Nonlinear Mesomechanics of Composites With Periodic Microstructure: First Report

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Summary

This work is concerned with modelling the mechanical deformation or constitutive behavior of composites comprised of a periodic microstructure under small displacement conditions at elevated temperature. A mesomechanics approach [1] is adopted which relates the micromechanical behavior of the heterogeneous composite with its in-service macroscopic behavior.

Two different methods, one based on a Fourier series approach and the other on a Green's function approach, are used in modelling the micromechanical behavior of the composite material. Although the constitutive formulations are based on a micromechanical approach, it should be stressed that the resulting equations are volume averaged to produce overall "effective" constitutive relations which relate the bulk, volume averaged, stress increment to the bulk, volume averaged, strain increment. As such, they are macromodels which can be used directly in nonlinear finite element programs such as MARC, ANSYS and ABAQUS or in boundary element programs such as BEST3D.

In developing the volume averaged or "effective" macromodels from the micromechanical models, both approaches (i.e. Fourier series and Green's function) will require the evaluation of volume integrals containing the spatially varying strain distributions throughout the composite material. By assuming that the strain distributions are spatially constant within each constituent phase—or within a given subvolume within each constituent phase—of the composite material, the volume integrals can be obtained in closed form. This simplified micromodel can then be volume averaged to obtain an "effective" macromodel suitable for use in the MARC, ANSYS and ABAQUS nonlinear finite element programs via user constitutive subroutines such as HYPELA and CMUSER. This "effective" macromodel can be used in a nonlinear finite element structural analysis to obtain the strain-temperature history at those points in the structure where thermomechanical cracking and damage are expected to occur, the so called "damage critical" points of the structure. The "exact" micromechanical models can then be subjected to the overall "effective" strain-temperature history obtained at the "damage critical" location and used outside of the finite element program to evaluate the heterogeneous stress-strain history throughout each constituent phase of the composite material. This variation must be known in order to evaluate the damage history variation throughout each constituent phase of the composite material.

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1   Introduction

The ultimate objective of this work is to produce a computer program to analyze the heterogeneous stress and strain history variation at the "damage critical" locations of a composite structure operating at elevated temperature. This report describes some of the theoretical foundations for the program. A mesomechanics [1] approach is adopted which relates the micromechanical behavior of the heterogeneous composite to its in-service macroscopic behavior.

Some composites are actually comprised of a periodic microstructure whilst others are possessed of an essentially randomly distributed microstructure. Pictures of metal matrix composites (tungsten-fiber-reinforced superalloys) which exhibit a periodic microstructure are shown in Fig. 1 which is taken from the article by Petrasek *et al* [2]. When the fibers in a composite material occupy a large volume fraction of the material, the induced deformation in one fiber interacts with and alters the induced deformation in the neighboring fibers. When the fibers are densely packed the interaction effect becomes dominant and must be accounted for in the constitutive formulation.

At NASA-Lewis Chamis and his colleagues [3,4,5] employ two different approaches for analyzing the behavior of structural composites. One method employs a sophisticated finite element analysis of a periodic microstructure. A unit cell in the periodic microstructure is modelled with one hundred and ninety two three-dimensional elements and the eight nearest neighbor cells of the fibrous composite are modelled with superelements. By applying the strain-temperature history at the "damage critical" location in the composite structure to the superelement model, the stress-strain history throughout the unit cell can be computed and used to estimate the maximum damage in the composite structure. This method will necessarily require large resources in computer time and memory to analyze the viscoplastic behavior of the composite structure under in-service thermomechanical loading conditions.

Another approach adopted by Chamis and his colleagues [3,4,5]—which avoids large computer resources—is to employ composite micromechanics theory to derive simplified relationships which describe the thermomechanical constitutive behavior of multilayered fibrous composites.

When suitable boundary conditions are applied to the superelement model of the periodic microstructure, it is possible to predict the elastic properties of the equivalent homogenized material. A comparison [5] with the homogenized elastic properties predicted by the simplified micromechanical equations generally shows good agreement with the superelement model except for the Poisson ratios. At high volume fractions ($\sim$ 60%) the longitudinal Poisson's ratio for unidirectional fibers predicted by the simplified equations is about 15% too small, whilst the transverse Poisson's ratio is about 30% too small. These anomalies occur because the interaction between the fibers is not accounted for in the simplified micromechanical model. This may be important when considering the highly nonlinear behavior of viscoplastic composites at elevated temperature.

Dvorak [6] and Dvorak and Bahei-El-Din *et al* [7,8,9] have also made great progress in modelling the micromechanical behavior of nonlinear composite materials and are embarked on a combined experimental and theoretical effort. The variation of the stress-strain history throughout the unit cell of a periodic microstructure is obtained with a finite element analysis in which the interaction effects of the surrounding cells is accounted for by applying periodic

boundary conditions to the surface of the unit cell.

Work on the theoretical foundations behind the homogenization of micromechanical constitutive models to produce bulk macroscopic models has been under way in France by Devriès and Léné [10], Léné [11,12], Duvaut [13], Renard and Marmonier [14], Léné and Leguillon [15] and Sanchez-Palencia [16,17]. These references give a good account of the work being conducted in France by other researchers. Much of this work is founded on the use of multivariable asymptotic techniques [18]. In an infinite periodic structure the stress-strain history in each unit periodic cell is, perforce, identical. Due to the finite size of the composite structure the effects of surface tractions and displacements on the surface will cause the stress-strain history to vary from cell to cell. If the unit cell is much smaller than the size of the structure this variation from cell to cell will be small. If L is a typical dimension of the unit periodic cell and D is a typical dimension of the composite structure, then the ratio L/D is a small parameter of the problem. The displacement variation throughout the unit cell will then depend on six spatial variables, $i.e.$,

$$u_i = u_i \left(x_1, x_2, x_3, x_1^\star, x_2^\star, x_3^\star \right)$$

where $x_i^\star = x_i L/D$. The spatial variables $x_i^\star$ take into account the slow variation of the displacement from cell to cell due to the finite size of the ratio L/D when u_i is a periodic function of the variables x_i. By expanding the displacement and other spatial variables of the problem into a series in powers of L/D and equating like powers in the perturbation expansion, it is possible to obtain the effect of the finite size of the structure on the deformation behavior in the unit cell. Due to the perturbative assumption of small L/D this method is not expected to be valid for thin composite sections or to be applicable at those places in the structure where surface effects or nonperiodic inclusions are important.

Rather than employing finite element techniques to determine the stress-strain history variation throughout the unit periodic cell, Aboudi [19] has recently developed a macroscopic formulation for periodic composites based on volume averaging a viscoplastic constitutive model over the unit periodic cell. This work expands the heterogeneous displacement throughout the constituent phases of the unit cell as linear and higher order functions of the coordinates. Good agreement with experimental results was achieved by volume averaging Bodner's [20] viscoplastic constitutive model over the unit periodic cell, but the method is general and any constitutive model may be used to represent the deformation behavior of the constituent phases. The limitation here is that large spatial gradients in the strain history may not be accurately modelled by linear or quadratic interpolation functions on the unit periodic cell.

Weng and his colleagues [21,22] have employed self-consistent methods to study the effect of inclusion size and volume fraction on the stress distribution in and around spheroidal inclusions embedded in an "effective" non-uniform matrix material, and the effect which this has on the overall "effective" macroscopic constitutive behavior of the composite. In the first paper they point out that the derivation of the fictitious body forces which represent the inelastic behavior of the heterogeneous composite material should be obtained from first principles rather than using their heuristic approach. In the second paper the Mori-Tanaka theorem [24] is used to represent the effect of the heterogeneous composite, and a similar procedure is followed in the present work to develop a self-consistent method for composites which exhibit a periodic microstructure. In addition, the present report also derives the

fictitious body forces for a periodic microstructure from first principles. In reference [23] Zhu and Weng have used a combined micromechanics and continuum theory approach to develop a creep deformation model for particle-strengthened metal matrix composites. They stress the fact that the creep resistance of the composite is underestimated when simplified metallurgical and mechanics approaches are adopted.

A comprehensive application of micromechanics to mechanical deformation problems is given by Mura [24] in his book "Micromechanics of Defects in Solids". This work was used by Nemat-Nasser and his colleagues who have exploited the mathematical simplicity of a periodic microstructure in order to develop elastic, plastic and creep constitutive models [25,26,27,28] for composite materials. The assumption of periodicity allows the heterogeneous stress, strain and displacement fields to be expanded in a Fourier series, which greatly simplifies the ensuing computations. This technique fully accounts for the interaction effects between neighboring fibers. Even when the composite is comprised of closely packed fibers distributed at random the method gives accurate results [25] for the "effective" elasticity tensor. When densely packed fibers form a large volume fraction of the composite material these interaction effects play a dominant rôle and must be included in the calculations. It appears that inclusion of the interaction effects can be as, or more, important than inclusion of the random nature of the microstructure when the fibers occupy a large volume fraction of the composite material. In this report we have developed the Fourier series approach in order to handle the viscoplastic behavior of the constituents in the unit periodic cell.

The nonlinear constitutive behavior of composites with a periodic microstructure can also be treated with a Green's function approach [29,30,31,32,33]. Here, the periodic heterogeneous material property variation—due to the fibers—is treated as a fictitious body force in the matrix material. The Green's function is used to evaluate the displacement due to a unit point force in the matrix material and the actual displacement at any point in the composite can then be determined by summing (integrating) the effect due to a volume distribution of fictitious periodic body forces. It is shown in Appendix B that this method is exactly equivalent to the Fourier series approach by invoking a mathematical technique known as the Poisson sum formula. The Green's function approach is more general in that the method can also handle the nonperiodic case where there may be inclusions in one unit cell but not in the neighboring cells. It is also able to handle surface effects, although the surface integrals which represent the surface effects in the Green's function method could be expanded in a Fourier series for thin composite sections.

The approach adopted in the present work is to develop homogenization techniques which can provide simplified macromodels for use in a nonlinear finite element program, similar in spirit to the simplified models used at NASA-Lewis, but which account for the viscoplastic interaction effects in the periodic structure and which allow surface effects for thin structures to be taken into account. Once the strain-temperature history at the "damage critical" location has been found from the finite element analysis, it can be used to "drive" the micromechanical relations in order to obtain the stress-strain history variation throughout the unit cell. These micromechanical relations are the same relations which are used to obtain the simplified homogenized constitutive model. When the unit cell is chosen to have the form shown in Fig. 2, it is clear that a periodic arrangement of such a microstructure allows for the analysis of laminated composite structures.

2 Overview of Theoretical Modelling Approaches

2.1 Outline of Approach

In order to develop a homogeneous macroscopic constitutive model from micromechanical principles it is necessary to know the stress-strain history throughout the unit cell of a periodic composite. Some of the approaches which are presently being given currency are described in the introduction.

The Fourier series and Green's function approaches can be used to compute the viscoplastic stress-strain history throughout the unit cell of a periodic composite and can be simplified to produce a model suitable for use in a nonlinear finite element program. In this report the Fourier series and Green's function approaches are developed and shown to be equivalent to each other by means of the Poisson sum formula. This equivalence holds only for an infinitely extended medium. When the medium has a finite size the effect of spatially varying displacements and tractions on the surface of the medium must be accounted for. This is easily accomplished with the Green's function method by retaining the appropriate surface integral contributions which are discarded in the case of an infinite medium. In the Fourier series approach the surface integral could be included, and, in the case of a thin composite section which has an infinite surface the integral can be expanded into a Fourier series. In fact, the methods can be combined, so that if inclusions are present in one unit cell and not in the neighboring cells, their effect can be taken into account in the Fourier series method by treating them with a Fourier integral or Green's function approach.

An overview of the present work is depicted in Fig. 3. Simplified versions of the micromechanical constitutive equations can be volume averaged to produce a macroscopic homogenized viscoplastic model. This can then be used in a nonlinear finite element program to analyze the structural behavior of a composite structure under in-service thermomechanical loading conditions. The finite element analysis yields the strain-temperature history at the "damage critical" location and this history can then be applied to the micromechanical equations to determine the heterogeneous stress-strain history throughout the unit cell.

A detailed flow chart in Fig. 4 shows the anticipated structural analysis procedure. Both the Fourier series and Green's function approaches can be used to create a coarse subvolume model. This coarse model can then be homogenized and included in a user defined constitutive subroutine in a nonlinear finite element program. The Green's function approach can also be used to derive a simple self-consistent model for use in the user subroutine.

2.2 Homogenized Macroscopic Equations

A periodic composite material is supposed acted upon by an imposed strain increment $\Delta\varepsilon_{ij}^0$ and responds in bulk with a stress increment $\Delta\sigma_{ij}^0$. These values are then equated to the respective volume averaged quantities in order to obtain the "effective" constitutive relation for the composite material, $i.e.$,

$$\Delta\sigma_{ij}^0 = \frac{1}{V}\iiint_V \Delta\sigma_{ij}(\mathbf{r})\,dV(\mathbf{r}) \quad \text{and} \quad \Delta\varepsilon_{ij}^0 = \frac{1}{V}\iiint_V \Delta\varepsilon_{ij}^T(\mathbf{r})\,dV(\mathbf{r}) \tag{1}$$

where V is the volume of the body.

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In section 4 it is shown that the volume averaged or "effective" constitutive relation for the composite material can be written as

$$\Delta\sigma_{ij}^0 = D_{ijkl}^m \Delta\varepsilon_{kl}^0 - \frac{1}{V_c}\iiint\limits_{V_c} \left\{ D_{ijkl}^m \Delta c_{kl}(\mathbf{r}) - \delta D_{ijkl}(\mathbf{r}) \left[\Delta\varepsilon_{kl}^T(\mathbf{r}) - \Delta c_{kl}(\mathbf{r}) \right] \right\} dV(\mathbf{r}) \quad (2)$$

where V_c is the volume of a unit periodic cell in the composite material, $\Delta\varepsilon_{kl}^T(\mathbf{r})$ is the total strain increment at point $\mathbf{r}$ in the periodic cell due to the imposed uniform total strain increment $\Delta\varepsilon_{kl}^0$ at the surface of the composite, and $\Delta c_{kl}(\mathbf{r})$ is the strain increment at point $\mathbf{r}$ in the periodic cell representing the deviation from isothermal elastic behavior. The fourth rank tensor $\delta D_{ijkl}(\mathbf{r})$ is defined by the relation

$$\delta D_{ijkl}(\mathbf{r}) = \vartheta(\mathbf{r}) \left(D_{ijkl}^f - D_{ijkl}^m \right) \quad (3)$$

where $\vartheta(\mathbf{r}) = 1$ in the fiber and $\vartheta(\mathbf{r}) = 0$ in the matrix, with D_{ijkl}^f denoting the elasticity tensor of the fiber and D_{ijkl}^m that of the matrix.

In the expression for the average or "effective" constitutive relation in equation 2, the quantities $\Delta\varepsilon_{kl}^0$, D_{ijkl}^m and $\delta D_{ijkl}(\mathbf{r})$ are given. The deviation strain increment $\Delta c_{kl}(\mathbf{r})$ can be obtained throughout the periodic cell as a function of position $\mathbf{r}$ by using an explicit forward difference method since the stress and state variables in a viscoplastic formulation will be known functions of position at the beginning of the increment. Everything is therefore known explicitly except the total strain increment $\Delta\varepsilon_{kl}^T(\mathbf{r})$.

2.3 Fourier Equation Overview

In the Fourier series approach described in section 4 we find that the total strain increment is determined by solving the integral equation,

$$\Delta\varepsilon_{kl}^T(\mathbf{r}) = \Delta\varepsilon_{kl}^0 + \frac{1}{V_c} \sum \sum_{n_p=0}^{\pm\infty} {\sum}' g_{klij}(\boldsymbol{\zeta}) \times$$

$$\times \iiint\limits_{V_c} e^{i\boldsymbol{\xi}\cdot(\mathbf{r}-\mathbf{r}')} \left\{ D_{ijrs}^m \Delta c_{rs}(\mathbf{r}') - \delta D_{ijrs}(\mathbf{r}') \left[\Delta\varepsilon_{rs}^T(\mathbf{r}') - \Delta c_{rs}(\mathbf{r}') \right] \right\} dV(\mathbf{r}') \quad (4)$$

where the fourth rank tensor $g_{klij}(\boldsymbol{\zeta})$ is given by

$$g_{klij}(\boldsymbol{\zeta}) = \frac{1}{2} \left(\zeta_j\zeta_l M_{ik}^{-1}(\boldsymbol{\zeta}) + \zeta_j\zeta_k M_{il}^{-1}(\boldsymbol{\zeta}) \right) \quad (5)$$

in which the Christoffel stiffness tensor $M_{ij}(\boldsymbol{\zeta})$, with inverse $M_{ij}^{-1}(\boldsymbol{\zeta})$ is defined (cf. [33]) by the relation,

$$M_{ij}(\boldsymbol{\zeta}) = D_{pijq}^m \zeta_p \zeta_q \quad (6)$$

with $\zeta_p = \xi_p/\sqrt{\xi_m\xi_m} = \xi_p/\xi$ being a unit vector in the direction of the Fourier wave vector $\boldsymbol{\xi}$, and $\xi = \sqrt{\xi_m\xi_m}$ denoting the magnitude of the vector $\boldsymbol{\xi}$. In equation 4 the sum is taken over integer values in which

$$\xi_1 = \frac{2\pi n_1}{L_1}, \quad \xi_2 = \frac{2\pi n_2}{L_2}, \quad \xi_3 = \frac{2\pi n_3}{L_3} \quad (7)$$

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and L_1, L_2, L_3 are the dimensions of the unit periodic cell in the x_1, x_2, x_3 directions, so that $V_c = L_1 L_2 L_3$. The values of n_1, n_2, n_3 are given by

$$n_p = 0, \pm 1, \pm 2, \pm 3, \dots, etc., \quad \text{for } p = 1, 2, 3 \tag{8}$$

and the prime on the triple summation signs indicates that the term with $n_1 = n_2 = n_3 = 0$ is excluded from the sum.

2.4 Green's Equation Overview

In the Green's function approach the total strain increment $\Delta\varepsilon_{kl}^T(\mathbf{r})$ is determined by solving a different integral equation, viz.,

$$\Delta\varepsilon_{kl}^T(\mathbf{r}) = \Delta\varepsilon_{kl}^0 + \iiint\limits_V U_{klmn}(\mathbf{r} - \mathbf{r}') \left\{ D_{mnrs}^m \Delta c_{rs}(\mathbf{r}') - \right.$$
$$\left. - \delta D_{mnrs}(\mathbf{r}') \left[\Delta\varepsilon_{rs}^T(\mathbf{r}') - \Delta c_{rs}(\mathbf{r}') \right] \right\} dV(\mathbf{r}') \tag{9}$$

where the fourth rank tensor $U_{klmn}(\mathbf{r} - \mathbf{r}')$ gives the kl component of the total strain increment at point $\mathbf{r}$ due to the mn component of a stress increment applied at point $\mathbf{r}'$ in the infinite matrix with elasticity tensor D_{mnrs}^m, i.e.,

$$U_{klmn}(\mathbf{r} - \mathbf{r}') = -\frac{1}{2} \left(\frac{\partial^2 G_{km}(\mathbf{r} - \mathbf{r}')}{\partial x_l \partial x_n} + \frac{\partial^2 G_{lm}(\mathbf{r} - \mathbf{r}')}{\partial x_k \partial x_n} \right) \tag{10}$$

and the volume integration in equation 9 extends over all the periodic cells in the composite material, i.e., over the entire composite.

The Green's function tensor is defined in Appendix A, equation A.26, by the Fourier integral [24,32,33]

$$G_{ij}(\mathbf{r} - \mathbf{r}') = \iiint\limits_{-\infty}^{\infty} \frac{d^3 \mathbf{K}}{(2\pi)^3} \frac{M_{ij}^{-1}(\boldsymbol{\zeta})}{K^2} e^{-i\mathbf{K}.(\mathbf{r} - \mathbf{r}')} \tag{11}$$

in which the tensor $\boldsymbol{\zeta}$ is now defined by the relation $\zeta_i = K_i/K$ with $K = \sqrt{K_q K_q}$ denoting the magnitude of the vector $\mathbf{K} = (K_1, K_2, K_3)$.

In Appendix B it is shown, by applying the Poisson sum formula, that equations 4 and 9 are identical, although the summation extends over the integer values n_1, n_2, n_3 in equation 4 and extends over the periodic cells in equation 9.

2.5 Integration of the Equations

Both equations 4 and 9 are implicit integral equations for the determination of the total strain increment $\Delta\varepsilon_{kl}^T(\mathbf{r})$, since this unknown quantity appears both on the left hand sides of the equations and on the right hand sides under the volume integrations.

The "effective" constitutive relation given in equation 2 and the total strain increment relation, given by either equation 4 or 9, contain the volume integration of the deviation strain increment $\Delta c_{kl}(\mathbf{r})$. In the periodic cell the deviation strain increment at any point $\mathbf{r}$

will be determined from a unified viscoplastic constitutive relation [34] appropriate to the constituent phase in which the point **r** resides. If a constituent phase is included at the fiber-matrix interface, a constitutive relation can also be proposed for this phase, and the resulting inelastic strain increment determined for inclusion in the volume integrals. This may be important for metal matrix composites where there can be chemical reactions between fiber and matrix at elevated temperatures, and for composites where the fiber has been coated to enhance overall composite properties.

Equations 2, 4 and 9 form the basic incremental constitutive equations for determining the "effective" overall deformation behavior of a composite material with a periodic microstructure. In order to update the stress state in each of the constituent phases in preparation for integrating the "effective" constitutive relation over the next increment, the constitutive relation,

$$\Delta\sigma_{ij}(\mathbf{r}) = D_{ijkl}(\mathbf{r})\left(\Delta\varepsilon_{kl}^T(\mathbf{r}) - \Delta c_{kl}(\mathbf{r})\right) \tag{12}$$

is used, where $D_{ijkl}(\mathbf{r}) = D_{ijkl}^f$ or D_{ijkl}^m according as the point **r** is in the fiber or matrix. This relation is used to update the stress $\sigma_{ij}(\mathbf{r})$ and, in turn, the internal viscoplastic state variables $q_i(\mathbf{r})$ at each point **r** in preparation for computing $\Delta c_{kl}(\mathbf{r})$ in the next increment.

The derivation of the preceding equations and some methods for their solution are discussed in the succeeding sections of this report. Numerical solutions will be obtained during the research effort from appropriate FORTRAN computer programs.

3 Periodic Microstructure

3.1 Volume Averaging

The periodic composite is supposed acted upon at its surface by a spatially linear displacement increment, $\Delta u_i^0(\mathbf{r})$, given by

$$\Delta u_i^0(\mathbf{r}) = x_j\Delta\varepsilon_{ij}^0 + x_j\Delta\omega_{ij}^0 \tag{13}$$

where $\Delta\varepsilon_{ij}^0$ and $\Delta\omega_{ij}^0$ are the spatially uniform strain and rotation increments at the surface of the composite.

If the matrix material was homogeneous and had no fibers embedded in it, the strain increment would be homogeneous and given by

$$\Delta\varepsilon_{ij}^0 = \frac{1}{2}\left(\frac{\partial\left(\Delta u_i^0\right)}{\partial x_j} + \frac{\partial\left(\Delta u_j^0\right)}{\partial x_i}\right) \tag{14}$$

Since this is constant, we may trivially volume average $\Delta\varepsilon_{ij}^0$ over the volume V of the homogeneous matrix material to obtain

$$\Delta\varepsilon_{ij}^0 = \frac{1}{V}\iiint\limits_V \frac{1}{2}\left(\frac{\partial\left(\Delta u_i^0(\mathbf{r})\right)}{\partial x_j} + \frac{\partial\left(\Delta u_j^0(\mathbf{r})\right)}{\partial x_i}\right)dV(\mathbf{r}) \tag{15}$$

which, by Gauss' divergence theorem, may be written as

$$\Delta\varepsilon_{ij}^0 = \frac{1}{V}\oiint\limits_S \tfrac{1}{2}\left(n_j(\mathbf{r})\Delta u_i^0(\mathbf{r}) + n_i(\mathbf{r})\Delta u_j^0(\mathbf{r})\right)dS(\mathbf{r}) \tag{16}$$

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where the integral extends over the surface of the material and $n_i(\mathbf{r})$ denotes the outwardly directed unit normal vector at point $\mathbf{r}$ on the surface. Thus, by applying the displacement increment $\Delta u_i^0(\mathbf{r})$ in equation 13 over the surface of the material to produce the surface strain increment given in equation 16, equations 15 and 16 show that the strain increment in the matrix material is spatially uniform.

If the displacement increment $\Delta u_i^0(\mathbf{r})$ in equation 13 is applied to the actual composite material, the total displacement increment within the material, $\Delta u_i^T(\mathbf{r})$, will vary in a periodic manner due to the assumed geometric periodicity of the composite material, so that

$$\Delta u_i^T(\mathbf{r}) = \Delta u_i^0(\mathbf{r}) + \Delta u_i(\mathbf{r}) \tag{17}$$

where $\Delta u_i^0(\mathbf{r})$ is the displacement increment which would be induced in the homogeneous matrix if the fiber phase were absent, and $\Delta u_i(\mathbf{r})$ is the perturbation or deviation from the homogeneous value due to the presence of the fibers.

Corresponding to these displacement increments, the total strain increment at any point $\mathbf{r}$ in the composite, $\Delta \varepsilon_{kl}^T(\mathbf{r})$, is given by the relation

$$\Delta \varepsilon_{kl}^T(\mathbf{r}) = \Delta \varepsilon_{kl}^0 + \Delta \varepsilon_{kl}(\mathbf{r}) \tag{18}$$

where

$$\Delta \varepsilon_{kl}^0 = \frac{1}{2} \left(\frac{\partial(\Delta u_k^0)}{\partial x_l} + \frac{\partial(\Delta u_l^0)}{\partial x_k} \right) \quad \text{and} \quad \Delta \varepsilon_{kl}(\mathbf{r}) = \frac{1}{2} \left(\frac{\partial(\Delta u_k)}{\partial x_l} + \frac{\partial(\Delta u_l)}{\partial x_k} \right) \tag{19}$$

with $\Delta \varepsilon_{kl}^0$ representing the spatially constant total strain increment which would be produced on the surface and in the interior of the homogeneous matrix if the fibers were absent, and $\Delta \varepsilon_{kl}(\mathbf{r})$ representing the deviation from the uniform value due to the presence of the fibers. Both the total strain increment $\Delta \varepsilon_{kl}^T(\mathbf{r})$ and the perturbed strain increment $\Delta \varepsilon_{kl}(\mathbf{r})$ vary throughout the composite in a periodic manner.

We *define* the volume averaged stress and strain increments as $\Delta \sigma_{ij}^0$ and $\Delta \varepsilon_{ij}^0$, respectively. The required "effective" constitutive equation for the composite material is then an expression relating the volume averaged stress and strain increments. For a function $f(\mathbf{r})$ which varies with position the volume average is defined by the relation

$$\langle f \rangle = \frac{1}{V} \iiint_V f(\mathbf{r}) \, dV(\mathbf{r}) \tag{20}$$

Since the composite is assumed to be comprised of a periodic aggregate of identical unit cells, we may write

$$\langle f \rangle = \frac{1}{V_c} \iiint_{V_c} f(\mathbf{r}) \, dV(\mathbf{r}) \tag{21}$$

where V_c denotes the volume of the unit periodic cell.

If we volume average the total strain increment in equation 18, we obtain

$$\langle \Delta \varepsilon_{kl}^T \rangle = \frac{1}{V_c} \iiint_{V_c} \Delta \varepsilon_{kl}^T(\mathbf{r}) \, dV(\mathbf{r}) = \Delta \varepsilon_{kl}^0 + \frac{1}{V_c} \iiint_{V_c} \Delta \varepsilon_{kl}(\mathbf{r}) \, dV(\mathbf{r}) \tag{22}$$

or

$$\langle \Delta \varepsilon_{kl}^T \rangle = \Delta \varepsilon_{kl}^0 + \langle \Delta \varepsilon_{kl} \rangle \tag{23}$$

But the volume averaged total strain increment is *defined* as $\Delta \varepsilon_{kl}^0$, so that $\langle \Delta \varepsilon_{kl}^T \rangle = \Delta \varepsilon_{kl}^0$ and

$$\langle \Delta \varepsilon_{kl} \rangle = 0 \tag{24}$$

which shows that the volume averaged perturbation strain increment, $\Delta \varepsilon_{kl}(\mathbf{r})$, is equal to zero.

3.2   Eigenstrain and Deviation Strain Increments

If the elasticity tensor is denoted by $D_{ijkl}(\mathbf{r})$ and the inelastic strain tensor by $\varepsilon_{kl}^P(\mathbf{r})$, then the constitutive equation at any point $\mathbf{r}$ in the composite material can be written as

$$\sigma_{ij}(\mathbf{r}) = D_{ijkl}(\mathbf{r}) \left(\varepsilon_{kl}^T(\mathbf{r}) - \varepsilon_{kl}^P(\mathbf{r}) - \alpha_{kl}(\mathbf{r}) \left(T - T_0 \right) \right) \tag{25}$$

where $\alpha_{kl}(\mathbf{r})$ is the coefficient of thermal expansion.

The incremental form of Hooke's law is

$$\Delta \sigma_{ij}(\mathbf{r}) = D_{ijkl}(\mathbf{r}) \left(\Delta \varepsilon_{kl}^T(\mathbf{r}) - \Delta c_{kl}(\mathbf{r}) \right) \tag{26}$$

where $\Delta c_{kl}(\mathbf{r})$ denotes the incremental strain representing the deviation from isothermal elastic conditions and is given by

$$\Delta c_{kl}(\mathbf{r}) = \Delta \varepsilon_{kl}^P(\mathbf{r}) + \alpha_{kl}^\star(\mathbf{r}) \Delta T \tag{27}$$

in which

$$\alpha_{kl}^\star(\mathbf{r}) \Delta T = \alpha_{kl}(\mathbf{r}) \Delta T + (T - T_0) \, \Delta \alpha_{kl}(\mathbf{r}) -$$
$$- D_{klij}^{-1}(\mathbf{r}) \, \Delta D_{ijmn}(\mathbf{r}) \left(\varepsilon_{mn}^T(\mathbf{r}) - \varepsilon_{mn}^P(\mathbf{r}) - \alpha_{mn}(\mathbf{r}) \left(T - T_0 \right) \right) \tag{28}$$

is the nonisothermal increment in strain. The tensors $\Delta D_{ijkl}(\mathbf{r})$ and $\Delta \alpha_{kl}(\mathbf{r})$ represent the incremental changes in the elasticity and thermal expansion tensors due to the temperature increment ΔT.

In a unified viscoplastic constitutive formulation [34] which is integrated by an explicit forward difference method, the inelastic strain increment $\Delta \varepsilon_{kl}^P(\mathbf{r})$ is a function of the current stress (at the beginning of the increment), $\sigma_{ij}(\mathbf{r})$, and the current values of the internal viscoplastic state variables, $q_i(\mathbf{r})$. For example, if

$$\dot{\varepsilon}_{ij}^P = f_{ij} \left(\sigma_{rs}, q_s \right) \tag{29}$$

then $\Delta \varepsilon_{ij}^P = f_{ij} \left(\sigma_{rs}, q_s \right) \Delta t$, and the inelastic strain increment is independent of the total strain increment $\Delta \varepsilon_{kl}^T(\mathbf{r})$. This independence of the inelastic strain increment on the total strain increment is no longer true if an implicit integration method (*e.g.* backward difference) or subincrementation method is used.

The elasticity tensor $D_{ijkl}(\mathbf{r})$ may be written as

$$D_{ijkl}(\mathbf{r}) = D_{ijkl}^m + \delta D_{ijkl}(\mathbf{r}) \tag{30}$$

where

$$\delta D_{ijkl}(\mathbf{r}) = \vartheta(\mathbf{r}) \left(D_{ijkl}^f - D_{ijkl}^m \right) \tag{31}$$

and $\vartheta(\mathbf{r}) = 1$ in the fiber and $\vartheta(\mathbf{r}) = 0$ in the matrix, the superscripts f and m referring to the elasticity tensor of the fiber and matrix, respectively. The constitutive equation at any point $\mathbf{r}$ can then be written, from equation 26, as

$$\Delta \sigma_{ij}(\mathbf{r}) = \left(D_{ijkl}^m + \delta D_{ijkl}(\mathbf{r}) \right) \left(\Delta \varepsilon_{kl}^0 + \Delta \varepsilon_{kl}(\mathbf{r}) - \Delta c_{kl}(\mathbf{r}) \right) \tag{32}$$

or

$$\begin{aligned} \Delta \sigma_{ij}(\mathbf{r}) &= D_{ijkl}^m \left(\Delta \varepsilon_{kl}^0 + \Delta \varepsilon_{kl}(\mathbf{r}) \right) - \\ &\quad - \left\{ D_{ijkl}^m \Delta c_{kl}(\mathbf{r}) - \delta D_{ijkl}(\mathbf{r}) \left[\Delta \varepsilon_{kl}^0 + \Delta \varepsilon_{kl}(\mathbf{r}) - \Delta c_{kl}(\mathbf{r}) \right] \right\} \end{aligned} \tag{33}$$

If the quantity in braces is set equal to $D_{ijkl}^m \Delta \varepsilon_{kl}^\star(\mathbf{r})$, that is, if

$$D_{ijkl}^m \Delta \varepsilon_{kl}^\star(\mathbf{r}) = D_{ijkl}^m \Delta c_{kl}(\mathbf{r}) - \delta D_{ijkl}(\mathbf{r}) \left[\Delta \varepsilon_{kl}^0 + \Delta \varepsilon_{kl}(\mathbf{r}) - \Delta c_{kl}(\mathbf{r}) \right] \tag{34}$$

then equation 26 can be written in the form,

$$\Delta \sigma_{ij}(\mathbf{r}) = D_{ijkl}^m \left(\Delta \varepsilon_{kl}^T(\mathbf{r}) - \Delta \varepsilon_{kl}^\star(\mathbf{r}) \right) = D_{ijkl}^m \left(\Delta \varepsilon_{kl}^0 + \Delta \varepsilon_{kl}(\mathbf{r}) - \Delta \varepsilon_{kl}^\star(\mathbf{r}) \right) \tag{35}$$

From the preceding equation it is evident that the eigenstrain increment, $\Delta \varepsilon_{kl}^\star(\mathbf{r})$, represents the incremental deviation from isothermal elastic behavior in the composite material when the elasticity tensor is taken to be a spatially constant tensor appropriate to that of the matrix phase.

Newton's law for continuing static equilibrium throughout the strain increment requires that

$$\frac{\partial(\Delta \sigma_{ij}(\mathbf{r}))}{\partial x_j} = 0 \tag{36}$$

Equations 35 and 36 then require that

$$\frac{\partial \left\{ D_{ijkl}^m \left(\Delta \varepsilon_{kl}^0 + \Delta \varepsilon_{kl}(\mathbf{r}) - \Delta \varepsilon_{kl}^\star(\mathbf{r}) \right) \right\}}{\partial x_j} = 0 \tag{37}$$

or, if $\Delta \varepsilon_{kl}^0$ is constant,

$$D_{ijkl}^m \frac{\partial(\Delta \varepsilon_{kl}(\mathbf{r}))}{\partial x_j} = D_{ijkl}^m \frac{\partial(\Delta \varepsilon_{kl}^\star(\mathbf{r}))}{\partial x_j} \tag{38}$$

4 Fourier Series Approach

4.1 Fourier Expansions

The application of Fourier series to the calculation of the "effective" overall constitutive behavior of periodic composites has been dealt with in detail by Nemat-Nasser and his colleagues [25,26,27,28]. This work is used in this section to develop constitutive relationships for viscoplastic composite materials under small displacement conditions.

