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Solutions of Differential Equations with Regular Coefficients by The Methods of Richmond and Runge-Kutta

C. R. Cockrell

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Langley Research Center
Hampton, Virginia 23665-5225

SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH REGULAR COEFFICIENTS BY THE METHODS OF RICHMOND AND RUNGE-KUTTA

ABSTRACT

Numerical solutions of the differential equation which describes the electric field within an inhomogeneous layer of permittivity, upon which a perpendicularly-polarized plane wave is incident, are considered. Richmond's method and the Runge-Kutta method are compared for linear and exponential profiles of permittivities. These two approximate solutions are also compared with the exact solutions.

INTRODUCTION

The treatment of perpendicularly-polarized plane wave incidence on inhomogeneous dielectric or plasma layers has been investigated by many authors [1-3]. Their efforts have been directed toward solving the differential equations which describe the electromagnetic behavior of fields within these layers. Solutions for continuous inhomogeneous layers require, in general, numerical techniques. Exact solutions for linear and exponential profiles are available [4].

Many methods have been used to solve the differential equations describing the fields within an inhomogeneous layer. Dividing the layer into a number of smaller segments and approximating the inhomogeneous profile by constant permittivity values within each segment, the multilayer boundary value problem is then solved. Approximate integral solutions are also available as are WKB asymptotic solutions where the inhomogeneous profile (or permittivity) gradient is small [5,4]. For these solutions the computation time is too long or the model is too crude to yield accurate results.

Practical solutions have been obtained using the Runge-Kutta method [6] and a step-by-step numerical method described by Richmond [7]. Both solutions have been programmed on a personal computer and the results compared. A comparison with exact solutions, linear and exponential, has also been made.

The primary purpose of this paper is to explain and demonstrate the Richmond and Runge-Kutta methods for solving a differential equation in which its parameters may or may not be known analytically. For definiteness, therefore, the differential equation which represents the electric field within a plane inhomogeneous layer of dielectric or plasma, upon which a perpendicularly-polarized plane wave has been assumed incident, has been considered. An efficient solution of this differential equation for this ideal model is needed since the same differential equation exists for realistic problems such as aperture antennas which are bounded by inhomogeneous dielectric or plasma layers.

INHOMOGENEOUS PLANE LAYER-PERPENDICULAR POLARIZATION

A plane wave is incident on a plane inhomogeneous layer of dielectric or plasma as shown in figure 1. The incident electric field which is polarized in the x-direction propagates in free space (region I) at an angle θ with the normal to the layer. The transmitted electric field which is also x-polarized exists into free space (region III) at an angle θ. The partial differential equation for the electric field in the free space regions and in the inhomogeneous layer is written as

$$\frac{\partial^2 E_x(y,z)}{\partial y^2} + \frac{\partial^2 E_x(y,z)}{\partial z^2} + k^2\ \varepsilon_r(z)\ E_x(y,z) = 0 \qquad (1)$$

where k is the free space wavenumber and $\varepsilon_r(z)$ is the relative dielectric constant of the medium. Since the relative dielectric constant in free space (regions I and III) equals one, the solutions of equation (1) in these regions are readily found:

$$E^I(y,z) = E_i\left[\ e^{jkz\cos\theta} + R\ e^{-jkz\cos\theta}\right]\ e^{jky\sin\theta} \qquad (2)$$

$$E_x^{III}(y,z) = E_t e^{jkz\cos\theta}\ e^{jky\sin\theta} \qquad (3)$$

where E_i is the intensity of the incident electric field, R is the reflection coefficient at the interface, and E_t is the transmitted electric field intensity. However, in region II, since $\varepsilon_r(z)$ can be an arbitrary function of z, the solution of equation (1) must be determined numerically, except for some specialized profiles such as linear and exponential.

