show the dramatic improvement in convergence which is realized by using tapered h- and p-version elements instead of uniform h-version elements.

Conclusions

The derivation, and validation, of a new, tapered, p-version beam element which both facilitates convergence checks and produces a better convergence rate than nontapered, h-version beam elements has been described. These two characteristics complement each other and, when combined, provide a powerful and versatile beam element which is easy to use. The shape functions on which the element is based were derived by using orthogonality relationships which produce element matrices that are extremely well-conditioned and of a form allowing explicit integration in the derivation of the element matrices. The latter feature eliminates the need for numerical quadrature; thus, roundoff error is reduced. The shape functions are hierarchical such that higher order element matrices can use the element matrices from previous lower order analyses. This simplifies the derivation, coding, and validation of the element. The present form of the beam element has been derived in a manner which allows for an infinite number of internal degrees of freedom. The beam element has been tested with up to 22nd order C^1-type shape functions and up to 20th order C^0-type shape functions with no evidence of ill conditioning or significant roundoff error.

NASA Langley Research Center
Hampton, VA 23665-5225
June 23, 1989

Appendix A

Expressions for Matrix Functions

Explicit expressions for the matrix functions are presented in this appendix. Each matrix function is a function of the element length l and the coefficients from the generic cross-sectional property polynomials, a_0 through a_4.

The definition of the S^0 set is

$$S^0_{i,j} = \int_0^l P_4(x) N_i^0 N_j^0 \, dx \tag{A1}$$

and the explicit integrations are

$$S^0_{1,1} = \frac{l(4a_4l^4 + 7a_3l^3 + 14a_2l^2 + 35a_1l + 140a_0)}{420} \tag{A2}$$

$$S^0_{1,2} = \frac{l(10a_4l^4 + 14a_3l^3 + 21a_2l^2 + 35a_1l + 70a_0)}{420} \tag{A3}$$

$$S^0_{2,2} = \frac{l(60a_4l^4 + 70a_3l^3 + 84a_2l^2 + 105a_1l + 140a_0)}{420} \tag{A4}$$

$$\begin{aligned} \underset{i=3,8}{S^0_{1,i}} = {} & -l\sqrt{2i-3}\,\big[(i-5)(33i^4 - 685i^3 + 4990i^2 - 14910i + 15492)a_4l^4 \\ & + 2(i-8)(23i^4 - 416i^3 + 2697i^2 - 7419i + 7305)a_3l^3 \\ & + 6(i-8)(i-7)(8i^3 - 90i^2 + 322i - 365)a_2l^2 \\ & + 4(i-8)(i-7)(i-6)(i-4)(4i-5)a_1l \\ & + 35(i-8)(i-7)(i-6)(i-5)(2i-5)a_0\big] \\ & /\big[168(587i^5 - 15825i^4 + 166115i^3 - 846075i^2 + 2087798i - 1994340)\big] \end{aligned} \tag{A5}$$

$$\underset{i=9,\infty}{S^0_{1,i}} = 0 \tag{A6}$$

$$\begin{aligned} \underset{i=3,8}{S^0_{2,i}} = {} & l\sqrt{2i-3}\,\big[(283i^5 - 7825i^4 + 84335i^3 - 442025i^2 + 1124532i - 1108260)a_4l^4 \\ & + 5(i-8)(50i^4 - 995i^3 + 7224i^2 - 22641i + 25770)a_3l^3 \\ & + 3(i-8)(i-7)(68i^3 - 905i^2 + 3907i - 5440)a_2l^2 \\ & + 3(i-8)(i-7)(i-6)(52i^2 - 413i + 785)a_1l \\ & + 35(i-8)(i-7)(i-6)(i-5)(4i-13)a_0\big] \\ & /\big[168(587i^5 - 15825i^4 + 166115i^3 - 846075i^2 + 2087798i - 1994340)\big] \end{aligned} \tag{A7}$$

$$\underset{i=9,\infty}{S^0_{2,i}} = 0 \tag{A8}$$

$$\begin{aligned} \underset{i=3,\infty}{S^0_{i,i}} = {} & l(2i-3)\big[3(7i^4 - 42i^3 + 7i^2 + 168i + 60)a_4l^4 \\ & + (2i+3)(2i-9)(7i^2 - 21i - 10)a_3l^3 \\ & + 2(2i+3)(2i-9)(5i^2 - 15i - 8)a_2l^2 \\ & + 4(2i+3)(2i+1)(2i-7)(2i-9)(a_1l + 2a_0)\big] \\ & /\big[16(2i+3)(2i+1)(2i-1)(2i-3)(2i-5)(2i-7)(2i-9)\big] \end{aligned} \tag{A9}$$

$$\underset{i=3,\infty}{S^0_{i,i+1}} = -l^2\sqrt{2i-3}\sqrt{2i-1}\big[6(i-1)(i^2-2i-5)a_4l^3 + (i-1)(7i^2-14i-36)a_3l^2 + 2(i-1)(2i+3)(2i-7)(a_2l+a_1)\big] / \big[16(2i+3)(2i+1)(2i-1)(2i-3)(2i-5)(2i-7)\big] \quad \text{(A10)}$$

$$\underset{i=3,\infty}{S^0_{i,i+2}} = -l\sqrt{2i-3}\sqrt{2i+1}\big[15(i^2-i-10)(i^2-i-8)a_4l^4 + 8(i+2)(i-3)(2i+5)(2i-7)a_3l^3 + 8(2i+5)(2i-7)(2i^2-2i-9)a_2l^2 + 8(2i+5)(2i+3)(2i-5)(2i-7)(a_1l+2a_0)\big] / \big[64(2i+5)(2i+3)(2i+1)(2i-1)(2i-3)(2i-5)(2i-7)\big] \quad \text{(A11)}$$

$$\underset{i=3,\infty}{S^0_{i,i+3}} = l^2\sqrt{2i-3}\sqrt{2i+3}\big[10i(i^2-7)a_4l^3 + i(13i^2-85)a_3l^2 + 4i(2i+5)(2i-5)(a_2l+a_1)\big] / \big[32(2i+5)(2i+3)(2i+1)(2i-1)(2i-3)(2i-5)\big] \quad \text{(A12)}$$

$$\underset{i=3,\infty}{S^0_{i,i+4}} = -l^3\sqrt{2i-3}\sqrt{2i+5}\big[(i+1)i(13i^2+13i-116)a_4l^2 + (i+1)i(2i+7)(2i-5)(3a_3l+2a_2)\big] / \big[32(2i+7)(2i+5)(2i+3)(2i+1)(2i-1)(2i-3)(2i-5)\big] \quad \text{(A13)}$$

$$\underset{i=3,\infty}{S^0_{i,i+5}} = \frac{l^4\sqrt{2i-3}\sqrt{2i+7}(i+2)(i+1)i(2a_4l+a_3)}{32(2i+7)(2i+5)(2i+3)(2i+1)(2i-1)(2i-3)} \quad \text{(A14)}$$

$$\underset{i=3,\infty}{S^0_{i,i+6}} = \frac{-\sqrt{2i-3}\sqrt{2i+9}(i+3)(i+2)(i+1)ia_4l^5}{64(2i+9)(2i+7)(2i+5)(2i+3)(2i+1)(2i-1)(2i-3)} \quad \text{(A15)}$$

$$\underset{i=3,\infty,\ j=7,\infty}{S^0_{i,i+j}} = 0 \quad \text{(A16)}$$

The definition of the S^{0P} set is

$$S^{0P}_{i,j} = \int_0^l P_4(x)(N^0_i)'(N^0_j)'\,dx \quad \text{(A17)}$$

and the explicit integrations are

$$S^{0P}_{1,1} = \frac{12a_4l^4+15a_3l^3+20a_2l^2+30a_1l+60a_0}{60l} \quad \text{(A18)}$$

$$S^{0P}_{1,2} = -\frac{12a_4l^4+15a_3l^3+20a_2l^2+30a_1l+60a_0}{60l} \quad \text{(A19)}$$

$$S^{0P}_{2,2} = \frac{12a_4l^4+15a_3l^3+20a_2l^2+30a_1l+60a_0}{60l} \quad \text{(A20)}$$

$$\underset{i=3,6}{S^{0P}_{1,i}} = \sqrt{2i-3}\big[3(29i^3-406i^2+1833i-2644)a_4l^3 + 3(i-6)(25i^2-202i+387)a_3l^2 + 2(i-6)(i-5)(26i-83)a_2l + 10(i-6)(i-5)(i-4)a_1\big] / \big[30(47i^3-628i^2+2729i-3792)\big] \quad \text{(A21)}$$

$$\underset{i=7,\infty}{S^{0P}_{1,i}} = 0 \tag{A22}$$

$$\begin{aligned}\underset{i=3,6}{S^{0P}_{2,i}} &= -\sqrt{2i-3}\big[3(29i^3-406i^2+1833i-2644)a_4l^3+3(i-6)(25i^2-202i+387)a_3l^2\\&\quad+2(i-6)(i-5)(26i-83)a_2l+10(i-6)(i-5)(i-4)a_1\big]\\&\quad/\big[30(47i^3-628i^2+2729i-3792)\big]\end{aligned} \tag{A23}$$

$$\underset{i=7,\infty}{S^{0P}_{2,i}} = 0 \tag{A24}$$

$$\begin{aligned}\underset{i=3,\infty}{S^{0P}_{i,i}} &= \big[(35i^4-210i^3+305i^2+30i-88)(2i-3)a_4l^4\\&\quad+2(2i+1)(2i-3)(2i-7)(5i^2-15i+7)a_3l^3\\&\quad+4(2i+1)(2i-3)(2i-7)(3i^2-9i+4)a_2l^2\\&\quad+4(2i+1)(2i-1)(2i-3)(2i-5)(2i-7)(a_1l+2a_0)\big]\\&\quad/\big[8l(2i+1)(2i-1)(2i-3)(2i-5)(2i-7)\big]\end{aligned} \tag{A25}$$

$$\begin{aligned}\underset{i=3,\infty}{S^{0P}_{i,i+1}} &= -\sqrt{2i-3}\sqrt{2i-1}\big[2(i-1)(7i^2-14i-8)a_4l^3\\&\quad+3(i-1)(5i^2-10i-6)a_3l^2\\&\quad+4(i-1)(2i+1)(2i-5)(a_2l+a_1)\big]\\&\quad/\big[8(2i+1)(2i-1)(2i-3)(2i-5)\big]\end{aligned} \tag{A26}$$

$$\begin{aligned}\underset{i=3,\infty}{S^{0P}_{i,i+2}} &= \sqrt{2i-3}\sqrt{2i+1}\big[2i(i-1)(7i^2-7i-26)a_4l^3\\&\quad+i(i-1)(2i+3)(2i-5)(3a_3l^2+2a_2l)\big]\\&\quad/\big[8(2i+3)(2i+1)(2i-1)(2i-3)(2i-5)\big]\end{aligned} \tag{A27}$$

$$\underset{i=3,\infty}{S^{0P}_{i,i+3}} = -\frac{\sqrt{2i-3}\sqrt{2i+3}(i+1)i(i-1)(2a_4l^3+a_3l^2)}{8(2i+3)(2i+1)(2i-1)(2i-3)} \tag{A28}$$

$$\underset{i=3,\infty}{S^{0P}_{i,i+4}} = \frac{\sqrt{2i-3}\sqrt{2i+5}(i+2)(i+1)i(i-1)a_4l^3}{16(2i+5)(2i+3)(2i+1)(2i-1)(2i-3)} \tag{A29}$$

