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Spatially Random Models, Estimation Theory, and Robot Arm Dynamics

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ABSTRACT

Spatially random models provide an alternative to the more traditional deterministic models used to describe robot arm dynamics. These alternative models can be used to establish a relationship between the methodologies of estimation theory and robot dynamics. A new class of algorithms for many of the fundamental robotics problems of inverse and forward dynamics, inverse kinematics, etc. can be developed that use computations typical in estimation theory. The algorithms make extensive use of the difference equations of Kalman filtering and Bryson-Frazier smoothing to conduct spatial recursions. The spatially random models are very easy to describe and are based on the assumption that all of the inertial (D'Alembert) forces in the system are represented by a spatially distributed white-noise model. The models can also be used to generate numerically the composite multibody system inertia matrix. This is done without resorting to the more common methods of deterministic modeling involving Lagrangian dynamics, Newton-Euler equations, etc. These methods make substantial use of human knowledge in derivation and manipulation of equations of motion for complex mechanical systems. In contrast, with the spatially random models, more primitive (i.e., simpler and less dependent on mathematical derivations) locally specified computations result in the emergence of a global collective system behavior equivalent to that obtained with the deterministic models.

1. INTRODUCTION

Recently, an equivalence has been discovered between estimation theory and recursive robot arm dynamics [1], as summarized in the following table.

TABLE 1

Equivalence Between Optimal Estimation and Recursive Robot Arm Dynamics

ESTIMATION		ROBOT DYNAMICS
States	$x(k)$	Spatial Forces
Co-States	$\lambda(k)$	Spatial Accelerations
Measurements	$\tau(k)$	Joint Moments
Transition Matrix	$\phi(k,k-1)$	Spatial Jacobian
Process Error Covariance	$M(k)$	Spatial Inertia Matrix
Known Input	$b(k)$	Bias Spatial Force
State-to-Output Map	$H(k)$	State-to-Joint-Axis Map

A spatial force $x(k)$ is a 6-dimensional vector consisting of three pure moment components and three force components. The argument k refers to a representative body k in a multibody system. Similarly, $\lambda(k)$ is a 6-dimensional vector of three angular acceleration components and three linear acceleration components. The joint moments $\tau(k)$ are due to external sources acting at the joints. The spatial transition matrix serves

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to propagate spatial forces within a body [1] in an inward direction from joint $k-1$ to joint k. Its transpose serves to propagate spatial accelerations in an opposite direction. The 6-by-6 matrix $M(k)$ represents the spatial inertia of body k about joint k. The state-to-output map $H(k)$ is a 1-by-6 vector that projects the spatial force into its component along the joint axis. The bias force $b(k)$ is due to nonlinear velocity and gravity dependent effects [1].

The spatial inertia matrix and the transition matrix associated with this system are defined as

$$M(k) = \begin{pmatrix} I(k) & m(k)\tilde{p}(k) \\ -m(k)\tilde{p}(k) & U \end{pmatrix}$$

$$\phi(k,k-1) = \begin{pmatrix} U & \tilde{L}(k) \\ 0 & U \end{pmatrix}$$

in which $I(k)$ is the body k inertia about joint k; $m(k)$ is the body k mass; $L(k)$ is the vector from joint k to joint $k-1$; and $p(k)$ is the vector from joint k to the body k mass center. The symbol U denotes the 3-by-3 identity.

A spatially random state space model for the multibody system is

$$x^-(k) = \phi(k,k-1)x^+(k-1) + \omega(k) \tag{1.1}$$

$$x^+(k) = x^-(k) \tag{1.2}$$

in which $x^-(k)$ is the value of the spatial force on the negative side of joint k, and $x^+(k-1)$ is the value of the spatial force on the positive side of joint $k-1$. The "+" superscript indicates that the corresponding force is evaluated at a point immediately adjacent to joint k and toward the base of the multibody system. Similarly, the "−" superscript indicates that the corresponding variable is evaluated on the negative side of joint k. Note that $x^+(k-1)$ and $x^-(k)$ refer to spatial forces that are acting on body k due to the adjacent bodies $k-1$ and $k+1$ respectively. Equation (1.2) expresses continuity of the spatial force in crossing a joint connecting two adjacent bodies.