Before attempting the solution of equation (1) in region II, it is reduced to an ordinary differential equation. This is readily accomplished by the method of separation of variables and the continuity of tangential fields throughout the media. The continuity requirement forces the y-variation in the fields to be the same as given in equations (2) and (3). The differential equation describing the z-variation of the electric field is written as

$$\frac{d^2 E_x^{II}(z)}{dz^2} + k^2(\ \varepsilon_r(z)\ -\ \sin^2\theta\ )E_x^{II}(z) = 0 \qquad (4)$$

Richmond's Method

One method of solving equation (4) for region II is reported in a paper by Richmond [7]. In his method a step-by-step numerical procedure is used to find the electric field within the layer at successive points. To begin the solution, the field at z=0 and z=h, which is a small distance within the layer, must be determined first. The field at z=0 is assumed to be unity and the field at z=h is found from the Maclaurin series expansion at z=0; i.e.,

$$E_x^{II}(h) = E_x(0) + \left.\frac{dE_x(z)}{dz}\right|_{z=0} h + \frac{1}{2}\left.\frac{d^2E_x(z)}{dz^2}\right|_{z=0} h^2 + \frac{1}{3\,!}\left.\frac{d^3E_x(z)}{dz^3}\right|_{z=0} h^3 + \cdot\cdot \quad (5)$$

where

$$\left.\begin{aligned} E_x(0) &= 1 \\ \left.\frac{dE_x(z)}{dz}\right|_{z=0} &= jk\cos\theta \\ \left.\frac{d^2E_x(z)}{dz^2}\right|_{z=0} &= -k^2(\varepsilon_c - \sin^2\theta) \\ \left.\frac{d^3E_x(z)}{dz^3}\right|_{z=0} &= -k^2\left[jk(\varepsilon_c - \sin^2\theta)\cos\theta + \frac{d\varepsilon_c}{dz}\right] \end{aligned}\right\} \quad (6)$$

and ε_c is the complex relative permittivity at the point $z=0_+$ and $\frac{d\varepsilon_c}{dz}$ is the derivative of the permittivity with respect to z at the point $z=0_+$. Substituting equation (6) into equation (5) yields

$$E_x^{II}(h) = 1 - \frac{k^2h^2}{2}(\varepsilon_+ - \sin^2\theta) - \frac{1}{6}k^2h^3\varepsilon_+'$$
$$+ j\left[kh\cos\theta + \frac{k^2h^2}{2}\varepsilon_+\tan\delta_+ - \frac{k^3h^3}{6}(\varepsilon_+ - \sin^2\theta)\cos\theta\right] \quad (7)$$

where ε_+ is the real relative permittivity, ε_+' is the derivative of the real relative permittivity with respect to z, and $\tan\delta_+$ is the loss tangent. The subscript + denotes evaluation at $z=0_+$.

The Taylor series expansion of the electric field about the point z=nh is written as

$$E_x^{II}(z) = E_x^{II}(nh) + \left.\frac{dE_x^{II}(z)}{dz}\right|_{z=nh}(z - nh) + \frac{1}{2}\left.\frac{d^2E_x^{II}(z)}{dz^2}\right|_{z=nh}(z - nh)^2 + \cdots \quad (8)$$

Evaluating equation (8) at z=nh+h and z=nh-h and adding the result, thus giving

$$E_x^{II}(nh+h) \cong 2E_x^{II}(nh) + h^2 \left.\frac{d^2E_x^{II}(z)}{dz^2}\right|_{z=nh} + \frac{h^4}{12}\left.\frac{d^4E_x^{II}(z)}{dz^4}\right|_{z=nh} - E_x^{II}(nh-h) \quad (9)$$

where

$$\left.\frac{d^2E_x^{II}(z)}{dz^2}\right|_{z=nh} = k^2(\varepsilon_r(z) - \sin^2\theta)\, E_x^{II}(z)\Big|_{z=nh} \quad (10)$$

$$\left.\frac{d^4E_x^{II}(z)}{dz^4}\right|_{z=nh} = k^4(\varepsilon_r(z)-\sin^2\theta)E_x^{II}(z) - 2k\,\varepsilon_r'(z)\, E_x^{II\prime}(z) - k^2\varepsilon_r''(z)\, E_x^{II}(z)\Big|_{z=nh} \quad (11)$$

with

$$E_x^{II\prime}(nh) = \frac{E_x^{II}(nh) - E_x^{II}(nh-h)}{h} \quad (12)$$

Employing the notation of Richmond, equation (9), with equation (10) substituted, the electric field in the layer becomes

$$E_{n+1} \cong \left[\, 2 - k^2h^2(\varepsilon_n - \sin^2\theta) \,\right] E_n - E_{n-1} + h^2 \frac{E_n^{IV}}{12} \quad (13)$$

where E_n^{IV} denotes equation (11). The dielectric or plasma layer is divided into N segment so that Nh equals D, the thickness of the layer. The criteria for choosing the appropriate step size h is discussed in reference 7. The solution of equation (13) is determined in a step-by-step procedure as n varies from unity to N; i.e., for n=1 the field at z=2h is computed from equation (13) since the fields at z=0 and z=h are given by equations (6) and (7) and the fourth derivative of the field at z=h is given by equations (10)-(12). By proceeding through the layer in this manner, the field at the region I and region II interface is determined. The field at this interface then enables one to determine the reflection coefficient.