$$\underset{i=3,\infty,\ j=5,\infty}{S^{0P}_{i,i+j}} = 0 \tag{A30}$$

The definition of the S^1 set is

$$S^1_{i,j} = \int_0^l P_2(x)N^1_iN^1_j\,dx \tag{A31}$$

and the explicit integrations are

$$S^1_{1,1} = \frac{l(19a_2l^2 + 54a_1l + 234a_0)}{630} \tag{A32}$$

$$S^1_{1,2} = \frac{l^2(17a_2l^2 + 42a_1l + 132a_0)}{2520} \tag{A33}$$

$$S^1_{1,3} = \frac{l(46a_2l^2 + 81a_1l + 162a_0)}{1260} \tag{A34}$$

$$S^1_{1,4} = -\frac{l^2(19a_2l^2 + 36a_1l + 78a_0)}{2520} \tag{A35}$$

$$S^1_{2,2} = \frac{l^3(4a_2l^2 + 9a_1l + 24a_0)}{2520} \tag{A36}$$

$$S^1_{2,3} = \frac{l^2(25a_2l^2 + 42a_1l + 78a_0)}{2520} \tag{A37}$$

$$S^1_{2,4} = -\frac{l^3(5a_2l^2 + 9a_1l + 18a_0)}{2520} \tag{A38}$$

$$S^1_{3,3} = \frac{l(145a_2l^2 + 180a_1l + 234a_0)}{630} \tag{A39}$$

$$S^1_{3,4} = -\frac{l^2(65a_2l^2 + 90a_1l + 132a_0)}{2520} \tag{A40}$$

$$S^1_{4,4} = \frac{l^3(10a_2l^2 + 15a_1l + 24a_0)}{2520} \tag{A41}$$

$$\begin{aligned}\underset{i=5,10}{S^1_{1,i}} &= \sqrt{2i-5}\big[(206i^5 - 7725i^4 + 114020i^3 - 827175i^2 + 2947034i - 4122960)a_2l^3 \\ &\quad + (i-10)(173i^4 - 5005i^3 + 53215i^2 - 245915i + 416292)a_1l^2 \\ &\quad + 2(i-10)(i-9)(i-7)(69i^2 - 1136i + 3892)a_0l\big] \\ &\quad /\big[1260(4759i^5 - 174190i^4 + 2512445i^3 - 17844110i^2 + 62404896i - 85980000)\big]\end{aligned} \tag{A42}$$

$$\underset{i=11,\infty}{S^1_{1,i}} = 0 \tag{A43}$$

$$\begin{aligned}\underset{i=5,10}{S^1_{2,i}} &= \sqrt{2i-5}\big[(62i^5 - 2315i^4 + 34000i^3 - 245305i^2 + 868878i - 1208400)a_2l^4 \\ &\quad + 2(i-10)(i-8)(i-6)(33i^2 - 468i + 1499)a_1l^3 \\ &\quad + (i-10)(i-9)(26i^3 - 661i^2 + 4961i - 11476)a_0l^2\big] \\ &\quad /\big[10080(434i^5 - 15875i^4 + 229060i^3 - 1628305i^2 + 5701086i - 7864620)\big]\end{aligned} \tag{A44}$$

$$\underset{i=11,\infty}{S^1_{2,i}} = 0 \tag{A45}$$

$$\begin{aligned} \underset{i=5,10}{S^1_{3,i}} &= -\sqrt{2i-5}\big[5(156i^5 - 5983i^4 + 90554i^3 - 675437i^2 + 2480302i - 3583344)a_2l^3 \\ &+ 5(i-10)(109i^4 - 3167i^3 + 33983i^2 - 159337i + 274980)a_1l^2 \\ &+ 2(i-10)(i-9)(i-7)(111i^2 - 1724i + 5908)a_0l\big] \\ &/\big[1260(4759i^5 - 174190i^4 + 2512445i^3 - 17844110i^2 + 62404896i - 85980000)\big] \end{aligned} \tag{A46}$$

$$\underset{i=11,\infty}{S^1_{3,i}} = 0 \tag{A47}$$

$$\begin{aligned} \underset{i=5,10}{S^1_{4,i}} &= -\sqrt{2i-5}\big[6(8i^5 - 275i^4 + 3670i^3 - 23685i^2 + 73642i - 87900)a_2l^4 \\ &+ 5(i-10)(22i^4 - 573i^3 + 5474i^2 - 22743i + 34716)a_1l^3 \\ &+ (i-10)(i-9)(176i^3 - 3141i^2 + 18371i - 35276)a_0l^2\big] \\ &/\big[10080(434i^5 - 15875i^4 + 229060i^3 - 1628305i^2 + 5701086i - 7864620)\big] \end{aligned} \tag{A48}$$

$$\underset{i=11,\infty}{S^1_{4,i}} = 0 \tag{A49}$$

$$\underset{i=5,\infty}{S^1_{i,i}} = \frac{(2i-5)l\big[(7i^2-35i-18)a_2l^2 + 3(2i+1)(2i-11)(a_1l+2a_0)\big]}{16(2i+1)(2i-1)(2i-3)(2i-5)(2i-7)(2i-9)(2i-11)} \tag{A50}$$

$$\underset{i=5,\infty}{S^1_{i,i+1}} = -\frac{\sqrt{2i-5}\sqrt{2i-3}(i-2)l^2(a_2l+a_1)}{16(2i+1)(2i-1)(2i-3)(2i-5)(2i-7)(2i-9)} \tag{A51}$$

$$\underset{i=5,\infty}{S^1_{i,i+2}} = -\frac{l\sqrt{2i-5}\sqrt{2i-1}\big[(17i^2-51i-126)a_2l^2 + 8(2i+3)(2i-9)(a_1l+2a_0)\big]}{64(2i+3)(2i+1)(2i-1)(2i-3)(2i-5)(2i-7)(2i-9)} \tag{A52}$$

$$\underset{i=5,\infty}{S^1_{i,i+3}} = \frac{3l^2\sqrt{2i-5}\sqrt{2i+1}(i-1)(a_2l+a_1)}{32(2i+3)(2i+1)(2i-1)(2i-3)(2i-5)(2i-7)} \tag{A53}$$

$$\underset{i=5,\infty}{S^1_{i,i+4}} = \frac{l\sqrt{2i-5}\sqrt{2i+3}\big[(i+4)(i-5)a_2l^2 + (2i+5)(2i-7)(a_1l+2a_0)\big]}{32(2i+5)(2i+3)(2i+1)(2i-1)(2i-3)(2i-5)(2i-7)} \tag{A54}$$

$$\underset{i=5,\infty}{S^1_{i,i+5}} = -\frac{l^2\sqrt{2i-5}\sqrt{2i+5}i(a_2l+a_1)}{32(2i+5)(2i+3)(2i+1)(2i-1)(2i-3)(2i-5)} \tag{A55}$$

$$\underset{i=5,\infty}{S^1_{i,i+6}} = \frac{\sqrt{2i-5}\sqrt{2i+7}(i+1)ia_2l^3}{64(2i+7)(2i+5)(2i+3)(2i+1)(2i-1)(2i-3)(2i-5)} \tag{A56}$$

$$\underset{i=5,\infty,\ j=7,\infty}{S^1_{i,i+j}} = 0 \tag{A57}$$

The definition of the S^{1P} set is

$$S^{1P}_{i,j} = \int_0^l P_4(x)(N_i^1)'(N_j^1)'\,dx \tag{A58}$$

and the explicit integrations are

$$S^{1P}_{1,1} = \frac{10a_4l^4 + 15a_3l^3 + 24a_2l^2 + 42a_1l + 84a_0}{70l} \tag{A59}$$

$$S^{1P}_{1,2} = \frac{5a_4l^4 + 7a_3l^3 + 10a_2l^2 + 14a_1l + 14a_0}{140} \tag{A60}$$

$$S^{1P}_{1,3} = -\frac{10a_4l^4 + 15a_3l^3 + 24a_2l^2 + 42a_1l + 84a_0}{70l} \tag{A61}$$

$$S^{1P}_{1,4} = -\frac{5a_4l^4 + 5a_3l^3 + 4a_2l^2 - 14a_0}{140} \tag{A62}$$

$$S^{1P}_{2,2} = \frac{l(8a_4l^4 + 11a_3l^3 + 16a_2l^2 + 28a_1l + 112a_0)}{840} \tag{A63}$$

$$S^{1P}_{2,3} = -\frac{5a_4l^4 + 7a_3l^3 + 10a_2l^2 + 14a_1l + 14a_0}{140} \tag{A64}$$

$$S^{1P}_{2,4} = -\frac{l(10a_4l^4 + 11a_3l^3 + 12a_2l^2 + 14a_1l + 28a_0)}{840} \tag{A65}$$

$$S^{1P}_{3,3} = \frac{10a_4l^4 + 15a_3l^3 + 24a_2l^2 + 42a_1l + 84a_0)}{70l} \tag{A66}$$

$$S^{1P}_{3,4} = \frac{5a_4l^4 + 5a_3l^3 + 4a_2l^2 - 14a_0}{140} \tag{A67}$$

$$S^{1P}_{4,4} = \frac{l(60a_4l^4 + 65a_3l^3 + 72a_2l^2 + 84a_1l + 112a_0)}{840} \tag{A68}$$

$$\begin{aligned}\underset{i=5,10}{S^{1P}_{1,i}} &= \sqrt{2i-5}\,[12(36i^5 - 1330i^4 + 19340i^3 - 138305i^2 + 486319i - 672680)a_4l^4\\ &\quad + 10(i-10)(i-6)(47i^3 - 993i^2 + 6803i - 15053)a_3l^3\\ &\quad + (i-10)(i-9)(419i^3 - 7744i^2 + 47129i - 94384)a_2l^2\\ &\quad + (i-10)(i-9)(i-8)(269i^2 - 3112i + 8823)a_1l\\ &\quad + 330(i-10)(i-9)(i-8)(i-7)(i-5)a_0]\\ &\quad /[840l(687i^5 - 25346i^4 + 368441i^3 - 2636686i^2 + 9287944i - 12883260)]\end{aligned} \tag{A69}$$

$$\underset{i=11,\infty}{S^{1P}_{1,i}} = 0 \tag{A70}$$

$$S^{1P}_{\substack{2,i\\ i=5,10}} = \sqrt{2i-5}\,[8(103i^5 - 3795i^4 + 55020i^3 - 392190i^2 + 1374332i - 1894305)a_4 l^4$$
$$+ (i-10)(991i^4 - 26735i^3 + 266005i^2 - 1157585i + 1860084)a_3 l^3$$
$$+ 4(i-10)(i-9)(i-6)(247i^2 - 3010i + 8857)a_2 l^2$$
$$+ 3(i-10)(i-9)(i-8)(159i^2 - 1792i + 4973)a_1 l$$
$$+ 6(i-10)(i-9)(i-8)(i-7)(158i - 783)a_0]$$
$$/[1680(1918i^5 - 71033i^4 + 1035968i^3 - 7434583i^2 + 26251650i - 36487620)] \tag{A71}$$

$$S^{1P}_{\substack{2,i\\ i=11,\infty}} = 0 \tag{A72}$$

$$S^{1P}_{\substack{3,i\\ i=5,10}} = -\sqrt{2i-5}\,[12(36i^5 - 1330i^4 + 19340i^3 - 138305i^2 + 486319i - 672680)a_4 l^4$$
$$+ 10(i-10)(i-6)(47i^3 - 993i^2 + 6803i - 15053)a_3 l^3$$
$$+ (i-10)(i-9)(419i^3 - 7744i^2 + 47129i - 94384)a_2 l^2$$
$$+ (i-10)(i-9)(i-8)(269i^2 - 3112i + 8823)a_1 l$$
$$+ 330(i-10)(i-9)(i-8)(i-7)(i-5)a_0]$$
$$/[840l(687i^5 - 25346i^4 + 368441i^3 - 2636686i^2 + 9287944i - 12883260)] \tag{A73}$$