The above is a linear model that reflects a balance of the forces that are acting on body k. The inertial forces are represented by a spatial white-noise process whose mean and covariance are

$$E[\omega(k)] = b(k) \quad \text{and} \quad E[\hat{\omega}(k)\hat{\omega}(k)^T] = M(k) \tag{1.3}$$

with $\hat{\omega}(k) = \omega(k) - b(k)$. The mean value of the inertial force $\omega(k)$ is set equal to the bias force $b(k)$. The covariance of the inertial force is set equal to the spatial inertia matrix. The output, or measurement, equation

$$\tau(k) = H(k)x^+(k) \tag{1.4}$$

completes a description of the stochastic model. In this model, the active joint moment $\tau(k)$ plays the role of the measurement in a linear state space system. Since the joint moments are known exactly, the corresponding measurement equation is free of measurement noise.

The above model can be cast in the more compact notation

$$X = \phi W \quad \text{and} \quad T = HX \tag{1.5}$$

where W, X, and T are the composite vectors $W = [\omega(1),\ldots,\omega(N)]$, $X = [x(1),\ldots,x(N)]$ and $T = [\tau(1),\ldots,\tau(N)]$. Here, N represents the total number of bodies in the multibody system. The composite process error vector W has a mean and covariance given by

$$E(W) = b \quad \text{and} \quad E[\widehat{W}\widehat{W}^*] = Q \tag{1.6}$$

where $b = [b(1), \ldots, b(N)]$. The $6N$-by-$6N$ block diagonal matrix Q is defined as $Q = diag[M(1), \ldots, M(N)]$. Its typical 6-by-6 diagonal block $M(k)$ is the spatial inertia of body k. The matrix ϕ is a causal (i.e., lower triangular) matrix defined as

$$\phi = \begin{pmatrix} U & 0 & \ldots & 0 \\ \phi(2,1) & U & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \phi(N,1) & \phi(N,2) & \ldots & U \end{pmatrix} \tag{1.7}$$

The closely related composite state-to-output map H in (1.5) is defined as $H = diag[H(1), \ldots, H(N)]$.

This model is now used to investigate a number of relationships between estimation theory and robot arm dynamics.

2. CONDITIONAL MEAN ESTIMATION

The estimation problem to be solved here is to estimate the process error vector W and the state X, given the measurements T. This corresponds to the dynamics problem of finding the inertial forces (due to accelerations) and the spatial forces, given the joint moments. The optimal estimates are obtained by means of the conditional expectations $E(W/T)$ and $E(X/T)$. It is relatively simple to compute these two conditional expectations, although care has to be exercised due to the non-zero mean of the inertial force W. By methods outlined in [2], it can be established that

$$E(X/T) = \phi b + G(T - H\phi b) \tag{2.1}$$

in which G is the "Kalman" gain

$$G = RH^*(HRH^*)^{-1} \quad \text{and} \quad R = \phi Q \phi^* \tag{2.2}$$

This is the estimate of the spatial forces given the applied joint moments. Note that the estimator equations have a predictor-corrector architecture. The prediction term is due to the bias force b in (2.1). This term "predicts" the cumulative spatial bias force on any given body due to the bias force acting on all of the preceding bodies. The covariance of the estimation error inherent in this "open-loop" predicted estimate is

$$E[(X - \phi b)(X - \phi b)^*] = \phi Q \phi^* = R \tag{2.3}$$

The prediction term is said to be open-loop because it is based only on the system model and does not depend on the measurement T. The effect of measurements is accounted for in the correction term involving the Kalman gain in (2.1). The Kalman gain determines the weight of the correction term, when this is added to the prediction term, to arrive at the final state estimate $E(X/T)$. The N-by-N matrix HRH^* that needs to be inverted to compute the Kalman gain turns out to be the composite multibody system inertia matrix.

To compute the covariance of the estimation error after correction has occurred, observe first that

$$X - E(X/T) = (I - GH)\phi W \tag{2.4}$$

is the estimation error. Its corresponding covariance is

$$P = (I - GH)R(I - GH)^* \tag{2.5}$$

Alternatively, this becomes

$$P = (I - GH)R = R(I - GH)^* = R - RH^*(HRH^*)^{-1}HR \tag{2.6}$$

Note that $HP = 0, PH^* = 0, HPH^* = 0$ which imply that the estimation error at the joints vanishes. This reflects the lack of measurement noise in the measurement Equation (1.4).