Runge-Kutta Method

The equations needed for solving second-order differential equations of the form

$$\frac{d^2y}{dz^2} = f(z,y) \tag{14}$$

by the Runge-Kutta method are given in reference 6. Equation (4) is written in this form as

$$\frac{d^2E_x^{II}(z)}{dz^2} = -\,k^2(\,\varepsilon_r(z) - \sin^2\theta\,)E_x^{II}(z) \tag{15}$$

Using the notation adapted earlier, the electric field at a general point z=nh where h is the spacing within the layer is represented as E_n and at z=nh+h as E_{n+1}. From the fourth-order Runge-Kutta method, the solution takes the form

$$E_{n+1} = E_n + h\left[E_n' + \frac{1}{6}\left(k_1 + 2k_2\right)\right] \tag{16}$$

$$E_{n+1}' = E_n' + \frac{1}{6}k_1 + \frac{2}{3}k_2 + \frac{1}{6}k_3 \tag{17}$$

where the primes denote the derivative with respect to z and

$$\left.\begin{aligned} k_1 &= h\,f\left(nh,\ E_n\right) \\ k_2 &= h\,f\left(nh + \frac{1}{2}h,\ E_n + \frac{h}{2}E_n' + \frac{h}{8}k_1\right) \\ k_3 &= h\,f\left(nh + h,\ E_n + h\,E_n' + \frac{h}{2}k_2\right) \end{aligned}\right\} \tag{18}$$

with

$$
\left.
\begin{aligned}
&f\left(nh,\ E_n\right) = -\,k^2\left(\ \varepsilon(nh)\ -\ \sin^2\theta\ \right)E_n \\
&f\left(nh+\ \tfrac{1}{2}\ h,\ E_n+\ \tfrac{h}{2}\ E'+\ \tfrac{h}{8}\ k_1\right) = \\
&\qquad\qquad -\,k^2\left[\varepsilon_r(nh+\ \tfrac{h}{2})\ -\ \sin^2\theta\right]\left[\ E_n+\ \tfrac{h}{2}\ E_n'+\ \tfrac{h}{8}\ k_1\right] \\
&f\left(nh+\ h,\ E_n+\ h\ E_n'+\ \tfrac{h}{2}\ k_2\right) = \\
&\qquad\qquad -\,k^2\left[\varepsilon_r(nh+\ h)\ -\ \sin^2\theta\right]\left[E_n+\ h\ E_n'+\ \tfrac{h}{2}\ k_2\right]
\end{aligned}
\right\}
\qquad (19)
$$

In this procedure, n varies from zero to N-1. The initial values E_0 and E_0' are given by the first two equations in equation (6).

DISCUSSION OF RESULTS

Linear and exponential profiles were chosen to demonstrate the Richmond and Runge-Kutta methods for solving the differential equations. A comparison of these methods is made. These two profiles were selected since they also have exact solutions as shown in the Appendix.

The particular linear profile chosen is shown in figure 2. The relative permittivity ε_r begins at a value of minus one (the exit point) and rises to a value of two at the front interface of the inhomogeneous layer. The real and imaginary parts of the electric fields at front interface for the Richmond (equation (13)) and the Runge-Kutta (equation (16)) methods are shown in Table I for 5° increments in the angle of incidence. Also shown are the complex values of the electric fields for the exact solutions (Airy-equation (A-8)). The approximate computational time for each value in the Richmond method was 8 seconds compared to 12 seconds for the Runge-Kutta method. Both methods compare very favorably with the exact solutions as can be readily seen in Table I.