$$S^{1P}_{\substack{3,i\\ i=11,\infty}} = 0 \tag{A74}$$

$$S^{1P}_{\substack{4,i\\ i=5,10}} = \sqrt{2i-5}\,[2(1412i^5 - 52945i^4 + 782650i^3 - 5698355i^2 + 20426118i - 28825620)a_4 l^4$$
$$+ 10(i-10)(247i^4 - 6828i^3 + 69745i^2 - 311950i + 515286)a_3 l^3$$
$$+ 2(i-10)(i-9)(949i^3 - 17989i^2 + 112234i - 230124)a_2 l^2$$
$$+ 3(i-10)(i-9)(i-8)(393i^2 - 4614i + 13261)a_1 l$$
$$+ 6(i-10)(i-9)(i-8)(i-7)(172i - 867)a_0]$$
$$/[1680(1918i^5 - 71033i^4 + 1035968i^3 - 7434583i^2 + 26251650i - 36487620)] \tag{A75}$$

$$S^{1P}_{\substack{4,i\\ i=11,\infty}} = 0 \tag{A76}$$

$$S^{1P}_{\substack{i,i\\ i=5,\infty}} = (2i-5)\,[3(7i^4 - 70i^3 + 175i^2 - 52)a_4 l^4 + (2i+1)(2i-11)(7i^2 - 35i + 18)a_3 l^3$$
$$+ 2(2i+1)(2i-11)(5i^2 - 25i + 12)a_2 l^2$$
$$+ 4(2i+1)(2i-1)(2i-9)(2i-11)(a_1 l + 2a_0)]$$
$$/[16l(2i+1)(2i-1)(2i-3)(2i-5)(2i-7)(2i-9)(2i-11)] \tag{A77}$$

$$S^{1P}_{\substack{i,i+1\\ i=5,\infty}} = -\sqrt{2i-5}\sqrt{2i-3}\,[6(i-2)(i^2 - 4i - 2)a_4 l^3 + (i-2)(7i^2 - 28i - 15)a_3 l^2$$
$$+ 2(i-2)(2i+1)(2i-9)(a_2 l + a_1)]$$
$$/[16(2i+1)(2i-1)(2i-3)(2i-5)(2i-7)(2i-9)] \tag{A78}$$

$$S^{1P}_{\substack{i,i+2\\ i=5,\infty}} = -\sqrt{2i-5}\sqrt{2i-1}\,[15(i^2 - 3i - 8)(i^2 - 3i - 6)a_4 l^4 + 8(i+1)(i-4)(2i+3)(2i-9)a_3 l^3$$
$$+ 8(2i+3)(2i-9)(2i^2 - 6i - 5)a_2 l^2 + 8(2i+3)(2i+1)(2i-7)(2i-9)(a_1 l + 2a_0)]$$
$$/[64l(2i+3)(2i+1)(2i-1)(2i-3)(2i-5)(2i-7)(2i-9)] \tag{A79}$$

$$\begin{aligned} S^{1P}_{\substack{i,i+3\\ i=5,\infty}} = {} & \sqrt{2i-5}\sqrt{2i+1}\big[10(i-1)(i^2-2i-6)a_4l^3 + (i-1)(13i^2-26i-72)a_3l^2 \\ & + 4(i-1)(2i+3)(2i-7)(a_2l+a_1)\big] \\ & /\big[32(2i+3)(2i+1)(2i-1)(2i-3)(2i-5)(2i-7)\big] \end{aligned} \tag{A80}$$

$$\begin{aligned} S^{1P}_{\substack{i,i+4\\ i=5,\infty}} = {} & -\sqrt{2i-5}\sqrt{2i+3}\big[i(i-1)(13i^2-13i-116)a_4l^3 \\ & + i(i-1)(2i+5)(2i-7)(3a_3l^2+2a_2l)\big] \\ & /\big[32(2i+5)(2i+3)(2i+1)(2i-1)(2i-3)(2i-5)(2i-7)\big] \end{aligned} \tag{A81}$$

$$S^{1P}_{\substack{i,i+5\\ i=5,\infty}} = \frac{\sqrt{2i-5}\sqrt{2i+5}(i+1)i(i-1)(2a_4l^3+a_3l^2)}{32(2i+5)(2i+3)(2i+1)(2i-1)(2i-3)(2i-5)} \tag{A82}$$

$$S^{1P}_{\substack{i,i+6\\ i=5,\infty}} = -\frac{\sqrt{2i-5}\sqrt{2i+7}(i+2)(i+1)i(i-1)a_4l^3}{64(2i+7)(2i+5)(2i+3)(2i+1)(2i-1)(2i-3)(2i-5)} \tag{A83}$$

$$S^{1P}_{\substack{i,i+j\\ i=5,\infty,\ j=7,\infty}} = 0 \tag{A84}$$

Finally, the definition of the S^{1PP} set is

$$S^{1PP}_{i,j} = \int_0^l P_4(x)(N_i^1)''(N_j^1)''\,dx \tag{A85}$$

and the explicit integrations are

$$S^{1PP}_{1,1} = \frac{3(44a_4l^4+49a_3l^3+56a_2l^2+70a_1l+140a_0)}{35l^3} \tag{A86}$$

$$S^{1PP}_{1,2} = \frac{38a_4l^4+42a_3l^3+49a_2l^2+70a_1l+210a_0}{35l^2} \tag{A87}$$

$$S^{1PP}_{1,3} = -\frac{3(44a_4l^4+49a_3l^3+56a_2l^2+70a_1l+140a_0)}{35l^3} \tag{A88}$$

$$S^{1PP}_{1,4} = \frac{94a_4l^4+105a_3l^3+119a_2l^2+140a_1l+210a_0}{35l^2} \tag{A89}$$

$$S^{1PP}_{2,2} = \frac{36a_4l^4+42a_3l^3+56a_2l^2+105a_1l+420a_0}{105l} \tag{A90}$$

$$S^{1PP}_{2,3} = -\frac{38a_4l^4+42a_3l^3+49a_2l^2+70a_1l+210a_0}{35l^2} \tag{A91}$$

$$S^{1PP}_{2,4} = \frac{78a_4l^4+84a_3l^3+91a_2l^2+105a_1l+210a_0}{105l} \tag{A92}$$

$$S^{1PP}_{3,3} = \frac{3(44a_4l^4+49a_3l^3+56a_2l^2+70a_1l+140a_0)}{35l^3} \tag{A93}$$

$$S^{1PP}_{3,4} = -\frac{94a_4l^4 + 105a_3l^3 + 119a_2l^2 + 140a_1l + 210a_0}{35l^2} \tag{A94}$$

$$S^{1PP}_{4,4} = \frac{204a_4l^4 + 231a_3l^3 + 266a_2l^2 + 315a_1l + 420a_0}{105l} \tag{A95}$$

$$\begin{aligned} \underset{i=5,8}{S^{1PP}_{1,i}} &= -\sqrt{2i-5}\big[3(45i^3 - 904i^2 + 5965i - 12902)a_4l^3 \\ &\quad + 3(i-8)(38i^2 - 463i + 1383)a_3l^2 + 2(i-8)(i-7)(41i-219)a_2l \\ &\quad + 28(i-8)(i-7)(i-6)a_1\big]/\big[7l^2(68i^3 - 1329i^2 + 8551i - 17970)\big] \end{aligned} \tag{A96}$$

$$\underset{i=9,\infty}{S^{1PP}_{1,i}} = 0 \tag{A97}$$

$$\begin{aligned} \underset{i=5,8}{S^{1PP}_{2,i}} &= \sqrt{2i-5}\big[2(50i^3 - 996i^2 + 6526i - 14061)a_4l^3 \\ &\quad + 12(i-8)(7i^2 - 85i + 255)a_3l^2 + (i-8)(i-7)(62i-345)a_2l \\ &\quad + 42(i-8)(i-7)(i-6)a_1\big]/\big[252l(4i^3 - 77i^2 + 483i - 995)\big] \end{aligned} \tag{A98}$$

$$\underset{i=9,\infty}{S^{1PP}_{2,i}} = 0 \tag{A99}$$

$$\begin{aligned} \underset{i=5,8}{S^{1PP}_{3,i}} &= \sqrt{2i-5}\big[3(45i^3 - 904i^2 + 5965i - 12902)a_4l^3 \\ &\quad + 3(i-8)(38i^2 - 463i + 1383)a_3l^2 + 2(i-8)(i-7)(41i-219)a_2l \\ &\quad + 28(i-8)(i-7)(i-6)a_1\big]/\big[7l^2(68i^3 - 1329i^2 + 8551i - 17970)\big] \end{aligned} \tag{A100}$$

$$\underset{i=9,\infty}{S^{1PP}_{3,i}} = 0 \tag{A101}$$

$$\begin{aligned} \underset{i=5,8}{S^{1PP}_{4,i}} &= \sqrt{2i-5}\big[2(74i^3 - 1485i^2 + 9811i - 21333)a_4l^3 \\ &\quad + 3(i-8)(2i-11)(19i-129)a_3l^2 + (i-8)(i-7)(76i-429)a_2l \\ &\quad + 42(i-8)(i-7)(i-6)a_1\big]/\big[252l(4i^3 - 77i^2 + 483i - 995)\big] \end{aligned} \tag{A102}$$

$$\underset{i=9,\infty}{S^{1PP}_{4,i}} = 0 \tag{A103}$$

$$\begin{aligned} \underset{i=5,\infty}{S^{1PP}_{i,i}} &= (2i-5)\big[(35i^4 - 350i^3 + 1145i^2 - 1350i + 432)a_4l^4 \\ &\quad + 2(2i-1)(2i-9)(5i^2 - 25i + 27)a_3l^3 + 4(2i-1)(2i-9)(3i^2 - 15i + 16)a_2l^2 \\ &\quad + 4(2i-1)(2i-3)(2i-7)(2i-9)(a_1l + 2a_0)\big] \\ &\quad /\big[8l^3(2i-1)(2i-3)(2i-5)(2i-7)(2i-9)\big] \end{aligned} \tag{A104}$$

$$\begin{aligned} \underset{i=5,\infty}{S^{1PP}_{i,i+1}} &= -\sqrt{2i-5}\sqrt{2i-3}\big[2(i-2)(7i^2 - 28i + 13)a_4l^3 + 3(i-2)(5i^2 - 20i + 9)a_3l^2 \\ &\quad + 4(i-2)(2i-1)(2i-7)(a_2l + a_1)\big]/\big[8l^2(2i-1)(2i-3)(2i-5)(2i-7)\big] \end{aligned} \tag{A105}$$

$$\begin{aligned} \underset{i=5,\infty}{S^{1PP}_{i,i+2}} &= \sqrt{2i-5}\sqrt{2i-1}\big[2(i-1)(i-2)(7i^2 - 21i - 12)a_4l^2 \\ &\quad + (i-1)(i-2)(2i+1)(2i-7)(3a_3l + 2a_2)\big] \\ &\quad /\big[8l(2i+1)(2i-1)(2i-3)(2i-5)(2i-7)\big] \end{aligned} \tag{A106}$$

$$\underset{i=5,\infty}{S^{1PP}_{i,i+3}} = -\frac{\sqrt{2i-5}\sqrt{2i+1}\,i(i-1)(i-2)(2a_4 l + a_3)}{8(2i+1)(2i-1)(2i-3)(2i-5)} \tag{A107}$$

$$\underset{i=5,\infty}{S^{1PP}_{i,i+4}} = \frac{\sqrt{2i-5}\sqrt{2i+3}(i+1)i(i-1)(i-2)a_4 l}{16(2i+3)(2i+1)(2i-1)(2i-3)(2i-5)} \tag{A108}$$

$$\underset{i=5,\infty,\ j=5,\infty}{S^{1PP}_{i,i+j}} = 0 \tag{A109}$$

Appendix B

Derivation of Vibrations of Uniform Beam

The derivation of the closed-form solutions to the vibrations of a uniform beam is presented in this appendix. First the bending vibration problem is solved, and then the torsional and axial vibration problems are solved.