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Due to the geometric periodicity of the composite we may expand $\Delta u_k(\mathbf{r})$ and $\Delta\varepsilon_{kl}^\star(\mathbf{r})$ in a Fourier series (cf., for example, Appendix 3 of Mura's book, [24]). This gives

$$\Delta u_k(\mathbf{r}) = \sum_{n_1=0}^{\pm\infty} \sum_{n_2=0}^{\pm\infty} \sum_{n_3=0}^{\pm\infty}{}' \Delta\hat{u}_k\left(n_1,n_2,n_3\right) e^{i\left(\frac{2\pi n_1}{L_1}x_1+\frac{2\pi n_2}{L_2}x_2+\frac{2\pi n_3}{L_3}x_3\right)} \tag{39}$$

where L_1, L_2, L_3 are the dimensions of a unit cell in the x_1, x_2, x_3 directions. The coefficients $\Delta\hat{u}_k$ in the Fourier expansion are determined by multiplying each side of equation 39 by $e^{-i\left(\frac{2\pi m_1}{L_1}x_1+\frac{2\pi m_2}{L_2}x_2+\frac{2\pi m_3}{L_3}x_3\right)}$ and integrating over the volume of the unit cell to give

$$\Delta\hat{u}_k\left(n_1,n_2,n_3\right) = \frac{1}{L_1L_2L_3} \int_{x_1=0}^{L_1} \int_{x_2=0}^{L_2} \int_{x_3=0}^{L_3} \Delta u_k(\mathbf{r}) e^{-i\left(\frac{2\pi n_1}{L_1}x_1+\frac{2\pi n_2}{L_2}x_2+\frac{2\pi n_3}{L_3}x_3\right)} dx_1\,dx_2\,dx_3 \tag{40}$$

where only the terms with $m_i = n_i$ survive in the summations.

Equations 39 and 40 can be written in shortened form as

$$\Delta u_k(\mathbf{r}) = \sum \sum_{n_p=0}^{\pm\infty} \sum{}' \Delta\hat{u}_k\left(\boldsymbol{\xi}\right) e^{i\boldsymbol{\xi}\cdot\mathbf{r}} \tag{41}$$

with coefficients $\Delta\hat{u}_k\left(\boldsymbol{\xi}\right)$ determined by the inverse relation

$$\Delta\hat{u}_k\left(\boldsymbol{\xi}\right) = \frac{1}{V_c} \iiint_{V_c} \Delta u_k(\mathbf{r}) e^{-i\boldsymbol{\xi}\cdot\mathbf{r}}\,dV(\mathbf{r}) \tag{42}$$

where

$$\boldsymbol{\xi} = \left(\xi_1,\xi_2,\xi_3\right), \qquad \mathbf{r} = \left(x_1,x_2,x_3\right), \qquad V_c = L_1L_2L_3 \tag{43}$$

with

$$\xi_i = \frac{2\pi n_i}{L_i} \qquad \text{(no sum on } i\text{)} \qquad \text{for } i = 1,2,3. \tag{44}$$

The strain increment $\Delta\varepsilon_{kl}^\star(\mathbf{r})$ can also be expanded in a Fourier series to give

$$\Delta\varepsilon_{kl}^\star(\mathbf{r}) = \sum \sum_{n_p=0}^{\pm\infty} \sum{}' \Delta\hat{\varepsilon}_{kl}^\star\left(\boldsymbol{\xi}\right) e^{i\boldsymbol{\xi}\cdot\mathbf{r}} \tag{45}$$

with coefficients $\Delta\hat{\varepsilon}_{kl}^\star$ determined by the inverse relation

$$\Delta\hat{\varepsilon}_{kl}^\star\left(\boldsymbol{\xi}\right) = \frac{1}{V_c} \iiint_{V_c} \Delta\varepsilon_{kl}^\star(\mathbf{r}) e^{-i\boldsymbol{\xi}\cdot\mathbf{r}}\,dV(\mathbf{r}) \tag{46}$$

In equations 41 and 45 the prime indicates that the term with $n_1 = n_2 = n_3 = 0$ is excluded from the summations, since $\Delta\hat{u}_k\left(n_1=0,n_2=0,n_3=0\right)$ represents a rigid body displacement increment and $\Delta\hat{\varepsilon}_{kl}^\star\left(n_1=0,n_2=0,n_3=0\right)$ represents a spatially uniform strain increment.

4.2 Equilibrium Equation

By substituting equation 41 into equation 19; equation 19 into the left hand side of equation 38; and equation 45 into the right hand side of equation 38, the equilibrium relationship becomes

$$D_{ijkl}^m \sum \sum_{n_p=0}^{\pm\infty} {\sum}' \frac{1}{2} \left(\Delta\hat{u}_k\left(\boldsymbol{\xi}\right) \xi_l\xi_j + \Delta\hat{u}_l\left(\boldsymbol{\xi}\right)\xi_k\xi_j \right) e^{i\boldsymbol{\xi}\cdot\mathbf{r}} \ =$$

$$-iD_{ijkl}^m \sum \sum_{n_p=0}^{\pm\infty} {\sum}' \Delta\hat{\varepsilon}_{kl}^{\star}\left(\boldsymbol{\xi}\right) \xi_j e^{i\boldsymbol{\xi}\cdot\mathbf{r}}$$

or

$$D_{ijkl}^m \xi_l\xi_j \Delta\hat{u}_k\left(\boldsymbol{\xi}\right) = -iD_{ijkl}^m \xi_j \Delta\hat{\varepsilon}_{kl}^{\star}\left(\boldsymbol{\xi}\right) \tag{47}$$

If $\xi = \sqrt{\xi_i\xi_i}$ denotes the magnitude of the vector $\boldsymbol{\xi}$, a unit vector $\boldsymbol{\zeta}$ in the direction of $\boldsymbol{\xi}$ can be written as $\zeta_i = \xi_i/\xi$. Equation 47 can therefore be written in the form,

$$\xi^2 D_{ijkl}^m \left(\frac{\xi_l}{\xi}\right) \left(\frac{\xi_j}{\xi}\right) \Delta\hat{u}_k\left(\boldsymbol{\xi}\right) = -iD_{ijkl}^m \xi_j \Delta\hat{\varepsilon}_{kl}^{\star}\left(\boldsymbol{\xi}\right)$$

or

$$\xi^2 \left(D_{ijkl}^m \zeta_l\zeta_j \right) \Delta\hat{u}_k\left(\boldsymbol{\xi}\right) = -iD_{ijkl}^m \xi_j \Delta\hat{\varepsilon}_{kl}^{\star}\left(\boldsymbol{\xi}\right) \tag{48}$$

The second rank tensor,

$$M_{ik}\left(\boldsymbol{\zeta}\right) = M_{ki}\left(\boldsymbol{\zeta}\right) = D_{ijkl}^m \zeta_l\zeta_j \tag{49}$$

is called the Christoffel stiffness tensor (*cf.* [33]) and equation 48 can be written as

$$\xi^2 M_{ik}\left(\boldsymbol{\zeta}\right) \Delta\hat{u}_k\left(\boldsymbol{\xi}\right) = -iD_{ijrs}^m \xi_j \Delta\hat{\varepsilon}_{rs}^{\star}\left(\boldsymbol{\xi}\right) \tag{50}$$

This equation can be inverted by premultiplying each side by the inverse tensor $\xi^{-2}\mathbf{M}^{-1}$ to give the Fourier expansion coefficients

$$\Delta\hat{u}_k\left(\boldsymbol{\xi}\right) = -iM_{ik}^{-1}\left(\boldsymbol{\zeta}\right) D_{ijrs}^m \xi_j \Delta\hat{\varepsilon}_{rs}^{\star}\left(\boldsymbol{\xi}\right) \xi^{-2} \tag{51}$$

The expansion coefficients can now be substituted into the Fourier expansion of $\Delta u_k(\mathbf{r})$ in equation 41 to give

$$\Delta u_k(\mathbf{r}) = -\sum \sum_{n_p=0}^{\pm\infty} {\sum}' i\xi^{-2} M_{ik}^{-1}\left(\boldsymbol{\zeta}\right) D_{ijrs}^m \xi_j \Delta\hat{\varepsilon}_{rs}^{\star}\left(\boldsymbol{\xi}\right) e^{i\boldsymbol{\xi}\cdot\mathbf{r}} \tag{52}$$

This result may now be substituted into equation 19, so that the perturbation strain increment may be written as

$$\Delta\varepsilon_{kl}(\mathbf{r}) = \sum \sum_{n_p=0}^{\pm\infty} {\sum}' \frac{1}{2} \left(\xi^{-2} M_{ik}^{-1}\left(\boldsymbol{\zeta}\right) \xi_j\xi_l + \xi^{-2} M_{il}^{-1}\left(\boldsymbol{\zeta}\right) \xi_j\xi_k \right) D_{ijrs}^m \Delta\hat{\varepsilon}_{rs}^{\star}\left(\boldsymbol{\xi}\right) e^{i\boldsymbol{\xi}\cdot\mathbf{r}} \tag{53}$$

If we define the fourth rank tensor $g_{klij}\left(\boldsymbol{\zeta}\right)$ by the relation

$$g_{klij}\left(\boldsymbol{\zeta}\right) = \frac{1}{2} \left(M_{ik}^{-1}\left(\boldsymbol{\zeta}\right) \zeta_j\zeta_l + M_{il}^{-1}\left(\boldsymbol{\zeta}\right) \zeta_j\zeta_k \right) \tag{54}$$

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then the perturbation strain increment can be written in the form

$$\Delta\varepsilon_{kl}(\mathbf{r}) = \sum \sum_{n_p=0}^{\pm\infty} {\sum}' g_{klij}(\boldsymbol{\zeta}) D_{ijrs}^m \Delta\hat{\varepsilon}_{rs}^\star (\boldsymbol{\xi}) e^{i\boldsymbol{\xi}\cdot\mathbf{r}} \qquad (55)$$

and by inserting the relation for the Fourier expansion coefficients $\Delta\hat{\varepsilon}_{rs}^\star$ from equation 46, we obtain

$$\Delta\varepsilon_{kl}(\mathbf{r}) = \frac{1}{V_c} \sum \sum_{n_p=0}^{\pm\infty} {\sum}' g_{klij}(\boldsymbol{\zeta}) \iiint_{V_c} D_{ijrs}^m \Delta\varepsilon_{rs}^\star (\mathbf{r}') e^{i\boldsymbol{\xi}\cdot(\mathbf{r}-\mathbf{r}')} dV(\mathbf{r}') \qquad (56)$$

where the integration extends over the volume, $V_c = L_1 L_2 L_3$, of the unit periodic cell.

From equation 18 the total strain increment is given by

$$\Delta\varepsilon_{kl}^T(\mathbf{r}) = \Delta\varepsilon_{kl}^0 + \frac{1}{V_c} \sum \sum_{n_p=0}^{\pm\infty} {\sum}' g_{klij}(\boldsymbol{\zeta}) \iiint_{V_c} D_{ijrs}^m \Delta\varepsilon_{rs}^\star (\mathbf{r}') e^{i\boldsymbol{\xi}\cdot(\mathbf{r}-\mathbf{r}')} dV(\mathbf{r}') \qquad (57)$$

which, from the definition of $D_{ijrs}^m \Delta\varepsilon_{rs}^\star(\mathbf{r}')$ in equation 34, may be written in the final form,

$$\begin{aligned} \Delta\varepsilon_{kl}^T(\mathbf{r}) = \ & \Delta\varepsilon_{kl}^0 + \frac{1}{V_c} \sum \sum_{n_p=0}^{\pm\infty} {\sum}' g_{klij}(\boldsymbol{\zeta}) \iiint_{V_c} e^{i\boldsymbol{\xi}\cdot(\mathbf{r}-\mathbf{r}')} \left\{ D_{ijrs}^m \Delta c_{rs}(\mathbf{r}') - \right. \\ & \left. - \delta D_{ijrs}(\mathbf{r}') \left[\Delta\varepsilon_{rs}^T(\mathbf{r}') - \Delta c_{rs}(\mathbf{r}') \right] \right\} dV(\mathbf{r}') \end{aligned} \qquad (58)$$

This implicit integral equation—equation 4 in section 2.1—must be solved to yield the total strain increment $\Delta\varepsilon_{kl}^T(\mathbf{r})$ at each point $\mathbf{r}$ in the unit periodic cell.

Instead of solving for $\Delta\varepsilon_{kl}^T(\mathbf{r})$ from this implicit integral equation, we could use equation 34 to eliminate $\Delta\varepsilon_{kl}^T(\mathbf{r})$ from equation 57 to give an equivalent integral equation for $\Delta\varepsilon_{kl}^\star(\mathbf{r})$, *viz.*,

$$\begin{aligned} D_{ijkl}^m \Delta\varepsilon_{kl}^\star(\mathbf{r}) = \ & D_{ijkl}^m \Delta c_{kl}(\mathbf{r}) - \delta D_{ijkl}(\mathbf{r}) \left[\Delta\varepsilon_{kl}^0 - \Delta c_{kl}(\mathbf{r}) \right] - \\ & - \delta D_{ijkl}(\mathbf{r}) \frac{1}{V_c} \sum \sum_{n_p=0}^{\pm\infty} {\sum}' g_{klmn}(\boldsymbol{\zeta}) \iiint_{V_c} D_{mnrs}^m \Delta\varepsilon_{rs}^\star (\mathbf{r}') e^{i\boldsymbol{\xi}\cdot(\mathbf{r}-\mathbf{r}')} dV(\mathbf{r}') \end{aligned} \qquad (59)$$

The incremental constitutive relation at any point $\mathbf{r}$ is given in equation 35, and this relation can be used to update the stress state at any point $\mathbf{r}$ in the unit cell once equation 59 is solved for $\Delta\varepsilon_{kl}^\star(\mathbf{r})$. Alternatively, equation 58 can be solved for $\Delta\varepsilon_{kl}^T(\mathbf{r})$ and inserted into equations 34 and 35. The overall "effective" constitutive relation for the composite material can be obtained by averaging equation 35 over the unit periodic cell. This gives

$$\langle \Delta\sigma_{ij} \rangle = \left\langle D_{ijkl}^m \left(\Delta\varepsilon_{kl}^0 + \Delta\varepsilon_{kl} - \Delta\varepsilon_{kl}^\star \right) \right\rangle$$

or

$$\langle \Delta\sigma_{ij} \rangle = D_{ijkl}^m \Delta\varepsilon_{kl}^0 + D_{ijkl}^m \langle \Delta\varepsilon_{kl} \rangle - D_{ijkl}^m \langle \Delta\varepsilon_{kl}^\star \rangle \qquad (60)$$

If we equate the volume averaged stress increment $\langle \Delta\sigma_{ij} \rangle$ and the overall bulk response stress increment $\Delta\sigma_{ij}^0$, *i.e.*, if $\langle \Delta\sigma_{ij} \rangle = \Delta\sigma_{ij}^0$, and we note from equation 24 that the volume averaged perturbation strain increment is zero, *i.e.* $\langle \Delta\varepsilon_{kl} \rangle = 0$, then the overall "effective" constitutive relationship is

$$\Delta\sigma_{ij}^0 = D_{ijkl}^m \Delta\varepsilon_{kl}^0 - D_{ijkl}^m \langle \Delta\varepsilon_{kl}^\star \rangle \qquad (61)$$

or

$$\Delta\sigma_{ij}^0 = D_{ijkl}^m \Delta\varepsilon_{kl}^0 - \frac{1}{V_c}\iiint\limits_{V_c} \left\{ D_{ijkl}^m \Delta c_{kl}\left(\mathbf{r}\right) - \delta D_{ijkl}\left(\mathbf{r}\right)\left[\Delta\varepsilon_{kl}^T\left(\mathbf{r}\right) - \Delta c_{kl}\left(\mathbf{r}\right)\right]\right\} dV(\mathbf{r}) \qquad (62)$$

which is the result presented in equation 2 of section 2.2. The procedure for integrating the overall "effective" constitutive relation then proceeds as follows.

4.3   Fourier Integration Algorithm

1. From a knowledge of the stress state throughout the unit periodic cell at the current time, t, calculate the inelastic strain increment $\Delta\varepsilon_{kl}^P\left(\sigma_{rs}, q_s, \mathbf{r}\right)$ from an appropriate unified viscoplastic constitutive relation. The viscoplastic constitutive relation will vary according as $\mathbf{r}$ is in the fiber or matrix phase, respectively.

2. Compute the eigenstrain $\Delta\varepsilon_{kl}^\star(\mathbf{r})$ throughout the unit periodic cell from the implicit integral equation 59 or from equations 34 and 58.

3. Compute the stress increment throughout the unit periodic cell from equation 35 and update the stress, strain and viscoplastic state variables according to the relations

$$\sigma_{ij}\left(\mathbf{r}, t + \Delta t\right) = \sigma_{ij}\left(\mathbf{r}, t\right) + \Delta\sigma_{ij}(\mathbf{r}),$$

$$\varepsilon_{ij}^T\left(\mathbf{r}, t + \Delta t\right) = \varepsilon_{ij}^T\left(\mathbf{r}, t\right) + \Delta\varepsilon_{ij}^T(\mathbf{r}),$$

$$q_i\left(\mathbf{r}, t + \Delta t\right) = q_i\left(\mathbf{r}, t\right) + \Delta q_i\left(\mathbf{r}\right).$$

4. Calculate the overall "effective" stress and strain increment for the composite from equation 61 and update the overall "effective" stress and strain from the relations

$$\sigma_{ij}^0\left(t + \Delta t\right) = \sigma_{ij}^0\left(t\right) + \Delta\sigma_{ij}^0,$$

$$\varepsilon_{ij}^0\left(t + \Delta t\right) = \varepsilon_{ij}^0\left(t\right) + \Delta\varepsilon_{ij}^0.$$

5. Repeat the preceding calculations for each incremental load step.

4.4   Implicit Integration Algorithm

The preceding algorithm makes use of the fact that the inelastic strain increment $\Delta\varepsilon_{kl}^P(\mathbf{r})$ is independent of the total strain increment $\Delta\varepsilon_{kl}^T(\mathbf{r})$ if an explicit forward difference method—such as Euler or Heun forward difference—is used to integrate the unified viscoplastic relations for the fiber and matrix phases. If an implicit method—such as backward difference or subincrementation—is used, the inelastic strain increment depends on the total strain increment. In this case the total strain increment must be obtained by iterating equation 58 in the form,

$$\Delta\varepsilon_{kl}^T(\mathbf{r}) = \Delta\varepsilon_{kl}^0 + \frac{1}{V_c}\sum\sum_{n_p=0}^{\pm\infty}{\sum}' g_{klij}\left(\boldsymbol{\zeta}\right)\iiint\limits_{V_c} e^{i\boldsymbol{\xi}\cdot(\mathbf{r}-\mathbf{r}')}\left\{D_{ijrs}^m\Delta c_{rs}\left(\mathbf{r}', \Delta\varepsilon_{pq}^T\left(\mathbf{r}'\right)\right) - \right.$$

$$\left. - \delta D_{ijrs}\left(\mathbf{r}'\right)\left[\Delta\varepsilon_{rs}^T\left(\mathbf{r}'\right) - \Delta c_{rs}\left(\mathbf{r}', \Delta\varepsilon_{pq}^T\left(\mathbf{r}'\right)\right)\right]\right\} dV(\mathbf{r}') \qquad (63)$$

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The first iterative guess can be taken as $\Delta \varepsilon_{kl}^T(\mathbf{r}) = \Delta \varepsilon_{kl}^0$, and the right hand side evaluated to give an improved guess for $\Delta \varepsilon_{kl}^T(\mathbf{r})$. This process is then continued with

$$\left\{ \Delta \varepsilon_{kl}^T(\mathbf{r}) \right\}_{\lambda+1} = \Delta \varepsilon_{kl}^0 + \frac{1}{V_c} \sum \sum_{n_p=0}^{\pm\infty} \sum{}' g_{klij}(\zeta) \iiint_{V_c} e^{i\boldsymbol{\xi}\cdot(\mathbf{r}-\mathbf{r}')} \left\{ D_{ijrs}^m \Delta c_{rs}\left(\mathbf{r}', \left\{\Delta\varepsilon_{pq}^T(\mathbf{r}')\right\}_\lambda\right) - \right.$$
$$\left. - \delta D_{ijrs}(\mathbf{r}') \left[\left\{\Delta\varepsilon_{rs}^T(\mathbf{r}')\right\}_\lambda - \Delta c_{rs}\left(\mathbf{r}', \left\{\Delta\varepsilon_{pq}^T(\mathbf{r}')\right\}_\lambda\right) \right] \right\} dV(\mathbf{r}') \qquad (64)$$

until the λth and $(\lambda+1)^{\text{th}}$ iterates of $\Delta \varepsilon_{kl}^T(\mathbf{r})$ converge.

Equation 59 is not so convenient for iteration as equation 58 when $\Delta \varepsilon_{rs}^P(\mathbf{r})$ depends on $\Delta \varepsilon_{kl}^T(\mathbf{r})$. It is always necessary to know the total strain increment $\Delta \varepsilon_{kl}^T(\mathbf{r})$ in order to calculate the inelastic strain increment $\Delta \varepsilon_{rs}^P\left(\mathbf{r}', \Delta\varepsilon_{pq}^T(\mathbf{r}')\right)$. But equation 34, *viz.*,

$$D_{ijkl}^m \Delta\varepsilon_{kl}^\star(\mathbf{r}) = D_{ijkl}^m \Delta c_{kl}\left(\mathbf{r}, \Delta\varepsilon_{pq}^T(\mathbf{r})\right) - \delta D_{ijkl}(\mathbf{r}) \left[\Delta\varepsilon_{kl}^T(\mathbf{r}) - \Delta c_{kl}\left(\mathbf{r}, \Delta\varepsilon_{pq}^T(\mathbf{r})\right) \right] \qquad (65)$$

is an implicit equation for $\Delta \varepsilon_{kl}^T(\mathbf{r})$ when the iterated quantity, $\Delta\varepsilon_{kl}^\star(\mathbf{r})$, is given. Equation 63 is therefore the appropriate equation to iterate when the inelastic strain increment depends on the total strain increment.

The procedure for solving the implicit integral equations in 58, 59 and 63 is described in section 8.

5 Green's Function Approach

5.1 Green's Solution of Navier's Equilibrium Equation

The equation of continuing static equilibrium for the composite material throughout an applied strain increment is given by

$$\frac{\partial(\Delta\sigma_{ij}(\mathbf{r}))}{\partial x_j} + \Delta f_i(\mathbf{r}) = 0 \qquad (66)$$

where $\Delta f_i(\mathbf{r})$ is the incremental body force per unit volume of the composite material. From equations 35 and 66 we obtain

$$D_{ijkl}^m \frac{\partial\left(\Delta\varepsilon_{kl}^T(\mathbf{r})\right)}{\partial x_j} = \frac{\partial}{\partial x_j}\left(D_{ijkl}^m \Delta\varepsilon_{kl}^\star(\mathbf{r}) \right) - \Delta f_i(\mathbf{r}) \qquad (67)$$

which is equivalent to equation 37 in the absence of the incremental body force $\Delta f_i(\mathbf{r})$. From this equation it is clear that the divergence of the stress variation produced by $\Delta\varepsilon_{kl}^\star(\mathbf{r})$ may be formally regarded as a fictitious body force increment, analogous to $\Delta f_i(\mathbf{r})$, which is applied to the homogeneous matrix material with elasticity tensor D_{ijkl}^m. The theory of elasticity for homogeneous materials is generally concerned with the solution of the homogeneous differential equation 67—Navier's equation—when the right hand side is zero. When body forces are present the standard method of solution is to obtain the displacement solution at $\mathbf{r}$ due to a unit body force applied at $\mathbf{r}'$. This solution is given by the Green's function $G_{ij}(\mathbf{r}-\mathbf{r}')$

which gives the displacement in the ith direction at $\mathbf{r}$ due to a unit point force applied in the jth direction at $\mathbf{r}'$. For a distributed incremental body force $\Delta f_i(\mathbf{r}')$ the displacement increment at $\mathbf{r}$ is obtained by summing the results for the distribution in the form

$$\Delta u_i(\mathbf{r}) = \iiint\limits_V G_{ij}(\mathbf{r} - \mathbf{r}')\Delta f_j(\mathbf{r}')\, dV(\mathbf{r}') \tag{68}$$

The integration extends over the whole volume, V, of the composite material which may be regarded as being of infinite extent.

When $\Delta f_j(\mathbf{r}') = 0$ we know that the displacement solution is $\Delta u_i^T(\mathbf{r}) = \Delta u_i^0(\mathbf{r})$, corresponding to an applied uniform strain increment $\Delta \varepsilon_{ij}^0$ on the infinite boundary of the homogeneous matrix. For an effective distributed body force increment given by the right hand side of equation 67, with $\Delta f_j(\mathbf{r}') = 0$, the solution for the total displacement increment $\Delta u_i^T(\mathbf{r})$ can be written as

$$\Delta u_i^T(\mathbf{r}) = \Delta u_i^0(\mathbf{r}) - \iiint\limits_V G_{ik}(\mathbf{r} - \mathbf{r}')\frac{\partial}{\partial x_l'}\left(D_{klmn}^m \Delta \varepsilon_{mn}^\star(\mathbf{r}')\right) dV(\mathbf{r}') \tag{69}$$

This corresponds to equations 17 and 39, the volume integral corresponding to the perturbed displacement increment $\Delta u_i(\mathbf{r})$ in 17.

For a material which is homogeneous with elasticity tensor D_{ijkl}^m the Green's function satisfies the differential relation (*cf.* Appendix A, equation A.11),

$$D_{ijkl}^m \frac{\partial^2 G_{km}(\mathbf{r} - \mathbf{r}')}{\partial x_j \partial x_l} + \delta_{im}\delta(\mathbf{r} - \mathbf{r}') = 0 \tag{70}$$

where δ_{im} is the Kronecker delta tensor given by $\delta_{im} = 1$ if $i = m$ and $\delta_{im} = 0$ if $i \neq m$, and $\delta(\mathbf{r} - \mathbf{r}')$ is the three dimensional Dirac delta function defined by the relation

$$\delta(\mathbf{r} - \mathbf{r}') = \delta(x_1 - x_1')\,\delta(x_2 - x_2')\,\delta(x_3 - x_3') \tag{71}$$

By applying the Fourier integral techniques in Appendix A, the Green's tensor is shown to have the Fourier integral form,

$$G_{ij}(\mathbf{r} - \mathbf{r}') = \iiint\limits_{-\infty}^{\infty} \frac{d^3\mathbf{K}}{(2\pi)^3}\,\frac{M_{ij}^{-1}(\boldsymbol{\zeta})}{K^2}\,e^{-i\mathbf{K}\cdot(\mathbf{r} - \mathbf{r}')} \tag{72}$$

in which the inverse Christoffel stiffness tensor (*cf.* [33]) $M_{ij}^{-1}(\boldsymbol{\zeta})$ is defined by

$$M_{ij}^{-1}(\boldsymbol{\zeta}) = \left(D_{pijq}^m \zeta_p \zeta_q\right)^{-1} \tag{73}$$

with $\zeta_p = K_p/\sqrt{K_m K_m} = K_p/K$ being a unit vector in the direction of the Fourier wave vector $\mathbf{K}$, and $K = \sqrt{K_m K_m}$ denoting the magnitude of the wave vector $\mathbf{K}$.

Making use of the relation

$$
\begin{aligned}
G_{ik}(\mathbf{r} - \mathbf{r}')\frac{\partial}{\partial x_l'}\left(D_{klmn}^m \Delta \varepsilon_{mn}^\star(\mathbf{r}')\right) &= \frac{\partial}{\partial x_l'}\left(G_{ik}(\mathbf{r} - \mathbf{r}')D_{klmn}^m \Delta \varepsilon_{mn}^\star(\mathbf{r}')\right) - \\
&\quad - \frac{\partial G_{ik}(\mathbf{r} - \mathbf{r}')}{\partial x_l'}D_{klmn}^m \Delta \varepsilon_{mn}^\star(\mathbf{r}')
\end{aligned} \tag{74}
$$

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we may write equation 69 in the form

$$\Delta u_i^T(\mathbf{r}) = \Delta u_i^0(\mathbf{r}) - \iiint\limits_V \frac{\partial}{\partial x_l'} \left(G_{ik}(\mathbf{r} - \mathbf{r}') D_{klmn}^m \Delta \varepsilon_{mn}^\star(\mathbf{r}') \right) dV(\mathbf{r}') +$$

$$+ \iiint\limits_V \frac{\partial G_{ik}(\mathbf{r} - \mathbf{r}')}{\partial x_l'} D_{klmn}^m \Delta \varepsilon_{mn}^\star(\mathbf{r}') \, dV(\mathbf{r}') \tag{75}$$

The first volume integral can be transformed into a surface integral *via* Gauss' divergence theorem, *viz.*,

$$\iiint\limits_V \frac{\partial}{\partial x_l'} \left(G_{ik}(\mathbf{r} - \mathbf{r}') D_{klmn}^m \Delta \varepsilon_{mn}^\star(\mathbf{r}') \right) dV(\mathbf{r}') = \iint\limits_S n_l(\mathbf{r}') \, G_{ik}(\mathbf{r} - \mathbf{r}') D_{klmn}^m \Delta \varepsilon_{mn}^\star(\mathbf{r}') \, dS(\mathbf{r}') \tag{76}$$

The surface integral extends over the entire outer surface of the "infinite" matrix material. Since this is assumed to be at an infinite distance, all the integration points $\mathbf{r}'$ in the surface integral are at an infinite distance from the field point $\mathbf{r}$ and $G_{ik}(\mathbf{r} - \mathbf{r}') = 0$. Thus, for an infinite body the first volume integral in equation 75 vanishes. This would not be the case for a finite body in which the field point $\mathbf{r}$ is close to the surface integration point $\mathbf{r}'$, and the volume (or surface) integral would need to be retained for these situations. In this case other surface integrals would arise (*cf.* Appendix D, equation D.27) due to the application of boundary incremental displacements or surface tractions on the surface of the material.

From the properties of the Green's function,

$$\frac{\partial G_{ik}(\mathbf{r} - \mathbf{r}')}{\partial x_l'} = -\frac{\partial G_{ik}(\mathbf{r} - \mathbf{r}')}{\partial x_l} \tag{77}$$

which follows since G_{ik} is a function of

$$\mathbf{r} - \mathbf{r}' = (x_1 - x_1', x_2 - x_2', x_3 - x_3') \tag{78}$$

Equation 75 may then be written alternatively as

$$\Delta u_i^T(\mathbf{r}) = \Delta u_i^0(\mathbf{r}) - \iiint\limits_V \frac{\partial G_{ik}(\mathbf{r} - \mathbf{r}')}{\partial x_l} D_{klmn}^m \Delta \varepsilon_{mn}^\star(\mathbf{r}') \, dV(\mathbf{r}') \tag{79}$$

But $\Delta \varepsilon_{ij}^T(\mathbf{r}) = \frac{1}{2} \left(\partial \left(\Delta u_i^T(\mathbf{r}) \right) / \partial x_j + \partial \left(\Delta u_j^T(\mathbf{r}) \right) / \partial x_i \right)$, so that by differentiating equation 79 with respect to x_i and x_j and taking half the sum, we obtain

$$\Delta \varepsilon_{ij}^T(\mathbf{r}) = \Delta \varepsilon_{ij}^0 + \iiint\limits_V U_{ijkl}(\mathbf{r} - \mathbf{r}') D_{klmn}^m \Delta \varepsilon_{mn}^\star(\mathbf{r}') \, dV(\mathbf{r}') \tag{80}$$

which, by means of equation 34, may be written as

$$\Delta \varepsilon_{ij}^T(\mathbf{r}) = \Delta \varepsilon_{ij}^0 + \iiint\limits_V U_{ijkl}(\mathbf{r} - \mathbf{r}') \left\{ D_{klrs}^m \Delta c_{rs}(\mathbf{r}') - \right.$$

$$\left. - \delta D_{klrs}(\mathbf{r}') \left[\Delta \varepsilon_{rs}^T(\mathbf{r}') - \Delta c_{rs}(\mathbf{r}') \right] \right\} dV(\mathbf{r}') \tag{81}$$

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An equivalent integral equation, involving the eigenstrain increment $\Delta\varepsilon_{ij}^{\star}(\mathbf{r})$, can also be obtained by using equation 34 to eliminate $\Delta\varepsilon_{ij}^{T}(\mathbf{r})$ from equation 80, which gives

$$D_{ijkl}^{m}\Delta\varepsilon_{kl}^{\star}(\mathbf{r}) = D_{ijkl}^{m}\Delta c_{kl}(\mathbf{r}) - \delta D_{ijkl}(\mathbf{r})\left[\Delta\varepsilon_{kl}^{0} - \Delta c_{kl}(\mathbf{r})\right] -$$

$$- \delta D_{ijkl}(\mathbf{r})\iiint\limits_{V} U_{klmn}\left(\mathbf{r}-\mathbf{r}'\right)D_{mnrs}^{m}\Delta\varepsilon_{rs}^{\star}\left(\mathbf{r}'\right)dV(\mathbf{r}') \qquad (82)$$

In the preceding equations the operator,

$$U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right) = -\frac{1}{2}\left(\frac{\partial^{2}G_{ik}\left(\mathbf{r}-\mathbf{r}'\right)}{\partial x_{j}\partial x_{l}} + \frac{\partial^{2}G_{jk}\left(\mathbf{r}-\mathbf{r}'\right)}{\partial x_{i}\partial x_{l}}\right) \qquad (83)$$

gives the ij component of the strain increment at point $\mathbf{r}$ due to an applied stress increment component kl at point $\mathbf{r}'$ in an infinite homogeneous medium with elasticity tensor D_{ijkl}^{m} and Green's function given by equation 72.

5.2 Equivalence of Perturbed Strain Increment

From equations 18, 56 and 80 we see that the perturbed strain increment, $\Delta\varepsilon_{kl}(\mathbf{r}) = \Delta\varepsilon_{kl}^{T}(\mathbf{r}) - \Delta\varepsilon_{kl}^{0}$, is given by the equivalent relations,

$$\Delta\varepsilon_{kl}(\mathbf{r}) = \frac{1}{V_{c}}\sum\sum_{n_{p}=0}^{\pm\infty}\sideset{}{'}\sum g_{klmn}\left(\boldsymbol{\zeta}\right)\iiint\limits_{V_{c}}D_{mnrs}^{m}\Delta\varepsilon_{rs}^{\star}\left(\mathbf{r}'\right)e^{i\boldsymbol{\xi}\cdot(\mathbf{r}-\mathbf{r}')}dV(\mathbf{r}') \qquad (84)$$

or

$$\Delta\varepsilon_{kl}(\mathbf{r}) = \iiint\limits_{V}U_{klmn}\left(\mathbf{r}-\mathbf{r}'\right)D_{mnrs}^{m}\Delta\varepsilon_{rs}^{\star}\left(\mathbf{r}'\right)dV(\mathbf{r}') \qquad (85)$$

The volume integral in the Fourier series representation extends over the volume, V_{c}, of the unit periodic cell and the summation extends over the integers $n_{p} = 0, \pm 1, \pm 2, \ldots, etc.$, where $p = 1, 2, 3$. In the Green's function approach the volume integral extends over the entire infinite medium, i.e., over all the periodic cells comprising the material. It is shown in Appendices B and C that the Fourier summation expression in equation 84 can be converted into the Green's function expression in equation 85 by means of the Poisson sum formula.