TABLE I

ELECTRIC FIELD DISTRIBUTION AT INTERFACE OF INHOMOGENEOUS LAYER-LINEAR PROFILE

m= 3k and b=-1

	Richmond		Runge-Kutta		Exact-Airy	
Angle	Real	Imag	Real	Imag	Real	Imag
0	0.9918	0.9178	0.9919	0.9179	0.9919	0.9179
5	0.9954	0.9155	0.9955	0.9156	0.9955	0.9156
10	1.0063	0.9086	1.0065	0.9086	1.0065	0.9086
15	1.0243	0.8969	1.0244	0.8969	1.0244	0.8969
20	1.0488	0.8800	1.0489	0.8800	1.0489	0.8800
25	1.0794	0.8577	1.0794	0.8577	1.0794	0.8577
30	1.1150	0.8296	1.1152	0.8296	1.1152	0.8296
35	1.1550	0.7953	1.1552	0.7953	1.1552	0.7953
40	1.1983	0.7543	1.1985	0.7543	1.1985	0.7543
45	1.2435	0.7065	1.2437	0.7065	1.2437	0.7065
50	1.2894	0.6515	1.2895	0.6516	1.2895	0.6516
55	1.3345	0.5895	1.3336	0.5895	1.3346	0.5895
60	1.3775	0.5206	1.3775	0.5206	1.3775	0.5206
65	1.4166	0.4452	1.4167	0.4452	1.4167	0.4452
70	1.4508	0.3640	1.4509	0.3640	1.4509	0.3640
75	1.4788	0.2777	1.4789	0.2777	1.4789	0.2777
80	1.4995	0.1874	1.4997	0.1874	1.4997	0.1874
85	1.5124	0.0944	1.5125	0.0944	1.5125	0.0944
90	1.5167	0.0000	1.5168	0.0000	1.5168	0.0000

The exponential profile selected for comparison is shown in figure 3. The relative permittivity ε_r begins at a value of one and rises to a value of e at the front interface of the inhomogeneous layer. The values of the real and imaginary parts of the electric fields at the front interface for the Richmond and Runge-Kutta methods are shown in Table II for 5° increments in the angle of incidence. Also shown are the complex values of the electric fields for two angles of incidence (0° and 30°) for the exact solution (Bessel-equation (A-17)); these two angles were selected because, for the parameters used in these computations, they yield integer order Bessel functions which are easily calculable. The approximate computational time for each value in the Richmond method was 9 seconds compared to 12 seconds for the Runge-Kutta method. Both methods compare very favorably with each other as well as to the two exact solutions as can be readily seen in Table II.

TABLE II

ELECTRIC FIELD DISTRIBUTION AT INTERFACE OF INHOMOGENEOUS LAYER-EXPONENTIAL PROFILE

$\beta = k$

	Richmond		Runge-Kutta		Exact-Bessel	
Angle	Real	Imag	Real	Imag	Real	Imag
0	0.3736	0.7411	0.3738	0.7411	0.3738	0.7411
5	0.3766	0.7393	0.3767	0.7393		
10	0.3851	0.7340	0.3853	0.7340		
15	0.3993	0.7249	0.3994	0.7249		
20	0.4187	0.7119	0.4188	0.7119		
25	0.4427	0.6945	0.4430	0.6945		
30	0.4711	0.6725	0.4713	0.6725	0.4722	0.6722
35	0.5028	0.6455	0.5030	0.6455		
40	0.5372	0.6130	0.5373	0.6130		
45	0.5731	0.5748	0.5733	0.5749		
50	0.6096	0.5308	0.6098	0.5308		
55	0.6456	0.4809	0.6457	0.4809		
60	0.6798	0.4251	0.6800	0.4252		
65	0.7112	0.3639	0.7113	0.3640		
70	0.7385	0.2976	0.7387	0.2978		
75	0.7610	0.2273	0.7611	0.2273		
80	0.7776	0.1535	0.7777	0.1535		
85	0.7878	0.0774	0.7880	0.0774		
90	0.7912	0.0000	0.7914	0.0000		

Reflection coefficients at the front interface of the inhomogeneous layers are also of interest. The equations required for computing the reflection coefficients are given in reference 7. Magnitude and phase of the reflection coefficients for the linear and exponential profiles discussed in this paper are shown in Tables III and IV.