Consider the translational displacement v only. Using the expressions for the kinetic and strain energies given in equations (2) and (3), respectively, and assuming harmonic motion give the following governing differential equation:

$$v'''' + \frac{\rho\omega_v^2}{E}v'' - \frac{\rho A\omega_v^2}{EI_{ZZ}}v = 0 \tag{B1}$$

When taking into account the cantilever conditions, the geometric boundary conditions are

$$v(0) = 0 \tag{B2}$$

$$v'(0) = 0 \tag{B3}$$

and the natural boundary conditions are

$$v''(l) = 0 \tag{B4}$$

$$v'''(l) + \frac{\rho\omega_v^2}{E}v'(l) = 0 \tag{B5}$$

The general solution to equation (B1) is

$$v(x) = k_1\, e^{b_1 x} + k_2\, e^{-b_1 x} + k_3\cos(b_2 x) + k_4\sin(b_2 x) \tag{B6}$$

where k_1, k_2, k_3, and k_4 are constants that depend upon the boundary conditions, and b_1 and b_2 are defined by

$$b_1 = \sqrt{\frac{\sqrt{A_2^2 + 4A_0} - A_2}{2}} \tag{B7}$$

$$b_2 = \sqrt{\frac{\sqrt{A_2^2 + 4A_0} + A_2}{2}} \tag{B8}$$

where

$$A_0 = \frac{\rho A\omega_v^2}{EI_{ZZ}} \tag{B9}$$

$$A_2 = \frac{\rho\omega_v^2}{E} \tag{B10}$$

Substituting the general solution given by equation (B6) into the boundary conditions (eqs. (B2) through (B5)) results in four linear equations for k_1 through k_4 that can be expressed in matrix form as follows:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ b_1 & -b_1 & 0 & b_2 \\ b_1^2\, e^{b_1 l} & b_1^2\, e^{-b_1 l} & -b_2^2\cos(b_2 l) & -b_2^2\sin(b_2 l) \\ b_1(b_1^2 + A_2)\, e^{b_1 l} & -b_1(b_1^2 + A_2)\, e^{-b_1 l} & b_2(b_2^2 - A_2)\sin(b_2 l) & -b_2(b_2^2 - A_2)\cos(b_2 l) \end{bmatrix} \begin{Bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{B11}$$

The bending natural frequencies of the beam are those which make the determinant of the 4 by 4 matrix in equation (B11) equal to zero. For the example described in the main text of this report, the first four frequencies are

$$f_{v1} = 27.4651 \text{ Hz}$$
$$f_{v2} = 171.6155 \text{ Hz}$$
$$f_{v3} = 478.2670 \text{ Hz}$$
$$f_{v4} = 930.8451 \text{ Hz}$$

Now consider only the torsional displacement θ. Taking equations (2) and (3) as the kinetic and strain energies, respectively, and assuming harmonic motion give the following governing differential equation:

$$\theta'' + \frac{\rho I_{XX}\omega_\theta^2}{GJ}\theta = 0 \tag{B12}$$

The boundary conditions are

$$\theta(0) = 0 \tag{B13}$$

$$\theta'(l) = 0 \tag{B14}$$

With $b^2 = \frac{\rho I_{XX}\omega_\theta^2}{GJ}$, the general solution to equation (B12) can be written as

$$\theta(x) = k_1 \cos(bx) + k_2 \sin(bx) \tag{B15}$$

Substituting equation (B15) into the boundary conditions leads to the 2 by 2 matrix system of linear equations:

$$\begin{bmatrix} 1 & 0 \\ -\sin(bl) & \cos(bl) \end{bmatrix} \begin{Bmatrix} k_1 \\ k_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{B16}$$

Setting the determinant equal to zero and solving for ω_θ yields

$$\omega_\theta = \frac{n\pi}{2l}\sqrt{\frac{GJ}{\rho I_{XX}}} \qquad (n = 1, 2, \ldots) \tag{B17}$$

For the beam of figure 3, the first torsional frequency is

$$f_{\theta 1} = 523.4749 \text{ Hz}$$

The governing differential equation for the axial displacement of the beam is of the same form as that for the torsional displacement given in equation (B12). Only the constant b is different. The natural frequencies can be shown to be

$$\omega_u = \frac{n\pi}{2l}\sqrt{\frac{E}{\rho}} \qquad (n = 1, 2, \ldots) \tag{B18}$$

For the beam of figure 3, the first axial frequency is

$$f_{u1} = 841.7997 \text{ Hz}$$

References

1. Cronkhite, J. D.; Berry, V. L.; and Dompka, R. V.: *Summary of the Modeling and Test Correlations of a NASTRAN Finite Element Vibrations Model for the AH-1G Helicopter.* NASA CR-178201, 1987.
2. Gabel, R.; Reed, D.; Ricks, R.; and Kesack, W.: Planning, Creating and Documenting a NASTRAN Finite Element Model of a Modern Helicopter. *Rotorcraft Dynamics 1984,* NASA CP-2400, 1985, pp. 307–323.
3. Toossi, Mostafa; Straub, Friedrich; Weisenburger, Richard; and Ferg, Douglas: AH-64A Shake Test and Correlation With NASTRAN Finite Element Model Predictions. *43rd Annual Forum Proceedings,* Volume I, American Helicopter Soc., c.1987, pp. 197–204.
4. Durno, Jason A.; Howland, Guy R.; and Twomey, William J.: Comparison of Black Hawk Shake Test Results With NASTRAN Finite Element Analysis. *43rd Annual Forum Proceedings,* Volume II, American Helicopter Soc., c.1987, pp. 511–526.
5. Lindberg, G. M.: Vibration of Non-Uniform Beams. *Aeronaut. Q.,* vol. XIV, pt. 4, Nov. 1963, pp. 387–395.
6. Sanger, D. J.: Transverse Vibration of a Class of Non-Uniform Beams. *J. Mech. Eng. Sci.,* vol. 10, no. 2, Apr. 1968, pp. 111–120.
7. Thomas, J.; and Dokumaci, E.: Improved Finite Elements for Vibration Analysis of Tapered Beams. *Aeronaut. Q.,* vol. 24, pt. 1, Feb. 1973, pp. 39–46.
8. To, C. W. S.: Higher Order Tapered Beam Finite Elements for Vibration Analysis. *J. Sound & Vibration,* vol. 63, no. 1, Mar. 8, 1979, pp. 33–50.
9. To, C. W. S.: A Linearly Tapered Beam Finite Element Incorporating Shear Deformation and Rotary Inertia for Vibration Analysis. *J. Sound & Vibration,* vol. 78, no. 4, Oct. 22, 1981, pp. 475–484.
10. Hodges, Dewey H.; Hopkins, A. Stewart; Kunz, Donald L.; and Hinnant, Howard E.: Introduction to GRASP—General Rotorcraft Aeromechanical Stability Program—A Modern Approach to Rotorcraft Modeling. *J. American Helicopter Soc.,* vol. 32, no. 2, Apr. 1987, pp. 78–90.
11. Craig, Roy R., Jr.: *Structural Dynamics—An Introduction to Computer Methods.* John Wiley & Sons, Inc., c.1981, pp. 198–199.
12. Hurty, Walter C.; and Rubinstein, Moshe F.: *Dynamics of Structures.* Prentice-Hall, Inc., c.1964, pp. 177–201.
13. Rivello, Robert M.: *Theory and Analysis of Flight Structures.* McGraw-Hill, Inc., c.1969, pp. 168 and 215.
14. *MSC/NASTRAN User's Manual—MSC/NASTRAN Version 65.* MSR-39, MacNeal-Schwendler Corp., Nov. 1985.
15. Gallagher, Richard H.: *Finite Element Analysis—Fundamentals.* Prentice-Hall, Inc., c.1975, pp. 111, 115, and 135–143.
16. Bathe, Klaus-Jürgen: *Finite Element Procedures in Engineering Analysis.* Prentice-Hall, Inc., c.1982, pp. 557–672.

Table I. Convergence of Frequencies for Uniform, h-Version Beam

Ndof	Nelem	f_{v1}, Hz	f_{v2}, Hz	f_{v3}, Hz	f_{v4}, Hz	$f_{\theta 1}$, Hz	f_{u1}, Hz
6	1	27.5953	270.3419			577.2137	928.2170
12	2	27.4784	173.0589	581.7178	1649.2712	537.0115	863.5679
18	3	27.4679	172.1745	484.1519	1081.3530	529.4733	851.4457
24	4	27.4660	171.8138	481.9206	944.1763	526.8447	847.2186
30	5	27.4655	171.7006	479.9624	941.5603	525.6302	845.2656
36	6	27.4653	171.6575	479.1306	936.7340	524.9711	844.2057
42	7	27.4652	171.6385	478.7476	934.2196	524.5739	843.5670
48	8	27.4652	171.6291	478.5541	932.8928	524.3162	843.1526
54	9	27.4652	171.6240	478.4485	932.1524	524.1396	842.8685
60	10	27.4652	171.6211	478.3871	931.7163	524.0132	842.6654
66	11	27.4652	171.6193	478.3496	931.4469	523.9198	842.5151
72	12	27.4652	171.6182	478.3256	931.2736	523.8487	842.4008
78	13	27.4652	171.6175	478.3097	931.1583	523.7934	842.3119
84	14	27.4652	171.6170	478.2988	931.0791	523.7495	842.2413
Exact . . .		27.4651	171.6155	478.2670	930.8451	523.4749	841.7997

Table II. Convergence of Frequencies for Uniform, p-Version Beam

Ndof	Idof	f_{v1}, Hz	f_{v2}, Hz	f_{v3}, Hz	f_{v4}, Hz	$f_{\theta 1}$, Hz	f_{u1}, Hz
6	0	27.5953	270.3419			577.2137	928.2170
10	1	27.4734	173.1390	906.1908		525.4401	844.9599
14	2	27.4652	172.5710	490.8033	2110.4920	523.5106	841.8572
18	3	27.4651	171.6203	490.0998	988.0348	523.4752	841.8003
22	4	27.4651	171.6178	478.4139	987.4482	523.4749	841.7997
26	5	27.4651	171.6155	478.4039	932.4859	523.4749	841.7997
30	6	27.4651	171.6155	478.2675	932.4643	523.4749	841.7997
34	7	27.4651	171.6155	478.2674	930.8613	523.4749	841.7997
38	8	27.4651	171.6155	478.2669	930.8613	523.4749	841.7997
42	9	27.4651	171.6155	478.2669	930.8452	523.4749	841.7997
46	10	27.4651	171.6155	478.2669	930.8452	523.4749	841.7997
50	11	27.4651	171.6155	478.2669	930.8451	523.4749	841.7997
54	12	27.4651	171.6155	478.2669	930.8451	523.4749	841.7997
58	13	27.4651	171.6155	478.2669	930.8451	523.4749	841.7997
Exact . . .		27.4651	171.6155	478.2670	930.8451	523.4749	841.7997