From equation 34 it is evident that if the elastic properties of the fiber are the same as that of the matrix, then $\delta D_{ijkl}(\mathbf{r}) = \vartheta(\mathbf{r})\left(D_{ijkl}^{f} - D_{ijkl}^{m}\right) = 0$, in which case

$$\Delta\varepsilon_{kl}^{\star}(\mathbf{r}) = \Delta c_{kl}(\mathbf{r}) \qquad (86)$$

is known explicitly without having to solve the integral equation. From equations 58 and 81 it can also be observed that $\Delta\varepsilon_{kl}^{T}(\mathbf{r})$ is known explicitly when $\delta D_{ijkl}(\mathbf{r}) = 0$. The explicit relation in equation 86 holds only when an explicit forward difference method is used to integrate the viscoplastic constitutive relations. For implicit integration methods in which the inelastic strain increment $\Delta\varepsilon_{kl}^{P}(\mathbf{r})$ depends on the total strain increment $\Delta\varepsilon_{kl}^{T}(\mathbf{r})$, equations 58 and 81 show that even when $\delta D_{ijkl}(\mathbf{r}) = 0$, the equation to determine $\Delta\varepsilon_{kl}^{T}(\mathbf{r})$ is still an implicit integral equation.

6 Self-Consistent Method

6.1 Outline of Self-Consistent Method

In this section we establish a self-consistent relationship between the overall "effective" stress increment, $\Delta\sigma_{ij}^0$, and the applied strain increment, $\Delta\varepsilon_{ij}^0$, for a matrix material which has cylindrical fibers embedded in it in a periodic fashion.

From equations 34 and 61, this relationship can be written as

$$\Delta\sigma_{ij}^0 = D_{ijkl}^m \Delta\varepsilon_{kl}^0 - \frac{1}{V_c}\iiint\limits_{V_c}\left\{D_{ijkl}^m \Delta c_{kl}(\mathbf{r}) - \delta D_{ijkl}(\mathbf{r})\left[\Delta\varepsilon_{kl}^T(\mathbf{r}) - \Delta c_{kl}(\mathbf{r})\right]\right\}dV(\mathbf{r}) \quad (87)$$

where the total strain increment is determined from equation 81 in the form,

$$\Delta\varepsilon_{kl}^T(\mathbf{r}) = \Delta\varepsilon_{kl}^0 + \iiint\limits_V U_{klmn}(\mathbf{r}-\mathbf{r}')\left\{D_{mnrs}^m \Delta c_{rs}(\mathbf{r}') - \right.$$
$$\left. - \delta D_{mnrs}(\mathbf{r}')\left[\Delta\varepsilon_{rs}^T(\mathbf{r}') - \Delta c_{rs}(\mathbf{r}')\right]\right\}dV(\mathbf{r}') \quad (88)$$

These equations can be solved in an approximate fashion by means of a self-consistent method in the following manner.

First, assume that the unit periodic cell consisting of a cylindrical fiber embedded in a unit matrix cell, Fig. 5, is replaced by a cylindrical fiber (of radius = a) embedded in a cylindrical matrix (of "effective" radius = b) as depicted in Fig. 6. The other unit cells outside the given unit cell—i.e., the rest of the composite—are then smeared out into a uniform matrix material whose overall "effective" constitutive properties are the volume average of the constitutive properties of the constrained unit periodic cell. The "effective" constitutive properties will be transversely isotropic if the fibers are arranged in hexagonal arrays or tetragonal if they are arranged in square arrays.

Second, assume that the total strain increment, $\Delta\varepsilon_{kl}^T(\mathbf{r})$, and the strain increment representing the deviation from isothermal elastic behavior, $\Delta c_{kl}(\mathbf{r})$, are spatially constant in the fiber and matrix phases of the unit cell. These constant values (different in the fiber and matrix of the unit cell) are taken to be the volume averages over the respective constituent volumes of the fiber and matrix phases of the unit cell.

The composite now consists of three constituent phases, viz., the fiber, matrix, and smeared out average phases. If the elasticity tensors of these phases are denoted by D_{ijkl}^f, D_{ijkl}^m and $\overline{D}_{ijkl}$, respectively, then the elasticity tensor at any point $\mathbf{r}$ in the composite can be written as

$$D_{klrs}(\mathbf{r}) = \overline{D}_{klrs} + \delta\overline{D}_{klrs}(\mathbf{r}) \quad (89)$$

where

$$\delta\overline{D}_{klrs}(\mathbf{r}) = \delta D_{klrs}^f = D_{klrs}^f - \overline{D}_{klrs} \quad (90)$$

if $\mathbf{r}$ is in the fiber;

$$\delta\overline{D}_{klrs}(\mathbf{r}) = \delta D_{klrs}^m = D_{klrs}^m - \overline{D}_{klrs} \quad (91)$$

if $\mathbf{r}$ is in the matrix; and

$$\delta\overline{D}_{klrs}(\mathbf{r}) = 0 \quad (92)$$

if $\mathbf{r}$ is in the surrounding smeared out "effective" material.

The fiber and matrix constituent phases now represent fictitious body forces in the infinite "effective" medium with elasticity tensor $\overline{D}_{ijkl}$, and the total strain increment is obtained from the solution of the integral equation,

$$\Delta\varepsilon_{ij}^{T}(\mathbf{r}) = \Delta\varepsilon_{ij}^{0} + \iiint_{V} U_{ijkl}\left(\mathbf{r} - \mathbf{r}'\right)\overline{D}_{klrs}\Delta\varepsilon_{rs}^{\star}\left(\mathbf{r}'\right)dV(\mathbf{r}') \tag{93}$$

in which

$$\overline{D}_{klrs}\Delta\varepsilon_{rs}^{\star}\left(\mathbf{r}'\right) = \overline{D}_{klrs}\Delta c_{rs}\left(\mathbf{r}'\right) - \delta\overline{D}_{klrs}\left(\mathbf{r}'\right)\left[\Delta\varepsilon_{rs}^{T}\left(\mathbf{r}'\right) - \Delta c_{rs}\left(\mathbf{r}'\right)\right] \tag{94}$$

6.2 Strain Increments in Three Phases

We now make the approximation that the strain increments in the three phases are spatially constant and equal to their respective volume averages, so that if $\mathbf{r}'$ is in the fiber $\Delta\varepsilon_{ij}^{T}\left(\mathbf{r}'\right)$ and $\Delta c_{ij}\left(\mathbf{r}'\right)$ are replaced by

$$\Delta\varepsilon_{ij}^{T}(f) = \frac{1}{V_{f}}\iiint_{V_{f}}\Delta\varepsilon_{ij}^{T}\left(\mathbf{r}'\right)dV(\mathbf{r}') \tag{95}$$

and

$$\Delta c_{ij}\left(f\right) = \frac{1}{V_{f}}\iiint_{V_{f}}\Delta c_{ij}\left(\mathbf{r}'\right)dV(\mathbf{r}') \tag{96}$$

so that, from equation 27,

$$\Delta c_{ij}(f) = \Delta\varepsilon_{ij}^{P}(f) + \alpha_{ij}^{\star f}\Delta T \tag{97}$$

with

$$\alpha_{ij}^{\star f}\Delta T = \alpha_{ij}^{f}\Delta T + (T - T_{0})\Delta\alpha_{ij}^{f} - \\ - \left(D_{ijkl}^{f}\right)^{-1}\Delta D_{klmn}^{f}\left(\varepsilon_{mn}^{T}(f) - \varepsilon_{mn}^{P}(f) - \alpha_{mn}^{f}\left(T - T_{0}\right)\right) \tag{98}$$

If $\mathbf{r}'$ is in the matrix the relations are replaced by

$$\Delta\varepsilon_{ij}^{T}\left(m\right) = \frac{1}{V_{m}}\iiint_{V_{m}}\Delta\varepsilon_{ij}^{T}\left(\mathbf{r}'\right)dV(\mathbf{r}') \tag{99}$$

and

$$\Delta c_{ij}\left(m\right) = \frac{1}{V_{m}}\iiint_{V_{m}}\Delta c_{ij}\left(\mathbf{r}'\right)dV(\mathbf{r}') \tag{100}$$

where

$$\Delta c_{ij}(m) = \Delta\varepsilon_{ij}^{P}(m) + \alpha_{ij}^{\star m}\Delta T \tag{101}$$

and

$$\alpha_{ij}^{\star m}\Delta T = \alpha_{ij}^{m}\Delta T + (T - T_{0})\Delta\alpha_{ij}^{m} - \\ - \left(D_{ijkl}^{m}\right)^{-1}\Delta D_{klmn}^{m}\left(\varepsilon_{mn}^{T}(m) - \varepsilon_{mn}^{P}(m) - \alpha_{mn}^{m}\left(T - T_{0}\right)\right) \tag{102}$$

If $\mathbf{r}'$ is in the smeared out "effective" or homogenized medium the corresponding results are

$$\overline{\Delta\varepsilon}_{ij}^T = \Delta\varepsilon_{ij}^0 = \frac{1}{V_s} \iiint\limits_{V_s} \Delta\varepsilon_{ij}^T \left(\mathbf{r}'\right) dV(\mathbf{r}') \tag{103}$$

and

$$\overline{\Delta c}_{ij} = \frac{1}{V_s} \iiint\limits_{V_s} \Delta c_{ij} \left(\mathbf{r}'\right) dV(\mathbf{r}') \tag{104}$$

where

$$\overline{\Delta c}_{ij} = \overline{\Delta\varepsilon}_{ij}^P + \overline{\alpha}_{ij}^\star \Delta T \tag{105}$$

and

$$\overline{\alpha}_{ij}^\star \Delta T = \overline{\alpha}_{ij}\Delta T + (T - T_0)\,\overline{\Delta\alpha}_{ij} - \\ - \overline{D}_{ijkl}^{-1}\,\overline{\Delta D}_{klmn}\left(\varepsilon_{mn}^0 - \overline{\varepsilon}_{mn}^P - \overline{\alpha}_{mn}\left(T - T_0\right)\right) \tag{106}$$

The volumes V_f, V_m and V_s refer to the volumes of the fiber, matrix and smeared out medium, respectively. If V_c is the volume of the unit cell and V denotes the total volume of the entire composite material, then

$$V_c = V_f + V_m \qquad \text{and} \qquad V = V_c + V_s = V_f + V_m + V_s \tag{107}$$

6.3 Applied, Homogenized and Volume Averaged Increments

At this point it is important to emphasize the following distinctions. First, the strain increment applied to the composite is denoted by $\Delta\varepsilon_{ij}^0$ which causes an incremental stress response $\Delta\sigma_{ij}^0$. To obtain the overall "effective" constitutive equation these are equated to the corresponding volume averaged quantities, $\left\langle \Delta\varepsilon_{ij}^T \right\rangle$ and $\left\langle \Delta\sigma_{ij} \right\rangle$. In the "effective" homogenized medium all quantities are denoted with overbars.

At any point $\mathbf{r}$ the appropriate constitutive relation is

$$\Delta\sigma_{ij}(\mathbf{r}) = D_{ijkl}(\mathbf{r}) \left(\Delta\varepsilon_{kl}^T(\mathbf{r}) - \Delta c_{kl}(\mathbf{r})\right) \tag{108}$$

If we volume average this relation over the unit cell we obtain

$$\left\langle \Delta\sigma_{ij} \right\rangle = \left\langle D_{ijkl}\Delta\varepsilon_{kl}^T \right\rangle - \left\langle D_{ijkl}\Delta c_{kl} \right\rangle \tag{109}$$

In the homogenized phase the constitutive relation can be written as

$$\overline{\Delta\sigma}_{ij} = \overline{D}_{ijkl}\overline{\Delta\varepsilon}_{kl}^T - \overline{D}_{ijkl}\overline{\Delta c}_{kl} \tag{110}$$

Since the strain increment $\overline{\Delta\varepsilon}_{kl}^T$ in the homogenized phase must correspond to the applied strain increment $\Delta\varepsilon_{kl}^0$—as in equation 103—and the homogenized stress increment $\overline{\Delta\sigma}_{ij}$ must correspond to the overall bulk stress increment $\Delta\sigma_{ij}^0$, we write the constitutive relation for the homogenized phase as

$$\Delta\sigma_{ij}^0 = \overline{D}_{ijkl}\Delta\varepsilon_{kl}^0 - \overline{D}_{ijkl}\overline{\Delta c}_{kl} \tag{111}$$

6.4 Requirement for Self-Consistency

For self-consistency we require that the volume average of the microscopic constitutive relation in equation 108 over the unit cell, *viz.* equation 109, should correspond to the constitutive relation for the overall "effective" homogenized medium in equation 111. That is,

$$\Delta\sigma_{ij}^0 = \left\langle D_{ijkl}\Delta\varepsilon_{kl}^T \right\rangle - \left\langle D_{ijkl}\Delta c_{kl} \right\rangle = \overline{D}_{ijkl}\Delta\varepsilon_{kl}^0 - \overline{D}_{ijkl}\overline{\Delta c}_{kl} \tag{112}$$

for self-consistency. Under the approximation that the strain increments $\Delta\varepsilon_{kl}^T$ and Δc_{kl} are spatially constant in the constituent phases, we obtain

$$\Delta\sigma_{ij}^0 = \frac{1}{V_c}\iiint\limits_{V_c} D_{ijkl}(\mathbf{r})\left(\Delta\varepsilon_{kl}^T(\mathbf{r}) - \Delta c_{kl}(\mathbf{r})\right) dV(\mathbf{r}) = \overline{D}_{ijkl}\Delta\varepsilon_{kl}^0 - \overline{D}_{ijkl}\overline{\Delta c}_{kl} \tag{113}$$

or

$$
\begin{aligned}
\Delta\sigma_{ij}^0 &= \frac{V_f}{V_c}D_{ijkl}^f\left(\Delta\varepsilon_{kl}^T(f) - \Delta c_{kl}(f)\right) + \frac{V_m}{V_c}D_{ijkl}^m\left(\Delta\varepsilon_{kl}^T(m) - \Delta c_{kl}(m)\right) \\
&= \overline{D}_{ijkl}\Delta\varepsilon_{kl}^0 - \overline{D}_{ijkl}\overline{\Delta c}_{kl}
\end{aligned}
\tag{114}
$$

At this point the elasticity tensor $\overline{D}_{ijkl}$ and the deviation strain increment $\overline{\Delta c}_{kl}$ in the homogenized medium are unknown quantities. In the next section we will solve the integral equations for the total strain increments in the fiber and matrix phases, $\Delta\varepsilon_{kl}^T(f)$ and $\Delta\varepsilon_{kl}^T(m)$, and we will find that these values depend on the quantities $\overline{D}_{ijkl}$, $\Delta\varepsilon_{kl}^0$ and $\overline{\Delta c}_{kl}$ in the surrounding homogenized medium. Then, by equating the coefficients of $\Delta\varepsilon_{kl}^0$ on both sides of equation 114 we obtain a relationship for the unknown elasticity tensor $\overline{D}_{ijkl}$ of the "effective" homogenized medium. The value of the unknown deviation strain increment $\overline{\Delta c}_{kl}$ in the homogenized medium can then be obtained by equating the terms independent of $\Delta\varepsilon_{kl}^0$ on the left hand side of equation 114 to the corresponding term $\overline{D}_{ijkl}\overline{\Delta c}_{kl}$ on the right hand side.

We now obtain the total strain increments $\Delta\varepsilon_{kl}^T(f)$ and $\Delta\varepsilon_{kl}^T(m)$ in the two phases of the unit cell. First, consider the total strain increment in the fiber phase.

6.5 Total Strain Increment in Fiber Phase

Equation 93 can be volume averaged over the fiber phase to give

$$\frac{1}{V_f}\iiint\limits_{V_f}\Delta\varepsilon_{ij}^T(\mathbf{r})\,dV(\mathbf{r}) = \Delta\varepsilon_{ij}^0 + \frac{1}{V_f}\iiint\limits_{V_f}dV(\mathbf{r})\iiint\limits_{V}U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right)\overline{D}_{klrs}\Delta\varepsilon_{rs}^\star\left(\mathbf{r}'\right)dV(\mathbf{r}') \tag{115}$$

where the field points $\mathbf{r}$ are in the fiber volume, V_f, and the integration points $\mathbf{r}'$ are in all three volume phases ($V = V_f + V_m + V_s$). Equation 115 can be written as

$$
\begin{aligned}
\Delta\varepsilon_{ij}^T(f) &= \Delta\varepsilon_{ij}^0 + \frac{1}{V_f}\Bigg[\iiint\limits_{V_f}dV(\mathbf{r})\bigg\{\iiint\limits_{V_f}U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right)\overline{D}_{klrs}\Delta\varepsilon_{rs}^\star(f)\,dV(\mathbf{r}') + \\
&\quad + \iiint\limits_{V_m}U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right)\overline{D}_{klrs}\Delta\varepsilon_{rs}^\star(m)\,dV(\mathbf{r}') + \iiint\limits_{V_s}U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right)\overline{D}_{klrs}\overline{\Delta c}_{rs}\,dV(\mathbf{r}')\bigg\}\Bigg]
\end{aligned}
\tag{116}
$$

in which $\overline{D}_{klrs}\Delta\varepsilon_{rs}^{\star}\left(\mathbf{r}'\right)$ has been replaced by

$$\overline{D}_{klrs}\Delta\varepsilon_{rs}^{\star}\left(f\right) = \overline{D}_{klrs}\Delta c_{rs}(f) - \delta D_{klrs}^{f}\left[\Delta\varepsilon_{rs}^{T}(f) - \Delta c_{rs}(f)\right] \tag{117}$$

and

$$\overline{D}_{klrs}\Delta\varepsilon_{rs}^{\star}\left(m\right) = \overline{D}_{klrs}\Delta c_{rs}(m) - \delta D_{klrs}^{m}\left[\Delta\varepsilon_{rs}^{T}(m) - \Delta c_{rs}(m)\right] \tag{118}$$

in the respective fiber and matrix phases, and by

$$\overline{D}_{klrs}\Delta\varepsilon_{rs}^{\star}(s) = \overline{D}_{klrs}\overline{\Delta c}_{rs} \tag{119}$$

in the smeared out "effective" medium where $\delta\overline{D}_{klrs}\left(\mathbf{r}'\right) = 0$.

In the first integral in equation 116 the field point $\mathbf{r}$ lies in the volume V_f, and since

$$\iiint\limits_{V} U_{ijkl}\left(\mathbf{r} - \mathbf{r}'\right)dV(\mathbf{r}')\,\overline{D}_{klrs} = S_{ijrs} \tag{120}$$

is Eshelby's tensor (*cf.* Appendix E, equation E.1 and [35]), which is a constant tensor independent of $\mathbf{r}$ when the field point $\mathbf{r}$ lies within the cylindrical volume V included in an infinite medium with elasticity tensor $\overline{D}_{klrs}$, we may write the first integral as

$$\iiint\limits_{V_f} U_{ijkl}\left(\mathbf{r} - \mathbf{r}'\right)dV(\mathbf{r}')\,\overline{D}_{klrs}\Delta\varepsilon_{rs}^{\star}(f) = S_{ijrs}\Delta\varepsilon_{rs}^{\star}(f) \tag{121}$$

The second volume integral extends over the volume $V_m = V_c - V_f$ of the matrix phase. Thus, for the second integral,

$$\iiint\limits_{V_m} U_{ijkl}\left(\mathbf{r} - \mathbf{r}'\right)dV(\mathbf{r}')\,\overline{D}_{klrs}\Delta\varepsilon_{rs}^{\star}(m)$$

$$= \iiint\limits_{V_c} U_{ijkl}\left(\mathbf{r} - \mathbf{r}'\right)dV(\mathbf{r}')\,\overline{D}_{klrs}\Delta\varepsilon_{rs}^{\star}(m) - \iiint\limits_{V_f} U_{ijkl}\left(\mathbf{r} - \mathbf{r}'\right)dV(\mathbf{r}')\,\overline{D}_{klrs}\Delta\varepsilon_{rs}^{\star}(m)$$

$$= S_{ijrs}\Delta\varepsilon_{rs}^{\star}(m) - S_{ijrs}\Delta\varepsilon_{rs}^{\star}(m) = 0 \tag{122}$$

since the field point $\mathbf{r}$ lies in the cylindrical volume V_f and therefore within the cylindrical volume V_c.

We now have to deal with the last integral in equation 116. This integral can be written as

$$\iiint\limits_{V_s} U_{ijkl}\left(\mathbf{r} - \mathbf{r}'\right)dV(\mathbf{r}')\,\overline{D}_{klrs}\overline{\Delta c}_{rs}$$

$$= -\iiint\limits_{V_s} \frac{1}{2}\left(\frac{\partial}{\partial x_j}\frac{\partial G_{ik}\left(\mathbf{r} - \mathbf{r}'\right)}{\partial x_l} + \frac{\partial}{\partial x_i}\frac{\partial G_{jk}\left(\mathbf{r} - \mathbf{r}'\right)}{\partial x_l}\right)dV(\mathbf{r}')\,\overline{D}_{klrs}\overline{\Delta c}_{rs}$$

$$= \iiint\limits_{V_s} \frac{1}{2}\left(\frac{\partial}{\partial x_j'}\frac{\partial G_{ik}\left(\mathbf{r} - \mathbf{r}'\right)}{\partial x_l} + \frac{\partial}{\partial x_i'}\frac{\partial G_{jk}\left(\mathbf{r} - \mathbf{r}'\right)}{\partial x_l}\right)dV(\mathbf{r}')\,\overline{D}_{klrs}\overline{\Delta c}_{rs}$$

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which can be transformed *via* Gauss' divergence theorem into two surface integrals: one over the outer "infinite" surface S of the smeared out "effective" medium; and the other over the inner surface of the "effective" medium, *i.e.*, over the surface S_c of the unit periodic cell. The volume integral then takes the form,

$$\iiint\limits_{V_s} U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right) dV(\mathbf{r}') \overline{D}_{klrs}\overline{\Delta c}_{rs}$$

$$= \iint\limits_{S} \frac{1}{2}\left(n_j^\star\left(\mathbf{r}'\right)\frac{\partial G_{ik}\left(\mathbf{r}-\mathbf{r}'\right)}{\partial x_l} + n_i^\star\left(\mathbf{r}'\right)\frac{\partial G_{jk}\left(\mathbf{r}-\mathbf{r}'\right)}{\partial x_l}\right) dS(\mathbf{r}') \overline{D}_{klrs}\overline{\Delta c}_{rs} +$$

$$+ \iint\limits_{S_c} \frac{1}{2}\left(n_j^\star\left(\mathbf{r}'\right)\frac{\partial G_{ik}\left(\mathbf{r}-\mathbf{r}'\right)}{\partial x_l} + n_i^\star\left(\mathbf{r}'\right)\frac{\partial G_{jk}\left(\mathbf{r}-\mathbf{r}'\right)}{\partial x_l}\right) dS(\mathbf{r}') \overline{D}_{klrs}\overline{\Delta c}_{rs}$$

where $n_i^\star\left(\mathbf{r}'\right)$ is a unit normal vector at point $\mathbf{r}'$ on the surfaces pointing away from the volume V_s. Now since the field point $\mathbf{r}$ lies in the fiber and the integration points $\mathbf{r}'$ on the surface S are infinitely removed, we have $\partial G_{ik}\left(\mathbf{r}-\mathbf{r}'\right)/\partial x_l \to 0$ on the outer surface S of the composite, and the first surface integral can be neglected. If we write $n_i\left(\mathbf{r}'\right) = -n_i^\star\left(\mathbf{r}'\right)$, then $n_i\left(\mathbf{r}'\right)$ is a unit normal vector pointing away from the volume V_c on the surface S_c of the unit cell, and we have, *via* Gauss' divergence theorem and equations 77 and 83,

$$\iiint\limits_{V_s} U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right) dV(\mathbf{r}') \overline{D}_{klrs}\overline{\Delta c}_{rs}$$

$$= -\iint\limits_{S_c} \frac{1}{2}\left(n_j\left(\mathbf{r}'\right)\frac{\partial G_{ik}\left(\mathbf{r}-\mathbf{r}'\right)}{\partial x_l} + n_i\left(\mathbf{r}'\right)\frac{\partial G_{jk}\left(\mathbf{r}-\mathbf{r}'\right)}{\partial x_l}\right) dS(\mathbf{r}') \overline{D}_{klrs}\overline{\Delta c}_{rs}$$

$$= -\iiint\limits_{V_c} \frac{1}{2}\left(\frac{\partial}{\partial x_j'}\frac{\partial G_{ik}\left(\mathbf{r}-\mathbf{r}'\right)}{\partial x_l} + \frac{\partial}{\partial x_i'}\frac{\partial G_{jk}\left(\mathbf{r}-\mathbf{r}'\right)}{\partial x_l}\right) dV(\mathbf{r}') \overline{D}_{klrs}\overline{\Delta c}_{rs}$$

$$= \iiint\limits_{V_c} \frac{1}{2}\left(\frac{\partial^2 G_{ik}\left(\mathbf{r}-\mathbf{r}'\right)}{\partial x_j \partial x_l} + \frac{\partial^2 G_{jk}\left(\mathbf{r}-\mathbf{r}'\right)}{\partial x_i \partial x_l}\right) dV(\mathbf{r}') \overline{D}_{klrs}\overline{\Delta c}_{rs}$$

$$= -\iiint\limits_{V_c} U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right) dV(\mathbf{r}') \overline{D}_{klrs}\overline{\Delta c}_{rs}$$

Since the field point $\mathbf{r}$ lies in the cylindrical volume V_c, the preceding equation takes the form,

$$\iiint\limits_{V_s} U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right) dV(\mathbf{r}') \overline{D}_{klrs}\overline{\Delta c}_{rs} = -\iiint\limits_{V_c} U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right) dV(\mathbf{r}') \overline{D}_{klrs}\overline{\Delta c}_{rs}$$

or

$$\iiint\limits_{V_s} U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right) dV(\mathbf{r}') \overline{D}_{klrs}\overline{\Delta c}_{rs} = -S_{ijrs}\overline{\Delta c}_{rs} \tag{123}$$

where S_{ijrs} is Eshelby's tensor for a cylinder with elasticity tensor $\overline{D}_{ijkl}$.

From equations 116, 121, 122 and 123 we obtain

$$\Delta\varepsilon_{ij}^T(f) = \Delta\varepsilon_{ij}^0 + \frac{1}{V_f}\iiint\limits_{V_f} \left\{S_{ijrs}\Delta\varepsilon_{rs}^\star(f) - S_{ijrs}\overline{\Delta c}_{rs}\right\} dV(\mathbf{r})$$

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and since the quantity within the braces is a constant tensor,

$$\Delta \varepsilon_{ij}^{T}(f) = \Delta \varepsilon_{ij}^{0} + S_{ijrs} \left(\Delta \varepsilon_{rs}^{\star}(f) - \overline{\Delta c_{rs}} \right) \tag{124}$$

Now

$$\overline{D}_{klrs} \Delta \varepsilon_{rs}^{\star}(f) = \overline{D}_{klrs} \Delta c_{rs}(f) - \delta D_{klrs}^{f} \left[\Delta \varepsilon_{rs}^{T}(f) - \Delta c_{rs}(f) \right]$$

so that

$$\Delta \varepsilon_{rs}^{\star}(f) = \Delta c_{rs}(f) - \overline{D}_{rspq}^{-1} \delta D_{pqmn}^{f} \left[\Delta \varepsilon_{mn}^{T}(f) - \Delta c_{mn}(f) \right]$$

and on substitution into equation 124 we obtain,

$$\Delta \varepsilon_{ij}^{T}(f) = \Delta \varepsilon_{ij}^{0} + S_{ijrs} \left(\Delta c_{rs}(f) - \overline{\Delta c_{rs}} - \overline{D}_{rspq}^{-1} \delta D_{pqmn}^{f} \left[\Delta \varepsilon_{mn}^{T}(f) - \Delta c_{mn}(f) \right] \right)$$

Given that

$$I_{ijkl} = \tfrac{1}{2} \left(\delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il} \right)$$

denotes the fourth rank identity tensor for symmetric second rank tensors, the preceding equation can be written as

$$\left[I_{ijmn} + S_{ijrs} \overline{D}_{rspq}^{-1} \delta D_{pqmn}^{f} \right] \Delta \varepsilon_{mn}^{T}(f) = \Delta \varepsilon_{ij}^{0} +$$
$$+ S_{ijrs} \left(\Delta c_{rs}(f) - \overline{\Delta c_{rs}} \right) + S_{ijrs} \overline{D}_{rspq}^{-1} \delta D_{pqmn}^{f} \Delta c_{mn}(f)$$

which, by premultiplying each side with the inverse of the tensor in square brackets, gives

$$\Delta \varepsilon_{ij}^{T}(f) = \left[I_{ijmn} + S_{ijrs} \overline{D}_{rspq}^{-1} \delta D_{pqmn}^{f} \right]^{-1} \left\{ \Delta \varepsilon_{mn}^{0} + S_{mnrs} \left(\Delta c_{rs}(f) - \overline{\Delta c_{rs}} \right) + \right.$$
$$\left. + S_{mnkl} \overline{D}_{klpq}^{-1} \delta D_{pqrs}^{f} \Delta c_{rs}(f) \right\} \tag{125}$$

The phase volume averaged stress increment in the fiber is then given by the relation

$$\Delta \sigma_{ij}(f) = D_{ijkl}^{f} \left(\Delta \varepsilon_{kl}^{T}(f) - \Delta c_{kl}(f) \right) \tag{126}$$

6.6 Total Strain Increment in Matrix Phase

Now consider a field point $\mathbf{r}$ in the matrix phase. From equation 93 we may write

$$\Delta \varepsilon_{ij}^{T}(\mathbf{r}) = \Delta \varepsilon_{ij}^{0} + \iiint_{V_f} U_{ijkl}(\mathbf{r} - \mathbf{r}') \, dV(\mathbf{r}') \, \overline{D}_{klrs} \Delta \varepsilon_{rs}^{\star}(f) +$$

$$+ \iiint_{V_m} U_{ijkl}(\mathbf{r} - \mathbf{r}') \, dV(\mathbf{r}') \, \overline{D}_{klrs} \Delta \varepsilon_{rs}^{\star}(m) + \iiint_{V_s} U_{ijkl}(\mathbf{r} - \mathbf{r}') \, dV(\mathbf{r}') \, \overline{D}_{klrs} \overline{\Delta c_{rs}} \tag{127}$$

Since $V_m = V_c - V_f$, the second integral can be written as

$$\iiint_{V_m} U_{ijkl}(\mathbf{r} - \mathbf{r}') \, dV(\mathbf{r}') \, \overline{D}_{klrs} \Delta \varepsilon_{rs}^{\star}(m)$$

$$= \iiint_{V_c} U_{ijkl}(\mathbf{r} - \mathbf{r}') \, dV(\mathbf{r}') \, \overline{D}_{klrs} \Delta \varepsilon_{rs}^{\star}(m) - \iiint_{V_f} U_{ijkl}(\mathbf{r} - \mathbf{r}') \, dV(\mathbf{r}') \, \overline{D}_{klrs} \Delta \varepsilon_{rs}^{\star}(m) \tag{128}$$

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and from equation 123 the last integral in equation 127 may be written as

$$\iiint\limits_{V_s} U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right)dV(\mathbf{r}')\,\overline{D}_{klrs}\overline{\Delta c}_{rs} = -\iiint\limits_{V_c} U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right)dV(\mathbf{r}')\,\overline{D}_{klrs}\overline{\Delta c}_{rs} \tag{129}$$

so that equation 127 is transformed into

$$\Delta\varepsilon_{ij}^T(\mathbf{r}) = \Delta\varepsilon_{ij}^0 + \iiint\limits_{V_f} U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right)dV(\mathbf{r}')\,\overline{D}_{klrs}\left(\Delta\varepsilon_{rs}^\star(f)-\Delta\varepsilon_{rs}^\star(m)\right) +$$

$$+ \iiint\limits_{V_c} U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right)dV(\mathbf{r}')\,\overline{D}_{klrs}\left(\Delta\varepsilon_{rs}^\star(m)-\overline{\Delta c}_{rs}\right) \tag{130}$$

Averaging equation 130 over the matrix phase gives

$$\Delta\varepsilon_{ij}^T(m) = \Delta\varepsilon_{ij}^0 + \frac{1}{V_m}\iiint\limits_{V_m} dV(\mathbf{r})\left\{\iiint\limits_{V_f} U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right)dV(\mathbf{r}')\,\overline{D}_{klrs}\left(\Delta\varepsilon_{rs}^\star(f)-\Delta\varepsilon_{rs}^\star(m)\right) +\right.$$

$$\left.+ \iiint\limits_{V_c} U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right)dV(\mathbf{r}')\,\overline{D}_{klrs}\left(\Delta\varepsilon_{rs}^\star(m)-\overline{\Delta c}_{rs}\right)\right\} \tag{131}$$

or, since $V_m = V_c - V_f$,

$$\Delta\varepsilon_{ij}^T(m) = \Delta\varepsilon_{ij}^0 + \frac{1}{V_m}\left[\iiint\limits_{V_c} dV(\mathbf{r})\iiint\limits_{V_f} U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right)dV(\mathbf{r}')\,\overline{D}_{klrs}\left(\Delta\varepsilon_{rs}^\star(f)-\Delta\varepsilon_{rs}^\star(m)\right) +\right.$$

$$+ \iiint\limits_{V_c} dV(\mathbf{r})\iiint\limits_{V_c} U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right)dV(\mathbf{r}')\,\overline{D}_{klrs}\left(\Delta\varepsilon_{rs}^\star(m)-\overline{\Delta c}_{rs}\right) -$$

$$- \iiint\limits_{V_f} dV(\mathbf{r})\iiint\limits_{V_f} U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right)dV(\mathbf{r}')\,\overline{D}_{klrs}\left(\Delta\varepsilon_{rs}^\star(f)-\Delta\varepsilon_{rs}^\star(m)\right) -$$

$$\left.- \iiint\limits_{V_f} dV(\mathbf{r})\iiint\limits_{V_c} U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right)dV(\mathbf{r}')\,\overline{D}_{klrs}\left(\Delta\varepsilon_{rs}^\star(m)-\overline{\Delta c}_{rs}\right)\right] \tag{132}$$

Now consider the first integral in equation 132. We may interchange the order of the volume integrals so that

$$\iiint\limits_{V_c} dV(\mathbf{r})\iiint\limits_{V_f} U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right)dV(\mathbf{r}') = \iiint\limits_{V_f} dV(\mathbf{r}')\iiint\limits_{V_c} U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right)dV(\mathbf{r}) \tag{133}$$

Now $\mathbf{r}$ and $\mathbf{r}'$ are dummy integration variables, so that on the right hand side of equation 133 the variables may be replaced with the integration variables $\mathbf{x}$ and $\mathbf{y}$, *viz.*,

$$\iiint\limits_{V_c} dV(\mathbf{r})\iiint\limits_{V_f} U_{ijkl}\left(\mathbf{r}-\mathbf{r}'\right)dV(\mathbf{r}') = \iiint\limits_{V_f} dV(\mathbf{y})\iiint\limits_{V_c} U_{ijkl}\left(\mathbf{x}-\mathbf{y}\right)dV(\mathbf{x}) \tag{134}$$

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But, from equation F.5 of Appendix F,

$$U_{ijkl}\left(\mathbf{x} - \mathbf{y}\right) = U_{ijkl}\left(\mathbf{y} - \mathbf{x}\right)$$

so that

$$\iiint_{V_c} dV(\mathbf{r}) \iiint_{V_f} U_{ijkl}\left(\mathbf{r} - \mathbf{r}'\right) dV(\mathbf{r}') = \iiint_{V_f} dV(\mathbf{y}) \iiint_{V_c} U_{ijkl}\left(\mathbf{y} - \mathbf{x}\right) dV(\mathbf{x}) \qquad (135)$$

and the dummy variables $\mathbf{y}$ and $\mathbf{x}$ may be replaced with the variables $\mathbf{r}$ and $\mathbf{r}'$ to give

$$\iiint_{V_c} dV(\mathbf{r}) \iiint_{V_f} U_{ijkl}\left(\mathbf{r} - \mathbf{r}'\right) dV(\mathbf{r}') = \iiint_{V_f} dV(\mathbf{r}) \iiint_{V_c} U_{ijkl}\left(\mathbf{r} - \mathbf{r}'\right) dV(\mathbf{r}') \qquad (136)$$

This relationship is discussed in Mura's book ([24], page 336) where it appears under the heading of the Tanaka-Mori theorem.