The conditional-mean estimate for the inertial forces is given by

$$E(W/T) = b + Q\phi^* H^*(HRH^*)^{-1}(T - H\phi b) \tag{2.7}$$

This estimate is made up of two elements. First is the element due to the bias force b. Second is the element due to the active moments T. To examine these two effects more closely, define the joint angle accelerations as

$$a = M^{-1}(T - H\phi b) \quad \text{where} \quad M = HRH^* \tag{2.8}$$

Observe that the matrix M, whose inversion is required to compute the joint angle accelerations, is the composite multibody system inertia matrix. In addition, observe [1] that the joint angle accelerations a and the spatial accelerations λ are related by

$$\lambda = \phi^* H^* a \tag{2.9}$$

Based on these definitions, the estimate for the inertial forces becomes

$$E(W/T) = b + Q\lambda \tag{2.10}$$

The covariance of the inertial force estimation error is obtained by arguments very similar to those used to arrive at (2.4). Observe first that $\widetilde{W} = W - E(W/T) = [I - Q\phi^* H^*(HRH^*)^{-1}H\phi]W$ is the estimation error. Its covariance is

$$E[\widetilde{W}\widetilde{W}^*] = Q - Q\phi^* H^* M^{-1} H\phi Q \tag{2.11}$$

The foregoing are "batch" solutions to the estimation problem, in the sense that all of the measurements are processed simultaneously. This implies that the composite system inertia matrix must be inverted in a batch mode. An alternative is provided by the sequential solution outlined below.

3. SEQUENTIAL ESTIMATION

The sequential solution processes the measurements (the applied moments) one at a time. In doing this, it does not require numerical inversion of the N-by-N system inertia matrix. Instead, the inertia matrix is factored as

$$M = (I + K)D(I + K^*) \tag{3.1}$$

in which D is an N-by-N diagonal matrix, and K is a lower-triangular matrix. The matrices K and D in this factorization are generated using a suitably defined Kalman filter. This factorization of a covariance into a product of a causal factor, a diagonal matrix, and the anti-causal adjoint factor is strongly reminiscent of the celebrated [5] Gohberg-Krein factorization. Applications of this result to estimation problems have been investigated by Kailath [6]. Once this factorization of the system inertia matrix is achieved, the corresponding inverse can be computed easily. The central result is that

$$(I + K)^{-1} = I - L \tag{3.2}$$

where L is a lower- triangular causal matrix generated by the same Kalman filter that generates K. This implies that the inertia matrix inverse can be expressed as

$$M^{-1} = (I - L^*)D^{-1}(I - L) \tag{3.3}$$

The central aim of this section is to outline how to obtain this result. Only the major results are presented. The detailed arguments leading to the results will be presented elsewhere by the author.

Result 3.1. *The state covariance matrix $R = \phi Q \phi^*$ can be expressed as*

$$R = r + \Phi r + r\Phi^* \tag{3.4}$$

Here, Φ is the system model matrix obtained by subtracting the $6N$-by-$6N$ identity from the matrix in (1.7). The matrix r is a $6N$-by-$6N$ block diagonal matrix $r = diag[r(1), \ldots, r(N)]$ whose blocks $r(k)$ satisfy the recursive relationships

$$\begin{gathered} r^+(0) = 0 \\ r^-(k) = \phi(k, k-1) r^+(k-1) \phi^T(k, k-1) + M(k) \\ r^+(k) = r^-(k) \end{gathered} \tag{3.5}$$

Define now the block-diagonal matrix $P = diag[P(1), \ldots, P(N)]$ whose diagonal blocks $P(k)$ satisfy the discrete Riccati equation

$$\begin{gathered} P^-(k) = \phi(k, k-1) P^+(k-1) \phi^T(k, k-1) + M(k) \\ D(k) = H(k) P^-(k) H^T(k) \\ P^+(k) = P^-(k) - P^-(k) H^T(k) H(k) P^-(k) / D(k) \end{gathered} \tag{3.6}$$

with the "initial" condition $P^+(0) = 0$.