Table III. Convergence of Frequencies for Modeling Tapered Beam With Uniform Beam Elements

Ndof	Nelem	f_{v1}, Hz	f_{v2}, Hz	f_{v3}, Hz	f_{v4}, Hz	$f_{\theta 1}$, Hz	f_{u1}, Hz
6	1	36.7115	358.0925			577.2137	928.2170
12	2	59.6562	172.5966	709.7736	1391.3137	934.4916	1255.9186
18	3	71.9805	187.9347	455.8576	1169.0279	1089.9564	1348.4876
24	4	77.8991	205.9470	441.9430	893.5114	1156.4921	1381.3936
30	5	80.9493	218.2811	455.8657	834.3421	1189.1758	1396.3582
36	6	82.6853	226.0285	470.7080	835.1067	1207.3211	1404.3645
42	7	83.7585	230.9909	481.8723	848.0075	1218.3655	1409.1399
48	8	84.4657	234.3112	489.7455	860.5784	1225.5667	1412.2157
54	9	84.9554	236.6301	495.3418	870.3900	1230.5161	1414.3128
60	10	85.3083	238.3102	499.4259	877.7242	1234.0613	1415.8068
66	11	85.5707	239.5654	502.4906	883.2469	1236.6867	1416.9088
72	12	85.7711	240.5273	504.8484	887.4925	1238.6846	1417.7450
78	13	85.9276	241.2805	506.7015	890.8284	1240.2401	1418.3946
84	14	86.0520	241.8811	508.1846	893.5011	1241.4746	1418.9092
90	15	86.1526	242.3676	509.3901	895.6784	1242.4707	1419.3238
150	25	86.5898	244.4943	514.7164	905.4243	1246.7811	1421.1123
300	50	86.7753	245.4031	517.0280	909.7627	1248.6001	1421.8642
450	75	86.8097	245.5723	517.4313	910.5846	1248.9370	1422.0033
600	100	86.8373	245.7077	517.8087	911.2465	1249.2228	1422.1216

Table IV. Convergence of Frequencies for Tapered, h-Version Beam

Ndof	Nelem	f_{v1}, Hz	f_{v2}, Hz	f_{v3}, Hz	f_{v4}, Hz	$f_{\theta 1}$, Hz	f_{u1}, Hz
6	1	86.9704	267.8185			1273.8317	1434.6550
12	2	86.9065	251.7443	533.1733	1203.8627	1271.8282	1432.6379
18	3	86.8561	247.4589	538.7430	940.3664	1263.2856	1428.3265
24	4	86.8434	246.2835	526.0757	958.9185	1258.0545	1425.9792
30	5	86.8396	245.9340	521.0733	934.7022	1255.1634	1424.7097
36	6	86.8384	245.8121	519.2780	921.7652	1253.4584	1423.9658
42	7	86.8379	245.7628	518.5613	916.4367	1252.3826	1423.4975
48	8	86.8376	245.7400	518.2391	914.1021	1251.6647	1423.1850
54	9	86.8375	245.7283	518.0774	912.9746	1251.1633	1422.9669
60	10	86.8374	245.7216	517.9880	912.3746	1250.8000	1422.8087
66	11	86.8374	245.7176	517.9345	912.0268	1250.5286	1422.6906
72	12	86.8374	245.7149	517.9004	911.8101	1250.3208	1422.6002
78	13	86.8373	245.7131	517.8775	911.6672	1250.1582	1422.5294
84	14	86.8373	245.7119	517.8615	911.5687	1250.0287	1422.4729

Table V. Convergence of Frequencies for Tapered, p-Version Beam

Ndof	Idof	f_{v1}, Hz	f_{v2}, Hz	f_{v3}, Hz	f_{v4}, Hz	$f_{\theta 1}$, Hz	f_{u1}, Hz
6	0	86.9704	267.8185			1273.8317	1434.6550
10	1	86.9555	249.6871	607.6043		1254.6750	1434.2370
14	2	86.8675	248.6826	545.9870	1173.2573	1249.4274	1422.2183
18	3	86.8417	246.2815	537.1999	1018.3391	1249.2122	1422.1270
22	4	86.8376	245.7577	520.7568	981.2253	1249.2122	1422.1223
26	5	86.8374	245.7151	517.9835	920.7022	1249.2093	1422.1164
30	6	86.8374	245.7151	517.9218	912.1356	1249.2074	1422.1149
34	7	86.8373	245.7127	517.8910	912.1356	1249.2067	1422.1145
38	8	86.8373	245.7101	517.8475	911.6613	1249.2065	1422.1145
42	9	86.8372	245.7087	517.8238	911.3808	1249.2064	1422.1144
46	10	86.8372	245.7080	517.8140	911.2864	1249.2064	1422.1144
50	11	86.8372	245.7078	517.8104	911.2582	1249.2064	1422.1144
54	12	86.8372	245.7077	517.8092	911.2498	1249.2064	1422.1144
58	13	86.8372	245.7077	517.8088	911.2473	1249.2064	1422.1144
62	14	86.8372	245.7077	517.8087	911.2466	1249.2064	1422.1144
66	15	86.8372	245.7077	517.8086	911.2464	1249.2064	1422.1144
70	16	86.8372	245.7077	517.8086	911.2463	1249.2064	1422.1144
74	17	86.8372	245.7077	517.8086	911.2463	1249.2064	1422.1144
78	18	86.8372	245.7077	517.8086	911.2463	1249.2064	1422.1144
82	19	86.8372	245.7077	517.8086	911.2463	1249.2064	1422.1144

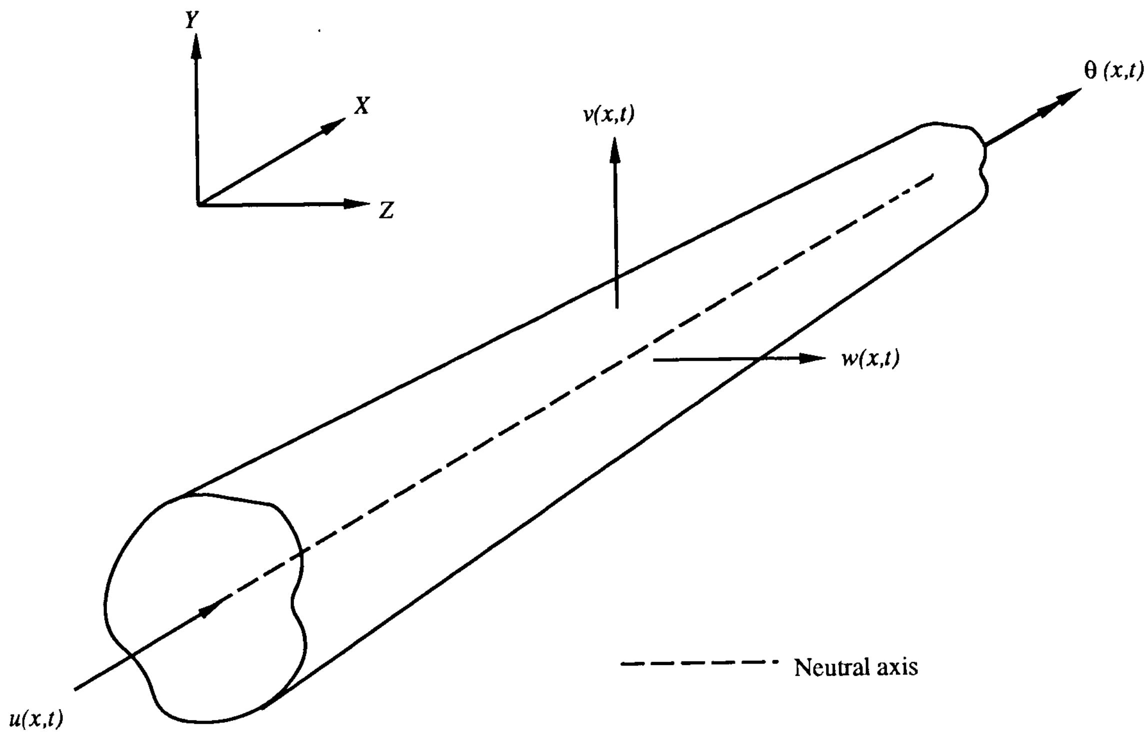

Figure 1. Beam element showing continuous displacements and coordinate system.

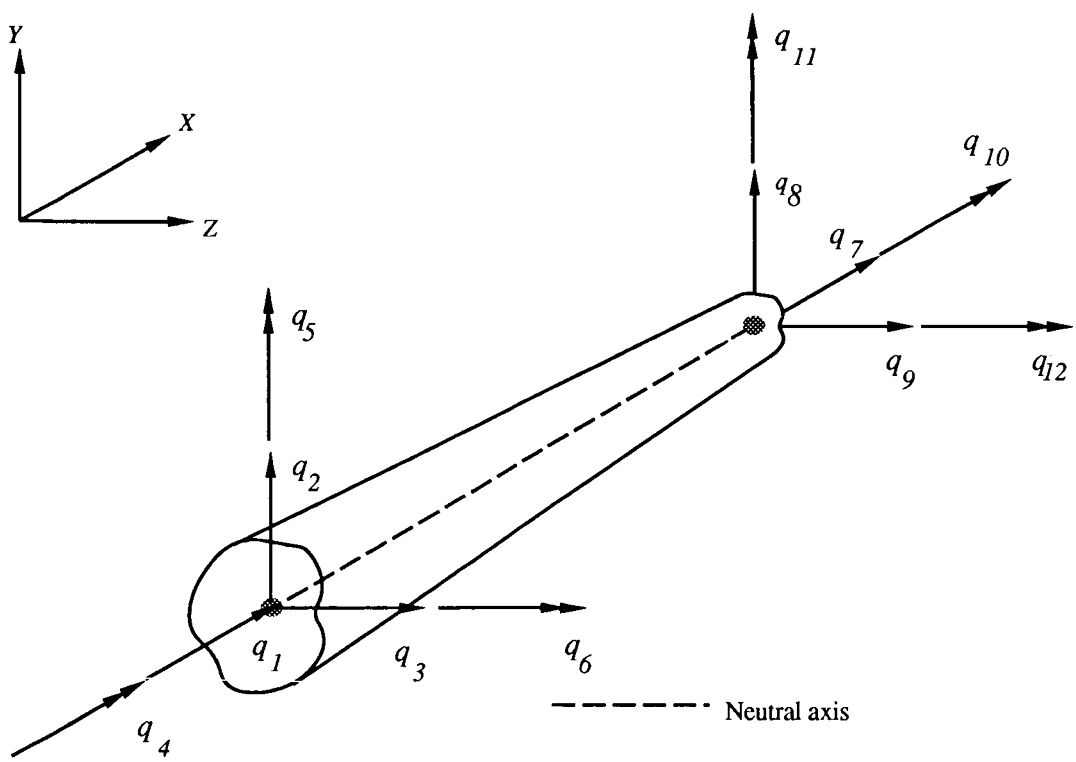

Figure 2. Beam element showing external discrete degrees of freedom.