From equation 136, the first integral in equation 132 is integrated over the field points $\mathbf{r}$ within the cylindrical volume V_f. Since these field points lie within the cylindrical integration volume V_c, the first integral in equation 132 may be written as

$$\frac{1}{V_m} \iiint_{V_f} dV(\mathbf{r}) \iiint_{V_c} U_{ijkl}\left(\mathbf{r} - \mathbf{r}'\right) dV(\mathbf{r}') \overline{D}_{klrs} \left(\Delta\varepsilon_{rs}^{\star}(f) - \Delta\varepsilon_{rs}^{\star}(m)\right)$$

$$= \frac{1}{V_m} \iiint_{V_f} S_{ijrs} \left(\Delta\varepsilon_{rs}^{\star}(f) - \Delta\varepsilon_{rs}^{\star}(m)\right) dV(\mathbf{r})$$

$$= \frac{V_f}{V_m} S_{ijrs} \left(\Delta\varepsilon_{rs}^{\star}(f) - \Delta\varepsilon_{rs}^{\star}(m)\right) \qquad (137)$$

In the second integral in equation 132 the field points $\mathbf{r}$ lie within the cylindrical volume V_c and so the second integral may be written as

$$\frac{1}{V_m} \iiint_{V_c} dV(\mathbf{r}) \iiint_{V_c} U_{ijkl}\left(\mathbf{r} - \mathbf{r}'\right) dV(\mathbf{r}') \overline{D}_{klrs} \left(\Delta\varepsilon_{rs}^{\star}(m) - \overline{\Delta c_{rs}}\right)$$

$$= \frac{1}{V_m} \iiint_{V_c} S_{ijrs} \left(\Delta\varepsilon_{rs}^{\star}(m) - \overline{\Delta c_{rs}}\right) dV(\mathbf{r})$$

$$= \frac{V_c}{V_m} S_{ijrs} \left(\Delta\varepsilon_{rs}^{\star}(m) - \overline{\Delta c_{rs}}\right) \qquad (138)$$

In the third integral the field points $\mathbf{r}$ lie within the cylindrical volume V_f and so

$$\frac{1}{V_m} \iiint_{V_f} dV(\mathbf{r}) \iiint_{V_f} U_{ijkl}\left(\mathbf{r} - \mathbf{r}'\right) dV(\mathbf{r}') \overline{D}_{klrs} \left(\Delta\varepsilon_{rs}^{\star}(f) - \Delta\varepsilon_{rs}^{\star}(m)\right)$$

$$= \frac{1}{V_m} \iiint_{V_f} S_{ijrs} \left(\Delta\varepsilon_{rs}^{\star}(f) - \Delta\varepsilon_{rs}^{\star}(m)\right) dV(\mathbf{r})$$

$$= \frac{V_f}{V_m} S_{ijrs} \left(\Delta\varepsilon_{rs}^{\star}(f) - \Delta\varepsilon_{rs}^{\star}(m)\right) \qquad (139)$$

Finally, in the fourth volume integral, the field points $\mathbf{r}$ lie in the cylindrical volume V_f. Since this lies within the cylindrical integration volume V_c, we have

$$\frac{1}{V_m}\iiint_{V_f} dV(\mathbf{r}) \iiint_{V_c} U_{ijkl}(\mathbf{r}-\mathbf{r}') \, dV(\mathbf{r}') \, \overline{D}_{klrs}\left(\Delta\varepsilon_{rs}^{\star}(m)-\overline{\Delta c}_{rs}\right)$$

$$= \frac{1}{V_m}\iiint_{V_f} S_{ijrs}\left(\Delta\varepsilon_{rs}^{\star}(m)-\overline{\Delta c}_{rs}\right) \, dV(\mathbf{r})$$

$$= \frac{V_f}{V_m}S_{ijrs}\left(\Delta\varepsilon_{rs}^{\star}(m)-\overline{\Delta c}_{rs}\right) \tag{140}$$

We thus obtain from equations 132, and 137 to 140,

$$\begin{aligned}
\Delta\varepsilon_{ij}^T(m) &= \Delta\varepsilon_{ij}^0 + \frac{V_f}{V_m}S_{ijrs}\left(\Delta\varepsilon_{rs}^{\star}(f)-\Delta\varepsilon_{rs}^{\star}(m)\right) + \\
&\quad + \frac{V_c}{V_m}S_{ijrs}\left(\Delta\varepsilon_{rs}^{\star}(m)-\overline{\Delta c}_{rs}\right) - \\
&\quad - \frac{V_f}{V_m}S_{ijrs}\left(\Delta\varepsilon_{rs}^{\star}(f)-\Delta\varepsilon_{rs}^{\star}(m)\right) - \\
&\quad - \frac{V_f}{V_m}S_{ijrs}\left(\Delta\varepsilon_{rs}^{\star}(m)-\overline{\Delta c}_{rs}\right) \\
&= \Delta\varepsilon_{ij}^0 + \frac{V_c-V_f}{V_m}S_{ijrs}\left(\Delta\varepsilon_{rs}^{\star}(m)-\overline{\Delta c}_{rs}\right)
\end{aligned}$$

or

$$\Delta\varepsilon_{ij}^T(m) = \Delta\varepsilon_{ij}^0 + S_{ijrs}\left(\Delta\varepsilon_{rs}^{\star}(m)-\overline{\Delta c}_{rs}\right) \tag{141}$$

This relation for the total strain increment in the matrix phase is similar to that for the fiber phase given in equation 124. By following the steps leading to equation 125 the expression for $\Delta\varepsilon_{ij}^T(m)$ can be put in the form

$$\begin{aligned}
\Delta\varepsilon_{ij}^T(m) &= \left[I_{ijmn}+S_{ijrs}\overline{D}_{rspq}^{-1}\delta D_{pqmn}^m\right]^{-1}\left\{\Delta\varepsilon_{mn}^0+S_{mnrs}\left(\Delta c_{rs}(m)-\overline{\Delta c}_{rs}\right) + \right.\\
&\quad \left. + S_{mnkl}\overline{D}_{klpq}^{-1}\delta D_{pqrs}^m\Delta c_{rs}(m)\right\} \tag{142}
\end{aligned}$$

The phase volume average stress increment in the matrix is then given by the relation

$$\Delta\sigma_{ij}(m) = D_{ijkl}^m\left(\Delta\varepsilon_{kl}^T(m)-\Delta c_{kl}(m)\right) \tag{143}$$

6.7 Overall "Effective" Constitutive Relation

As stated in section 6.4, for self-consistency we require that the volume average of the constrained micromechanical constitutive relation over the unit periodic cell should correspond to that for the "effective" homogenized medium. From equation 114 we require that

$$\begin{aligned}
\Delta\sigma_{ij}^0 &= \frac{V_f}{V_c}D_{ijkl}^f\left(\Delta\varepsilon_{kl}^T(f)-\Delta c_{kl}(f)\right) + \frac{V_m}{V_c}D_{ijkl}^m\left(\Delta\varepsilon_{kl}^T(m)-\Delta c_{kl}(m)\right) \\
&= \overline{D}_{ijkl}\Delta\varepsilon_{kl}^0 - \overline{D}_{ijkl}\overline{\Delta c}_{kl} \tag{144}
\end{aligned}$$

where the total strain increments in the fiber and matrix phases are given by equations 125 and 142 as

$$\Delta\varepsilon_{ij}^{T}(f) = \left[I_{ijmn} + S_{ijrs}\overline{D}_{rspq}^{-1}\,\delta D_{pqmn}^{f}\right]^{-1}\left\{\Delta\varepsilon_{mn}^{0} + S_{mnrs}\left(\Delta c_{rs}(f) - \overline{\Delta c_{rs}}\right) + \right.$$
$$\left. + S_{mnkl}\overline{D}_{klpq}^{-1}\,\delta D_{pqrs}^{f}\Delta c_{rs}(f)\right\} \tag{145}$$

and

$$\Delta\varepsilon_{ij}^{T}(m) = \left[I_{ijmn} + S_{ijrs}\overline{D}_{rspq}^{-1}\,\delta D_{pqmn}^{m}\right]^{-1}\left\{\Delta\varepsilon_{mn}^{0} + S_{mnrs}\left(\Delta c_{rs}(m) - \overline{\Delta c_{rs}}\right) + \right.$$
$$\left. + S_{mnkl}\overline{D}_{klpq}^{-1}\,\delta D_{pqrs}^{m}\Delta c_{rs}(m)\right\} \tag{146}$$

with the deviation strain increments defined in equations 97, 101, 105 as

$$\Delta c_{ij}(f) = \Delta\varepsilon_{ij}^{P}(f) + \alpha_{ij}^{\star f}\Delta T \tag{147}$$

$$\Delta c_{ij}(m) = \Delta\varepsilon_{ij}^{P}(m) + \alpha_{ij}^{\star m}\Delta T \tag{148}$$

and

$$\overline{\Delta c}_{ij} = \overline{\Delta\varepsilon}_{ij}^{P} + \overline{\alpha}_{ij}^{\star}\Delta T \tag{149}$$

By inserting equations 90, 91, 145, 146 147, 148 and 149 into equation 144 the relationship for self-consistency requires that

$$\Delta\sigma_{ij}^{0} = A_{ijkl}\Delta\varepsilon_{kl}^{0} - \left\{A_{ijkl}S_{klrs}\overline{\Delta\varepsilon}_{rs}^{P} - B_{ijrs}\Delta\varepsilon_{rs}^{P}(f) - C_{ijrs}\Delta\varepsilon_{rs}^{P}(m)\right\} -$$
$$- \left\{A_{ijkl}S_{klrs}\overline{\alpha}_{rs}^{\star} - B_{ijrs}\alpha_{rs}^{\star f} - C_{ijrs}\alpha_{rs}^{\star m}\right\}\Delta T$$
$$= \overline{D}_{ijkl}\Delta\varepsilon_{kl}^{0} - \overline{D}_{ijkl}\overline{\Delta\varepsilon}_{kl}^{P} - \overline{D}_{ijkl}\overline{\alpha}_{kl}^{\star}\Delta T \tag{150}$$

in which

$$A_{ijkl} = \frac{V_f}{V_c}D_{ijrs}^{f}\left[I_{rskl} + S_{rspq}\overline{D}_{pqmn}^{-1}\left(D_{mnkl}^{f} - \overline{D}_{mnkl}\right)\right]^{-1} +$$
$$+ \frac{V_m}{V_c}D_{ijrs}^{m}\left[I_{rskl} + S_{rspq}\overline{D}_{pqmn}^{-1}\left(D_{mnkl}^{m} - \overline{D}_{mnkl}\right)\right]^{-1} \tag{151}$$

$$B_{ijkl} = \frac{V_f}{V_c}D_{ijrs}^{f}\left[I_{rskl} + S_{rspq}\overline{D}_{pqmn}^{-1}\left(D_{mngh}^{f} - \overline{D}_{mngh}\right)\right]^{-1} \times$$
$$\times \left[S_{ghkl} + S_{ghab}\overline{D}_{abcd}^{-1}\left(D_{cdkl}^{f} - \overline{D}_{cdkl}\right)\right] - \frac{V_f}{V_c}D_{ijkl}^{f} \tag{152}$$

and

$$C_{ijkl} = \frac{V_m}{V_c}D_{ijrs}^{m}\left[I_{rskl} + S_{rspq}\overline{D}_{pqmn}^{-1}\left(D_{mngh}^{m} - \overline{D}_{mngh}\right)\right]^{-1} \times$$
$$\times \left[S_{ghkl} + S_{ghab}\overline{D}_{abcd}^{-1}\left(D_{cdkl}^{m} - \overline{D}_{cdkl}\right)\right] - \frac{V_m}{V_c}D_{ijkl}^{m} \tag{153}$$

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These results for B_{ijkl} and C_{ijkl} can be simplified somewhat. We write B_{ijkl} as

$$B_{ijkl} = \frac{V_f}{V_c} y_{ijkl} \tag{154}$$

where

$$y_{ijkl} = D^f_{ijpq} \left(I_{pqrs} + X_{pqrs}\right)^{-1} \left(S_{rskl} + X_{rskl}\right) - D^f_{ijkl} \tag{155}$$

and

$$X_{pqrs} = S_{pqmn} \overline{D}^{-1}_{mngh} \delta D^f_{ghrs} \tag{156}$$

We then obtain

$$y_{ijkl} + D^f_{ijkl} = D^f_{ijpq} \left(I_{pqrs} + X_{pqrs}\right)^{-1} \left(S_{rskl} + X_{rskl}\right) \tag{157}$$

or

$$\left(D^f_{ijkl}\right)^{-1} \left(y_{klmn} + D^f_{klmn}\right) = \left(I_{ijkl} + X_{ijkl}\right)^{-1} \left(S_{klmn} + X_{klmn}\right) \tag{158}$$

This result simplifies to

$$\left(D^f_{ijkl}\right)^{-1} y_{klmn} + I_{ijmn} = \left(I_{ijkl} + X_{ijkl}\right)^{-1} \left(S_{klmn} + X_{klmn}\right) \tag{159}$$

which can be premultiplied by $\left(I_{pqij} + X_{pqij}\right)$ to give

$$\left(I_{pqij} + X_{pqij}\right) \left(D^f_{ijkl}\right)^{-1} y_{klmn} = S_{pqmn} - I_{pqmn} \tag{160}$$

from which

$$y_{ijkl} = D^f_{ijpq} \left(I_{pqrs} + X_{pqrs}\right)^{-1} \left(S_{rskl} - I_{rskl}\right) \tag{161}$$

From this result we find that B_{ijkl} and C_{ijkl} can be written in the simplified forms

$$B_{ijkl} = \frac{V_f}{V_c} D^f_{ijpq} \left[I_{pqgh} + S_{pqrs}\overline{D}^{-1}_{rsmn} \left(D^f_{mngh} - \overline{D}_{mngh}\right)\right]^{-1} \left(S_{ghkl} - I_{ghkl}\right) \tag{162}$$

and

$$C_{ijkl} = \frac{V_m}{V_c} D^m_{ijpq} \left[I_{pqgh} + S_{pqrs}\overline{D}^{-1}_{rsmn} \left(D^m_{mngh} - \overline{D}_{mngh}\right)\right]^{-1} \left(S_{ghkl} - I_{ghkl}\right) \tag{163}$$

Equating the coefficients of $\Delta\varepsilon^0_{kl}$ in equation 150 for self-consistency then requires that

$$\overline{D}_{ijkl} = A_{ijkl} \tag{164}$$

which, from equation 151, produces the implicit relation

$$\overline{D}_{ijkl} = \frac{V_f}{V_c} D^f_{ijrs} \left[I_{rskl} + S_{rspq}\overline{D}^{-1}_{pqmn} \left(D^f_{mnkl} - \overline{D}_{mnkl}\right)\right]^{-1} +$$
$$+ \frac{V_m}{V_c} D^m_{ijrs} \left[I_{rskl} + S_{rspq}\overline{D}^{-1}_{pqmn} \left(D^m_{mnkl} - \overline{D}_{mnkl}\right)\right]^{-1} \tag{165}$$

The value of homogenized "effective" elasticity tensor $\overline{D}_{ijkl}$ may be obtained from this implicit relationship by iteration. Naturally, when the self-consistent method is embedded in a nonlinear finite element program, this iteration would be done outside of the code and the explicit values of $\overline{D}_{ijkl}$ would be used in the program.

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For self-consistency we also require from equation 150 that

$$A_{ijpq}S_{pqkl}\overline{\Delta\varepsilon}_{kl}^{P} - B_{ijkl}\Delta\varepsilon_{kl}^{P}(f) - C_{ijkl}\Delta\varepsilon_{kl}^{P}(m) = \overline{D}_{ijkl}\overline{\Delta\varepsilon}_{kl}^{P} \tag{166}$$

and

$$A_{ijpq}S_{pqkl}\overline{\alpha}_{kl}^{\star} - B_{ijkl}\Delta\alpha_{kl}^{\star f} - C_{ijkl}\Delta\alpha_{kl}^{\star m} = \overline{D}_{ijkl}\overline{\alpha}_{kl}^{\star} \tag{167}$$

which, by setting $A_{ijpq} = \overline{D}_{ijpq}$, reduce to

$$\overline{\Delta\varepsilon}_{ij}^{P} = \left[\overline{D}_{ijpq}\left(S_{pqrs} - I_{pqrs}\right)\right]^{-1}\left(B_{rskl}\Delta\varepsilon_{kl}^{P}(f) + C_{rskl}\Delta\varepsilon_{kl}^{P}(m)\right) \tag{168}$$

and

$$\overline{\alpha}_{ij}^{\star} = \left[\overline{D}_{ijpq}\left(S_{pqrs} - I_{pqrs}\right)\right]^{-1}\left(B_{rskl}\alpha_{kl}^{\star f} + C_{rskl}\alpha_{kl}^{\star m}\right) \tag{169}$$

The overall "effective" constitutive relation for the homogenized composite in equation 150 is now easily computed.

If a forward difference algorithm is used to evaluate the viscoplastic strain increments, the only implicit equation which occurs in the formulation is that for the elasticity tensor of the homogenized medium given in equation 165. It is, perhaps, ironic that in deriving the highly nonlinear viscoplastic constitutive relationship for the homogenized medium, the only iterative procedure required is that for the elasticity tensor. This implicit elasticity relationship also occurs in the subvolume method due to the occurrence of the tensor δD_{ijkl} in the volume integration. The implicit nature of $\overline{D}_{ijkl}$ is due to the fact that the homogenized elasticity tensor is found by volume averaging the constrained elastic properties of the unit periodic cell, and these constrained properties, in turn, depend on the elasticity tensor $\overline{D}_{ijkl}$ of the homogenized constraining medium.

The constitutive relations given in equations 126 and 143 are used to update the stress-strain history in the constituent phases, whilst equation 150 is used to update the stress-strain history in the homogenized self-consistent medium. These relations, which contain $\Delta\varepsilon_{ij}^{T}(f)$, $\Delta\varepsilon_{ij}^{T}(m)$ and $\overline{D}_{ijkl}$, depend on the Eshelby tensor S_{ijrs} for the homogeneous smeared out medium. which is defined in equation 123 as

$$S_{ijrs} = \iiint\limits_{V} U_{ijkl}\left(\mathbf{r} - \mathbf{r}'\right) dV(\mathbf{r}')\overline{D}_{klrs} \tag{170}$$

when the field point $\mathbf{r}$ lies within the cylindrical volume, V. The "effective" homogeneous smeared out medium for a composite with cylindrical fibers will exhibit transverse isotropy if the fibers are arranged in hexagonal arrays, and it is shown in Appendix E that the Eshelby tensor for a transversely isotropic cylinder, whose x_3 cylindrical axis is normal to the plane of transverse isotropy, has the component form,

$$S_{1111} = \frac{5\overline{D}_{1111} + \overline{D}_{1122}}{8\overline{D}_{1111}} \tag{171}$$

$$S_{2222} = S_{1111} \tag{172}$$

$$S_{1122} = \frac{3\overline{D}_{1122} - \overline{D}_{1111}}{8\overline{D}_{1111}} \tag{173}$$

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$$S_{2233} = \frac{\overline{D}_{1133}}{2\overline{D}_{1111}} \tag{174}$$

$$S_{1133} = S_{2233} \tag{175}$$

$$S_{2211} = S_{1122} \tag{176}$$

$$S_{1212} = S_{1221} = \frac{3\overline{D}_{1111} - \overline{D}_{1122}}{8\overline{D}_{1111}} \tag{177}$$

$$S_{2323} = S_{2332} = S_{1313} = S_{1331} = \tfrac{1}{4} \tag{178}$$

with all other $S_{ijkl} = 0$. If the fibers are arranged in tetragonal arrays, the Eshelby tensor will exhibit tetragonal symmetry. This case is currently being worked out.

7 Integration of Self-Consistent Model

Fourth rank tensors can be written in Voigt notation as matrices and second rank tensors as vectors. (cf. Appendix 2 of Mura's book, [24]). For example, with the notational changes,

$$\Delta\sigma_1 = \Delta\sigma_{11}, \quad \Delta\sigma_2 = \Delta\sigma_{22}, \quad \Delta\sigma_3 = \Delta\sigma_{33}, \quad \Delta\sigma_4 = \Delta\sigma_{23}, \quad \Delta\sigma_5 = \Delta\sigma_{13}, \quad \Delta\sigma_6 = \Delta\sigma_{12}$$

and

$$\Delta\varepsilon_1 = \Delta\varepsilon_{11}, \quad \Delta\varepsilon_2 = \Delta\varepsilon_{22}, \quad \Delta\varepsilon_3 = \Delta\varepsilon_{33}, \quad \Delta\varepsilon_4 = 2\Delta\varepsilon_{23}, \quad \Delta\varepsilon_5 = 2\Delta\varepsilon_{13}, \quad \Delta\varepsilon_6 = 2\Delta\varepsilon_{12}$$

Hooke's law for an isotropic elastic medium can be written as

$$\begin{Bmatrix} \Delta\sigma_1 \\ \Delta\sigma_2 \\ \Delta\sigma_3 \\ \Delta\sigma_4 \\ \Delta\sigma_5 \\ \Delta\sigma_6 \end{Bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{Bmatrix} \Delta\varepsilon_1 \\ \Delta\varepsilon_2 \\ \Delta\varepsilon_3 \\ \Delta\varepsilon_4 \\ \Delta\varepsilon_5 \\ \Delta\varepsilon_6 \end{Bmatrix} \tag{179}$$

For a transversely isotropic medium—such as the smeared out "effective" matrix for hexagonal fiber arrays—the relationship can be written as

$$\begin{Bmatrix} \Delta\sigma_1 \\ \Delta\sigma_2 \\ \Delta\sigma_3 \\ \Delta\sigma_4 \\ \Delta\sigma_5 \\ \Delta\sigma_6 \end{Bmatrix} = \begin{bmatrix} \overline{D}_{1111} & \overline{D}_{1122} & \overline{D}_{1133} & 0 & 0 & 0 \\ \overline{D}_{1122} & \overline{D}_{1111} & \overline{D}_{1133} & 0 & 0 & 0 \\ \overline{D}_{1133} & \overline{D}_{1133} & \overline{D}_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{D}_{1313} & 0 & 0 \\ 0 & 0 & 0 & 0 & \overline{D}_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\dfrac{\overline{D}_{1111} - \overline{D}_{1122}}{2}\right) \end{bmatrix} \begin{Bmatrix} \Delta\varepsilon_1 \\ \Delta\varepsilon_2 \\ \Delta\varepsilon_3 \\ \Delta\varepsilon_4 \\ \Delta\varepsilon_5 \\ \Delta\varepsilon_6 \end{Bmatrix} \tag{180}$$

from which the isotropic results can be recovered by taking $\overline{D}_{1111} = 2\mu\nu(1-\nu)/(1-2\nu)$, $\overline{D}_{3333} = 2\mu\nu(1-\nu)/(1-2\nu)$, $\overline{D}_{1122} = 2\mu\nu/(1-2\nu)$, $\overline{D}_{1133} = 2\mu\nu/(1-2\nu)$, and $\overline{D}_{1313} = \mu$.

The Eshelby tensor [35] relates the constrained strain increment, $\Delta\varepsilon_{ij}^c$, in an inclusion which undergoes a transformation or eigenstrain increment, $\Delta\varepsilon_{kl}^{\star}$, in an infinite medium with elasticity tensor $\overline{D}_{ijkl}$, in the form

$$\Delta\varepsilon_{ij}^c = S_{ijkl}\Delta\varepsilon_{kl}^{\star} \tag{181}$$

In Voigt notation we have

$$\Delta\varepsilon_i^c = S_{ij}\Delta\varepsilon_j^{\star}$$

where the Eshelby matrix takes the form,

$$[S_{ij}] = \begin{bmatrix} \dfrac{5\overline{D}_{1111} + \overline{D}_{1122}}{8\overline{D}_{1111}} & \dfrac{3\overline{D}_{1122} - \overline{D}_{1111}}{8\overline{D}_{1111}} & \dfrac{\overline{D}_{1133}}{2\overline{D}_{1111}} & 0 & 0 & 0 \\[3ex] \dfrac{3\overline{D}_{1122} - \overline{D}_{1111}}{8\overline{D}_{1111}} & \dfrac{5\overline{D}_{1111} + \overline{D}_{1122}}{8\overline{D}_{1111}} & \dfrac{\overline{D}_{1133}}{2\overline{D}_{1111}} & 0 & 0 & 0 \\[3ex] 0 & 0 & 0 & 0 & 0 & 0 \\[2ex] 0 & 0 & 0 & 2\left(\dfrac{1}{4}\right) & 0 & 0 \\[2ex] 0 & 0 & 0 & 0 & 2\left(\dfrac{1}{4}\right) & 0 \\[2ex] 0 & 0 & 0 & 0 & 0 & 2\left(\dfrac{3\overline{D}_{1111} - \overline{D}_{1122}}{8\overline{D}_{1111}}\right) \end{bmatrix} \tag{182}$$

The integration of the self-consistent model then proceeds as follows.

1. Initialize the starting variables: time $t = 0$; temperature $T = T_0$; overall "effective" stress and strain $\sigma_i^0 = \varepsilon_i^0 = \overline{\varepsilon}_i^P = 0$ for $i = 1$ to 6; stress and strain in the respective phases $\sigma_i(f) = \varepsilon_i^T(f) = \varepsilon_i^P(f) = 0$, and $\sigma_i(m) = \varepsilon_i^T(m) = \varepsilon_i^P(m) = 0$ for $i = 1$ to 6; equilibrium stress in the respective phases $\Omega_i(f) = \Omega_i(m) = 0$ for $i = 1$ to 6; drag stress in the respective phases $K(f) = K_0(f)$ and $K(m) = K_0(m)$.

2. Compute the overall "effective" elasticity matrix iteratively from the relation

$$\overline{D}_{ij} = \frac{V_f}{V_c} D_{ik}^f \left[\delta_{kj} + S_{kl}\overline{D}_{lm}^{-1}\left(D_{mj}^f - \overline{D}_{mj}\right)\right]^{-1} +$$
$$+ \frac{V_m}{V_c} D_{ik}^m \left[\delta_{kj} + S_{kl}\overline{D}_{lm}^{-1}\left(D_{mj}^m - \overline{D}_{mj}\right)\right]^{-1}$$

where δ_{kj} is the Kronecker delta matrix, $\delta_{kj} = 1$ for $k = j$ and $\delta_{kj} = 0$ for $k \neq j$, and the Eshelby matrix S_{ij} is given in equation 182.

3. Start the loading history step. Evaluate the inelastic strain and state variable increments in the fiber and matrix phases from the unified viscoplastic constitutive relations. Any unified viscoplastic model may be used. Such relations may have the following form:

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In the fiber the inelastic strain increment is

$$\Delta\varepsilon_i^P(f) = \left(\frac{s_i(f) - \Omega_i(f)}{K(f)}\right) \left|\frac{\sqrt{\frac{2}{3}\left(\frac{3}{2}s_q(f) - \Omega_q(f)\right)\left(\frac{3}{2}s_q(f) - \Omega_q(f)\right)}}{K(f)}\right|^{n^f-1}$$

where the equilibrium or back stress increment is calculated from the relation

$$\Delta\Omega_i(f) = \varrho_1^f \Delta\varepsilon_i^P(f) - \varrho_2^f \sqrt{\frac{2}{3}\Delta\varepsilon_q^P(f)\Delta\varepsilon_q^P(f)}\,\Omega_i(f)$$

for $i = 1$ to 6, and the drag stress increment from

$$\Delta K(f) = \left[\varrho_3^f - \varrho_4^f\left(K(f) - K_0(f)\right)\right]\sqrt{\frac{2}{3}\Delta\varepsilon_q^P(f)\Delta\varepsilon_q^P(f)}$$

In the matrix a similar set of constitutive relations can be used, so that

$$\Delta\varepsilon_i^P(m) = \left(\frac{s_i(m) - \Omega_i(m)}{K(m)}\right) \left|\frac{\sqrt{\frac{2}{3}\left(\frac{3}{2}s_q(m) - \Omega_q(f)\right)\left(\frac{3}{2}s_q(m) - \Omega_q(m)\right)}}{K(m)}\right|^{n^m-1}$$

$$\Delta\Omega_i(m) = \varrho_1^m \Delta\varepsilon_i^P(m) - \varrho_2^m \sqrt{\frac{2}{3}\Delta\varepsilon_q^P(m)\Delta\varepsilon_q^P(m)}\,\Omega_i(m)$$

for $i = 1$ to 6, and

$$\Delta K(m) = \left[\varrho_3^m - \varrho_4^m\left(K(m) - K_0(m)\right)\right]\sqrt{\frac{2}{3}\Delta\varepsilon_q^P(m)\Delta\varepsilon_q^P(m)}$$

The quantities $n^f, n^m, \varrho_p^f, \varrho_p^m$, for $p = 1$ to 4, are material constants associated with the unified viscoplastic constitutive relations.

The deviatoric stress in the fiber phase is defined by

$$s_i(f) = \sigma_i(f) - \frac{1}{3}\left(\sigma_1(f) + \sigma_2(f) + \sigma_3(f)\right) \qquad \text{for} \qquad i = 1, 2, 3$$

and

$$s_i(f) = \sigma_i(f) \qquad \text{for} \qquad i = 4, 5, 6.$$

Similar relations apply to the matrix phase, *viz.*,

$$s_i(m) = \sigma_i(m) - \frac{1}{3}\left(\sigma_1(m) + \sigma_2(m) + \sigma_3(m)\right) \qquad \text{for} \qquad i = 1, 2, 3$$

and

$$s_i(m) = \sigma_i(m) \qquad \text{for} \qquad i = 4, 5, 6.$$

4. Compute the "effective" inelastic and thermal strain increments from the relations

$$\overline{\Delta\varepsilon}_i^P = \left[\overline{D}_{ip}\left(S_{pq} - I_{pq}\right)\right]^{-1}\left(B_{qk}\Delta\varepsilon_k^P(f) + C_{qk}\Delta\varepsilon_k^P(m)\right)$$

and

$$\overline{\alpha}_i^\star \Delta T = \left[\overline{D}_{ip}\left(S_{pq} - I_{pq}\right)\right]^{-1}\left(B_{qk}\alpha_k^{\star f} + C_{qk}\alpha_k^{\star m}\right)\Delta T$$

where

$$B_{qk} = \frac{V_f}{V_c} D_{qi}^f\left[\delta_{ip} + S_{im}\overline{D}_{mn}^{-1}\left(D_{np}^f - \overline{D}_{np}\right)\right]^{-1}\left(S_{pk} - \delta_{pk}\right)$$

and

$$C_{qk} = \frac{V_m}{V_c} D_{qi}^m\left[\delta_{ip} + S_{im}\overline{D}_{mn}^{-1}\left(D_{np}^m - \overline{D}_{np}\right)\right]^{-1}\left(S_{pk} - \delta_{pk}\right)$$

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5. Evaluate the deviation strain increments from the relations

$$\Delta c_q(f) = \Delta\varepsilon_q^P(f) + \alpha_q^{\star f}\Delta T$$

$$\Delta c_q(m) = \Delta\varepsilon_q^P(m) + \alpha_q^{\star m}\Delta T$$

and

$$\overline{\Delta c_q} = \overline{\Delta\varepsilon_q}^P + \overline{\alpha}_q^{\star}\Delta T$$

6. Evaluate the phase volume averaged total strain increments

$$
\begin{aligned}
\Delta\varepsilon_i^T(f) &= \left[\delta_{ij} + S_{ip}\overline{D}_{pq}^{-1}\left(D_{qj}^f - \overline{D}_{qj}\right)\right]^{-1}\left\{\Delta\varepsilon_j^0 + \right. \\
&\quad \left. + S_{jq}\left(\Delta c_q(f) - \overline{\Delta c_q}\right) + S_{jp}\overline{D}_{pq}^{-1}\left(D_{qr}^f - \overline{D}_{qr}\right)\Delta c_r(f)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta\varepsilon_i^T(m) &= \left[\delta_{ij} + S_{ip}\overline{D}_{pq}^{-1}\left(D_{qj}^m - \overline{D}_{qj}\right)\right]^{-1}\left\{\Delta\varepsilon_j^0 + \right. \\
&\quad \left. + S_{jq}\left(\Delta c_q(m) - \overline{\Delta c_q}\right) + S_{jp}\overline{D}_{pq}^{-1}\left(D_{qr}^m - \overline{D}_{qr}\right)\Delta c_r(m)\right\}
\end{aligned}
$$

7. Calculate the stress increments in the fiber and matrix phases from the relations

$$\Delta\sigma_i(f) = D_{ij}^f\left(\Delta\varepsilon_j^T(f) - \Delta c_j(f)\right)$$

and

$$\Delta\sigma_i(m) = D_{ij}^m\left(\Delta\varepsilon_j^T(m) - \Delta c_j(m)\right)$$

8. Compute the overall "effective" stress increment from the relation

$$\Delta\sigma_i^0 = \overline{D}_{ij}\left(\Delta\varepsilon_j^0 - \overline{\Delta\varepsilon}_j^P - \overline{\alpha}_j^{\star}\Delta T\right)$$

9. Update the variables:

$$
\begin{aligned}
\sigma_i\left(f, t + \Delta t\right) &= \sigma_i\left(f, t\right) + \Delta\sigma_i(f) \\
\sigma_i\left(m, t + \Delta t\right) &= \sigma_i\left(m, t\right) + \Delta\sigma_i(m) \\
\Omega_i\left(f, t + \Delta t\right) &= \Omega_i\left(f, t\right) + \Delta\Omega_i(f) \\
\Omega_i\left(m, t + \Delta t\right) &= \Omega_i\left(m, t\right) + \Delta\Omega_i(m) \\
K\left(f, t + \Delta t\right) &= K\left(f, t\right) + \Delta K(f) \\
K\left(m, t + \Delta t\right) &= K\left(m, t\right) + \Delta K(m) \\
\varepsilon_i^P\left(f, t + \Delta t\right) &= \varepsilon_i^P\left(f, t\right) + \Delta\varepsilon_i^P(f) \\
\varepsilon_i^P\left(m, t + \Delta t\right) &= \varepsilon_i^P\left(m, t\right) + \Delta\varepsilon_i^P(m) \\
\varepsilon_i^T\left(f, t + \Delta t\right) &= \varepsilon_i^T\left(f, t\right) + \Delta\varepsilon_i^T(f) \\
\varepsilon_i^T\left(m, t + \Delta t\right) &= \varepsilon_i^T\left(m, t\right) + \Delta\varepsilon_i^T(m) \\
\sigma_i^0\left(t + \Delta t\right) &= \sigma_i^0\left(t\right) + \Delta\sigma_i^0 \\
\varepsilon_i^0\left(t + \Delta t\right) &= \varepsilon_i^0\left(t\right) + \Delta\varepsilon_i^0 \\
T(t + \Delta t) &= T(t) + \Delta T
\end{aligned}
$$

10. Start new load step.

8 Subvolume Method

8.1 Approximate Integration of Integral Equations

The determination of the stress and strain increments throughout the composite material requires the solution of the integral equations

$$
D_{ijkl}^m \Delta \varepsilon_{kl}^{\star}(\mathbf{r}) = D_{ijkl}^m \Delta c_{kl}(\mathbf{r}) - \delta D_{ijkl}(\mathbf{r}) \left[\Delta \varepsilon_{kl}^0 - \Delta c_{kl}(\mathbf{r}) \right] -
$$
$$
- \delta D_{ijkl}(\mathbf{r}) \frac{1}{V_c} \sum \sum_{n_p=0}^{\pm \infty} \sum{}' g_{klmn}(\boldsymbol{\zeta}) \iiint_{V_c} D_{mnrs}^m \Delta \varepsilon_{rs}^{\star}(\mathbf{r}') e^{i\boldsymbol{\xi} \cdot (\mathbf{r}-\mathbf{r}')} dV(\mathbf{r}') \tag{183}
$$

or

$$
D_{ijkl}^m \Delta \varepsilon_{kl}^{\star}(\mathbf{r}) = D_{ijkl}^m \Delta c_{kl}(\mathbf{r}) - \delta D_{ijkl}(\mathbf{r}) \left[\Delta \varepsilon_{kl}^0 - \Delta c_{kl}(\mathbf{r}) \right] -
$$
$$
- \delta D_{ijkl}(\mathbf{r}) \iiint_V U_{klmn}(\mathbf{r} - \mathbf{r}') D_{mnrs}^m \Delta \varepsilon_{rs}^{\star}(\mathbf{r}') dV(\mathbf{r}') \tag{184}
$$

at each field point $\mathbf{r}$ in the unit periodic cell.