Result 3.2. *The matrices R and P above are related by*

$$R = P + \Phi P + P\Phi^* + \Phi PHD^{-1}HP\Phi^* \tag{3.7}$$

Result 3.3. *The system inertia matrix M factors as in (3.1), with the causal matrix K and the diagonal matrix D defined as*

$$K = H\Phi PH^*D^{-1} \quad \text{and} \quad D = HPH^* \tag{3.8}$$

Define now the transition matrix $\psi(k,m)$ by means of the Kalman filtering equations

$$\begin{aligned} \psi(k,k) &= I \\ \psi(k^-, k-1^+) &= \phi(k, k-1) \\ \psi(k^+, k^-) &= I - g(k)H(k) \end{aligned} \tag{3.9}$$

in which $g(k)$ is the Kalman gain

$$g(k) = P^-(k)H^T(k)D^{-1}(k) \tag{3.10}$$

Define also the related composite matrix

$$\Psi = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \psi(2,1) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \psi(N,1) & \psi(N,2) & \dots & 0 \end{pmatrix} \tag{3.11}$$

These two definitions can be used to establish the following identity.

Result 3.4. *The "open-loop" and "closed-loop" transition matrices Φ and Ψ are related by*

$$\Psi = (I - gH)\Phi \tag{3.12}$$

where $g = [g(1), \dots, g(N)]$ is the matrix of Kalman gains.

Result 3.5. *The lower triangular factor $I + K$ can be inverted as*

$$(I+K)^{-1} = I - L \tag{3.13}$$

in which L is the lower triangular matrix

$$L = H\Psi PH^*D^{-1} \tag{3.14}$$

This also implies that $K = L + KL$, $K = L + LK$, and $LK = KL$.

The above sequence of results is the necessary ingredient to establish the recursive factorization of the inverse of the composite system inertia matrix as in (3.3).

4. FILTERING AND SMOOTHING

Typically, the composite system inertia matrix is inverted to solve what is referred to as the forward dynamics problem. This problem consists of computing a set of joint angle accelerations given a corresponding set of applied joint moments. The joint angle accelerations a and the applied joint moments T are related by

$$a = (I - L^*)D^{-1}(I - L)T \tag{4.1}$$

where $a = [a(1), \dots, a(N)]$ is the vector of joint angle accelerations. This states that the joint moments must be processed by means of a two-stage computation. The first stage represents filtering and is characterized by the factor $(I - L)$.

The second stage represents smoothing and is characterized by the factor $(I - L^*)$.

Filtering. This stage produces an "innovations" process defined as

$$e^- = (I - L)T \tag{4.2}$$

It produces also the filtered state estimate

$$Z = \Psi P H^* D^{-1} T = \Psi g T \tag{4.3}$$

The components $z(k)$ of $Z = [z(1), \ldots, z(N)]$ satisfy the Kalman filter equations [1]

$$z^-(k) = \phi(k, k-1) z^+(k-1) + b(k) \tag{4.4}$$

$$z^+(k) = z^-(k) + g(k) e^-(k) \tag{4.5}$$

in which $e^-(k)$ are the elements of the innovations vector $e^- = [e^-(1), \ldots, e^-(N)]$. Multiplication of the innovations process by the inverse of the diagonal matrix D produces the residuals

$$e^+ = D^{-1} e^- \tag{4.6}$$

These residuals are processed in the smoothing stage that follows.

Smoothing. This corresponds to multiplication of the residuals by the anti-causal factor $(I - L^*)$ to obtain the joint angle accelerations, i.e.,

$$a = (I - L^*) e^+ \tag{4.7}$$

A spatial difference equation which is based on (4.7) can be obtained by re-introducing the co-state variables defined earlier. The co-state variables λ and the residuals e^+ are related by

$$\lambda = \Psi^* H^* e^+ \tag{4.8}$$

Use of this in (4.7) implies that

$$a = e^+ - g^* \lambda \tag{4.9}$$

This last relationship expresses the joint angle accelerations in terms of the residuals and the co-state variables. Furthermore, (4.8) can be used to infer that the co-state variables satisfy the difference equation

$$\lambda^+(k-1) = \phi^T(k, k-1) \lambda^-(k) \tag{4.10}$$

$$\lambda^-(k) = \lambda^+(k) + H^T(k) e^+(k) \tag{4.11}$$

with the terminal condition $\lambda^+(N) = 0$. These equations are referred to as the Bryson-Frazier smoother equations [4]. Their application to problems in robot dynamics is discussed in more detail in [1].

5. COVARIANCE ANALYSIS

The aim here is develop formulas to compute the covariance of several relevant quantities (state, state estimation error, innovations, etc.) discussed in previous sections. The stochastic model (1.5) is assumed as a starting point. As in earlier discussions, the results are stated without proof.