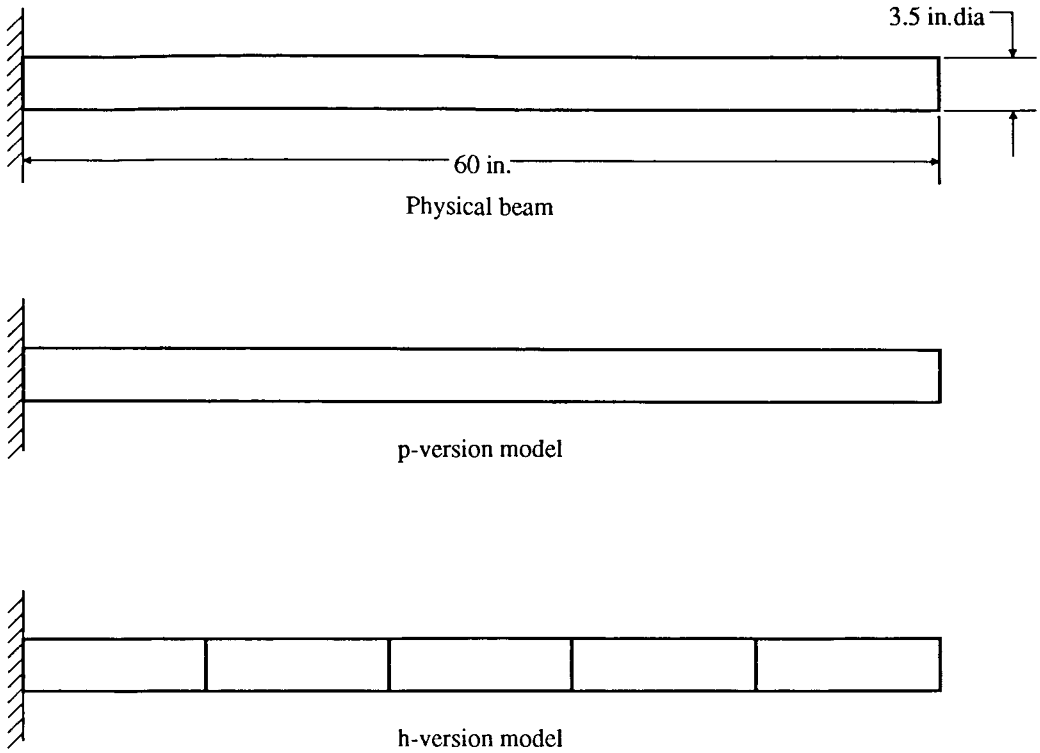

Figure 3. Uniform cantilevered beam and representative h- and p-version finite-element models.

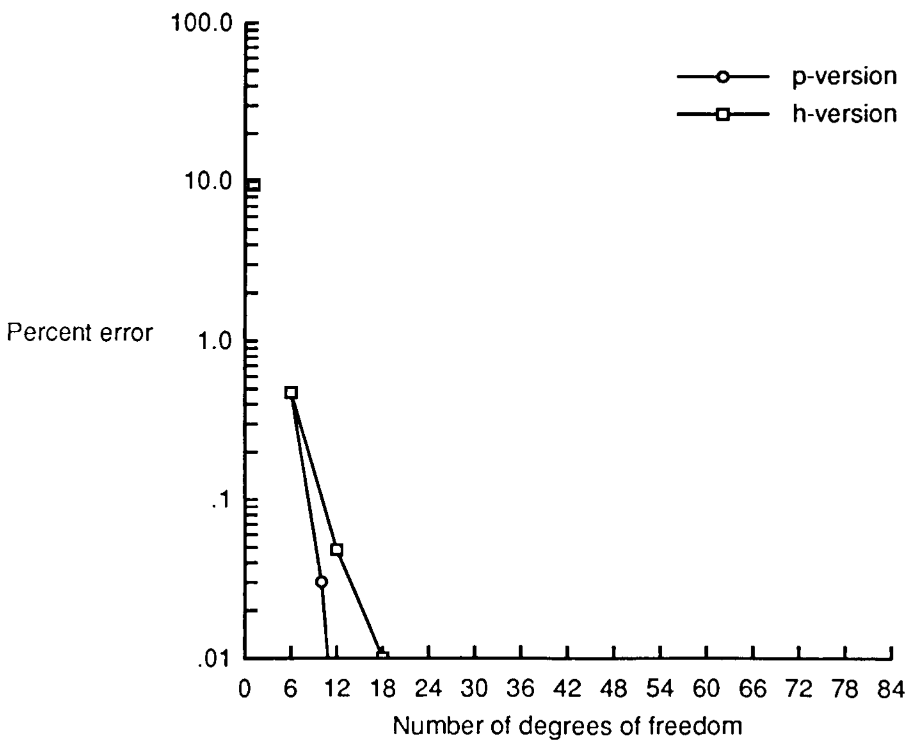

Figure 4. Percent error versus number of degrees of freedom for first bending frequency of uniform beam.

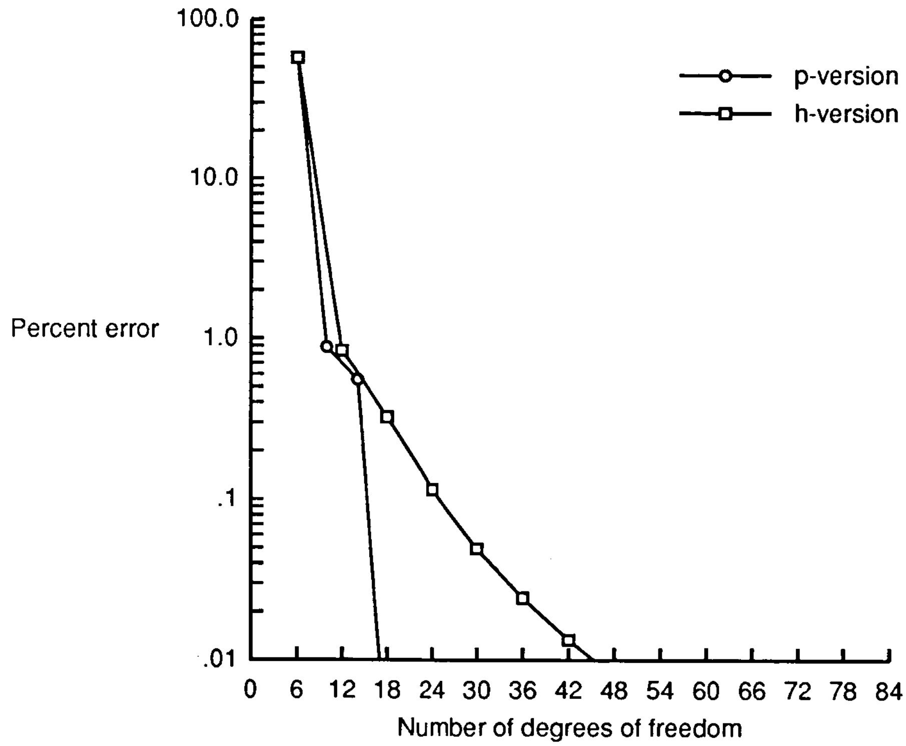

Figure 5. Percent error versus number of degrees of freedom for second bending frequency of uniform beam.

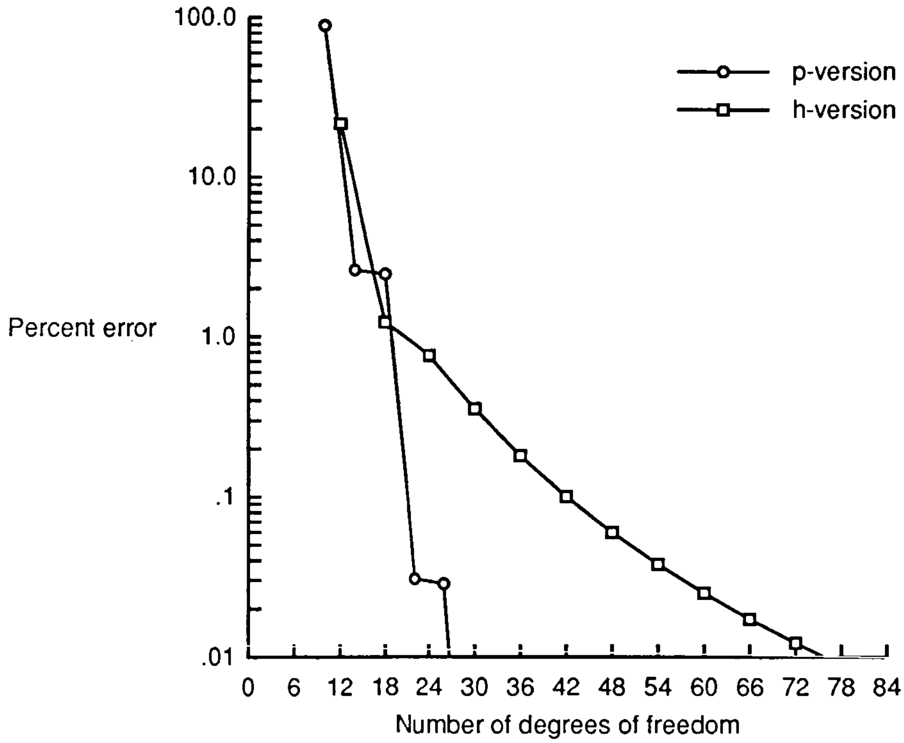

Figure 6. Percent error versus number of degrees of freedom for third bending frequency of uniform beam.

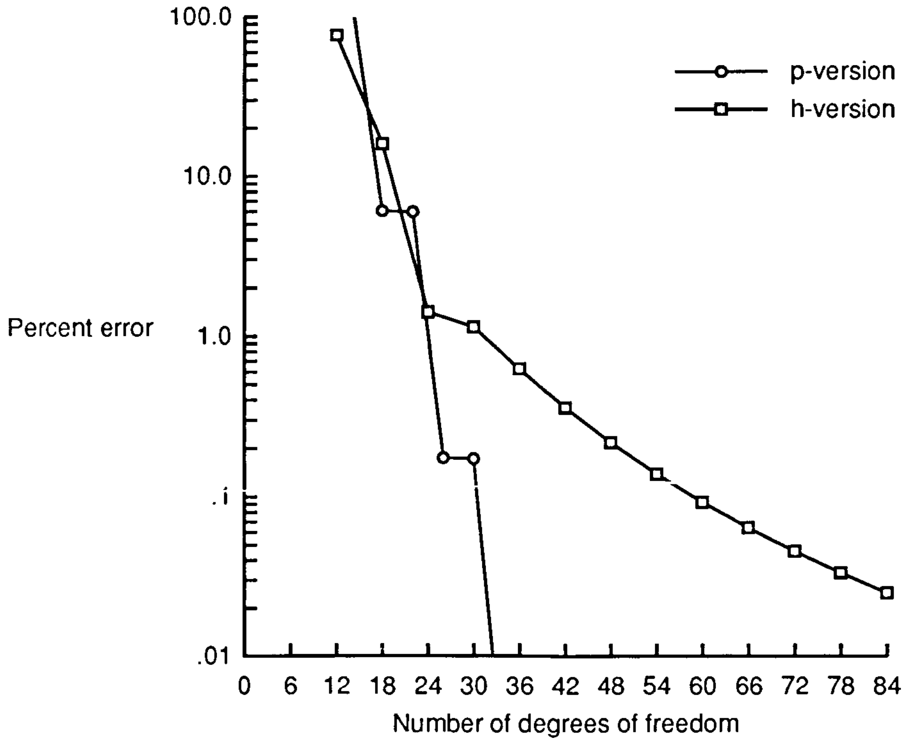

Figure 7. Percent error versus number of degrees of freedom for fourth bending frequency of uniform beam.