Nemat-Nasser and his colleagues [25,26,27,28] have demonstrated the efficacy of dividing the unit cell into a number of subvolumes and assuming that $\Delta \varepsilon_{mn}^{\star}(\mathbf{r}')$ is replaced by

$$
\Delta \varepsilon_{mn}^{\star}(\mathbf{r}') \equiv \Delta \varepsilon_{mn}^{\star \beta} = \frac{1}{V_\beta} \iiint_{V_\beta} \Delta \varepsilon_{mn}^{\star}(\mathbf{r}') \, dV(\mathbf{r}') \tag{185}
$$

corresponding to its average value in the βth subvolume.

Let there be N subvolumes in the unit cell, with M subvolumes in the fiber and $N - M$ subvolumes in the matrix. Then the preceding integral equations can be written as

$$
D_{ijkl}^m \Delta \varepsilon_{kl}^{\star}(\mathbf{r}) = D_{ijkl}^m \Delta c_{kl}(\mathbf{r}) - \delta D_{ijkl}(\mathbf{r}) \left[\Delta \varepsilon_{kl}^0 - \Delta c_{kl}(\mathbf{r}) \right] -
$$
$$
- \delta D_{ijkl}(\mathbf{r}) \frac{1}{V_c} \sum \sum_{n_p=0}^{\pm \infty} \sum{}' g_{klmn}(\boldsymbol{\zeta}) e^{i\boldsymbol{\xi} \cdot \mathbf{r}} \sum_{\beta=1}^{N} \left(\iiint_{V_\beta} e^{-i\boldsymbol{\xi} \cdot \mathbf{r}'} dV(\mathbf{r}') \right) D_{mnrs}^m \Delta \varepsilon_{rs}^{\star \beta} \tag{186}
$$

and

$$
D_{ijkl}^m \Delta \varepsilon_{kl}^{\star}(\mathbf{r}) = D_{ijkl}^m \Delta c_{kl}(\mathbf{r}) - \delta D_{ijkl}(\mathbf{r}) \left[\Delta \varepsilon_{kl}^0 - \Delta c_{kl}(\mathbf{r}) \right] -
$$
$$
- \delta D_{ijkl}(\mathbf{r}) \sum_{q=1}^{\infty} \sum_{\beta=1}^{N} \left(\iiint_{V_{\beta q}} U_{klmn}(\mathbf{r} - \mathbf{r}') \, dV(\mathbf{r}') \right) D_{mnrs}^m \Delta \varepsilon_{rs}^{\star \beta} \tag{187}
$$

where $V_{\beta q}$ denotes the βth subvolume in the qth unit periodic cell, and it is assumed that the field point $\mathbf{r}$ is in the first periodic cell for which $q = 1$.

These equations can be volume averaged over the αth subvolume in the unit periodic cell to give

$$
D_{ijkl}^m \Delta \varepsilon_{kl}^{\star \alpha} = D_{ijkl}^m \Delta c_{kl}^{\alpha} - \delta D_{ijkl}^{\alpha} \left[\Delta \varepsilon_{kl}^0 - \Delta c_{kl}^{\alpha} \right] - \delta D_{ijkl}^{\alpha} \times
$$
$$
\times \frac{1}{V_c} \sum \sum_{n_p=0}^{\pm \infty} \sum{}' g_{klmn}(\boldsymbol{\zeta}) \left(\frac{1}{V_\alpha} \iiint_{V_\alpha} e^{i\boldsymbol{\xi} \cdot \mathbf{r}} dV(\mathbf{r}) \right) \sum_{\beta=1}^{N} V_\beta \left(\frac{1}{V_\beta} \iiint_{V_\beta} e^{-i\boldsymbol{\xi} \cdot \mathbf{r}'} dV(\mathbf{r}') \right) D_{mnrs}^m \Delta \varepsilon_{rs}^{\star \beta} \tag{188}
$$

and

$$D^m_{ijkl}\Delta\varepsilon^{\star\alpha}_{kl} = D^m_{ijkl}\Delta c^\alpha_{kl} - \delta D^\alpha_{ijkl}\left[\Delta\varepsilon^0_{kl} - \Delta c^\alpha_{kl}\right] -$$

$$- \delta D^\alpha_{ijkl}\sum_{q=1}^\infty\sum_{\beta=1}^N\left(\frac{1}{V_\alpha}\iiint_{V_\alpha}dV(\mathbf{r})\iiint_{V_{\beta q}}U_{klmn}\left(\mathbf{r}-\mathbf{r}'\right)dV(\mathbf{r}')\right)D^m_{mnrs}\Delta\varepsilon^{\star\beta}_{rs} \qquad (189)$$

In these equations the deviation strain increments Δc^α_{kl} are evaluated from the unified viscoplastic constitutive relation for the αth subvolume based on the stress value $\sigma^\alpha_{ij}(f)$ or $\sigma^\alpha_{ij}(m)$ in the subvolume, according as the αth subvolume is in the fiber or the matrix phase, respectively. The notation δD^α_{ijkl} also denotes the value of $D^f_{ijkl} - D^m_{ijkl}$ or 0 according as the αth subvolume is in the fiber or matrix phase, respectively.

If we use Nemat-Nasser's notation and write

$$Q^\alpha(\boldsymbol{\xi}) = \frac{1}{V_\alpha}\iiint_{V_\alpha}e^{i\boldsymbol{\xi}\cdot\mathbf{r}}dV(\mathbf{r}) \qquad (190)$$

and denote

$$f^\alpha = \frac{V_\alpha}{V_c} \qquad (191)$$

as the volume fraction of the αth subvolume, then the preceding equations may be written as

$$\sum_{\beta=1}^N\left[D^m_{ijrs}\delta^{\alpha\beta} + \delta D^\alpha_{ijkl}\sum_{n_p=0}^{\pm\infty}\sum{}'g_{klmn}(\boldsymbol{\zeta})D^m_{mnrs}f^\beta Q^\alpha(\boldsymbol{\xi})Q^\beta(-\boldsymbol{\xi})\right]\Delta\varepsilon^{\star\beta}_{rs}$$

$$= D^m_{ijkl}\Delta c^\alpha_{kl} - \delta D^\alpha_{ijkl}\left[\Delta\varepsilon^0_{kl} - \Delta c^\alpha_{kl}\right] \qquad (192)$$

and

$$\sum_{\beta=1}^N\left[D^m_{ijrs}\delta^{\alpha\beta} + \delta D^\alpha_{ijkl}\sum_{q=1}^\infty\left(\frac{1}{V_\alpha}\iiint_{V_\alpha}dV(\mathbf{r})\iiint_{V_{\beta q}}U_{klmn}\left(\mathbf{r}-\mathbf{r}'\right)dV(\mathbf{r}')\right)D^m_{mnrs}\right]\Delta\varepsilon^{\star\beta}_{rs}$$

$$= D^m_{ijkl}\Delta c^\alpha_{kl} - \delta D^\alpha_{ijkl}\left[\Delta\varepsilon^0_{kl} - \Delta c^\alpha_{kl}\right] \qquad (193)$$

where $\delta^{\alpha\beta} = 1$ if $\alpha = \beta$ and $\delta^{\alpha\beta} = 0$ if $\alpha \neq \beta$, and no sums on α, β are intended unless explicitly stated.

Now $\delta D^\alpha_{ijkl} = 0$ if the αth subvolume resides in the matrix. In this case equations 188 and 189 show that

$$\Delta\varepsilon^{\star\alpha}_{kl} = \Delta c^\alpha_{kl} \qquad \text{for} \quad M < \alpha \leq N \qquad (194)$$

Thus, only M unknowns (associated with the subvolumes in the fiber) are involved in $\Delta\varepsilon^{\star\beta}_{rs}$ and the $N - M$ known quantities (associated with the subvolumes in the matrix) given by equation 194 may be taken over to the right hand side of the equations. Equations 192 and 193 may therefore be written in the form

$$\sum_{\beta=1}^{M} \left[D_{ijrs}^{m} \delta^{\alpha\beta} + \delta D_{ijkl}^{\alpha} \sum \sum_{n_p=0}^{\pm\infty} \sum{}' g_{klmn}(\boldsymbol{\zeta}) D_{mnrs}^{m} f^{\beta} Q^{\alpha}(\boldsymbol{\xi}) Q^{\beta}(-\boldsymbol{\xi}) \right] \Delta\varepsilon_{rs}^{\star\beta}$$

$$= D_{ijkl}^{m} \Delta c_{kl}^{\alpha} - \delta D_{ijkl}^{\alpha} \left[\Delta\varepsilon_{kl}^{0} - \Delta c_{kl}^{\alpha} \right] -$$

$$- \sum_{\beta=M+1}^{N} \delta D_{ijkl}^{\alpha} \sum \sum_{n_p=0}^{\pm\infty} \sum{}' g_{klmn}(\boldsymbol{\zeta}) D_{mnrs}^{m} f^{\beta} Q^{\alpha}(\boldsymbol{\xi}) Q^{\beta}(-\boldsymbol{\xi}) \Delta c_{rs}^{\beta} \qquad (195)$$

and

$$\sum_{\beta=1}^{M} \left[D_{ijrs}^{m} \delta^{\alpha\beta} + \delta D_{ijkl}^{\alpha} \sum_{q=1}^{\infty} \left(\frac{1}{V_\alpha} \iiint_{V_\alpha} dV(\mathbf{r}) \iiint_{V_{\beta q}} U_{klmn}(\mathbf{r}-\mathbf{r}') \, dV(\mathbf{r}') \right) D_{mnrs}^{m} \right] \Delta\varepsilon_{rs}^{\star\beta}$$

$$= D_{ijkl}^{m} \Delta c_{kl}^{\alpha} - \delta D_{ijkl}^{\alpha} \left[\Delta\varepsilon_{kl}^{0} - \Delta c_{kl}^{\alpha} \right] -$$

$$- \sum_{\beta=M+1}^{N} \delta D_{ijkl}^{\alpha} \sum_{q=1}^{\infty} \left(\frac{1}{V_\alpha} \iiint_{V_\alpha} dV(\mathbf{r}) \iiint_{V_{\beta q}} U_{klmn}(\mathbf{r}-\mathbf{r}') \, dV(\mathbf{r}') \right) D_{mnrs}^{m} \Delta c_{rs}^{\beta} \qquad (196)$$

for $\alpha = 1$ to M.

By defining the fourth rank tensor $A_{ijrs}^{\alpha\beta}$ and the second rank tensor b_{ij}^{α} in the Fourier series representation as

$$A_{ijrs}^{\alpha\beta} = D_{ijrs}^{m} \delta^{\alpha\beta} + \delta D_{ijkl}^{\alpha} \sum \sum_{n_p=0}^{\pm\infty} \sum{}' g_{klmn}(\boldsymbol{\zeta}) D_{mnrs}^{m} f^{\beta} Q^{\alpha}(\boldsymbol{\xi}) Q^{\beta}(-\boldsymbol{\xi}) \qquad (197)$$

and

$$b_{ij}^{\alpha} = D_{ijkl}^{m} \Delta c_{kl}^{\alpha} - \delta D_{ijkl}^{\alpha} \left[\Delta\varepsilon_{kl}^{0} - \Delta c_{kl}^{\alpha} \right] -$$

$$- \sum_{\beta=M+1}^{N} \delta D_{ijkl}^{\alpha} \sum \sum_{n_p=0}^{\pm\infty} \sum{}' g_{klmn}(\boldsymbol{\zeta}) D_{mnrs}^{m} f^{\beta} Q^{\alpha}(\boldsymbol{\xi}) Q^{\beta}(-\boldsymbol{\xi}) \Delta c_{rs}^{\beta} \qquad (198)$$

or, in the Green's function representation, as

$$A_{ijrs}^{\alpha\beta} = D_{ijrs}^{m} \delta^{\alpha\beta} + \delta D_{ijkl}^{\alpha} \sum_{q=1}^{\infty} \left(\frac{1}{V_\alpha} \iiint_{V_\alpha} dV(\mathbf{r}) \iiint_{V_{\beta q}} U_{klmn}(\mathbf{r}-\mathbf{r}') \, dV(\mathbf{r}') \right) D_{mnrs}^{m} \qquad (199)$$

and

$$b_{ij}^{\alpha} = D_{ijkl}^{m} \Delta c_{kl}^{\alpha} - \delta D_{ijkl}^{\alpha} \left[\Delta\varepsilon_{kl}^{0} - \Delta c_{kl}^{\alpha} \right] -$$

$$- \sum_{\beta=M+1}^{N} \delta D_{ijkl}^{\alpha} \sum_{q=1}^{\infty} \left(\frac{1}{V_\alpha} \iiint_{V_\alpha} dV(\mathbf{r}) \iiint_{V_{\beta q}} U_{klmn}(\mathbf{r}-\mathbf{r}') \, dV(\mathbf{r}') \right) D_{mnrs}^{m} \Delta c_{rs}^{\beta} \qquad (200)$$

the integral equations can be cast in the form,

$$\sum_{\beta=1}^{M} A_{ijrs}^{\alpha\beta}\Delta\varepsilon_{rs}^{\star\beta} = b_{ij}^{\alpha} \qquad \text{for} \quad \alpha = 1 \text{ to } M \tag{201}$$

In Voigt notation the fourth rank tensor $A_{ijrs}^{\alpha\beta}$ can be written as a matrix $A_{pq}^{\alpha\beta}$, and the second rank tensors $\Delta\varepsilon_{rs}^{\star\beta}$ and b_{ij}^{α} can be written as vectors $\Delta\varepsilon_q^{\star\beta}$ and b_p^{α}, so that

$$\sum_{\beta=1}^{M} A_{pq}^{\alpha\beta}\Delta\varepsilon_q^{\star\beta} = b_p^{\alpha} \qquad \text{for} \quad \alpha = 1 \text{ to } M \tag{202}$$

This represents a system of $6M$ linear equations for the unknown values $\Delta\varepsilon_q^{\star\beta}$, each matrix element $\alpha\beta$ of the matrix $\mathbf{A}$ consisting of a 6×6 submatrix, in the form

$$\begin{bmatrix} [\mathbf{A}_{11}] & \cdots & \cdots & \cdots & [\mathbf{A}_{1M}] \\ [\mathbf{A}_{21}] & \cdots & \cdots & \cdots & [\mathbf{A}_{2M}] \\ \vdots & \ddots & \cdots & \cdots & \vdots \\ \vdots & \cdots & [\mathbf{A}_{\alpha\beta}] & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ [\mathbf{A}_{M1}] & \cdots & \cdots & \cdots & [\mathbf{A}_{MM}] \end{bmatrix} \begin{Bmatrix} \{\Delta\varepsilon^{\star 1}\} \\ \{\Delta\varepsilon^{\star 2}\} \\ \vdots \\ \{\Delta\varepsilon^{\star\beta}\} \\ \vdots \\ \{\Delta\varepsilon^{\star M}\} \end{Bmatrix} = \begin{Bmatrix} \{\mathbf{b}^1\} \\ \{\mathbf{b}^2\} \\ \vdots \\ \{\mathbf{b}^\beta\} \\ \vdots \\ \{\mathbf{b}^M\} \end{Bmatrix} \tag{203}$$

where the submatrix elements are defined as

$$[\mathbf{A}_{\alpha\beta}] = \begin{bmatrix} A_{11}^{\alpha\beta} & A_{12}^{\alpha\beta} & A_{13}^{\alpha\beta} & A_{14}^{\alpha\beta} & A_{15}^{\alpha\beta} & A_{16}^{\alpha\beta} \\ A_{21}^{\alpha\beta} & A_{22}^{\alpha\beta} & A_{23}^{\alpha\beta} & A_{24}^{\alpha\beta} & A_{25}^{\alpha\beta} & A_{26}^{\alpha\beta} \\ A_{31}^{\alpha\beta} & A_{32}^{\alpha\beta} & A_{33}^{\alpha\beta} & A_{34}^{\alpha\beta} & A_{35}^{\alpha\beta} & A_{36}^{\alpha\beta} \\ A_{41}^{\alpha\beta} & A_{42}^{\alpha\beta} & A_{43}^{\alpha\beta} & A_{44}^{\alpha\beta} & A_{45}^{\alpha\beta} & A_{46}^{\alpha\beta} \\ A_{51}^{\alpha\beta} & A_{52}^{\alpha\beta} & A_{53}^{\alpha\beta} & A_{54}^{\alpha\beta} & A_{55}^{\alpha\beta} & A_{56}^{\alpha\beta} \\ A_{61}^{\alpha\beta} & A_{62}^{\alpha\beta} & A_{63}^{\alpha\beta} & A_{64}^{\alpha\beta} & A_{65}^{\alpha\beta} & A_{66}^{\alpha\beta} \end{bmatrix} \tag{204}$$

and the corresponding column vectors as

$$\{\Delta\varepsilon^{\star\beta}\} = \begin{Bmatrix} \Delta\varepsilon_1^{\star\beta} \\ \Delta\varepsilon_2^{\star\beta} \\ \Delta\varepsilon_3^{\star\beta} \\ \Delta\varepsilon_4^{\star\beta} \\ \Delta\varepsilon_5^{\star\beta} \\ \Delta\varepsilon_6^{\star\beta} \end{Bmatrix} \qquad \text{and} \qquad \{\mathbf{b}^\beta\} = \begin{Bmatrix} b_1^\beta \\ b_2^\beta \\ b_3^\beta \\ b_4^\beta \\ b_5^\beta \\ b_6^\beta \end{Bmatrix} \tag{205}$$

This system can be solved by standard Gaussian elimination. However, if M subvolumes are included in the fiber, these equations represent a $6M$ system of equations, whose solution may pose storage problems on the computer.

8.2 Solution of Integral Equations by Iteration

An alternative is to use equations 188 and 189 in an iterative fashion. As a first guess the integral terms in 188 and 189 can be neglected and we obtain

$$D_{ijkl}^m \Delta\varepsilon_{kl}^{\star\alpha} = D_{ijkl}^m \Delta c_{kl}^\alpha - \delta D_{ijkl}^\alpha \left[\Delta\varepsilon_{kl}^0 - \Delta c_{kl}^\alpha \right] \qquad \text{for} \qquad \alpha = 1 \text{ to } M$$

corresponding to the subvolumes in the fiber, and

$$D_{ijkl}^m \Delta\varepsilon_{kl}^{\star\alpha} = D_{ijkl}^m \Delta c_{kl}^\alpha \qquad \text{for} \qquad \alpha = M+1 \text{ to } N$$

corresponding to the subvolumes in the matrix. These relations can then be substituted into the integral terms to yield an improved "Rayleigh-Born" approximation to $D_{ijkl}^m \Delta\varepsilon_{kl}^{\star\alpha}$ for $\alpha = 1$ to N. This process can then be repeated until $D_{ijkl}^m \Delta\varepsilon_{kl}^{\star\alpha}$ converges to within a user specified tolerance. In essence we solve the equations

$$D_{ijkl}^m \left\{ \Delta\varepsilon_{kl}^{\star\alpha} \right\}_{\lambda+1} = D_{ijkl}^m \Delta c_{kl}^\alpha - \delta D_{ijkl}^\alpha \left[\Delta\varepsilon_{kl}^0 - \Delta c_{kl}^\alpha \right] -$$

$$- \sum_{\beta=1}^M \delta D_{ijkl}^\alpha \sum \sum_{n_p=0}^{\pm\infty} {\sum}' g_{klmn}(\zeta) D_{mnrs}^m f^\beta Q^\alpha(\boldsymbol{\xi}) Q^\beta(-\boldsymbol{\xi}) \left\{ \Delta\varepsilon_{rs}^{\star\beta} \right\}_\lambda -$$

$$- \sum_{\beta=M+1}^N \delta D_{ijkl}^\alpha \sum \sum_{n_p=0}^{\pm\infty} {\sum}' g_{klmn}(\zeta) D_{mnrs}^m f^\beta Q^\alpha(\boldsymbol{\xi}) Q^\beta(-\boldsymbol{\xi}) \Delta c_{rs}^\beta \qquad (206)$$

or

$$D_{ijkl}^m \left\{ \Delta\varepsilon_{kl}^{\star\alpha} \right\}_{\lambda+1} = D_{ijkl}^m \Delta c_{kl}^\alpha - \delta D_{ijkl}^\alpha \left[\Delta\varepsilon_{kl}^0 - \Delta c_{kl}^\alpha \right] -$$

$$- \sum_{\beta=1}^M \delta D_{ijkl}^\alpha \sum_{q=1}^\infty \left(\frac{1}{V_\alpha} \iiint_{V_\alpha} dV(\mathbf{r}) \iiint_{V_{\beta q}} U_{klmn}(\mathbf{r}-\mathbf{r}')\, dV(\mathbf{r}') \right) D_{mnrs}^m \left\{ \Delta\varepsilon_{rs}^{\star\beta} \right\}_\lambda -$$

$$- \sum_{\beta=M+1}^N \delta D_{ijkl}^\alpha \sum_{q=1}^\infty \left(\frac{1}{V_\alpha} \iiint_{V_\alpha} dV(\mathbf{r}) \iiint_{V_{\beta q}} U_{klmn}(\mathbf{r}-\mathbf{r}')\, dV(\mathbf{r}') \right) D_{mnrs}^m \Delta c_{rs}^\beta \qquad (207)$$

until the $(\lambda+1)^{\text{th}}$ iterate differs insignificantly from the λth iterate.

In the solution of the composite problem, two constituent phases, namely the fiber and matrix phases, have been considered. For composites with a third chemically degrading phase separating the fiber from the matrix, the preceding solutions may be modified by assuming that, in the summations from $\beta = 1$ to M, some of the subvolumes, say from $\beta = L$ to M, pertain to the degraded material. It will then be necessary to postulate a viscoplastic constitutive relation for this chemically degrading phase.

The total strain increment in the αth subvolume in the unit cell is then obtained by averaging equations 57 and 80 over the αth subvolume to give,

$$\Delta\varepsilon_{kl}^{T\alpha} = \Delta\varepsilon_{kl}^0 + \sum_{\beta=1}^M \sum \sum_{n_p=0}^{\pm\infty} {\sum}' g_{klmn}(\zeta) f^\beta Q^\alpha(\boldsymbol{\xi}) Q^\beta(-\boldsymbol{\xi}) D_{mnrs}^m \Delta\varepsilon_{rs}^{\star\beta} +$$

$$+ \sum_{\beta=M+1}^N \sum \sum_{n_p=0}^{\pm\infty} {\sum}' g_{klmn}(\zeta) f^\beta Q^\alpha(\boldsymbol{\xi}) Q^\beta(-\boldsymbol{\xi}) D_{mnrs}^m \Delta c_{rs}^\beta \qquad (208)$$

or

$$\Delta\varepsilon_{kl}^{T\alpha} = \Delta\varepsilon_{kl}^{0} + \sum_{\beta=1}^{M}\sum_{q=1}^{\infty}\left(\frac{1}{V_\alpha}\iiint_{V_\alpha}dV(\mathbf{r})\iiint_{V_{\beta q}}U_{klmn}\left(\mathbf{r}-\mathbf{r}'\right)dV(\mathbf{r}')\right)D_{mnrs}^{m}\Delta\varepsilon_{rs}^{\star\beta} +$$

$$+ \sum_{\beta=M+1}^{N}\sum_{q=1}^{\infty}\left(\frac{1}{V_\alpha}\iiint_{V_\alpha}dV(\mathbf{r})\iiint_{V_{\beta q}}U_{klmn}\left(\mathbf{r}-\mathbf{r}'\right)dV(\mathbf{r}')\right)D_{mnrs}^{m}\Delta c_{rs}^{\beta} \qquad (209)$$

for $\alpha = 1$ to N.

The constitutive relation, required to update the stress and state variables in each subvolume, is then given for the αth subvolume as the average of equation 35 in the form,

$$\Delta\sigma_{ij}^{\alpha} = D_{ijkl}^{m}\left(\Delta\varepsilon_{kl}^{T\alpha} - \Delta\varepsilon_{kl}^{\star\alpha}\right) \qquad (210)$$

If we assume that $N = 2$, with one subvolume in the fiber and the other in the matrix, then the theory is similar to the self-consistent model in which the strain increments in the constituent phases are assumed to be spatially constant and equal to their respective constituent volume averages. However, the interaction effects of the nearest neighboring cells are fully accounted for since geometric periodicity is assumed in the integral equation formulation and the material outside the unit cell has not been smeared into an "effective" uniform material.

In both the Fourier series and the Green's function formulations integrals of the form

$$\iiint_{V_\alpha}e^{i\mathbf{K}\cdot\mathbf{r}}\,dV(\mathbf{r}) \qquad (211)$$

need to be evaluated over the subvolume, V_α. These Laue interference integrals [39] can be evaluated exactly if each subvolume consists of a circular or oblong cylinder. In the case of a circular cylindrical fiber, each subvolume within the fiber would consist of an infinite cylinder with a cross-section in the shape of an element of area in cylindrical coordinates, comprised of two circular arcs with constant radii, r_1 and r_2, and two radial segments along the lines of constant θ_1 and θ_2. An attempt will be made to evaluate equation 211 for this type of cross-section. If this proves too unwieldy, the subvolumes within the cylindrical fiber can be taken to be cylinders themselves, with the cylindrical fiber represented as a "bundle of sticks". We assume that the actual fiber is comprised of subvolumes of the correct shape, but we make an approximation in performing the volume integration over a circular cylindrical subvolume.

9 Concluding Remarks

This document is the first annual report on NASA Grant NAG3-882. Much of the work on which this report is based exists only as a mélange in the literature and we have therefore attempted to write the report in enough mathematical detail that it can be worked through without reference to the literature. In the second year we shall work out the required integrals

in the formulations and program the methods in FORTRAN subroutines suitable for inclusion in nonlinear finite element programs. In the third year we will determine the material constants for various composite materials and provide a comparison of the present theory with finite element and experimental results.

Our aim is to produce an end product which can be used in nonlinear finite element and boundary element programs for analyzing the structural behavior of composite materials under thermomechanical loading conditions at elevated temperature.

The viscoplastic behavior of periodic composites is analyzed by means of implicit integral equations. These integral equations arise when the problem of determining the stress-strain variation throughout a unit periodic cell in the periodic composite is solved by a Fourier series or Green's function approach. In this report we show that the Fourier series and Green's function approaches are mathematically equivalent by means of the Poisson sum formula. By applying simplifying assumptions the integral equations can be solved in an approximate fashion and used in structural analysis programs to analyze the overall behavior of the composite. When the strain-temperature history at the "damage critical" location has been determined from the structural analysis, this can be used to "drive" the "exact" integral equations to determine the stress-strain history variation throughout a unit periodic cell located at the critical location.

The unit cell in the periodic structure can be formulated to analyze fibrous, laminated and particulate composites. By retaining the effects due to the application of displacements and tractions at the surface of the composite it is also possible to analyze the behavior of thin walled composite sections such as are found in turbine engine combustor liners and blades. When this is done the integral equations which must be solved are basically those which are used in boundary element programs. In the constitutive subroutine which we plan to embed in the nonlinear finite element program to analyze the overall macroscopic behavior of the composite, we effectively have a boundary element equation (specialized for the case of a periodic composite) which we solve in an approximate fashion for the stress at the Gaussian integration point when the boundary displacement on the element is prescribed by the finite element program.

When the effects of damage are included in the constitutive formulations it will be possible to embed the subroutine in an optimization program such as ADS in order to determine optimum composite configurations.

10   Acknowledgment

Appendix A

Properties of the Green's Function

Consider a point force $f_k(\mathbf{r}')$ acting at the point $\mathbf{r}'$ in an infinite medium with elasticity tensor D_{ijkl}. From the definition of the Green's function the displacement at the field point $\mathbf{r}$ due to the point force $f_k(\mathbf{r}')$ at $\mathbf{r}'$ is

$$u_i(\mathbf{r}) = G_{ij}(\mathbf{r} - \mathbf{r}')f_j(\mathbf{r}') \tag{A.1}$$

so that the infinitesimal strain at $\mathbf{r}$ is

$$\varepsilon_{im}(\mathbf{r}) = \frac{1}{2}\left(\frac{\partial G_{ij}(\mathbf{r} - \mathbf{r}')}{\partial x_m} + \frac{\partial G_{mj}(\mathbf{r} - \mathbf{r}')}{\partial x_i}\right)f_j(\mathbf{r}') \tag{A.2}$$

and the associated stress is

$$\sigma_{kp}(\mathbf{r}) = D_{kpim}\varepsilon_{im}(\mathbf{r}) \tag{A.3}$$

or

$$\sigma_{kp}(\mathbf{r}) = D_{kpim}\frac{1}{2}\left(\frac{\partial G_{ij}(\mathbf{r} - \mathbf{r}')}{\partial x_m} + \frac{\partial G_{mj}(\mathbf{r} - \mathbf{r}')}{\partial x_i}\right)f_j(\mathbf{r}') \tag{A.4}$$

Since the elasticity tensor D_{kpim} is symmetric with respect to the indices i and m, the last relation can be written as

$$\sigma_{kp}(\mathbf{r}) = D_{kpim}\frac{\partial G_{ij}(\mathbf{r} - \mathbf{r}')}{\partial x_m}f_j(\mathbf{r}') \tag{A.5}$$

For static equilibrium, we must have

$$f_k(\mathbf{r}') = -\iint_S n_p(\mathbf{r})\sigma_{pk}(\mathbf{r})\,dS(\mathbf{r}) \tag{A.6}$$

where S denotes any closed surface in the infinite medium with an outward unit normal $n_p(\mathbf{r})$ which surrounds the point of application of the point force $f_k(\mathbf{r}')$. An application of Gauss' divergence theorem gives

$$f_k(\mathbf{r}') = -\iiint_V \frac{\partial\sigma_{kp}(\mathbf{r})}{\partial x_p}\,dV(\mathbf{r}) = -\iiint_V D_{kpim}\frac{\partial^2 G_{ij}(\mathbf{r} - \mathbf{r}')}{\partial x_m \partial x_p}f_j(\mathbf{r}')\,dV(\mathbf{r}) \tag{A.7}$$

Writing

$$f_k(\mathbf{r}') = f_j(\mathbf{r}')\iiint_V \delta_{kj}\delta(\mathbf{r} - \mathbf{r}')\,dV(\mathbf{r}) \tag{A.8}$$

then gives the equilibrium requirement that

$$f_j(\mathbf{r}')\iiint_V \left\{D_{kpim}\frac{\partial^2 G_{ij}(\mathbf{r} - \mathbf{r}')}{\partial x_m \partial x_p} + \delta_{kj}\delta(\mathbf{r} - \mathbf{r}')\right\}dV(\mathbf{r}) = 0 \tag{A.9}$$

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Since $f_j(\mathbf{r}')$ and V are arbitrary, then

$$D_{kpim}\frac{\partial^2 G_{ij}(\mathbf{r}-\mathbf{r}')}{\partial x_m \partial x_p} + \delta_{kj}\delta(\mathbf{r}-\mathbf{r}') = 0 \tag{A.10}$$

is the differential relation satisfied by the Green's tensor function. When multiplied by f_j, this is just Navier's equation of elasticity with the displacement $u_i(\mathbf{r}) = G_{ij}(\mathbf{r}-\mathbf{r}')f_j(\mathbf{r}')$ and the body force set equal to $\delta_{kj}f_j(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')$.