TABLE III

REFLECTION COEFFICIENTS AT INTERFACE OF INHOMOGENEOUS LAYER-LINEAR PROFILE

m= 3k and b=-1

	Richmond		Runge-Kutta	
Angle	Magnitude	Phase in radians	Magnitude	Phase in radians
0	0.2925	-.0548	0.2929	-.0825
5	0.2932	-.0480	0.2936	-.0757
10	0.2954	-.0277	0.2957	-.0553
15	0.2991	0.0030	0.2992	-.0211
20	0.3044	0.0542	0.3043	0.0268
25	0.3116	0.1159	0.3112	0.0889
30	0.3208	0.1919	0.3201	0.1655
35	0.3324	0.2828	0.3314	0.2571
40	0.3473	0.3897	0.3457	0.3644
45	0.3659	0.5127	0.3638	0.4885
50	0.3896	0.6540	0.3870	0.6308
55	0.4202	0.8154	0.4169	0.7934
60	0.4604	1.0007	0.4564	0.9797
65	0.5137	1.2139	0.5092	1.1946
70	0.5862	1.4642	0.5811	1.4463
75	0.6840	1.7644	0.6788	1.7485
80	0.8090	2.1351	0.8048	2.1221
85	0.9384	2.5981	0.9367	2.5901
90	1.0000	3.1416	1.0000	3.1416

TABLE IV

REFLECTION COEFFICIENTS AT INTERFACE OF INHOMOGENEOUS LAYER-EXPONENTIAL PROFILE

$\beta = k$

Angle	Richmond Magnitude	Richmond Phase in radians	Runge-Kutta Magnitude	Runge-Kutta Phase in radians
0	0.2788	-2.4876	0.2825	-2.4801
5	0.2800	-2.4863	0.2837	-2.4791
10	0.2836	-2.4833	0.2874	-2.4762
15	0.2898	-2.4780	0.2935	-2.4714
20	0.2986	-2.4710	0.3024	-2.4651
25	0.3105	-2.4631	0.3144	-2.4575
30	0.3258	-2.4540	0.3297	-2.4494
35	0.3449	-2.4450	0.3489	-2.4414
40	0.3686	-2.4372	0.3727	-2.4345
45	0.3979	-2.4320	0.4021	-2.4302
50	0.4340	-2.4311	0.4382	-2.4303
55	0.4784	-2.4373	0.4828	-2.4374
60	0.5334	-2.4542	0.5376	-2.4553
65	0.6009	-2.4868	0.6051	-2.4889
70	0.6831	-2.5427	0.6969	-2.5455
75	0.7791	-2.6306	0.7823	-2.6338
80	0.8808	-2.7601	0.8828	-2.7631
85	0.9656	-2.9348	0.9663	-2.9367
90	1.0000	-3.1416	1.0000	-3.1416

CONCLUDING REMARKS

The numerical techniques of Richmond and Runge-Kutta were shown to be quite efficient for solving the second-order differential equation which describes the field within a plane inhomogeneous layer of permittivity, upon which a perpendicularly-polarized plane wave has been assumed incident. Richmond's method seems to be little more efficient than the Runge-Kutta method, at least for the two relative permittivity profiles considered.

Although the techniques of Richmond and Runge-Kutta were applied to analytical profiles, they should be equally valid for arbitrary relative permittivity profiles or even profiles which are known discretely. This is important because realistic relative permittivity profiles may take on almost any shape.

APPENDIX

Exact solutions of equation (4) are available for only a few relative permittivity profiles [3,4]. A brief derivation of the solutions of equation (4) for linear and exponential profiles is given here.

For the linear profile, the relative permittivity is assumed to vary as

$$\varepsilon_r(z) = mz + b \tag{A-1}$$

where m is the slope of the profile and b is the beginning value of the profile (at z=0). Substituting equation (A-1) into equation (4) yields

$$\frac{d^2E_x(z)}{dz^2} + k^2(mz + b - \sin^2\theta) E_x(z) = 0 \tag{A-2}$$

By making the substitution

$$z = Mv + B \tag{A-3}$$

equation (A-2) becomes

$$\frac{d^2E_x(v)}{dv^2} + \left\{ M^2k^2\left[m M v + m B + b - \sin^2\theta \right]\right\} E_x(v) = 0 \tag{A-4}$$

By equating the term in braces to -v and solving for M and B, it is found that

$$\frac{d^2E_x(v)}{dv^2} - v E_x(v) = 0 \tag{A-5}$$

where

$$\left.\begin{aligned} M &= - \left(\frac{1}{mk^2} \right)^{\frac{1}{3}} \\ B &= \frac{\sin^2\theta - b}{m} \end{aligned}\right\} \tag{A-6}$$

From equations (A-3) and (A-6), it follows that

$$v = -\left(mz + b - \sin^2\theta \right)\left(\frac{k}{m} \right)^{\frac{2}{3}} \tag{A-7}$$