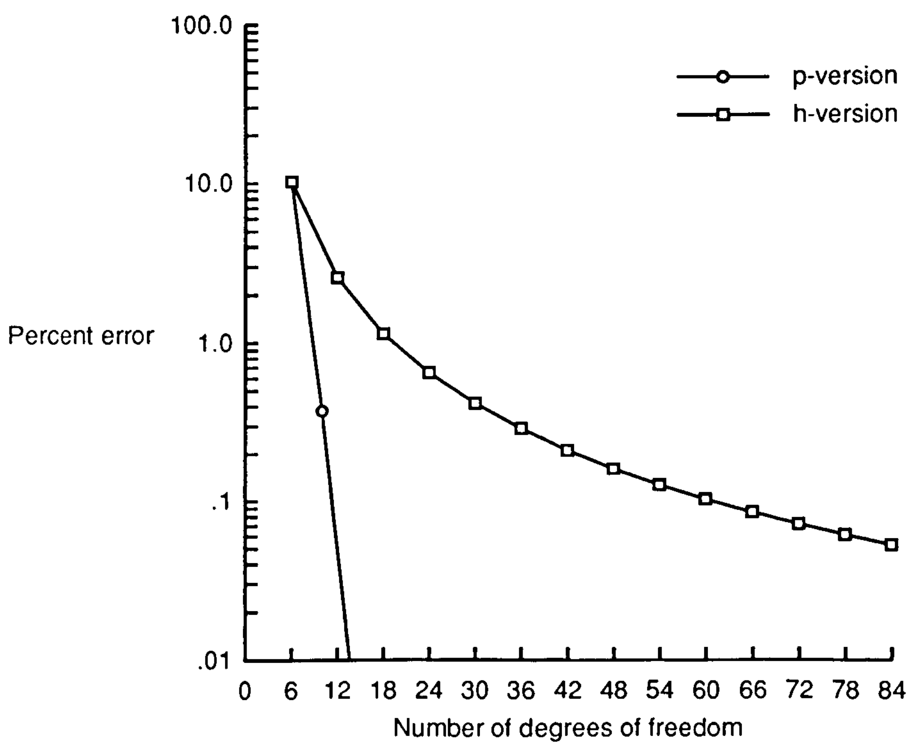

Figure 8. Percent error versus number of degrees of freedom for first torsional frequency of uniform beam.

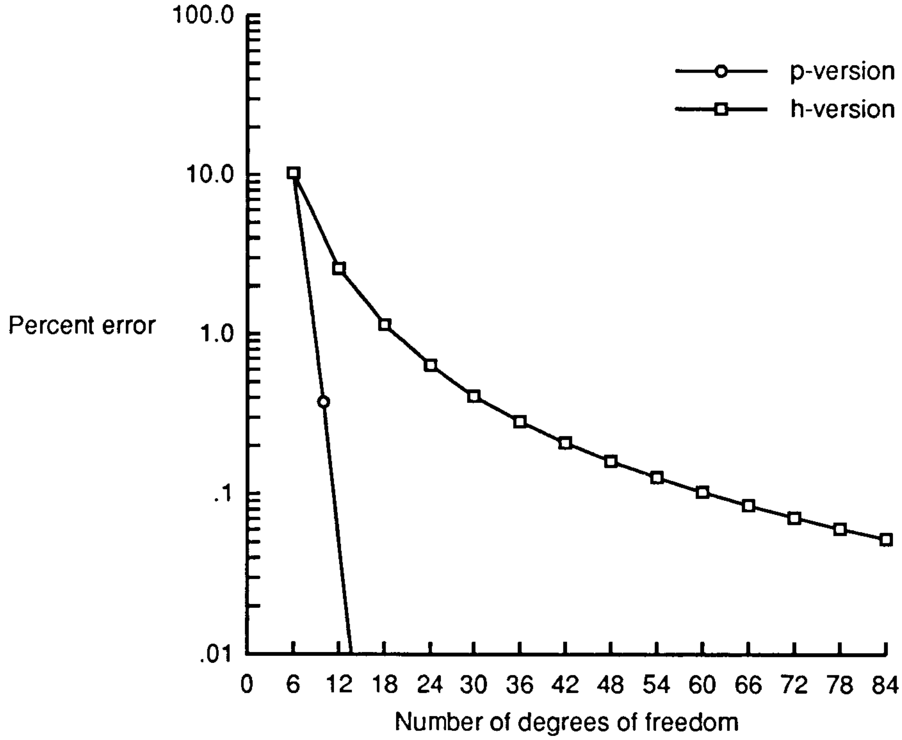

Figure 9. Percent error versus number of degrees of freedom for first axial frequency of uniform beam.

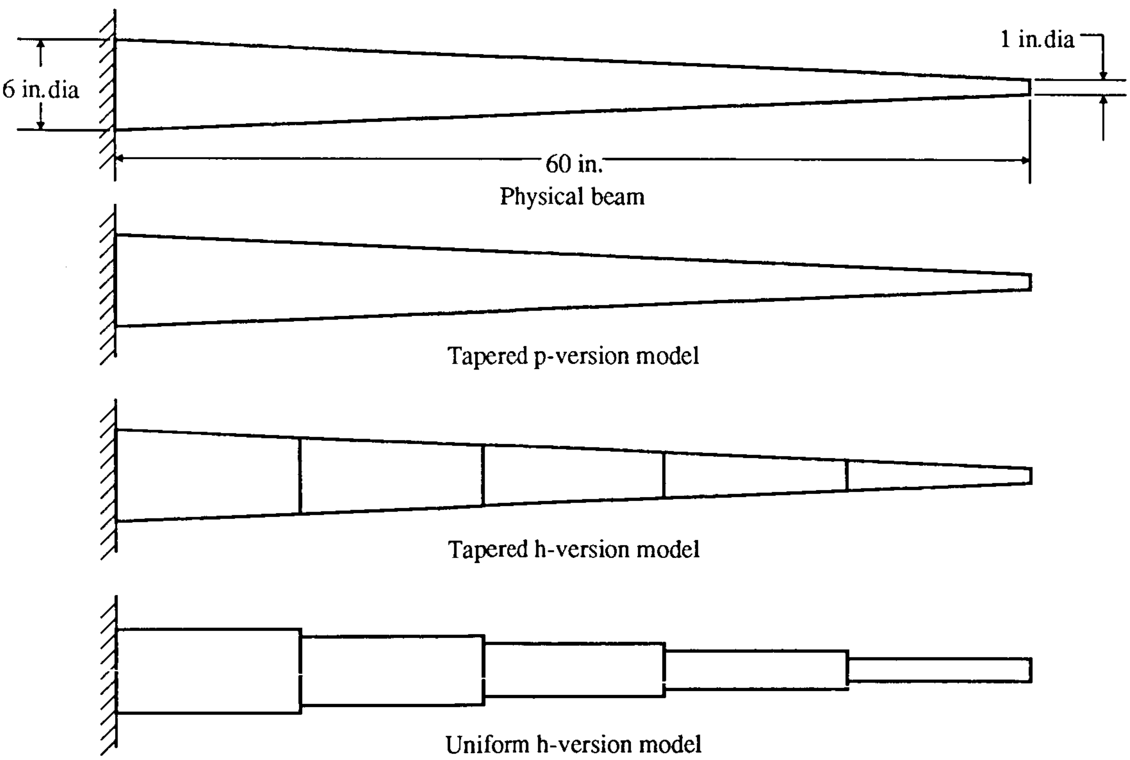

Figure 10. Tapered cantilevered beam and representative h- and p-version finite-element models.

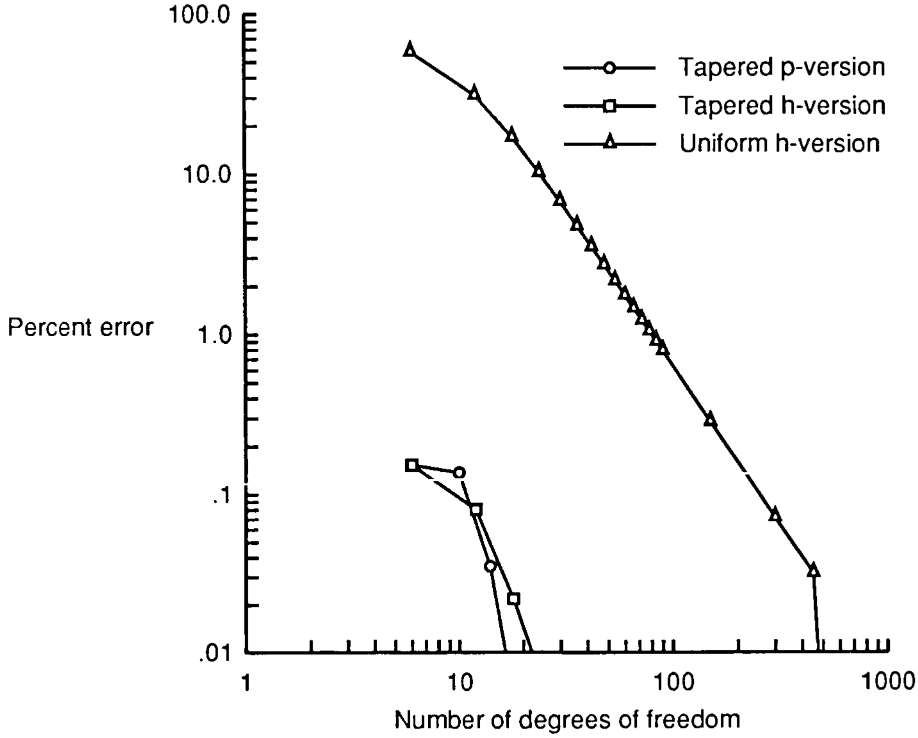

Figure 11. Percent error versus number of degrees of freedom for first bending frequency of tapered beam.

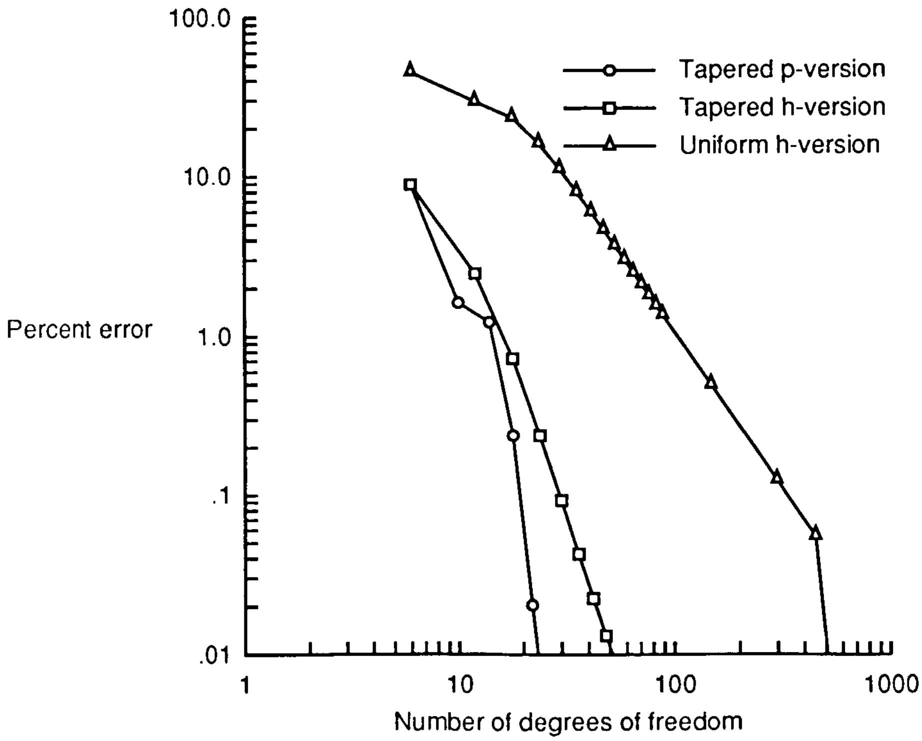

Figure 12. Percent error versus number of degrees of freedom for second bending frequency of tapered beam.