Rearranging the indices, this differential relation can be expressed as

$$D_{ijkl}\frac{\partial^2 G_{kp}(\mathbf{r})}{\partial x_l \partial x_j} + \delta_{ip}\delta(\mathbf{r}) = 0 \tag{A.11}$$

The solution to the differential equation can be found by applying Fourier integral techniques. On multiplying the differential relation by $e^{iK_q x_q}\,dx_1\,dx_2\,dx_3$, i.e., by $e^{i\mathbf{K}\cdot\mathbf{r}}\,dV(\mathbf{r})$, and integrating over all space, we obtain

$$D_{ijkl}\iiint\limits_{-\infty}^{\infty}\frac{\partial^2 G_{kp}(\mathbf{r})}{\partial x_l \partial x_j}e^{iK_q x_q}\,dx_1\,dx_2\,dx_3 + \delta_{ip}\iiint\limits_{-\infty}^{\infty}\delta(x_1)\,\delta(x_2)\,\delta(x_3)e^{iK_q x_q}\,dx_1\,dx_2\,dx_3 = 0 \tag{A.12}$$

From the sifting properties of the Dirac delta function the last integral is unity, so that

$$\iiint\limits_{-\infty}^{\infty}\left\{D_{ijk1}\frac{\partial}{\partial x_1}\left(\frac{\partial G_{kp}(\mathbf{r})}{\partial x_j}\right) + D_{ijk2}\frac{\partial}{\partial x_2}\left(\frac{\partial G_{kp}(\mathbf{r})}{\partial x_j}\right) + \right.$$
$$\left. + D_{ijk3}\frac{\partial}{\partial x_3}\left(\frac{\partial G_{kp}(\mathbf{r})}{\partial x_j}\right)\right\}e^{iK_q x_q}\,dx_1\,dx_2\,dx_3 + \delta_{ip} = 0 \tag{A.13}$$

Integration by parts severally with respect to x_1, x_2, x_3 then gives

$$\iint\limits_{-\infty}^{\infty}\left\{\left[D_{ijk1}e^{iK_1 x_1}\frac{\partial G_{kp}(\mathbf{r})}{\partial x_j}\right]_{x_1=-\infty}^{\infty}e^{i(K_2 x_2 + K_3 x_3)}\,dx_2\,dx_3 + \left[D_{ijk2}e^{iK_2 x_2}\frac{\partial G_{kp}(\mathbf{r})}{\partial x_j}\right]_{x_2=-\infty}^{\infty}\times\right.$$
$$\left.\times e^{i(K_1 x_1 + K_3 x_3)}\,dx_1\,dx_3 + \left[D_{ijk3}e^{iK_3 x_3}\frac{\partial G_{kp}(\mathbf{r})}{\partial x_j}\right]_{x_3=-\infty}^{\infty}e^{i(K_1 x_1 + K_2 x_2)}\,dx_1\,dx_2\right\} -$$
$$- \iiint\limits_{-\infty}^{\infty}\left\{D_{ijk1}\frac{\partial G_{kp}(\mathbf{r})}{\partial x_j}iK_1 + D_{ijk2}\frac{\partial G_{kp}(\mathbf{r})}{\partial x_j}iK_2 + D_{ijk3}\frac{\partial G_{kp}(\mathbf{r})}{\partial x_j}iK_3\right\}e^{iK_q x_q}\,dx_1\,dx_2\,dx_3 +$$
$$+ \delta_{ip} = 0 \tag{A.14}$$

The surface integrals are zero since $\partial G_{kp}(\mathbf{r})/\partial x_j$ vanishes at the infinite lower and upper limits of integration, so that one integration by parts yields the result,

$$-D_{ijkl}\iiint\limits_{-\infty}^{\infty}iK_l\frac{\partial G_{kp}(\mathbf{r})}{\partial x_j}e^{i\mathbf{K}\cdot\mathbf{r}}\,dV(\mathbf{r}) + \delta_{ip} = 0 \tag{A.15}$$

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A second integration by parts yields

$$+ D_{ijkl} \iiint\limits_{-\infty}^{\infty} i^2 K_l K_j G_{kp}(\mathbf{r}) e^{i\mathbf{K}\cdot\mathbf{r}} \, dV(\mathbf{r}) + \delta_{ip} = 0 \tag{A.16}$$

or

$$D_{ijkl} K_l K_j \widehat{G}_{kp}(\mathbf{K}) = \delta_{ip} \tag{A.17}$$

where

$$\widehat{G}_{kp}(\mathbf{K}) = \iiint\limits_{-\infty}^{\infty} G_{kp}(\mathbf{r}) e^{i\mathbf{K}\cdot\mathbf{r}} \, dV(\mathbf{r}) \tag{A.18}$$

is the Fourier integral transform of $G_{kp}(\mathbf{r})$. By writing $\boldsymbol{\zeta}$ as a unit vector in the direction of the wave vector $\mathbf{K}$, we have

$$\zeta_j = \frac{K_j}{\sqrt{K_q K_q}} = \frac{K_j}{K} \tag{A.19}$$

in which $K = \sqrt{K_q K_q}$ is the magnitude of the $\mathbf{K}$ vector. Then

$$D_{ijkl} \frac{K_l}{K} \frac{K_j}{K} K^2 \widehat{G}_{kp}(\mathbf{K}) = \delta_{ip} \tag{A.20}$$

or

$$K^2 D_{ijkl} \zeta_l \zeta_j \widehat{G}_{kp}(\mathbf{K}) = \delta_{ip} \tag{A.21}$$

The Christoffel stiffness tensor $\mathbf{M}$ (*cf.* [33]) is defined by the relation

$$M_{ik}(\boldsymbol{\zeta}) = D_{ijkl} \zeta_l \zeta_j \tag{A.22}$$

so that

$$K^2 M_{ik}(\boldsymbol{\zeta}) \widehat{G}_{kp}(\mathbf{K}) = \delta_{ip} \tag{A.23}$$

Premultiplying both sides by the tensor $K^{-2}\mathbf{M}^{-1}$ gives

$$\widehat{G}_{ij}(\mathbf{K}) = K^{-2} M_{ij}^{-1}(\boldsymbol{\zeta}) \tag{A.24}$$

The Fourier inverse of equation A.18 gives

$$G_{ij}(\mathbf{r}) = (2\pi)^{-3} \iiint\limits_{-\infty}^{\infty} e^{-i\mathbf{K}\cdot\mathbf{r}} \widehat{G}_{ij}(\mathbf{K}) \, \mathrm{d}^3\mathbf{K} \tag{A.25}$$

where $\mathrm{d}^3\mathbf{K} = dK_1 \, dK_2 \, dK_3$, so that we finally obtain the Green's function in the Fourier integral form

$$G_{ij}(\mathbf{r}) = \iiint\limits_{-\infty}^{\infty} \frac{\mathrm{d}^3\mathbf{K}}{(2\pi)^3} \frac{M_{ij}^{-1}(\boldsymbol{\zeta})}{K^2} e^{-i\mathbf{K}\cdot\mathbf{r}} \tag{A.26}$$

with

$$M_{ij}^{-1}(\boldsymbol{\zeta}) = (D_{iklj} \zeta_l \zeta_k)^{-1} \tag{A.27}$$

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This representation of the Green's function yields explicit results for isotropic and transversely isotropic materials (*cf.* Mura's book, [24]). For cubic and general anisotropy the Fourier integral representation must be used.

Often, we are concerned with volume integrals of the Green's function and its derivatives with respect to $\mathbf{r}$, such as $U_{klmn}(\mathbf{r})$. It is then advantageous to use the Fourier integral representation even for isotropic and transversely isotropic materials. The advantage is gained by reversing the order of the wave vector and volume integrations, whereby many of the integrations can be carried out explicitly.

Sir William Thomson (Lord Kelvin) obtained an explicit form for the Green's function of an isotropic elastic material in 1848. As an example we may deduce the Kelvin result for the Green's function of an isotropic material from the Fourier integral relation. For an isotropic material

$$D_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{A.28}$$

and so $M_{ik}(\boldsymbol{\zeta}) = D_{ijkl}\zeta_l\zeta_j$ has the form

$$M_{ik}(\boldsymbol{\zeta}) = (\lambda\zeta_i\zeta_k + \mu\delta_{ik}\zeta_l\zeta_l + \mu\zeta_i\zeta_k) = \left((\lambda + \mu)\zeta_i\zeta_k + \mu\delta_{ik}\right) \tag{A.29}$$

since

$$\zeta_l\zeta_l = \zeta_1^2 + \zeta_2^2 + \zeta_3^2 = \left(\frac{K_1}{K}\right)^2 + \left(\frac{K_2}{K}\right)^2 + \left(\frac{K_3}{K}\right)^2 = 1 \tag{A.30}$$

The inverse tensor $M_{ik}^{-1}(\boldsymbol{\zeta})$ is given by the relation

$$M_{ik}^{-1}(\boldsymbol{\zeta}) = \left((\lambda + \mu)\zeta_i\zeta_k + \mu\delta_{ik}\right)^{-1} = \frac{\delta_{ik}}{\mu} - \frac{\lambda + \mu}{\mu(\lambda + 2\mu)}\zeta_i\zeta_k \tag{A.31}$$

which is easily verified by showing that

$$M_{ij}^{-1}M_{jk} = \delta_{ik} \tag{A.32}$$

From the preceding relations

$$\begin{aligned}
M_{ij}^{-1}M_{jk} &= \left(\frac{\delta_{ij}}{\mu} - \frac{\lambda + \mu}{\mu(\lambda + 2\mu)}\zeta_i\zeta_j\right)\left((\lambda + \mu)\zeta_j\zeta_k + \mu\delta_{jk}\right) \\
&= \frac{\lambda + \mu}{\mu}\zeta_i\zeta_k + \delta_{ik} - \frac{(\lambda + \mu)^2}{\mu(\lambda + 2\mu)}\zeta_i\zeta_k - \frac{(\lambda + \mu)\mu}{\mu(\lambda + 2\mu)}\zeta_i\zeta_k = \delta_{ik}
\end{aligned} \tag{A.33}$$

as required.

The Green's function is therefore obtained in the form

$$G_{ij}(\mathbf{r}) = (2\pi)^{-3}\iiint\limits_{-\infty}^{\infty}\frac{1}{\mu}\frac{\delta_{ij}}{K^2}\,e^{-i\mathbf{K}\cdot\mathbf{r}}\,d^3\mathbf{K} - (2\pi)^{-3}\iiint\limits_{-\infty}^{\infty}\frac{\lambda + \mu}{\mu(\lambda + 2\mu)}\frac{K_iK_j}{K^4}\,e^{-i\mathbf{K}\cdot\mathbf{r}}\,d^3\mathbf{K} \tag{A.34}$$

From a table of Fourier transforms (*cf.* [36]) we find that

$$r = -\frac{1}{\pi^2}\iiint\limits_{-\infty}^{\infty}\frac{1}{K^4}e^{-i\mathbf{K}\cdot\mathbf{r}}\,d^3\mathbf{K} \tag{A.35}$$

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which may be differentiated with respect to x_i and x_j to give

$$\frac{\partial^2 r}{\partial x_i \partial x_j} = \frac{1}{\pi^2} \iiint\limits_{-\infty}^{\infty} \frac{K_i K_j}{K^4} e^{-i\mathbf{K} \cdot \mathbf{r}} \, \mathrm{d}^3 \mathbf{K} \qquad (A.36)$$

By contracting the i and j indices we obtain

$$\frac{\partial^2 r}{\partial x_j \partial x_j} = \frac{1}{\pi^2} \iiint\limits_{-\infty}^{\infty} \frac{1}{K^2} e^{-i\mathbf{K} \cdot \mathbf{r}} \, \mathrm{d}^3 \mathbf{K} \qquad (A.37)$$

The Green's function may therefore be written as

$$G_{ij}(\mathbf{r}) = \frac{1}{(2\pi)^3} \left(\frac{\delta_{ij}}{\mu} \pi^2 \frac{\partial^2 r}{\partial x_q \partial x_q} - \frac{\lambda + \mu}{\mu(\lambda + 2\mu)} \pi^2 \frac{\partial^2 r}{\partial x_i \partial x_j} \right) \qquad (A.38)$$

or

$$G_{ij}(\mathbf{r}) = \frac{1}{8\pi\mu r} \left\{ 2\delta_{ij} - \frac{\lambda + \mu}{\lambda + 2\mu} \left(\delta_{ij} - \frac{x_i x_j}{r^2} \right) \right\} \qquad (A.39)$$

where the relations

$$\frac{\partial^2 r}{\partial x_q \partial x_q} = \frac{2}{r} \qquad \text{and} \qquad \frac{\partial^2 r}{\partial x_i \partial x_j} = \frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3} \qquad (A.40)$$

obtained by differentiating $r = \sqrt{x_q x_q}$, have been used.

Appendix B

Relationship Between Fourier Series and Green's Function Approaches

In the composite material the total strain increment $\Delta\varepsilon_{kl}^T(\mathbf{r})$ is periodic in $\mathbf{r}$ and is defined by the relationship

$$\Delta\varepsilon_{kl}^T(\mathbf{r}) = \Delta\varepsilon_{kl}^0 + \Delta\varepsilon_{kl}(\mathbf{r}) \tag{B.1}$$

where $\Delta\varepsilon_{kl}^0$ is the strain increment applied to the composite's boundary which is equal to the volume average of $\Delta\varepsilon_{kl}^T(\mathbf{r})$ over the unit periodic cell, and $\Delta\varepsilon_{kl}(\mathbf{r})$ is the deviation or perturbation from the average value due to the presence of the fibers.

From equations 84 and 85 the perturbed strain increment is given in the Fourier series and Green's function approaches by the equivalent relations,

$$\Delta\varepsilon_{kl}(\mathbf{r}) = \frac{1}{V_c} \sum \sum_{n_p=0}^{\pm\infty} \sum{}' g_{klij}(\boldsymbol{\zeta}) \iiint_{V_c} D_{ijrs}^m \Delta\varepsilon_{rs}^\star(\mathbf{r}') \, e^{i\boldsymbol{\xi}\cdot(\mathbf{r}-\mathbf{r}')} \, dV(\mathbf{r}') \tag{B.2}$$

or

$$\Delta\varepsilon_{kl}(\mathbf{r}) = \iiint_V U_{klij}(\mathbf{r}-\mathbf{r}') \, D_{ijrs}^m \Delta\varepsilon_{rs}^\star(\mathbf{r}') \, dV(\mathbf{r}') \tag{B.3}$$

We now show that these equations are equivalent and that the Green's function relation is the Poisson sum transformation of the Fourier series relation.

From the definition of $g_{klmn}(\boldsymbol{\zeta})$ in equation 54 we may write

$$g_{klij}(\boldsymbol{\zeta}) = \tfrac{1}{2}\left(M_{ik}^{-1}(\boldsymbol{\zeta})\zeta_j\zeta_l + M_{il}^{-1}(\boldsymbol{\zeta})\zeta_j\zeta_k\right) \tag{B.4}$$

or

$$g_{klij}(\zeta_1,\zeta_2,\zeta_3) = \tfrac{1}{2}\left(M_{ik}^{-1}(\zeta_1,\zeta_2,\zeta_3)\zeta_j\zeta_l + M_{il}^{-1}(\zeta_1,\zeta_2,\zeta_3)\zeta_j\zeta_k\right) \tag{B.5}$$

where

$$\zeta_i = \frac{\xi_i}{\xi} = \frac{\dfrac{2\pi n_i}{L_i}}{\sqrt{\left(\dfrac{2\pi n_1}{L_1}\right)^2 + \left(\dfrac{2\pi n_2}{L_2}\right)^2 + \left(\dfrac{2\pi n_3}{L_3}\right)^2}} \qquad \text{(no sum on } i\text{)} \qquad \text{for } i = 1,2,3. \tag{B.6}$$

We may therefore write

$$g_{klij}(\boldsymbol{\zeta}) = g_{klij}\Big(\zeta_1(n_1,n_2,n_3),\, \zeta_2(n_1,n_2,n_3),\, \zeta_3(n_1,n_2,n_3)\Big) = f_{klij}(n_1,n_2,n_3) \tag{B.7}$$

The perturbation strain increment can then be written in the form

$$\begin{aligned}
\Delta\varepsilon_{kl}(\mathbf{r}) &= \frac{1}{L_1 L_2 L_3} \sum_{n_1=0}^{\pm\infty} \sum_{n_2=0}^{\pm\infty} \sum_{n_3=0}^{\pm\infty}{}' f_{klij}(n_1,n_2,n_3) \iiint_{V_c} D_{ijrs}^m \Delta\varepsilon_{rs}^\star(\mathbf{r}') \times \\
&\quad \times e^{i\left(\frac{2\pi n_1}{L_1}(x_1-x_1') + \frac{2\pi n_2}{L_2}(x_2-x_2') + \frac{2\pi n_3}{L_3}(x_3-x_3')\right)} \, dx_1' \, dx_2' \, dx_3'
\end{aligned} \tag{B.8}$$

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or as

$$\Delta\varepsilon_{kl}(\mathbf{r}) = \frac{1}{L_1 L_2 L_3} \sum_{n_1=0}^{\pm\infty} \sum_{n_2=0}^{\pm\infty} \sideset{}{'}\sum_{n_3=0}^{\pm\infty} h_{kl}(n_1, n_2, n_3) \tag{B.9}$$

where

$$h_{kl}(n_1, n_2, n_3) = f_{klij}(n_1, n_2, n_3) \iiint\limits_{V_c} D^m_{ijrs} \Delta\varepsilon^\star_{rs}(\mathbf{r}') \times$$

$$\times e^{i\left(\frac{2\pi n_1}{L_1}(x_1 - x_1') + \frac{2\pi n_2}{L_2}(x_2 - x_2') + \frac{2\pi n_3}{L_3}(x_3 - x_3')\right)} dx_1' \, dx_2' \, dx_3' \tag{B.10}$$

By the Poisson sum formula (*cf.* Morse and Feshbach's "Theoretical Physics", [37])

$$\sum_{n_1=0}^{\pm\infty} \sum_{n_2=0}^{\pm\infty} \sideset{}{'}\sum_{n_3=0}^{\pm\infty} h_{kl}(n_1, n_2, n_3) = \sum_{m_1=0}^{\pm\infty} \sum_{m_2=0}^{\pm\infty} \sum_{m_3=0}^{\pm\infty} \frac{L_1 L_2 L_3}{(2\pi)^3} \iiint\limits_{-\infty}^{\infty} d^3\mathbf{K}\, e^{i(m_1 K_1 L_1 + m_2 K_2 L_2 + m_3 K_3 L_3)} \times$$

$$\times h_{kl}\left(\frac{K_1 L_1}{2\pi}, \frac{K_2 L_2}{2\pi}, \frac{K_3 L_3}{2\pi}\right) \tag{B.11}$$

where the sum over the integers n_1, n_2, n_3 is replaced by the sum over the integers m_1, m_2, m_3 in the Fourier integrals. The sum over m_i includes the case where $m_1 = m_2 = m_3 = 0$.

We now have the alternative sum,

$$\Delta\varepsilon_{kl}(\mathbf{r}) = \frac{1}{L_1 L_2 L_3} \sum_{n_1=0}^{\pm\infty} \sum_{n_2=0}^{\pm\infty} \sideset{}{'}\sum_{n_3=0}^{\pm\infty} h_{kl}(n_1, n_2, n_3)$$

$$= \sum_{m_1=0}^{\pm\infty} \sum_{m_2=0}^{\pm\infty} \sum_{m_3=0}^{\pm\infty} \iiint\limits_{-\infty}^{\infty} \frac{d^3\mathbf{K}}{(2\pi)^3} e^{i(m_1 K_1 L_1 + m_2 K_2 L_2 + m_3 K_3 L_3)} f_{klij}\left(\frac{K_1 L_1}{2\pi}, \frac{K_2 L_2}{2\pi}, \frac{K_3 L_3}{2\pi}\right) \times$$

$$\times \iiint\limits_{V_c} D^m_{ijrs} \Delta\varepsilon^\star_{rs}(\mathbf{r}') e^{i\left(K_1(x_1 - x_1') + K_2(x_2 - x_2') + K_3(x_3 - x_3')\right)} dx_1' \, dx_2' \, dx_3' \tag{B.12}$$

or

$$\Delta\varepsilon_{kl}(\mathbf{r}) = \sum_{m_1=0}^{\pm\infty} \sum_{m_2=0}^{\pm\infty} \sum_{m_3=0}^{\pm\infty} \iiint\limits_{-\infty}^{\infty} \frac{d^3\mathbf{K}}{(2\pi)^3} f_{klij}\left(\frac{K_1 L_1}{2\pi}, \frac{K_2 L_2}{2\pi}, \frac{K_3 L_3}{2\pi}\right) e^{i(K_1 x_1 + K_2 x_2 + K_3 x_3)} \times$$

$$\times \iiint\limits_{V_c} D^m_{ijrs} \Delta\varepsilon^\star_{rs}(\mathbf{r}') e^{-i\left(K_1(x_1' - m_1 L_1) + K_2(x_2' - m_2 L_2) + K_3(x_3' - m_3 L_3)\right)} dx_1' \, dx_2' \, dx_3'$$

$$\tag{B.13}$$

Due to the geometric periodicity of the unit cell we may write

$$\Delta\varepsilon^\star_{rs}(\mathbf{r}') = \Delta\varepsilon^\star_{rs}(x_1', x_2', x_3') = \Delta\varepsilon^\star_{rs}(x_1' - m_1 L_1, x_2' - m_2 L_2, x_3' - m_3 L_3) \tag{B.14}$$

and

$$dx_1' \, dx_2' \, dx_3' = d(x_1' - m_1 L_1) \, d(x_2' - m_2 L_2) \, d(x_3' - m_3 L_3) \tag{B.15}$$

so that by making the change of variable

$$(x_1' - m_1 L_1, x_2' - m_2 L_2, x_3' - m_3 L_3) = (x_1'', x_2'', x_3'') = \mathbf{r}'' \tag{B.16}$$

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we obtain

$$\Delta\varepsilon_{kl}(\mathbf{r}) = \iiint\limits_{-\infty}^{\infty} \frac{d^3\mathbf{K}}{(2\pi)^3} f_{klij}\left(\frac{K_1 L_1}{2\pi}, \frac{K_2 L_2}{2\pi}, \frac{K_3 L_3}{2\pi}\right) e^{i(K_1 x_1 + K_2 x_2 + K_3 x_3)} \times$$

$$\times \sum_{m_1=0}^{\pm\infty} \sum_{m_2=0}^{\pm\infty} \sum_{m_3=0}^{\pm\infty} \int\int\limits_{V_c(m_1,m_2,m_3)}\int D^m_{ijrs}\Delta\varepsilon^\star_{rs}(\mathbf{r}'') e^{-i\mathbf{K}\cdot\mathbf{r}''} dV(\mathbf{r}'') \qquad \text{(B.17)}$$

where the volume integration extends over the volume $V_c(m_1, m_2, m_3)$ of the unit cell whose center is at the point $(m_1 L_1, m_2 L_2, m_3 L_3)$. Since m_1, m_2, m_3 range over all integer values, the summation of the volume integrals extends to *all* the cells in the periodic lattice, *i.e.*, it extends over the entire volume, V, of the composite medium. The expression for $\Delta\varepsilon_{kl}(\mathbf{r})$ thus takes the form

$$\Delta\varepsilon_{kl}(\mathbf{r}) = \iiint\limits_{-\infty}^{\infty} \frac{d^3\mathbf{K}}{(2\pi)^3} f_{klij}\left(\frac{K_1 L_1}{2\pi}, \frac{K_2 L_2}{2\pi}, \frac{K_3 L_3}{2\pi}\right) e^{i\mathbf{K}\cdot\mathbf{r}} \times$$

$$\times \iiint\limits_{V} D^m_{ijrs}\Delta\varepsilon^\star_{rs}(\mathbf{r}'') e^{-i\mathbf{K}\cdot\mathbf{r}''} dV(\mathbf{r}'') \qquad \text{(B.18)}$$

By interchanging the order of the volume and wave vector integrals and noting that $\mathbf{r}''$ can be replaced by $\mathbf{r}'$ since it is a dummy integration variable, we obtain

$$\Delta\varepsilon_{kl}(\mathbf{r}) = \iiint\limits_{V} dV(\mathbf{r}') \iiint\limits_{-\infty}^{\infty} \frac{d^3\mathbf{K}}{(2\pi)^3} f_{klij}\left(\frac{K_1 L_1}{2\pi}, \frac{K_2 L_2}{2\pi}, \frac{K_3 L_3}{2\pi}\right) e^{i\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}')} D^m_{ijrs}\Delta\varepsilon^\star_{rs}(\mathbf{r}') \quad \text{(B.19)}$$

Introducing $\left(\frac{K_1 L_1}{2\pi}, \frac{K_2 L_2}{2\pi}, \frac{K_3 L_3}{2\pi}\right)$ in place of (n_1, n_2, n_3) in the expression for

$$f_{klij}(n_1, n_2, n_3) = g_{klij}\Big(\zeta_1(n_1, n_2, n_3), \zeta_2(n_1, n_2, n_3), \zeta_3(n_1, n_2, n_3)\Big) \qquad \text{(B.20)}$$

then gives

$$g_{klij}\left(\zeta_1\left(\frac{K_1 L_1}{2\pi}, \frac{K_2 L_2}{2\pi}, \frac{K_3 L_3}{2\pi}\right), \zeta_2\left(\frac{K_1 L_1}{2\pi}, \frac{K_2 L_2}{2\pi}, \frac{K_3 L_3}{2\pi}\right), \zeta_3\left(\frac{K_1 L_1}{2\pi}, \frac{K_2 L_2}{2\pi}, \frac{K_3 L_3}{2\pi}\right)\right)$$

$$= \frac{1}{2}\left(\frac{M^{-1}_{ik}(\zeta)}{K^2} K_j K_l + \frac{M^{-1}_{il}(\zeta)}{K^2} K_j K_k\right) \qquad \text{(B.21)}$$

with

$$\zeta_i = \frac{K_i}{K} = \frac{K_i}{\sqrt{K_q K_q}} \qquad \text{(B.22)}$$

and the perturbed strain increment takes the form

$$\Delta\varepsilon_{kl}(\mathbf{r}) = \iiint\limits_{V} dV(\mathbf{r}') \iiint\limits_{-\infty}^{\infty} \frac{d^3\mathbf{K}}{(2\pi)^3} \frac{1}{2}\left(\frac{M^{-1}_{ik}(\zeta)}{K^2} K_j K_l + \frac{M^{-1}_{il}(\zeta)}{K^2} K_j K_k\right) e^{i\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}')} D^m_{ijrs}\Delta\varepsilon^\star_{rs}(\mathbf{r}')$$

$$\text{(B.23)}$$

But, from Appendix A,

$$G_{ik}\left(\mathbf{r} - \mathbf{r}'\right) = \iiint\limits_{-\infty}^{\infty} \frac{\mathrm{d}^3\mathbf{K}}{(2\pi)^3} \frac{M_{ik}^{-1}(\zeta)}{K^2}\, e^{-i\mathbf{K}.(\mathbf{r}-\mathbf{r}')} = \iiint\limits_{-\infty}^{\infty} \frac{\mathrm{d}^3\mathbf{K}}{(2\pi)^3} \frac{M_{ik}^{-1}(\zeta)}{K^2}\, e^{i\mathbf{K}.(\mathbf{r}-\mathbf{r}')} \qquad (B.24)$$

since $G_{ik}\left(\mathbf{r} - \mathbf{r}'\right) = G_{ik}\left(\mathbf{r}' - \mathbf{r}\right)$, and therefore

$$\frac{\partial^2 G_{ik}\left(\mathbf{r} - \mathbf{r}'\right)}{\partial x_j \partial x_l} = -\iiint\limits_{-\infty}^{\infty} \frac{\mathrm{d}^3\mathbf{K}}{(2\pi)^3} \frac{M_{ik}^{-1}(\zeta)}{K^2} K_j K_l\, e^{i\mathbf{K}.(\mathbf{r}-\mathbf{r}')} \qquad (B.25)$$

Inserting the last relation into the expression for $\Delta\varepsilon_{kl}(\mathbf{r})$ then shows that

$$\Delta\varepsilon_{kl}(\mathbf{r}) = -\iiint\limits_{V} dV(\mathbf{r}')\,\frac{1}{2}\left(\frac{\partial^2 G_{ik}\left(\mathbf{r} - \mathbf{r}'\right)}{\partial x_j \partial x_l} + \frac{\partial^2 G_{il}\left(\mathbf{r} - \mathbf{r}'\right)}{\partial x_j \partial x_k}\right) D_{ijrs}^m \Delta\varepsilon_{rs}^{\star}(\mathbf{r}') \qquad (B.26)$$

From the definition of the tensor $U_{klmn}\left(\mathbf{r} - \mathbf{r}'\right)$ in equation 83, we see that

$$\Delta\varepsilon_{kl}(\mathbf{r}) = \iiint\limits_{V} U_{klij}\left(\mathbf{r} - \mathbf{r}'\right) D_{ijrs}^m \Delta\varepsilon_{rs}^{\star}(\mathbf{r}')\, dV(\mathbf{r}') \qquad (B.27)$$

which is the result obtained with the Green's function approach.

The Fourier series expression for the perturbation strain increment is thus identical to the Green's function expression and the two are linked *via* the Poisson sum formula.

Appendix C

Poisson Sum Formula

In Fig. 7 the function $f(x) = \left[\left(x - \frac{1}{2}L\right)/\frac{1}{2}L\right]^3$ is shown on the unit cell extending from $x = 0$ to $x = L$. The corresponding function defined on the nth unit cell to the right is given by $f(x + nL)$ and the periodic function $q(x)$, which is comprised of the functions $f(x + nL)$ defined on all of the unit cells extending from $x = -\infty$ to $x = +\infty$, is given by

$$q(x) = \sum_{n=-\infty}^{\infty} f(x + nL) \tag{C.1}$$

Each function, $f(x + nL)$, is defined only over the corresponding n^{th} periodic cell and is taken to be zero outside of the cell. Each function can therefore be represented as a Fourier integral and the periodic function $q(x)$ can be written as a sum of Fourier integrals,

$$q(x) = \sum_{n=-\infty}^{\infty} f(x + nL) = \sum_{n=-\infty}^{\infty} \text{Fourier integral of } f(x + nL) \tag{C.2}$$

By setting $x = 0$ in both summations we obtain the Poisson sum formula. This method is outlined at the end of this Appendix.

The Poisson sum formula can also be derived by expanding the periodic function $q(x)$ into a Fourier expansion and showing that the Fourier integral sum, when $x = 0$, is the sum of the coefficients in the Fourier series expansion.

Since $q(x)$ is a periodic function of period L, it may be expanded into a Fourier series in the form

$$q(x) = \sum_{m=-\infty}^{\infty} a_m e^{-\frac{i2\pi mx}{L}} \tag{C.3}$$

where

$$a_m = \frac{1}{L} \int_0^L e^{\frac{i2\pi mx'}{L}} q\left(x'\right) dx' \tag{C.4}$$

The object is to show that the Fourier series

$$q(x) = \frac{1}{L} \sum_{m=-\infty}^{\infty} e^{-\frac{i2\pi mx}{L}} \int_0^L e^{\frac{i2\pi mx'}{L}} q\left(x'\right) dx' \tag{C.5}$$

represents a sum of Fourier integrals. This is easily accomplished by introducing the expression

$$q\left(x'\right) = \sum_{n=-\infty}^{\infty} f\left(x' + nL\right)$$

into the Fourier expansion and changing the integration variable by means of the relation

$$y = x' + nL \tag{C.6}$$

In this way we obtain

$$q(x) = \sum_{n=-\infty}^{\infty} f(x + nL) = \frac{1}{L} \sum_{m=-\infty}^{\infty} e^{-\frac{i2\pi mx}{L}} \sum_{n=-\infty}^{\infty} \int_{y=(n-1)L}^{nL} e^{\frac{i2\pi m(y-nL)}{L}} f(y)\, dy \tag{C.7}$$

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The exponential function $\exp(i2\pi my/L)$ is a periodic function with period L, so that

$$e^{\frac{i2\pi m(y-nL)}{L}} = e^{\frac{i2\pi my}{L}} \tag{C.8}$$

If we set $x = 0$ and note that the sum over the integration limits is equivalent to summing over the entire axis of x from $x = -\infty$ to $x = +\infty$, we obtain

$$q(0) = \sum_{n=-\infty}^{\infty} f(nL) = \sum_{m=-\infty}^{\infty} \frac{1}{L} \int_{-\infty}^{\infty} e^{\frac{i2\pi my}{L}} f(y)\, dy \tag{C.9}$$

Putting $L = 1$ gives

$$q(0) = \sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i2\pi my} f(y)\, dy \tag{C.10}$$

and by changing the integration variable to $K = 2\pi y/L$, we obtain

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} \frac{L}{2\pi} \int_{-\infty}^{\infty} e^{imKL} f\left(\frac{KL}{2\pi}\right) dK \tag{C.11}$$

In three dimensions this result takes the form

$$\sum_{n_1=0}^{\pm\infty} \sum_{n_2=0}^{\pm\infty} \sum_{n_3=0}^{\pm\infty} f(n_1, n_2, n_3) = \sum_{m_1=0}^{\pm\infty} \sum_{m_2=0}^{\pm\infty} \sum_{m_3=0}^{\pm\infty} \frac{L_1 L_2 L_3}{(2\pi)^3} \iiint_{-\infty}^{\infty} d^3\mathbf{K}\ e^{i(m_1 K_1 L_1 + m_2 K_2 L_2 + m_3 K_3 L_3)} \times$$

$$\times f\left(\frac{K_1 L_1}{2\pi}, \frac{K_2 L_2}{2\pi}, \frac{K_3 L_3}{2\pi}\right) \tag{C.12}$$

which is the form used in Appendix B. The cubic function defined here for illustration purposes has the property that the constant a_0 in the Fourier expansion is zero, since $\int_0^L f(x)\, dx = 0$. This term may therefore be omitted from the summation on the left and the summation signs primed to denote the omission of the term with $n_1 = n_2 = n_3 = 0$.

It is now possible to show that the Poisson sum formula follows from the Fourier integral sum in equation C.2.

We have the Fourier integral sum representation

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} f(y)\delta(y - m)\, dy \tag{C.13}$$

where the Dirac delta function is given by the Fourier integral

$$\delta(y - m) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(y-m)z}\, dz \tag{C.14}$$

Then the Fourier integral of $f(m)$ is

$$f(m) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-imz}\, dz \int_{-\infty}^{\infty} e^{izy} f(y)\, dy \tag{C.15}$$

Putting $z = 2\pi\alpha$ gives

$$f(m) = \int_{-\infty}^{\infty} e^{-i2\pi m\alpha} \, d\alpha \int_{-\infty}^{\infty} e^{i2\pi\alpha y} f(y) \, dy \qquad (C.16)$$

The Fourier series expansion which culminates in equation C.10 shows that

$$f(\alpha) = \int_{-\infty}^{\infty} e^{i2\pi\alpha y} f(y) \, dy \qquad (C.17)$$

so that equation C.16 becomes

$$f(m) = \int_{-\infty}^{\infty} e^{-i2\pi m\alpha} f(\alpha) \, d\alpha \qquad (C.18)$$

We may therefore write

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{m=\infty}^{-\infty} f(m) = \sum_{m=-\infty}^{\infty} f(-m) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i2\pi m\alpha} f(\alpha) \, d\alpha \qquad (C.19)$$

This is the Poisson sum formula in equation C.10, from which equations C.11 and C.12 follow.

Appendix D

Integral Equation for Displacement Increment
In Neighborhood of Free Surface

The static equilibrium equation for a medium with elasticity tensor D_{ijkl}^m is obtained from equation 37 as

$$D_{ijkl}^m \frac{\partial\left(\Delta\varepsilon_{kl}^T(\mathbf{r})\right)}{\partial x_j} = \frac{\partial}{\partial x_j}\left\{D_{ijkl}^m \Delta\varepsilon_{kl}^\star(\mathbf{r})\right\} \tag{D.1}$$

in which $\Delta\varepsilon_{kl}^T(\mathbf{r}) = \Delta\varepsilon_{kl}^0(\mathbf{r}) + \Delta\varepsilon_{kl}(\mathbf{r})$.