The solutions of equation (A-5) are the two independent Airy functions, Ai(v) and Bi(v) [4]. The general solution of equation (A-5), therefore is written as

$$E_x(v) = C_1 Ai(v) + C_2 Bi(v) \tag{A-8}$$

where C_1 and C_2 are constants. These constants are determined by applying the appropriate boundary conditions at z=0. The boundary conditions are given by the first two equations in equation (6) but which must be expressed in terms of the new independent v. The evaluation of the constants is given in matrix form as

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{D}\begin{pmatrix} Bi'(v_o) & -Bi(v_o) \\ -Ai'(v_o) & Ai(v_o) \end{pmatrix}\begin{pmatrix} 1 \\ -j\left(\frac{k}{m} \right)^{\frac{1}{3}}\cos\theta \end{pmatrix} \tag{A-9}$$

where

$$\left.\begin{aligned} D &= Ai(v_o)Bi'(v_o) - Ai'(v_o)Bi(v_o) \\ v_o &= -\left(\frac{k}{m} \right)^{\frac{2}{3}}\left(b - \sin^2\theta \right) \end{aligned}\right\} \tag{A-10}$$

and the primes denote the derivative with respect to v. For the solutions in this paper m=3k and b=-1. Once the C's are determined, the electric field at any point in the layer is computed by using equation (8).

For the exponential profile, the relative permittivity is assumed to vary as

$$\varepsilon_r(z) = e^{\beta z} \tag{A-11}$$

Substituting equation (A-11) into equation (4) yields

$$\frac{d^2E_x(z)}{dz^2} + k^2\left(e^{\beta z} - \sin^2\theta \right)E_x(z) = 0 \tag{A-12}$$

Following the substitution given in reference 4,

$$v = \frac{2k}{\beta} e^{\frac{\beta z}{2}} \qquad \text{(A-13)}$$

the differential equation is now written as

$$v^2 \frac{d^2E_x(v)}{dv^2} + v \frac{dE_x(v)}{dv} + (v^2 - p^2) E_x(v) = 0 \qquad \text{(A-14)}$$

where

$$p^2 = \frac{4k^2}{\beta^2} \sin^2\theta \qquad \text{(A-15)}$$

If p is zero or a positive integer, the general solution of equation (A-14) becomes

$$E_x(v) = B_1J_p(v) + B_2Y_p(v) \qquad \text{(A-16)}$$

where $J_p(v)$ is the Bessel function of the first kind, $Y_p(v)$ is the Bessel function of the second kind, and the B's are constants. Upon applying the appropriate boundary conditions at z=0, the matrix for the B's becomes

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \frac{1}{\mathbb{D}} \begin{pmatrix} Y_p'(v_o) & -Y_p(v_o) \\ -J_p'(v_o) & J_p(v_o) \end{pmatrix} \begin{pmatrix} 1 \\ j\cos\theta \end{pmatrix} \qquad \text{(A-17)}$$

where

$$\left.\begin{aligned} \mathbb{D} &= J_p(v_o)Y_p'(v_o) - J_p'(v_o)Y_p(v_o) \\ v_o &= \frac{2k}{\beta} \end{aligned}\right\} \qquad \text{(A-18)}$$

and the primes denote the derivative with respect to v. For the solutions in this paper, β=k with kD=1. Once the B's are determined, the electric field at any point in the layer is computed by using equation (16). If p is not zero or a positive integer, the general solution of equation (A-14) becomes

$$E_x(v) = D_1J_p(v) + D_2J_{-p}(v) \qquad \text{(A-19)}$$

where the constants D_1 and D_2 are determined in the manner just described.