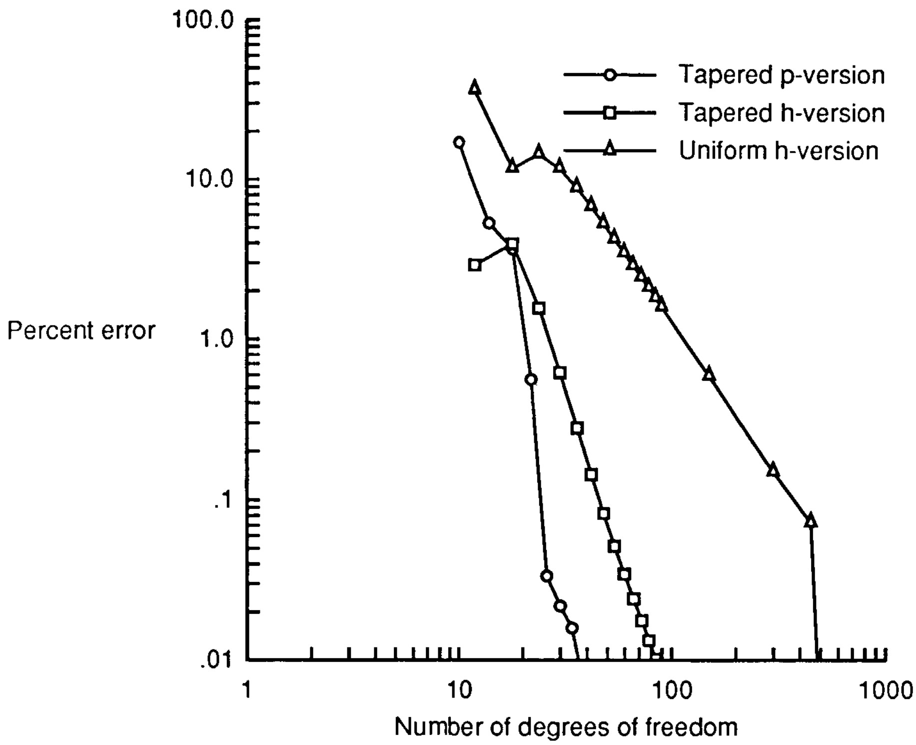

Figure 13. Percent error versus number of degrees of freedom for third bending frequency of tapered beam.

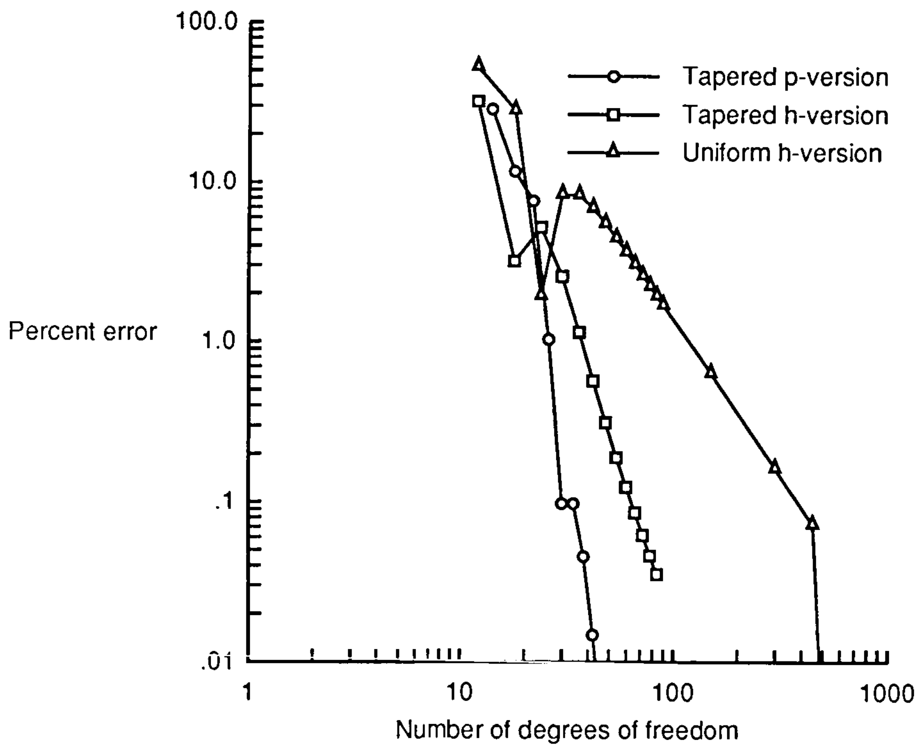

Figure 14. Percent error versus number of degrees of freedom for fourth bending frequency of tapered beam.

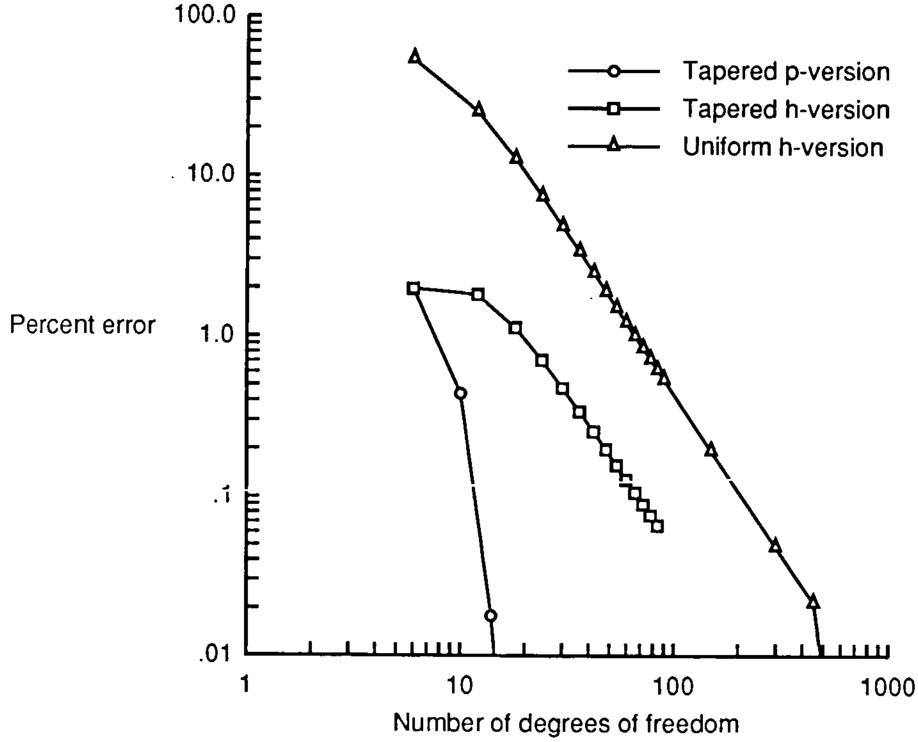

Figure 15. Percent error versus number of degrees of freedom for first torsional frequency of tapered beam.

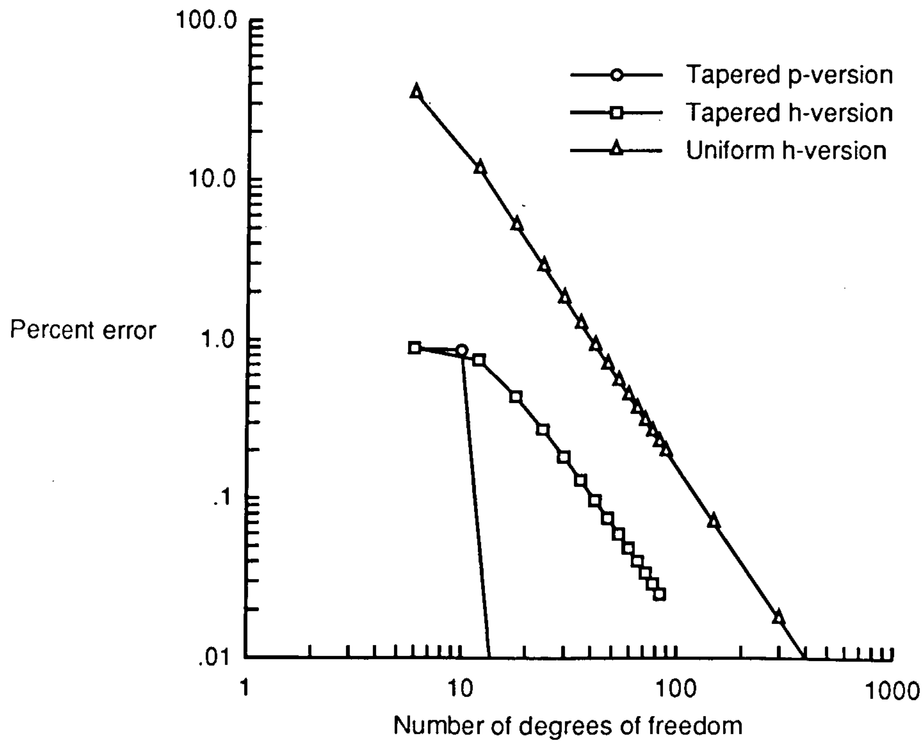

Figure 16. Percent error versus number of degrees of freedom for first axial frequency of tapered beam.

NASA
National Aeronautics and Space Administration

Report Documentation Page

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA TP-2931 AVSCOM TR-89-B-002		

4. Title and Subtitle Derivation of a Tapered p-Version Beam Finite Element	5. Report Date August 1989
	6. Performing Organization Code
7. Author(s) Howard E. Hinnant	8. Performing Organization Report No. L-16577
9. Performing Organization Name and Address Aerostructures Directorate USAARTA-AVSCOM Langley Research Center Hampton, VA 23665-5225	10. Work Unit No. 505-63-51-01
	11. Contract or Grant No.
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, DC 20546-0001 and U.S. Army Aviation Systems Command St. Louis, MO 63120-1798	13. Type of Report and Period Covered Technical Paper
	14. Army Project No. 1L162211A47AB

15. Supplementary Notes
Howard E. Hinnant: Aerostructures Directorate, USAARTA-AVSCOM, Hampton, Virginia.

16. Abstract
A tapered, p-version beam finite element suitable for dynamic applications is derived. The taper in the element is represented by allowing the area moments of inertia to vary as quartic polynomials along the length of the beam and the cross-sectional area to vary as a quadratic polynomial. The p-version finite-element characteristics are implemented through a set of polynomial shape functions. The lower order shape functions are identical to the classical cubic and linear shape functions normally associated with a beam element. The higher order shape functions are a hierarchical set of polynomials that are integrals of orthogonal polynomials. Explicit expressions for the mass and stiffness matrices are presented for an arbitrary value of p. The element has been verified to be numerically stable with shape functions through 22nd order.

17. Key Words (Suggested by Authors(s))	18. Distribution Statement
Finite-element analysis Beam p-version Taper	Unclassified–Unlimited Subject Category 39

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	43	A03

NASA FORM 1626 OCT 86 NASA-Langley, 1989

For sale by the National Technical Information Service, Springfield, Virginia 22161-2171