In this equilibrium relation we will not assume that the strain increment $\Delta\varepsilon_{kl}^0(\mathbf{r})$ applied at the surface of the composite is constant and will take it as a spatial variable. If we set

$$\Delta f_i(\mathbf{r}) = \frac{\partial}{\partial x_j}\left\{D_{ijkl}^m \Delta\varepsilon_{kl}^\star(\mathbf{r})\right\} \tag{D.2}$$

and note that

$$\Delta\varepsilon_{kl}^T(\mathbf{r}) = \frac{1}{2}\left(\frac{\partial\left(\Delta u_k^T(\mathbf{r})\right)}{\partial x_l} + \frac{\partial\left(\Delta u_l^T(\mathbf{r})\right)}{\partial x_k}\right) \tag{D.3}$$

the equilibrium equation may be written in the form

$$D_{ijkl}^m \frac{\partial^2\left(\Delta u_l^T\right)}{\partial x_j \partial x_k} = \Delta f_i \tag{D.4}$$

where the symmetry of D_{ijkl}^m with respect to the indices k and l has been used. On denoting the operator $\mathcal{F}_{il}$ by the relationship

$$\mathcal{F}_{il} = D_{ijkl}^m \frac{\partial^2}{\partial x_j \partial x_k} \tag{D.5}$$

the equilibrium equation is

$$\mathcal{F}_{il}\Delta u_l^T = \Delta f_i \tag{D.6}$$

Now consider the integral

$$I(\boldsymbol{\phi}, \boldsymbol{\psi}) = \iiint\limits_V \phi_i\left(\mathbf{r}'\right)\mathcal{F}_{ij}\psi_j\left(\mathbf{r}'\right)\,dV(\mathbf{r}') \tag{D.7}$$

for any two field variables $\phi_i(\mathbf{r})$ and $\psi_i(\mathbf{r})$. These field variables may be tensors of any rank. For example, if ϕ and ψ were second rank tensors, then $\mathbf{I}(\boldsymbol{\phi}, \boldsymbol{\psi})$ would be a second rank tensor integral

$$I_{pq}(\boldsymbol{\phi}, \boldsymbol{\psi}) = \iiint\limits_V \phi_{pi}\left(\mathbf{r}'\right)\mathcal{F}_{ij}\psi_{jq}\left(\mathbf{r}'\right)\,dV(\mathbf{r}') \tag{D.8}$$

From the definition of the operator $\mathcal{F}_{ij}$ we have

$$I(\boldsymbol{\phi}, \boldsymbol{\psi}) = \iiint\limits_V \phi_i\left(\mathbf{r}'\right) D^m_{ipqj} \frac{\partial}{\partial x'_p} \left(\frac{\partial \psi_j\left(\mathbf{r}'\right)}{\partial x'_q} \right) dV(\mathbf{r}') \tag{D.9}$$

and since

$$\frac{\partial}{\partial x'_p}\left(\phi_i\left(\mathbf{r}'\right) D^m_{ipqj} \frac{\partial \psi_j\left(\mathbf{r}'\right)}{\partial x'_q} \right) = \frac{\partial \phi_i\left(\mathbf{r}'\right)}{\partial x'_p} D^m_{ipqj} \frac{\partial \psi_j\left(\mathbf{r}'\right)}{\partial x'_q} + \phi_i\left(\mathbf{r}'\right) D^m_{ipqj} \frac{\partial}{\partial x'_p}\left(\frac{\partial \psi_j\left(\mathbf{r}'\right)}{\partial x'_q} \right) \tag{D.10}$$

the integral becomes

$$I(\boldsymbol{\phi}, \boldsymbol{\psi}) = \iiint\limits_V \frac{\partial}{\partial x'_p}\left(\phi_i\left(\mathbf{r}'\right) D^m_{ipqj} \frac{\partial \psi_j\left(\mathbf{r}'\right)}{\partial x'_q} \right) dV(\mathbf{r}') - \iiint\limits_V \frac{\partial \phi_i\left(\mathbf{r}'\right)}{\partial x'_p} D^m_{ipqj} \frac{\partial \psi_j\left(\mathbf{r}'\right)}{\partial x'_q} dV(\mathbf{r}') \tag{D.11}$$

The first integral can be transformed into a surface integral *via* Gauss' divergence theorem, so that

$$I(\boldsymbol{\phi}, \boldsymbol{\psi}) = \iint\limits_S n_p\left(\mathbf{r}'\right) \phi_i\left(\mathbf{r}'\right) D^m_{ipqj} \frac{\partial \psi_j\left(\mathbf{r}'\right)}{\partial x'_q} dS(\mathbf{r}') - \iiint\limits_V \frac{\partial \phi_i\left(\mathbf{r}'\right)}{\partial x'_p} D^m_{ipqj} \frac{\partial \psi_j\left(\mathbf{r}'\right)}{\partial x'_q} dV(\mathbf{r}') \tag{D.12}$$

By interchanging the arguments $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ it is evident that

$$I(\boldsymbol{\psi}, \boldsymbol{\phi}) = \iint\limits_S n_p\left(\mathbf{r}'\right) \psi_i\left(\mathbf{r}'\right) D^m_{ipqj} \frac{\partial \phi_j\left(\mathbf{r}'\right)}{\partial x'_q} dS(\mathbf{r}') - \iiint\limits_V \frac{\partial \psi_i\left(\mathbf{r}'\right)}{\partial x'_p} D^m_{ipqj} \frac{\partial \phi_j\left(\mathbf{r}'\right)}{\partial x'_q} dV(\mathbf{r}') \tag{D.13}$$

Now the elasticity tensor D^m_{ipqj} is symmetric with respect to its indices, so that the interchanges

$$ip \to pi, \qquad qj \to jq, \qquad ip \leftrightarrow (qj \text{ or } jq), \qquad qj \leftrightarrow (ip \text{ or } pi) \tag{D.14}$$

leave the elasticity tensor unaltered. This shows that the volume integrals in $I(\boldsymbol{\phi}, \boldsymbol{\psi})$ and $I(\boldsymbol{\psi}, \boldsymbol{\phi})$ are identical, so that Green's identity ([38], page 434) can be written as

$$I(\boldsymbol{\phi}, \boldsymbol{\psi}) - I(\boldsymbol{\psi}, \boldsymbol{\phi}) = \iint\limits_S n_p\left(\mathbf{r}'\right) \left(\phi_i\left(\mathbf{r}'\right) D^m_{ipqj} \frac{\partial \psi_j\left(\mathbf{r}'\right)}{\partial x'_q} - \psi_i\left(\mathbf{r}'\right) D^m_{ipqj} \frac{\partial \phi_j\left(\mathbf{r}'\right)}{\partial x'_q} \right) dS(\mathbf{r}') \tag{D.15}$$

or

$$\iiint\limits_V \left(\phi_i \mathcal{F}_{ij} \psi_j - \psi_i \mathcal{F}_{ij} \phi_j \right) dV = \iint\limits_S n_p \left(\phi_i D^m_{ipqj} \frac{\partial \psi_j}{\partial x'_q} - \psi_i D^m_{ipqj} \frac{\partial \phi_j}{\partial x'_q} \right) dS \tag{D.16}$$

Now choose the field variables $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ as a vector and second rank tensor in the forms

$$\phi_i\left(\mathbf{r}'\right) = \Delta u_i^T\left(\mathbf{r}'\right) - \Delta u_i^0\left(\mathbf{r}'\right) = \Delta u_i\left(\mathbf{r}'\right) \tag{D.17}$$

and

$$\psi_{ij}\left(\mathbf{r}'\right) = G_{ij}\left(\mathbf{r} - \mathbf{r}'\right) \tag{D.18}$$

where $\phi_i\left(\mathbf{r}'\right)$ is the perturbation displacement increment $\Delta u_i\left(\mathbf{r}'\right)$ and $G_{ij}\left(\mathbf{r} - \mathbf{r}'\right)$ is the Green's tensor function satisfying the differential equation (*cf.* Appendix A),

$$\mathcal{F}_{ij} G_{jk}\left(\mathbf{r} - \mathbf{r}'\right) + \delta_{ik}\delta\left(\mathbf{r} - \mathbf{r}'\right) = 0 \tag{D.19}$$

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The integral relation then becomes

$$
\iiint\limits_{V} \left\{ \left(\Delta u_i^T(\mathbf{r}') - \Delta u_i^0(\mathbf{r}') \right) \mathcal{F}_{ij} G_{jk}(\mathbf{r} - \mathbf{r}') - \right.
$$

$$
\left. - G_{ik}(\mathbf{r} - \mathbf{r}') \mathcal{F}_{ij} \left(\Delta u_j^T(\mathbf{r}') - \Delta u_j^0(\mathbf{r}') \right) \right\} dV(\mathbf{r}')
$$

$$
= \iint\limits_{S} n_p(\mathbf{r}') \left\{ \left(\Delta u_i^T(\mathbf{r}') - \Delta u_i^0(\mathbf{r}') \right) D_{ipqj}^m \frac{\partial G_{jk}(\mathbf{r} - \mathbf{r}')}{\partial x_q'} - \right.
$$

$$
\left. - G_{ik}(\mathbf{r} - \mathbf{r}') D_{ipqj}^m \left(\frac{\partial \left(\Delta u_j^T(\mathbf{r}') \right)}{\partial x_q'} - \frac{\partial \left(\Delta u_j^0(\mathbf{r}') \right)}{\partial x_q'} \right) \right\} dS(\mathbf{r}') \qquad (D.20)
$$

Replacing $\mathcal{F}_{ij} G_{jk}(\mathbf{r} - \mathbf{r}')$ by $-\delta_{ik}\delta(\mathbf{r} - \mathbf{r}')$ in the first term of the volume integral gives

$$
\Delta u_k^T(\mathbf{r}) - \Delta u_k^0(\mathbf{r}) + \iiint\limits_{V} G_{ik}(\mathbf{r} - \mathbf{r}') \mathcal{F}_{ij} \left(\Delta u_j^T(\mathbf{r}') - \Delta u_j^0(\mathbf{r}') \right) dV(\mathbf{r}')
$$

$$
= \iint\limits_{S} n_p(\mathbf{r}') \left\{ G_{ik}(\mathbf{r} - \mathbf{r}') D_{ipqj}^m \left(\frac{\partial \left(\Delta u_j^T(\mathbf{r}') \right)}{\partial x_q'} - \frac{\partial \left(\Delta u_j^0(\mathbf{r}') \right)}{\partial x_q'} \right) - \right.
$$

$$
\left. - \left(\Delta u_i^T(\mathbf{r}') - \Delta u_i^0(\mathbf{r}') \right) D_{ipqj}^m \frac{\partial G_{jk}(\mathbf{r} - \mathbf{r}')}{\partial x_q'} \right\} dS(\mathbf{r}') \qquad (D.21)
$$

From the definition of the operator $\mathcal{F}_{ij}$ we obtain the relation

$$
\mathcal{F}_{ij} \Delta u_j^T(\mathbf{r}) = \Delta f_i(\mathbf{r}) = \frac{\partial}{\partial x_j} \left\{ D_{ijkl}^m \Delta \varepsilon_{kl}^\star(\mathbf{r}) \right\} \qquad (D.22)
$$

and by inserting this result into the integral equation and noting that

$$
D_{ipqj}^m \frac{\partial \left(\Delta u_j^0(\mathbf{r}') \right)}{\partial x_q'} = D_{ipqj}^m \frac{1}{2} \left(\frac{\partial \left(\Delta u_j^0(\mathbf{r}') \right)}{\partial x_q'} + \frac{\partial \left(\Delta u_q^0(\mathbf{r}') \right)}{\partial x_j'} \right) = D_{ipqj}^m \Delta \varepsilon_{qj}^0(\mathbf{r}') \qquad (D.23)
$$

we find that the total displacement increment is

$$
\Delta u_k^T(\mathbf{r}) = \Delta u_k^0(\mathbf{r}) - \iiint\limits_{V} G_{ik}(\mathbf{r} - \mathbf{r}') \frac{\partial}{\partial x_j'} \left\{ D_{ijrs}^m \left(\Delta \varepsilon_{rs}^\star(\mathbf{r}') - \Delta \varepsilon_{rs}^0(\mathbf{r}') \right) \right\} dV(\mathbf{r}') +
$$

$$
+ \iint\limits_{S} n_p(\mathbf{r}') \left\{ G_{ik}(\mathbf{r} - \mathbf{r}') D_{ipqj}^m \left(\Delta \varepsilon_{qj}^T(\mathbf{r}') - \Delta \varepsilon_{qj}^0(\mathbf{r}') \right) - \right.
$$

$$
\left. - \left(\Delta u_i^T(\mathbf{r}') - \Delta u_i^0(\mathbf{r}') \right) D_{ipqj}^m \frac{\partial G_{jk}(\mathbf{r} - \mathbf{r}')}{\partial x_q'} \right\} dS(\mathbf{r}') \qquad (D.24)
$$

In the volume integral an integration by parts with Gauss' divergence theorem using the relationship

$$
\frac{\partial G_{ik}(\mathbf{r} - \mathbf{r}')}{\partial x_j'} = - \frac{\partial G_{ik}(\mathbf{r} - \mathbf{r}')}{\partial x_j} \qquad (D.25)
$$

gives

$$\iiint_V G_{ik}\left(\mathbf{r}-\mathbf{r}'\right)\frac{\partial}{\partial x_j'}\left\{D_{ijrs}^m\left(\Delta\varepsilon_{rs}^\star\left(\mathbf{r}'\right)-\Delta\varepsilon_{rs}^0\left(\mathbf{r}'\right)\right)\right\}dV(\mathbf{r}')$$

$$=\iiint_V\frac{\partial}{\partial x_j'}\left\{G_{ik}\left(\mathbf{r}-\mathbf{r}'\right)D_{ijrs}^m\left(\Delta\varepsilon_{rs}^\star\left(\mathbf{r}'\right)-\Delta\varepsilon_{rs}^0\left(\mathbf{r}'\right)\right)\right\}dV(\mathbf{r}')-$$

$$-\iiint_V\frac{\partial G_{ik}\left(\mathbf{r}-\mathbf{r}'\right)}{\partial x_j'}D_{ijrs}^m\left(\Delta\varepsilon_{rs}^\star\left(\mathbf{r}'\right)-\Delta\varepsilon_{rs}^0\left(\mathbf{r}'\right)\right)dV(\mathbf{r}')$$

$$=\iint_S n_j\left(\mathbf{r}'\right)G_{ik}\left(\mathbf{r}-\mathbf{r}'\right)D_{ijrs}^m\left(\Delta\varepsilon_{rs}^\star\left(\mathbf{r}'\right)-\Delta\varepsilon_{rs}^0\left(\mathbf{r}'\right)\right)dS(\mathbf{r}')+$$

$$+\iiint_V\frac{\partial G_{ik}\left(\mathbf{r}-\mathbf{r}'\right)}{\partial x_j}D_{ijrs}^m\left(\Delta\varepsilon_{rs}^\star\left(\mathbf{r}'\right)-\Delta\varepsilon_{rs}^0\left(\mathbf{r}'\right)\right)dV(\mathbf{r}') \qquad (D.26)$$

This result may now be substituted into the integral equation to produce

$$\Delta u_k^T(\mathbf{r}) = \Delta u_k^0(\mathbf{r})-\iiint_V\frac{\partial G_{ik}\left(\mathbf{r}-\mathbf{r}'\right)}{\partial x_j}D_{ijrs}^m\left(\Delta\varepsilon_{rs}^\star\left(\mathbf{r}'\right)-\Delta\varepsilon_{rs}^0\left(\mathbf{r}'\right)\right)dV(\mathbf{r}')+$$

$$+\iint_S n_j\left(\mathbf{r}'\right)\left\{G_{ik}\left(\mathbf{r}-\mathbf{r}'\right)D_{ijrs}^m\left(\Delta\varepsilon_{rs}^T\left(\mathbf{r}'\right)-\Delta\varepsilon_{rs}^\star\left(\mathbf{r}'\right)\right)+\right.$$

$$\left.+\left(\Delta u_i^T\left(\mathbf{r}'\right)-\Delta u_i^0\left(\mathbf{r}'\right)\right)D_{ijrs}^m\frac{\partial G_{sk}\left(\mathbf{r}-\mathbf{r}'\right)}{\partial x_r}\right\}dS(\mathbf{r}') \qquad (D.27)$$

In the first two terms of the surface integral we observe from equation 35 that

$$n_j\left(\mathbf{r}'\right)D_{ijrs}^m\left(\Delta\varepsilon_{rs}^T\left(\mathbf{r}'\right)-\Delta\varepsilon_{rs}^\star\left(\mathbf{r}'\right)\right)=n_j\Delta\sigma_{ij}\left(\mathbf{r}'\right)=\Delta t_i\left(\mathbf{r}'\right) \qquad (D.28)$$

represents the incremental surface traction on the surface of the composite. Equation D.27 represents the well known Somigliana identity ([38], page 93) for the displacement increment. In the case where the composite is assumed to be of infinite extent the surface integrals in the preceding integral equation vanish, and if $\Delta\varepsilon_{rs}^0\left(\mathbf{r}'\right)$ is assumed to be spatially constant, the total displacement increment is given by the relationship,

$$\Delta u_k^T(\mathbf{r})=\Delta u_k^0(\mathbf{r})-\iiint_V\frac{\partial G_{ik}\left(\mathbf{r}-\mathbf{r}'\right)}{\partial x_j}D_{ijrs}^m\Delta\varepsilon_{rs}^\star\left(\mathbf{r}'\right)dV(\mathbf{r}') \qquad (D.29)$$

which corresponds to equation 79 and is the form used in the main report. However, equation D.27 must be used when the surface is not infinitely removed and if $\Delta\varepsilon_{rs}^0\left(\mathbf{r}'\right)$ is not assumed to be constant.

In a finite element context it will normally be assumed that the fibers are very small in comparison with the dimensions of the finite element. At the Gaussian integration point in the finite element it is then permissible to neglect the contribution from the surface integrals since the surface of the finite element is assumed to be many periodic cells away at "infinity". In some situations, however, this may not be a valid assumption. Some turbine blades and

turbine engine combustor liners are fabricated from thin sections in which the central passages are hollow to allow cooling air to pass through the component. In the thin cross sections of such components the surface integrals must be retained in the constitutive formulation.

Suppose, for example, that the total displacement increment at the node points of a finite element are given. From these nodal values and a knowledge of the element's displacement interpolation functions it is then possible to compute the total displacement increment $\Delta u_k^0(\mathbf{r})$ on the surface of the element and the total strain increment $\Delta \varepsilon_{rs}^0(\mathbf{r})$ at any point. Since $\Delta u_i^T(\mathbf{r}') = \Delta u_i^0(\mathbf{r}')$ on the surface of the finite element, the last term in the integral equation vanishes and the total displacement increment is determined from

$$
\Delta u_k^T(\mathbf{r}) = \Delta u_k^0(\mathbf{r}) - \iiint_V \frac{\partial G_{ik}(\mathbf{r}-\mathbf{r}')}{\partial x_j} D_{ijrs}^m \left(\Delta \varepsilon_{rs}^\star(\mathbf{r}') - \Delta \varepsilon_{rs}^0(\mathbf{r}') \right) dV(\mathbf{r}') +
$$

$$
+ \iint_S n_j(\mathbf{r}') G_{ik}(\mathbf{r}-\mathbf{r}') D_{ijrs}^m \left(\Delta \varepsilon_{rs}^T(\mathbf{r}') - \Delta \varepsilon_{rs}^\star(\mathbf{r}') \right) dS(\mathbf{r}') \qquad \text{(D.30)}
$$

in which the terms in the surface integral represent the contribution to the total displacement increment due to the incremental traction,

$$
\Delta t_i(\mathbf{r}') = n_j(\mathbf{r}') D_{ijrs}^m \left(\Delta \varepsilon_{rs}^T(\mathbf{r}') - \Delta \varepsilon_{rs}^\star(\mathbf{r}') \right) \qquad \text{(D.31)}
$$

on the surface of the element. This surface traction is needed to maintain the displacement increment equality $\Delta u_i^T(\mathbf{r}') = \Delta u_i^0(\mathbf{r}')$, which is imposed at the element's surface.

By differentiating $\Delta u_k^T(\mathbf{r})$ with respect to x_k and x_l and taking half the sum, the total strain increment is subject to the integral equation

$$
\Delta \varepsilon_{kl}^T(\mathbf{r}) = \Delta \varepsilon_{kl}^0(\mathbf{r}) + \iiint_V U_{klij}(\mathbf{r}-\mathbf{r}') D_{ijrs}^m \left(\Delta \varepsilon_{rs}^\star(\mathbf{r}') - \Delta \varepsilon_{rs}^0(\mathbf{r}') \right) dV(\mathbf{r}') +
$$

$$
+ \iint_S n_j(\mathbf{r}') \frac{1}{2} \left(\frac{\partial G_{ik}(\mathbf{r}-\mathbf{r}')}{\partial x_l} + \frac{\partial G_{il}(\mathbf{r}-\mathbf{r}')}{\partial x_k} \right) D_{ijrs}^m \left(\Delta \varepsilon_{rs}^T(\mathbf{r}') - \Delta \varepsilon_{rs}^\star(\mathbf{r}') \right) dS(\mathbf{r}')
$$

$$
\text{(D.32)}
$$

in which

$$
D_{ijkl}^m \Delta \varepsilon_{kl}^\star(\mathbf{r}') = D_{ijkl}^m \Delta c_{kl}(\mathbf{r}') - \delta D_{ijkl}(\mathbf{r}') \left[\Delta \varepsilon_{kl}^T(\mathbf{r}') - \Delta c_{kl}(\mathbf{r}') \right] \qquad \text{(D.33)}
$$

and this integral equation should be used for thin sections of composite material where surface effects are important. This implicit integral equation is similar to that for the infinite medium but contains a correction term for the surface effects in the last integral. This surface integral will become less important—due to the derivatives of the Green's function—when the integration points $\mathbf{r}'$ are far removed from the field point $\mathbf{r}$ and it vanishes for an infinite medium.

In the preceding development it was assumed that the displacement increment $\Delta u_i^0(\mathbf{r}')$ was known, by interpolation with the element displacement polynomials, from the nodal values. This forces the incremental surface traction $\Delta t_i(\mathbf{r}')$ to adopt a periodic distribution in order to maintain $\Delta u_i^T(\mathbf{r}') = \Delta u_i^0(\mathbf{r}')$ on the surface of the element. We could, alternatively, assume that the surface traction increment is zero on the free surface of the element, in which case

the total displacement increment $\Delta u_k^T(\mathbf{r})$ will exhibit a periodic variation on the the surface and the surface takes on the appearance of a frilled structure.

If we therefore assume that the finite element is thin (see Fig. 8); that the surfaces are free of surface traction; and that the surfaces at the ends of the finite element are sufficiently far removed from the Gaussian integration point, the first term in the surface integral in the integral equation is zero and in lieu of equation D.30 the relationship for the total displacement increment now takes the form,

$$
\Delta u_k^T(\mathbf{r}) = \Delta u_k^0(\mathbf{r}) - \iiint_V \frac{\partial G_{ik}(\mathbf{r} - \mathbf{r}')}{\partial x_j} D_{ijrs}^m \left(\Delta \varepsilon_{rs}^{\star}(\mathbf{r}') - \Delta \varepsilon_{rs}^0(\mathbf{r}') \right) dV(\mathbf{r}') +
$$

$$
+ \iint_S n_j(\mathbf{r}') \left(\Delta u_i^T(\mathbf{r}') - \Delta u_i^0(\mathbf{r}') \right) D_{ijrs}^m \frac{\partial G_{sk}(\mathbf{r} - \mathbf{r}')}{\partial x_r} dS(\mathbf{r}') \qquad \text{(D.34)}
$$

The solution to this integral equation gives a periodic total displacement increment, $\Delta u_k^T(\mathbf{r})$, which, on the surface of the composite, will exhibit frilling.

It is clear that during the finite element analysis frilling will not occur in the element. The interpolation functions normally used in isoparametric elements are linear and quadratic, and cannot adopt the required periodic behavior. However, the *stiffness* of the finite element—as computed at the Gaussian integration points with the composite constitutive model—will reflect that the fact that the constitutive properties are computed *as though the element were free to take on a frilled appearance*. When the "damage critical" strain-temperature history is used to determine the stress-strain history variation throughout the unit periodic cell outside of the finite element program, the preceding integral equation will allow the frilled appearance of the composite to be calculated.

Appendix E

Evaluation of the Eshelby Tensor

The Eshelby tensor S_{iplm} is defined by the relation

$$S_{iplm} = -\frac{1}{2}\left\{\frac{\partial^2}{\partial x_i \partial x_k}\iiint_V G_{pj}\left(\mathbf{r}-\mathbf{r}'\right) dV(\mathbf{r}') + \frac{\partial^2}{\partial x_p \partial x_k}\iiint_V G_{ij}\left(\mathbf{r}-\mathbf{r}'\right) dV(\mathbf{r}')\right\}\overline{D}_{jklm} \quad (E.1)$$

or as

$$S_{iplm} = \iiint_V U_{iprs}\left(\mathbf{r}-\mathbf{r}'\right) dV(\mathbf{r}')\overline{D}_{rslm} \quad (E.2)$$

where the field point $\mathbf{r}$ lies within the volume, V, and where the volume extends over an infinite cylinder of radius a in a medium with elasticity tensor $\overline{D}_{ijkl}$. Although the Green's function for transversely isotropic materials is known [24], it is more convenient to work with the Fourier integral representation of the Green's function as given in Appendix A.

Introduction of the Fourier integral representation,

$$G_{ik}\left(\mathbf{r}-\mathbf{r}'\right) = \iiint_{-\infty}^{\infty} \frac{d^3\mathbf{K}}{(2\pi)^3}\frac{M_{ik}^{-1}(\boldsymbol{\zeta})}{K^2}e^{-i\mathbf{K}.(\mathbf{r}-\mathbf{r}')} \quad (E.3)$$

where $\zeta_i = K_i/K = K_i/\sqrt{K_q K_q}$, into one of the volume integrals in the definition of S_{iplm} gives, on reversing the order of the volume and wave vector integrations,

$$L_{kgij} = \frac{\partial^2}{\partial x_k \partial x_g}\iiint_V G_{ij}\left(\mathbf{r}-\mathbf{r}'\right) dV(\mathbf{r}') \quad (E.4)$$

or

$$L_{kgij} = \frac{\partial^2}{\partial x_k \partial x_g}\iiint_{-\infty}^{\infty} \frac{d^3\mathbf{K}}{(2\pi)^3}\frac{M_{ij}^{-1}(\boldsymbol{\zeta})}{K^2}e^{-i\mathbf{K}.\mathbf{r}}\iiint_V e^{i\mathbf{K}.\mathbf{r}'} dV(\mathbf{r}') \quad (E.5)$$

The Laue interference integral [39] extends over the cylindrical volume and can be written as

$$I = \iiint_V e^{i\mathbf{K}.\mathbf{r}'} dV(\mathbf{r}') = \iiint_V e^{i\left(K_1 x_1' + K_2 x_2' + K_3 x_3'\right)} dx_1' dx_2' dx_3' \quad (E.6)$$

Let $x_1' = \varrho\cos\theta$, $x_2' = \varrho\sin\theta$. Then in cylindrical coordinates

$$I = \int_{-\infty}^{\infty}\int_0^a\int_0^{2\pi} e^{iK_3 x_3'}e^{i\left(K_1\varrho\cos\theta + K_2\varrho\sin\theta\right)} dx_3'\,\varrho\,d\varrho\,d\theta \quad (E.7)$$

Since

$$\int_{-\infty}^{\infty} e^{iK_3 x_3'} dx_3' = 2\pi\delta\left(K_3\right) \quad (E.8)$$

where $\delta\left(K_3\right)$ is the Dirac delta function, the integral takes the form

$$I = 2\pi\delta\left(K_3\right)\int_0^a\int_0^{2\pi} e^{i\left(K_1\varrho\cos\theta + K_2\varrho\sin\theta\right)}\varrho\,d\varrho\,d\theta \quad (E.9)$$

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Let $\varrho = \dfrac{q}{\sqrt{K_1^2 + K_2^2}}$, $d\varrho = \dfrac{dq}{\sqrt{K_1^2 + K_2^2}}$. Then, if $K = \sqrt{K_1^2 + K_2^2}$,

$$I = \frac{2\pi\delta\left(K_3\right)}{\sqrt{K_1^2 + K_2^2}} \int_0^{aK} \int_0^{2\pi} e^{i\left(\frac{qK_1\cos\theta}{\sqrt{K_1^2+K_2^2}} + \frac{qK_2\sin\theta}{\sqrt{K_1^2+K_2^2}}\right)} q\, dq\, d\theta \tag{E.10}$$

If we now set $\dfrac{K_1}{\sqrt{K_1^2 + K_2^2}} = \cos\theta'$, $\dfrac{K_2}{\sqrt{K_1^2 + K_2^2}} = \sin\theta'$, then

$$I = \frac{2\pi\delta\left(K_3\right)}{\sqrt{K_1^2 + K_2^2}} \int_0^{aK} \int_0^{2\pi} e^{iq\cos(\theta-\theta')} q\, dq\, d\theta \tag{E.11}$$

Since the integration extends over a whole circumference, it is immaterial where the origin of θ is placed. The integral may therefore be written as

$$\begin{aligned}
I &= \frac{2\pi\delta\left(K_3\right)}{\sqrt{K_1^2 + K_2^2}} \int_0^{aK} q\, dq \int_0^{2\pi} e^{iq\cos\theta}\, d\theta \\
&= \frac{2\pi\delta\left(K_3\right)}{\sqrt{K_1^2 + K_2^2}} \int_0^{aK} q\, dq\, 2\pi J_0\left(q\right)
\end{aligned} \tag{E.12}$$

or as

$$I = 4\pi^2\delta\left(K_3\right) a\, \frac{J_1\left(a\sqrt{K_1^2 + K_2^2}\right)}{\sqrt{K_1^2 + K_2^2}} \tag{E.13}$$

where J_0 and J_1 denote the usual Bessel functions of order zero and one.

The integral L_{kgij} can therefore be written as

$$L_{kgij} = \iiint_{-\infty}^{\infty} \frac{d^3\mathbf{K}}{(2\pi)^3} \frac{M_{ij}^{-1}(\zeta)}{K^2} \frac{\partial^2 e^{-i\mathbf{K}\cdot\mathbf{r}}}{\partial x_k \partial x_g} 4\pi^2\delta\left(K_3\right) a\, \frac{J_1\left(a\sqrt{K_1^2 + K_2^2}\right)}{\sqrt{K_1^2 + K_2^2}} \tag{E.14}$$

Now

$$\frac{\partial^2 e^{-i\mathbf{K}\cdot\mathbf{r}}}{\partial x_k \partial x_g} = -K_k K_g\, e^{-i\mathbf{K}\cdot\mathbf{r}} \tag{E.15}$$

so that

$$\begin{aligned}
L_{kgij} &= -\frac{1}{2\pi} \iiint_{-\infty}^{\infty} \frac{dK_1\, dK_2\, dK_3}{K_1^2 + K_2^2 + K_3^2} K_k K_g\, M_{ij}^{-1}(\zeta) \times \\
&\quad \times e^{-iK_3 x_3}\, e^{-i(K_1 x_1 + K_2 x_2)}\, \delta\left(K_3\right) a\, \frac{J_1\left(a\sqrt{K_1^2 + K_2^2}\right)}{\sqrt{K_1^2 + K_2^2}}
\end{aligned} \tag{E.16}$$

If $k = 3$ or $g = 3$, the Dirac delta function $\delta\left(K_3\right)$ gives zero values for the integral. Hence, the non-zero values of L_{kgij} are given by $k = 1, 2$ and $g = 1, 2$.

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Invoking the sifting properties of the Dirac delta function, *viz.*,

$$\int_{-\infty}^{\infty} f(K_1, K_2, K_3) \, \delta(K_3) \, dK_3 = f(K_1, K_2, 0) \tag{E.17}$$

then gives

$$L_{kgij} = -\frac{1}{2\pi} \iint_{-\infty}^{\infty} dK_1 \, dK_2 \, M_{ij}^{-1}(\zeta_1, \zeta_2, \zeta_3 = 0) \left(\frac{K_k K_g}{K_1^2 + K_2^2} \right) e^{-i(K_1 x_1 + K_2 x_2)} a \frac{J_1\left(a\sqrt{K_1^2 + K_2^2}\right)}{\sqrt{K_1^2 + K_2^2}} \tag{E.18}$$

where the unit vector $\boldsymbol{\zeta}$ is now defined by the relations

$$\zeta_1 = \frac{K_1}{\sqrt{K_1^2 + K_2^2}}, \qquad \zeta_2 = \frac{K_2}{\sqrt{K_1^2 + K_2^2}}, \qquad \zeta_3 = 0 \tag{E.19}$$

If we put

$$\zeta_1 = \frac{K_1}{K} = \frac{K_1}{\sqrt{K_1^2 + K_2^2}} = \cos\theta, \qquad \zeta_2 = \frac{K_2}{K} = \frac{K_2}{\sqrt{K_1^2 + K_2^2}} = \sin\theta \tag{E.20}$$

and set $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, then in cylindrical coordinates,

$$L_{kgij} = -\frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} K \, dK \, d\theta \, M_{ij}(\zeta_1, \zeta_2) \, \zeta_k \zeta_g \, e^{-iKr\cos(\theta-\phi)} \frac{aJ_1(aK)}{K} \tag{E.21}$$

The integration with respect to θ extends over a complete circumference, so that

$$L_{kgij} = -\frac{1}{2\pi} \int_0^{2\pi} M_{ij}^{-1}(\zeta_1, \zeta_2) \, \zeta_k \zeta_g \, d\theta \int_0^\infty a e^{-iKr\cos\theta} J_1(aK) \, dK \tag{E.22}$$

Since S_{iplm} is real, the real part of the preceding integral involving the integration over K is

$$N = \int_0^\infty a\cos(Kr\cos\theta) J_1(aK) \, dK \tag{E.23}$$

Setting $z = r\cos\theta$, and noting that $\cos(Kz) = \cos(-Kz)$, we need be concerned only with positive values of z. Now if the field point $\mathbf{r}$ lies within the cylindrical volume, then $0 \le z \le a$. But, from Gradshteyn and Ryzhik [40],

$$N = \int_0^\infty a\cos(Kz) J_1(aK) \, dK = \frac{a\cos\left\{\sin^{-1}(z/a)\right\}}{\sqrt{a^2 - z^2}} \qquad \text{for } 0 \le z \le a \tag{E.24}$$

If $\psi = \sin^{-1}(z/a)$, then

$$N = \frac{a\cos\psi}{\sqrt{a^2 - z^2}} = \frac{\cos\psi}{\left[\dfrac{\sqrt{a^2 - z^2}}{a} \right]} = \frac{\cos\psi}{\cos\psi} = 1 \tag{E.25}$$

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Thus,

$$L_{kgij} = -\frac{1}{2\pi} \int_0^{2\pi} M_{ij}^{-1}(\zeta_1, \zeta_2) \zeta_k \zeta_g \, d\theta \qquad (E.26)$$

independent of position **r** in the cylinder as expected from Eshelby's result. In this integral we have $\zeta_1 = \cos\theta$, $\zeta_2 = \sin\theta$, $\zeta_3 = 0$, $M_{ij}^{-1}(\zeta_1,\zeta_2) = \left(\zeta_m \overline{D}_{mijn} \zeta_n\right)^{-1}$ and k and g are restricted to the values 1 and 2. The Eshelby tensor may now be written as

$$S_{iplm} = \frac{1}{4\pi} \overline{D}_{kjlm} \left\{ \int_0^{2\pi} M_{pj}^{-1}(\zeta_1,\zeta_2) \zeta_i \zeta_k \, d\theta + \int_0^{2\pi} M_{ij}^{-1}(\zeta_1,\zeta_2) \zeta_p \zeta_k \, d\theta \right\} \qquad (E.27)$$

When $\zeta_3 = 0$ the Christoffel stiffness tensor for a transversely isotropic material, M_{ij}, and its inverse, M_{ij}^{-1}, (which applies to the homogenized medium of a composite with fibers arranged in hexagonal arrays) have the component forms

$$M_{11} = \overline{D}_{1111}\zeta_1^2 + \tfrac{1}{2}\left(\overline{D}_{1111} - \overline{D}_{1122}\right)\zeta_2^2 \qquad (E.28)$$

$$M_{12} = M_{21} = \tfrac{1}{2}\left(\overline{D}_{1111} + \overline{D}_{1122}\right)\zeta_1\zeta_2 \qquad (E.29)$$

$$M_{13} = M_{31} = 0 \qquad (E.30)$$

$$M_{22} = \tfrac{1}{2}\left(\overline{D}_{1111} - \overline{D}_{1122}\right)\zeta_1^2 + \overline{D}_{1111}\zeta_2^2 \qquad (E.31)$$

$$M_{23} = M_{32} = 0 \qquad (E.32)$$

$$M_{33} = \overline{D}_{1313} \qquad (E.33)$$

and

$$M_{11}^{-1} = \frac{\tfrac{1}{2}\left(\overline{D}_{1111} - \overline{D}_{1122}\right)\zeta_1^2 + \overline{D}_{1111}\zeta_2^2}{\tfrac{1}{2}\overline{D}_{1111}\left(\overline{D}_{1111} - \overline{D}_{1122}\right)} \qquad (E.34)$$

$$M_{12}^{-1} = M_{21}^{-1} = -\frac{\overline{D}_{1111} + \overline{D}_{1122}}{\overline{D}_{1111}\left(\overline{D}_{1111} - \overline{D}_{1122}\right)} \qquad (E.35)$$

$$M_{13}^{-1} = M_{31}^{-1} = 0 \qquad (E.36)$$

$$M_{22}^{-1} = \frac{\overline{D}_{1111}\zeta_1^2 + \tfrac{1}{2}\left(\overline{D}_{1111} - \overline{D}_{1122}\right)\zeta_2^2}{\tfrac{1}{2}\overline{D}_{1111}\left(\overline{D}_{1111} - \overline{D}_{1122}\right)} \qquad (E.37)$$

$$M_{23}^{-1} = M_{32}^{-1} = 0 \qquad (E.38)$$

$$M_{33}^{-1} = \frac{1}{\overline{D}_{1313}} \qquad (E.39)$$

The Eshelby tensor can now be determined by integration in the form,

$$S_{1111} = \frac{5\overline{D}_{1111} + \overline{D}_{1122}}{8\overline{D}_{1111}} \qquad (E.40)$$

$$S_{2222} = S_{1111} \qquad (E.41)$$

$$S_{1122} = \frac{3\overline{D}_{1122} - \overline{D}_{1111}}{8\overline{D}_{1111}} \qquad (E.42)$$

$$S_{2233} = \frac{\overline{D}_{1133}}{2\overline{D}_{1111}} \qquad (E.43)$$

$$S_{1133} = S_{2233} \tag{E.44}$$

$$S_{2211} = S_{1122} \tag{E.45}$$

$$S_{1212} = S_{1221} = \frac{3\overline{D}_{1111} - \overline{D}_{1122}}{8\overline{D}_{1111}} \tag{E.46}$$

$$S_{2323} = S_{2332} = S_{1331} = S_{1313} = S_{1313} = \tfrac{1}{4} \tag{E.47}$$

The Eshelby tensor for tetragonal materials—which applies to the homogenized medium of a composite with a square array of fibers—is currently being worked out.