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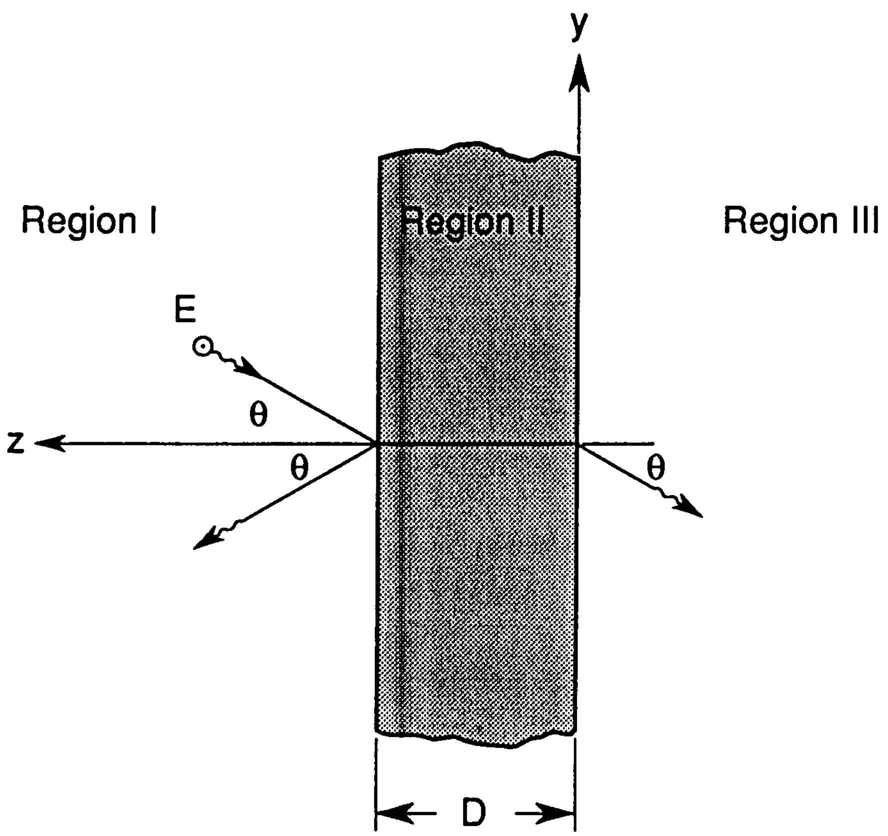

Figure 1.-Perpendicularly-polarized plane wave incident on an inhomogeneous plasma layer.

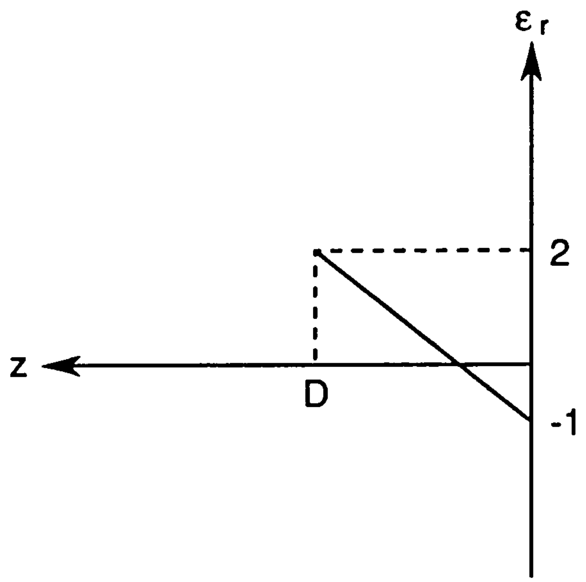

Figure 2.-Linear profile.

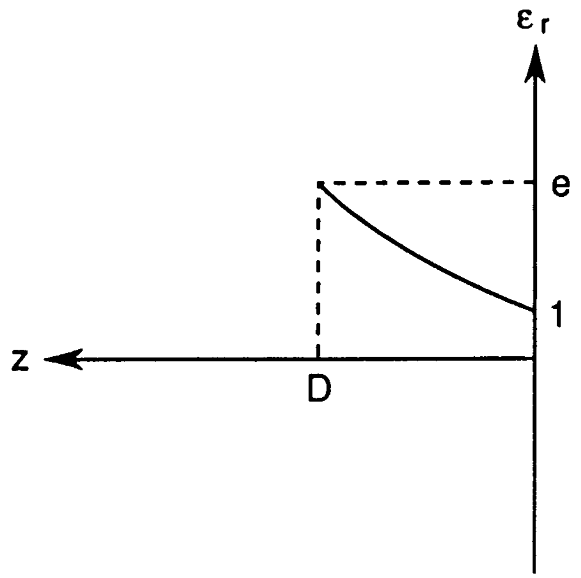

Figure 3.-Exponential profile.

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16. Abstract

Numerical solutions of the differential equation which describe the electric field within an inhomogeneous layer of permittivity, upon which a perpendicularly-polarized plane wave is incident, are considered. Richmond's method and the Runge-Kutta method are compared for linear and exponential profiles of permittivities. These two approximate solutions are also compared with the exact solutions.

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