Result 5.1. *The composite system inertia matrix M is the covariance of the measurement process, i.e.,*

$$M = E(TT^*) = HRH^* \tag{5.1}$$

This result has an interesting interpretation. It states that the collective system behavior, as represented by the system inertia, emerges from the covariance of the output T of the spatially random model (1.5). It therefore provides a means to compute the inertia matrix numerically by direct simulation of the stochastic model. From such a simulation, the inertia matrix would emerge (without conducting the more traditional manual derivation of the equations of motion).

Result 5.2. *The spatial inertia matrix P produced by the Riccati equation is equal to the covariance of the state estimation error, i.e.,*

$$E[(X-Z)(X-Z)^*] = P + \Psi P + P\Psi^* \tag{5.2}$$

The corresponding mean-square estimation error is

$$E[(X-Z)^*(X-Z)] = Tr[P] \tag{5.3}$$

Result 5.3. *The innovations process has a covariance given by*

$$E[(e^-)(e^-)^*] = D \tag{5.4}$$

Result 5.4. *The covariance of the co-states is*

$$E(\lambda\lambda^*) = \Psi^* H^* D^{-1} H \Psi = \Lambda + \Lambda\Psi + \Psi^*\Lambda \tag{5.5}$$

in which $\Lambda = diag[\Lambda(1), \ldots, \Lambda(N)]$. The diagonal blocks $\Lambda(k)$ satisfy

$$\Lambda^-(k) = [I - g(k)H(k)]^T \Lambda^+(k)[I - g(k)H(k)] + H^T(k)H(k)/D(k) \tag{5.6}$$

$$\Lambda^+(k-1) = \phi^T(k,k-1)\Lambda^-(k)\phi(k,k-1) \tag{5.7}$$

with the terminal condition $\Lambda^+(N) = 0$.

6. CLOSED-FORM INERTIA MATRIX INVERSE

The foregoing results can be used to obtain in closed form the inverse of the composite multibody system inertia. This is done in terms of the covariance matrices P and Λ of the previous section.

Result 6.1. *The inverse of the system inertia matrix can be expressed as*

$$M^{-1} = D^{-1} + g^*\Lambda g + g^*\Psi^*(\Lambda g - H^* D^{-1}) + (g^*\Lambda - D^{-1}H)\Psi g \tag{6.1}$$

Alternatively, it can be expressed as $M^{-1} = D^{-1}HSH^*D^{-1}$ where

$$S = P + P\Lambda P + P\Psi^*(\Lambda P - I) + (P\Lambda - I)\Psi P \tag{6.2}$$

This result is quite similar to that obtained in [1] by more detailed methods. The result has an interesting potential application in robot dynamics analysis and in control design. The equations of motion for the multibody system representing a robot arm are typically written, neglecting bias forces and accelerations due to nonlinear velocity and gravity dependent effects, in the form

$$Ma = T \tag{6.3}$$

where a is the set of joint angle accelerations, and T is the set of applied joint moments. The primary reason for the widespread use of such an equation is that many of the known methods for deriving equations of motion result in a matrix equation of this form. The equation consistently involves the presence of a composite system inertia matrix. There is, however, nothing intrinsic in the multibody dynamics problem that would make the presence of an inertia matrix in the equations of motion completely inevitable. In fact, Result 6.1 shows how to compute the inverse of the inertia matrix directly, without having to evaluate the inertia matrix first and then having to invert it. It is therefore possible, by using this result, to arrive directly, without numerical inversion of the inertia matrix, at a set of motion equations of the form

$$a = M^{-1}T \tag{6.4}$$

This is potentially a very useful result, since the system in (6.4) is much easier to work with, in simulation and control design, for instance, than the equivalent system in (6.3).

7. CONCLUDING REMARKS

The use of spatially distributed random models has been explored in analyzing robot arm dynamics. Based on such models a previously undiscovered relationship has been established between estimation theory and recursive robot arm dynamics. Many of the fundamental problems in robot dynamics can be approached using the techniques of estimation theory. The interaction between these two areas has not been recognized before and leads to many useful insights, such as the equivalence of covariance and spatial inertia. The numerical properties of the new algorithms emerging from the estimation approach to robot dynamics are under investigation.

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