The results for an infinite isotropic cylinder may be recovered by taking

$$\overline{D}_{1111} = 2\mu(1 - \nu)/(1 - 2\nu), \quad \overline{D}_{1122} = 2\mu\nu/(1 - 2\nu), \quad \text{and} \quad \overline{D}_{1133} = 2\mu\nu/(1 - 2\nu) \tag{E.48}$$

where μ is the Lamé shear modulus and ν is Poisson's ratio. For an infinite isotropic cylinder the Eshelby tensor reduces to

$$S_{1111} = \frac{5 - 4\nu}{8(1 - \nu)} \tag{E.49}$$

$$S_{2222} = S_{1111} \tag{E.50}$$

$$S_{1122} = \frac{4\nu - 1}{8(1 - \nu)} \tag{E.51}$$

$$S_{2233} = \frac{\nu}{2(1 - \nu)} \tag{E.52}$$

$$S_{1133} = S_{2233} \tag{E.53}$$

$$S_{2211} = S_{1122} \tag{E.54}$$

$$S_{1212} = S_{1221} = \frac{3 - 4\nu}{8(1 - \nu)} \tag{E.55}$$

$$S_{2323} = S_{1313} = S_{1331} = S_{2332} = \tfrac{1}{4} \tag{E.56}$$

The Eshelby tensor for both isotropic and transversely isotropic materials can also be deduced from equations 17.27, 17.30 and 17.31 of Mura's book, [24], by setting $\varrho = 0$ in his notation.

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Appendix F

Proof that $U_{ijkl}(\mathbf{x} - \mathbf{y}) = U_{ijkl}(\mathbf{y} - \mathbf{x})$

From the definition of $U_{ijkl}(\mathbf{x} - \mathbf{y})$ we have

$$U_{ijkl}(\mathbf{x} - \mathbf{y}) = -\frac{1}{2}\left(\frac{\partial^2 G_{ik}(\mathbf{x} - \mathbf{y})}{\partial x_j \partial x_l} + \frac{\partial^2 G_{jk}(\mathbf{x} - \mathbf{y})}{\partial x_i \partial x_l}\right) \tag{F.1}$$

But $\dfrac{\partial G_{ik}(\mathbf{x} - \mathbf{y})}{\partial x_l} = -\dfrac{\partial G_{ik}(\mathbf{x} - \mathbf{y})}{\partial y_l}$, so that

$$\frac{\partial^2 G_{ik}(\mathbf{x} - \mathbf{y})}{\partial x_j \partial x_l} = \frac{\partial^2 G_{ik}(\mathbf{x} - \mathbf{y})}{\partial y_j \partial y_l} \tag{F.2}$$

The operator can therefore be written as

$$U_{ijkl}(\mathbf{x} - \mathbf{y}) = -\frac{1}{2}\left(\frac{\partial^2 G_{ik}(\mathbf{x} - \mathbf{y})}{\partial y_j \partial y_l} + \frac{\partial^2 G_{jk}(\mathbf{x} - \mathbf{y})}{\partial y_i \partial y_l}\right) \tag{F.3}$$

But $G_{ik}(\mathbf{x} - \mathbf{y}) = G_{ik}(\mathbf{y} - \mathbf{x})$, so that

$$U_{ijkl}(\mathbf{x} - \mathbf{y}) = -\frac{1}{2}\left(\frac{\partial^2 G_{ik}(\mathbf{y} - \mathbf{x})}{\partial y_j \partial y_l} + \frac{\partial^2 G_{jk}(\mathbf{y} - \mathbf{x})}{\partial y_i \partial y_l}\right) \tag{F.4}$$

or

$$U_{ijkl}(\mathbf{x} - \mathbf{y}) = U_{ijkl}(\mathbf{y} - \mathbf{x}) \tag{F.5}$$

as required.

Appendix G

Differentiation of Singular Integrals

In the text and Appendices we have taken derivatives of the volume integrals and written, for example,

$$
\begin{aligned}
I_{kq} &= \frac{\partial}{\partial x_q} \iiint\limits_V \frac{\partial G_{ik}\left(\mathbf{r} - \mathbf{r}'\right)}{\partial x_j} D^m_{ijrs} \Delta\varepsilon^\star_{rs}\left(\mathbf{r}'\right) \, dV(\mathbf{r}') \\
&= \iiint\limits_V \frac{\partial^2 G_{ik}\left(\mathbf{r} - \mathbf{r}'\right)}{\partial x_q \partial x_j} D^m_{ijrs} \Delta\varepsilon^\star_{rs}\left(\mathbf{r}'\right) \, dV(\mathbf{r}')
\end{aligned} \tag{G.1}
$$

If the integration volume V contains the field point $\mathbf{r}$ the integrand $\partial G_{ik}\left(\mathbf{r} - \mathbf{r}'\right)/\partial x_j$ is singular at the point $\mathbf{r}' = \mathbf{r}$, and the above operation in which the derivative is taken inside the integral must be treated with caution, as pointed out by Bui, [41] and Born and Wolf, [42]. We should, in fact, isolate a small spherical volume, D, about the singular point $\mathbf{r}' = \mathbf{r}$ and evaluate the integral according to Bui's procedure, *viz.*,

$$
\begin{aligned}
I_{kq} &= \iiint\limits_{V-D} \frac{\partial^2 G_{ik}\left(\mathbf{r} - \mathbf{r}'\right)}{\partial x_q \partial x_j} D^m_{ijrs} \Delta\varepsilon^\star_{rs}\left(\mathbf{r}'\right) \, dV(\mathbf{r}') + \\
&\quad + \frac{\partial}{\partial x_q} \iiint\limits_D \frac{\partial G_{ik}\left(\mathbf{r} - \mathbf{r}'\right)}{\partial x_j} D^m_{ijrs} \Delta\varepsilon^\star_{rs}\left(\mathbf{r}'\right) \, dV(\mathbf{r}') \\
&= \iiint\limits_{V-D} \frac{\partial^2 G_{ik}\left(\mathbf{r} - \mathbf{r}'\right)}{\partial x_q \partial x_j} D^m_{ijrs} \Delta\varepsilon^\star_{rs}\left(\mathbf{r}'\right) \, dV(\mathbf{r}') - \\
&\quad - \iiint\limits_D \frac{\partial}{\partial x'_q}\left(\frac{\partial G_{ik}\left(\mathbf{r} - \mathbf{r}'\right)}{\partial x_j}\right) \, dV(\mathbf{r}') \, D^m_{ijrs} \Delta\varepsilon^\star_{rs}(\mathbf{r})
\end{aligned} \tag{G.2}
$$

where we have used the fact that, if the spherical volume D about the point $\mathbf{r}$ is small enough, the strain increment can be considered constant and taken to have the value at the center of the sphere, $\Delta\varepsilon^\star_{rs}(\mathbf{r})$. The integral may therefore be written as

$$
\begin{aligned}
I_{kq} &= \iiint\limits_{V'-D} \frac{\partial^2 G_{ik}\left(\mathbf{r} - \mathbf{r}'\right)}{\partial x_q \partial x_j} D^m_{ijrs} \Delta\varepsilon^\star_{rs}\left(\mathbf{r}'\right) \, dV(\mathbf{r}') - \\
&\quad - \iint\limits_S n_q\left(\mathbf{r}'\right) \frac{\partial G_{ik}\left(\mathbf{r} - \mathbf{r}'\right)}{\partial x_j} \, dS(\mathbf{r}') \, D^m_{ijrs} \Delta\varepsilon^\star_{rs}(\mathbf{r})
\end{aligned} \tag{G.3}
$$

The first volume integral is evaluated in the principal value sense as $D \to 0$.

Rather than using the preceding operations outlined by Bui, we may treat $G_{ij}\left(\mathbf{r} - \mathbf{r}'\right)$ as a Fourier integral. The preceding operations are not then necessary and the derivative can be taken inside the integral. That is, equation G.1 is valid when the Fourier integral representation of the Green's function is used.

To demonstrate the validity of equation G.1, consider the singular integral used by Bui. He considers the derivative of the integral

$$F(x) = \int_{-1}^{1} \frac{dt}{t-x} = \log\left|\frac{1-x}{1+x}\right| \tag{G.4}$$

where $-1 < x < 1$. Since the integral is known, its derivative is simply found as

$$\frac{dF}{dx} = \frac{1}{x-1} - \frac{1}{x+1} \tag{G.5}$$

Notice that the integrand is singular at the point $t = x$. Bui demonstrates that in order to take the derivative of the integral we must write it in its principal value sense,

$$F(x) = \lim_{\epsilon \to 0} \left(\int_{t=-1}^{x-\epsilon} \frac{dt}{t-x} + \int_{t=x+\epsilon}^{1} \frac{dt}{t-x} \right) \tag{G.6}$$

and the derivative dF/dx must be evaluated by noting that both limits and the integrand are functions of x. Using Leibnitz's rule for differentiating an integral whose limits depend on x gives

$$\begin{aligned}
\frac{dF}{dx} &= \lim_{\epsilon \to 0} \left(\frac{d(x-\epsilon)}{dx} \frac{1}{x-\epsilon-x} + \int_{-1}^{x-\epsilon} \frac{d}{dx}\left(\frac{1}{t-x}\right) dt \right) + \\
&\quad + \lim_{\epsilon \to 0} \left(-\frac{d(x+\epsilon)}{dx} \frac{1}{x+\epsilon-x} + \int_{x+\epsilon}^{1} \frac{d}{dx}\left(\frac{1}{t-x}\right) dt \right) \\
&= \lim_{\epsilon \to 0} \left(\frac{1}{-\epsilon} - \frac{1}{\epsilon} + \int_{-1}^{x-\epsilon} \frac{dt}{(t-x)^2} + \int_{x+\epsilon}^{1} \frac{dt}{(t-x)^2} \right) \\
&= \lim_{\epsilon \to 0} \left(-\frac{2}{\epsilon} - \left[\frac{1}{t-x}\right]_{-1}^{x-\epsilon} - \left[\frac{1}{t-x}\right]_{x+\epsilon}^{1} \right) \tag{G.7}
\end{aligned}$$

or

$$\frac{dF}{dx} = \frac{1}{x-1} - \frac{1}{x+1} \tag{G.8}$$

To avoid the convected terms which arise from differentiating an integral whose limits depend on x, consider representing the integrand as a Fourier integral. We have, from Gradshteyn and Rhyzik [43], the Fourier integral representation,

$$\frac{1}{t-x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{2}} \, i \, \mathrm{sgn}(K) \, e^{-iK(t-x)} \, dK \tag{G.9}$$

The singular integral $F(x)$ can then be written as

$$\begin{aligned}
F(x) &= \int_{-1}^{1} dt \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{2}} \, i \, \mathrm{sgn}(K) \, e^{-iK(t-x)} \, dK \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{2}} \, i \, \mathrm{sgn}(K) \, e^{iKx} \, dK \int_{-1}^{1} e^{-iKt} \, dt \tag{G.10}
\end{aligned}$$

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If we now differentiate with respect to x in the normal manner we obtain

$$
\begin{aligned}
\frac{dF}{dx} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dK \sqrt{\frac{\pi}{2}}\, i\, \mathrm{sgn}(K)\, e^{iKx}\, iK \left[\frac{e^{-iKt}}{(-iK)} \right]_{t=-1}^{1} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dK \sqrt{\frac{\pi}{2}}\, i\, \mathrm{sgn}(K)\, e^{iKx} \left(e^{iK} - e^{-iK} \right) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dK \sqrt{\frac{\pi}{2}}\, i\, \mathrm{sgn}(K) \left(e^{iK(x+1)} - e^{-iK(1-x)} \right)
\end{aligned}
\tag{G.11}
$$

A comparison of this integral with equation G.9 shows that this Fourier integral has the inverse relation,

$$
\frac{dF}{dx} = \frac{1}{x-1} - \frac{1}{x+1}
\tag{G.12}
$$

which is the correct result.

Thus, by expanding the integrand of a singular integral as a Fourier integral, reversing the integrals, taking the normal derivative, and inverting the resulting Fourier integral, we obtain the correct derivative of the singular integral. It is then clear that if the Green's function is represented in Fourier integral form the procedure of Bui is not required. In fact, the Eshelby tensor in Appendix E is obtained by taking the derivative of the Fourier integral, and the correct result is obtained.

Appendix H

Origin of Self-Consistency

Many researchers in the mechanics literature suggest that the self-consistent method has its origins in the present century. It would appear that the method is, however, very old and has its origins in the last century. In the Lorentz-Lorenz theory ([42], page 87 and [45]) of 1880 the electric dipole moment **p** in a dielectric is related to the electric field **E**′ by the constitutive relation **p** = α**E**′, where α is the polarizability. The polarizability α is related to the refractive index n and the number of molecules per unit volume, N. If **E** is the mean or volume averaged field applied to the dielectric the actual field at any point is given by

$$\mathbf{E}' = \mathbf{E} + \frac{4\pi N}{3}\mathbf{p} \tag{H.1}$$

where $4\pi N\mathbf{p}/3$ denotes the perturbation or deviation from the average electric field. As shown on page 85 of reference [42] this value is estimated by smearing the effects of the molecules outside a spherical volume enclosing the point at which the field is observed. An analogous formula for statical fields had been derived even earlier by Clausius in 1879 and Mossotti in 1850.

Twersky [44] observes:

> In the biography of John William Strutt (third Baron Rayleigh) by his son Robert John (the fourth baron), the son quotes the father on the verse that faces the initial contents page of the first four of Lord Rayleigh's six volumes of *Scientific Papers*: "When I was bringing out my Scientific Papers I proposed a motto from the Psalms, 'The works of the Lord are Great, sought out of all them that have pleasure therein'. The Secretary to the Press suggested with many apologies that the reader might suppose that *I* was the Lord." The Secretary need not have been so apologetic. The second verse of Psalm 111 should have been augmented with the next three lines: "His work is honourable and glorious, and his righteousness endureth forever. He hath made his wonderful works to be remembered." Departing from King James' translation, we may read in the Hebrew of the last verse of this psalm the most important of all the Rayleigh principles of mathematical physics—that the wise beginning of work in this field is to assume that the problem had been considered by Rayleigh and to study his works: "The beginning of wisdom is reverence for the Lord; very good sense have all who do so."

Rayleigh [45] tackled the problem in his paper "On The Influence of Obstacles Arranged in Rectangular Order Upon the Properties of a Medium" and was probably the first person to define when the self-consistent method, *viz.* the Lorentz-Lorenz formula, could be expected to break down. At the end of his paper he states:

> The general conclusion as regards the optical application is that, even if we may neglect dispersion, we must not expect such formulæ as (the Lorentz-Lorenz equation) to be more than approximately correct in the case of dense fluid and solid bodies.

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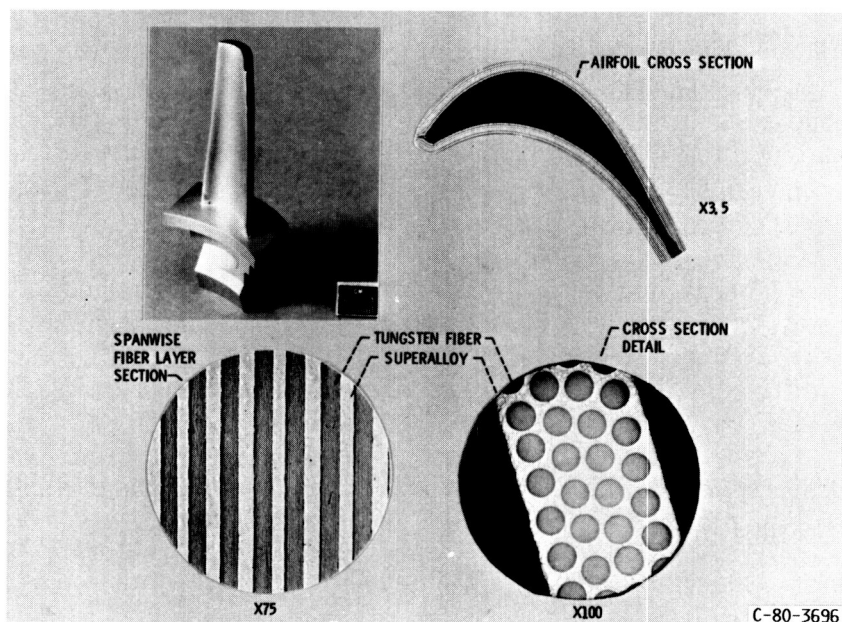
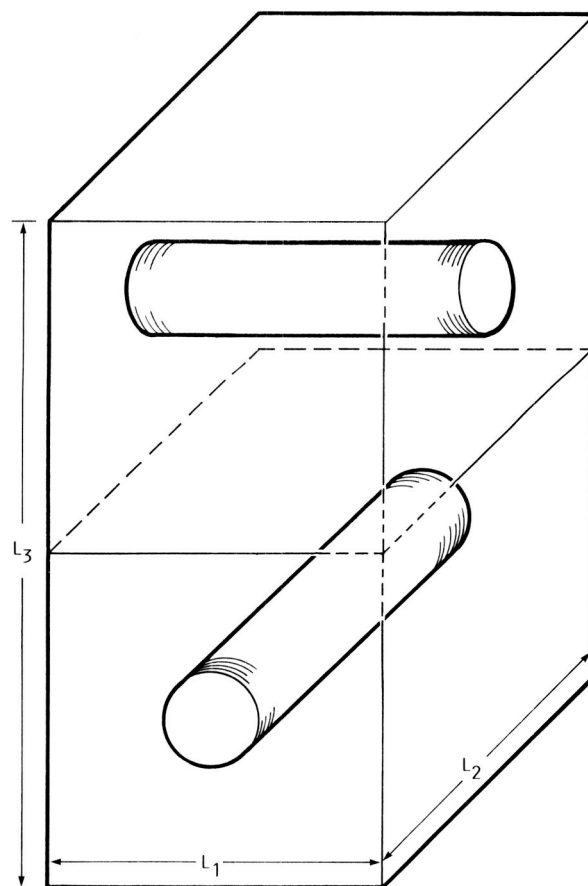FIGURE 1. - TURBINE BLADE WITH PERIODIC MICROSTRUCTURE.

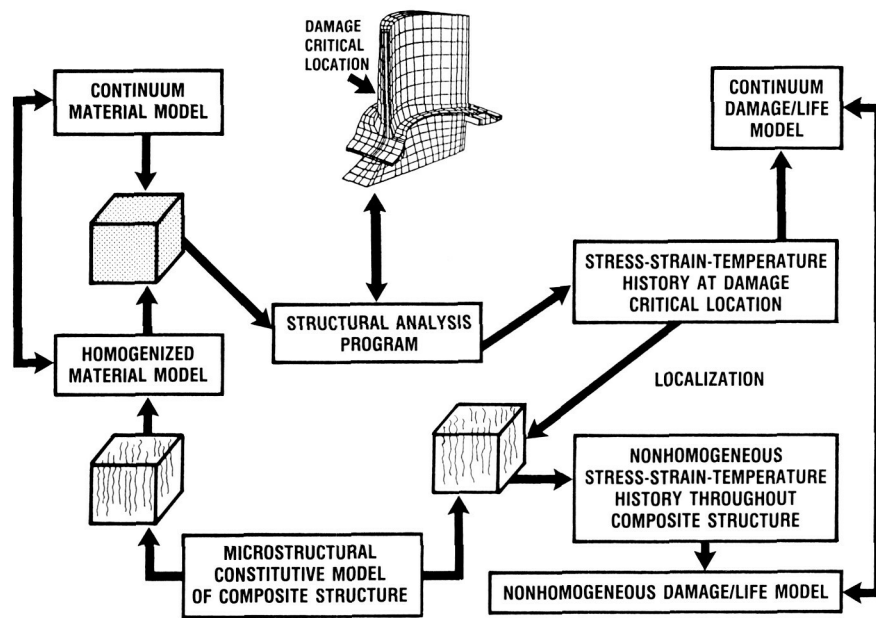FIGURE 2. - UNIT PERIODIC CELL FOR LAMINATED COMPOSITE.

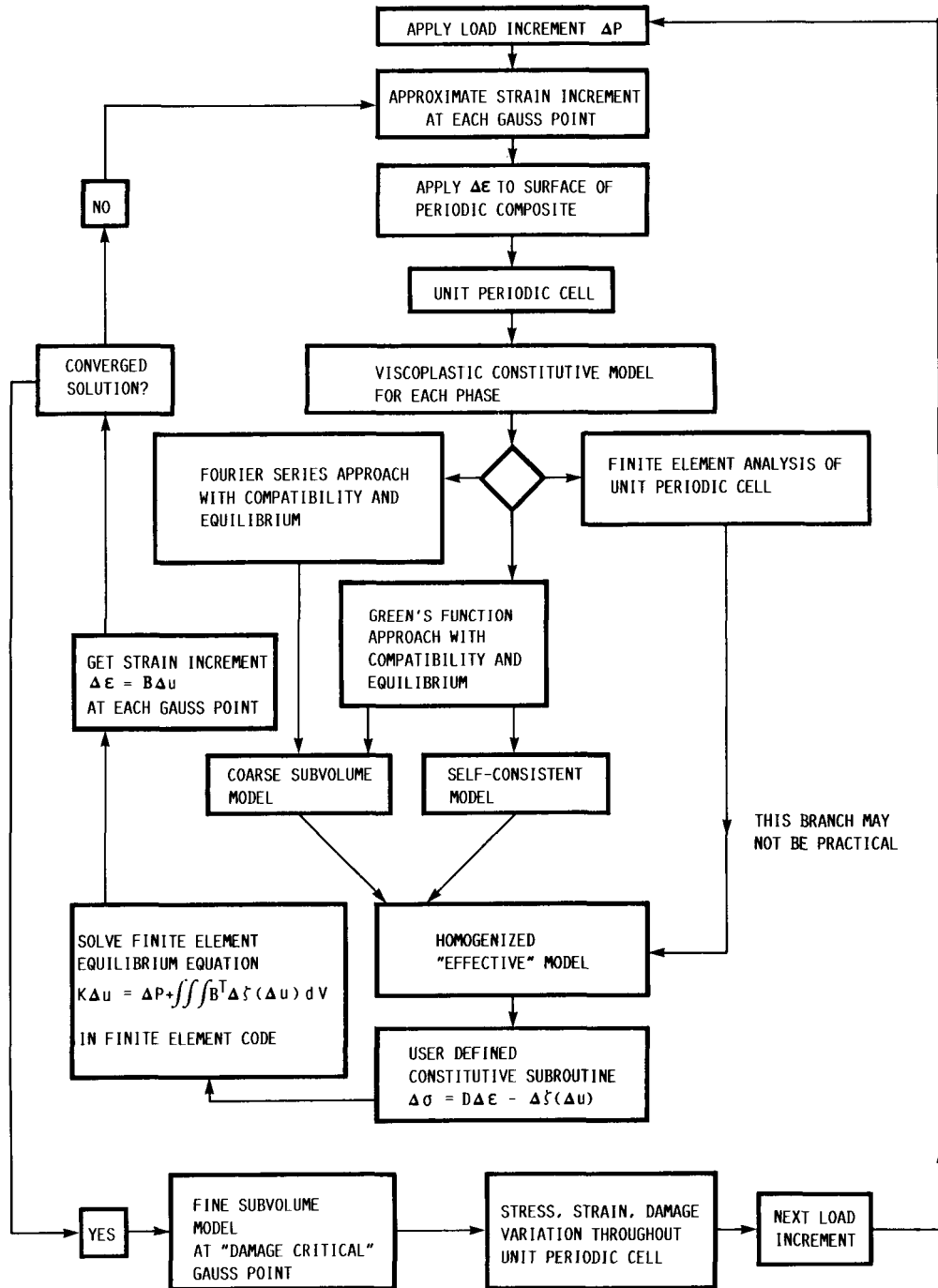FIGURE 3. - OVERVIEW OF PERIODIC ARRAY ANALYSIS.

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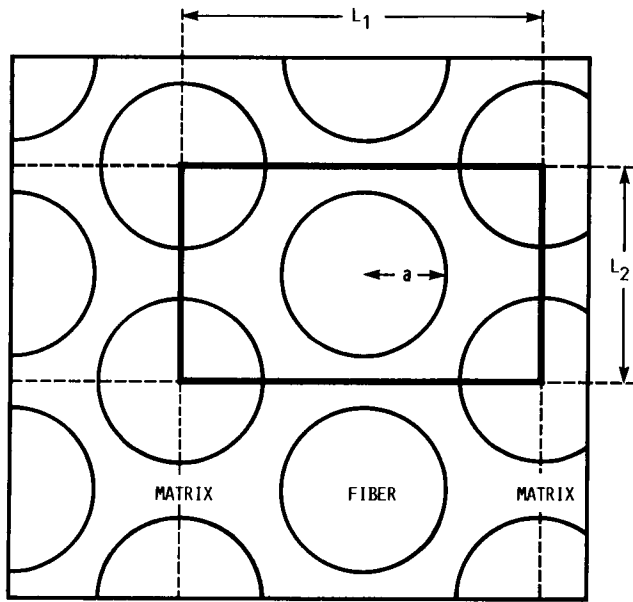FIGURE 4. - FLOWCHART OF FINITE ELEMENT IMPLEMENTATION.

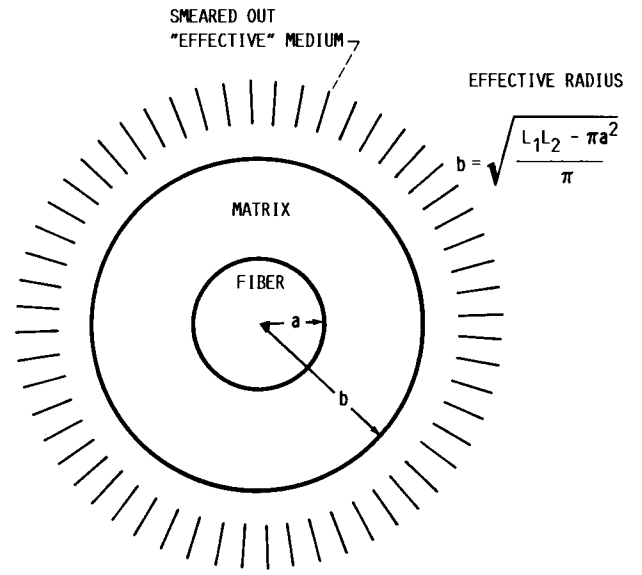FIGURE 5. – PERIODIC UNIT CELL IN HEXAGONAL FIBER ARRAY SURROUNDED BY NEAREST NEIGHBOR CELLS.


FIGURE 6. –. PERIODIC CELL IS REPLACED BY A SURROUNDING MATRIX OF "EFFECTIVE" RADIUS b AND THE NEIGHBOR CELLS ARE REPLACED BY A SMEARED OUT "EFFECTIVE" MEDIUM WHOSE AVERAGE CONSTITUTIVE PROPERTIES ARE THE VOLUME AVERAGE OF THE CONSTITUTIVE PROPERTIES IN THE FIBER AND MATRIX, WHEN CONSTRAINED BY THE "EFFECTIVE" MEDIUM.

THE FUNCTION $f(x) = \left(\dfrac{x - \dfrac{L}{2}}{\dfrac{L}{2}}\right)^3$ IS DEFINED OVER THE PERIODIC
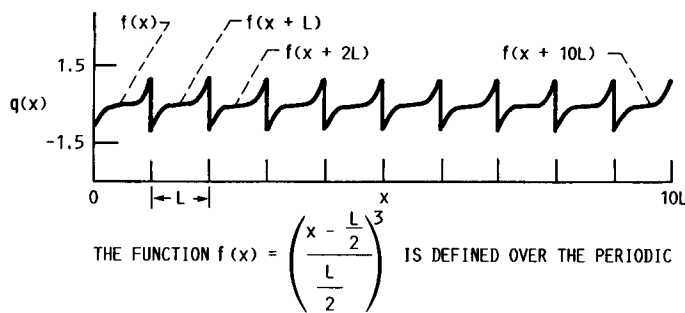
UNIT CELL OF LENGTH L, FROM x = 0 TO x = L. THE PERIODIC FUNCTION q(x) EXTENDING FROM $-\infty$ TO ∞ IS OBTAINED BY SUMMING THE FUNCTIONS ON THE UNIT CELLS IN THE FORM

$$q(x) = \sum_{n=-\infty}^{\infty} f(x + nL).$$

FIGURE 7. – REPRESENTATION OF A PERIODIC FUNCTION AS A SUM OVER UNIT CELLS.
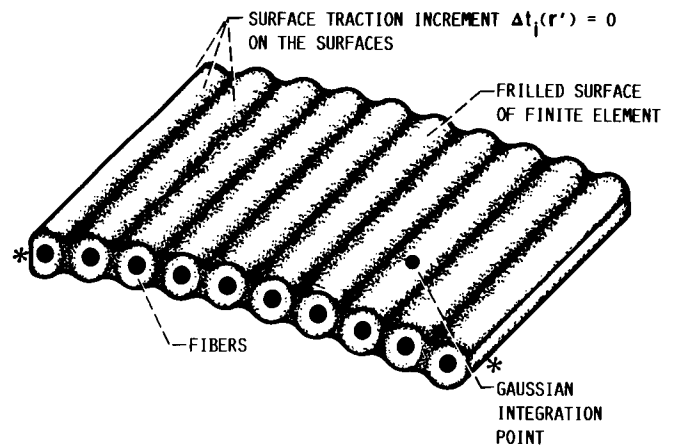
*THESE SURFACES ARE FAR ENOUGH AWAY FROM THE GAUSSIAN INTEGRATION POINTS TO IGNORE THE SURFACE TRACTION CONTRIBUTION TO THE TOTAL DISPLACEMENT INCREMENT $\Delta u_i^T(r)$

FIGURE 8. – REPRESENTATION OF A FINITE ELEMENT FOR THIN COMPOSITE SECTIONS WHEN SURFACE IS FREE OF APPLIED TRACTION.

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15. Supplementary Notes

16. Abstract

This work is concerned with modeling the mechanical deformation or constitutive behavior of composites comprised of a periodic microstructure under small displacement conditions at elevated temperature. A mesomechanics approach [1] is adopted which relates the micromechanical behavior of the heterogeneous composite with its in-service macroscopic behavior. Two different methods, one based on a Fourier series approach and the other on a Green's function approach, are used in modelling the micromechanical behavior of the composite material. Although the constitutive formulations are based on a micromechanical approach, it should be stressed that the resulting equations are volume averaged to produce overall "effective" constitutive relations which relate the bulk, volume averaged, stress increment to the bulk, volume averaged, strain increment. As such, they are macromodels which can be used directly in nonlinear finite element programs such as MARC, ANSYS and ABAQUS or in boundary element programs such as BEST3D. In developing the volume averaged or "effective" macromodels from the micromechanical models, both approaches (i.e. Fourier series and Green's function) will require the evaluation of volume integrals containing the spatially varying strain distributions throughout the composite material. By assuming that the strain distributions are spatially constant within each constituent phase—or within a given subvolume within each constituent phase—of the composite material, the volume integrals can be obtained in closed form. This simplified micromodel can then be volume averaged to obtain an "effective" macromodel suitable for use in the MARC, ANSYS and ABAQUS nonlinear finite element programs via user constitutive subroutines such as HYPELA and CMUSER. This "effective" macromodel can be used in a nonlinear finite element structural analysis to obtain the strain-temperature history at those points in the structure where thermomechanical cracking and damage are expected to occur, the so called "damage critical" points of the structure. The "exact" micromechanical models can then be subjected to the overall "effective" strain-temperature history obtained at the "damage critical" location and used outside of the finite element program to evaluate the heterogeneous stress-strain history throughout each constituent phase of the composite material. This variation must be known in order to evaluate the damage history variation throughout each constituent phase of the composite material.

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