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# **The Upper-Branch Stability of Compressible Boundary Layer Flows**

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## **ABSTRACT**

The upper-branch linear and nonlinear stability of compressible boundary layer flows is studied using the approach of Smith & Bodonyi (1982) for a similar incompressible problem. Both pressure gradient boundary layers and Blasius flow are considered with and without heat transfer. and the neutral eigenrelations incorporating compressibility effects are obtained explicitly. The compressible nonlinear viscous critical layer equations are derived and solved numerically and the results indicate some solutions with positive phase shift across the critical layer. Various limiting cases are investigated including the case of much larger disturbance amplitudes and this indicates the structure for the strongly nonlinear critical layer of the Benney-Bergeron (1969) type. Finally we show also how a match with the inviscid neutral inflexional modes arising from the generalized inflexion point criterion, is achieved.

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## §1 Introduction.

The motivation for the present work arises from the need to consider the stability of compressible boundary layer flows but within the modern high Reynolds number asymptotic framework. Many of the earlier classical approaches to the stability of incompressible boundary layer flows (see for example Reid (1965)) have now been largely superseded by the more self consistent asymptotic methods based on triple-deck or multi-deck ideas. Smith (1979a,b) was the first to apply such techniques to stability problems and he showed how the triple-deck structure governs the lower branch stability of boundary layer flows. This then allowed for weakly non-linear effects, strongly nonlinear effects, effects of the non-parallelism of the basic flow to be assessed systematically. Subsequent papers have applied the same basic concepts but with more complicated structures to consider the upper-branch stability of incompressible boundary layer flows, see Smith & Bodonyi (1980), Smith & Bodonyi (1982), Bodonyi, Smith & Gajjar (1983); the lower- and upper-branch stability of three-dimensional boundary layer flows, see Stewart & Smith (1987), Bassom & Gajjar (1988).

As far as the stability of compressible boundary layer flows is concerned there have been several contributions by Lees & Lin (1946), Lees & Reshotko (1962), Reshotko (1962), albeit on classical lines. Pate & Schueler (1969) and Kendall (1975) present the relatively few experimental studies of compressible boundary layer transition. Most of this and subsequent work is extensively reviewed by Reshotko (1976), and Mack (1984,1986). However an important distinction between compressible and incompressible boundary layer flows is the existence of unstable linear inviscid modes satisfying the generalized inflexion point criterion, see Lees & Lin (1946), or Mack (1984). This would suggest that many of the linear results obtained by Lees & Lin are still valid.

Our main objective in this paper is to extend the work of Smith & Bodonyi (1982) who considered the upper-branch stability of incompressible boundary layer flows, to compressible flows. We address upper-branch stability rather than lower-branch stability because the disturbance structure for upper-branch modes is more closely related to that for the inviscid modes than is the triple-deck structure. The importance of the upper-branch and inviscid modes is highlighted by the fact that these modes have shorter scales and higher

growth rates than the lower-branch modes. For a discussion of some aspects of the lower-branch linear instability properties of compressible boundary layers see the work by Smith (1987). Malik (1987) presents some recent computational results for hypersonic boundary layers.

There have been very few studies of nonlinear effects in the stability of compressible boundary layer flows and in this paper we address at least some aspects of the nonlinear theory relevant for modes on the upper-branch. We show that the neutral modes are now governed by the properties of the compressible critical layer equation. The present work provides the foundation for studying nonlinear inviscid modes of either the generalized inflexion point type or the Benney-Bergeron (1969) type. In fact the scalings and structures for studying the latter can be deduced from the limiting cases that we consider.

Starting with the Smith & Bodonyi (1982) paper, hereafter referred to SB, the most obvious strategy for studying the effects due to compressibility is to gradually increase the Mach number until significant changes from the incompressible theory occurs. For general pressure gradient boundary layers we find that the first major contributions due to compressibility occur when the Mach number becomes  $O(1)$ , for flows with and without heat transfer. In the outermost potential flow region the controlling equation is now the Prandtl-Glauert equation. Also if there is heat transfer then the density and temperature disturbances become singular in the vicinity of the critical layer. This then leads to an additional contribution to the disturbances shearing stress which balances the usual contribution at the wall and gives rise to the dispersion relations. The total contribution to the disturbance shearing stress at the critical layer is now proportional to the quantity  $D(\rho_B DU_B)$  occurring in the generalized inflexion point criterion. Here  $D = \partial/\partial Y$ , where  $Y$  is the boundary layer coordinate, and  $U_B, \rho_B$  are the basic streamwise velocity and density respectively. For linear neutral modes to exist we require that the flow must be subsonic and that  $D(\rho_B DU_B)$  must be negative. Expressions for the neutral frequencies and wavenumbers are obtained explicitly for both the linear and nonlinear theories. In the nonlinear theory these neutral modes depend on the amplitude of the disturbance indirectly through the phase shift across the critical layer, as in SB, but now the phase shift has to be determined by solving a generalization of the Haberman (1972) equation

heat transfer. The nonlinear theory for pressure gradient boundary layers is considered in section 3 where we also derive the compressible critical layer equation. The properties of this equation are studied in section 4 and finally we conclude with some brief additional comments in section 5.

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## §2 Basic Equations and Linear Theory.

The Navier-Stokes equations are nondimensionalized as follows with the suffix  $\infty$  denoting local free stream values and an asterisk denoting dimensional quantities. We introduce non-dimensional cartesian coordinates  $(x_*, y_*) = L(x, y)$ , corresponding velocity components  $(u_*, v_*) = U_\infty(u, v)$ , pressure  $p_* = \rho_\infty U_\infty^2 p$ , time  $t_* = (L/U_\infty)t$ , density  $\rho_* = \rho_\infty \rho$ , temperature  $T_* = T_\infty T$ , coefficient of viscosity  $\mu_* = \mu_\infty \mu$ , coefficient of bulk viscosity  $\mu'_* = \mu'_\infty \mu'$ . This gives the non dimensional equations;

(continuity:)

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0, \quad (2.1a)$$

(momentum equations:)

$$\rho \left[ \frac{\partial u_\beta}{\partial t} + u_\alpha \frac{\partial u_\beta}{\partial x_\alpha} \right] = -\frac{\partial p}{\partial x_\beta} + \frac{1}{Re} \left[ \frac{\partial}{\partial x_\beta} (\mu' - \frac{2}{3}\mu) \nabla \cdot u + \frac{\partial}{\partial x_\alpha} (\mu e_{\alpha\beta}) \right], \quad (2.1b)$$

(equation of state:)

$$\gamma M_\infty^2 p = \rho T, \quad (2.1c)$$

(energy equation:)

$$\begin{aligned} \rho \left[ \frac{\partial p}{\partial t} + u_\alpha \frac{\partial p}{\partial x_\alpha} \right] &= \gamma p \left[ \frac{\partial \rho}{\partial t} + u_\alpha \frac{\partial \rho}{\partial x_\alpha} \right] \\ + \rho \frac{(\gamma - 1)}{Re} \left[ \frac{1}{2} \mu e_{\alpha\beta} e_{\alpha\beta} + (\mu' - \frac{2}{3}\mu) \left( \frac{\partial u_\alpha}{\partial x_\alpha} \right)^2 \right] &+ \frac{\rho}{Re M_\infty^2} \frac{\partial}{\partial x_\alpha} \left( \frac{\mu}{\sigma_p} \frac{\partial T}{\partial x_\alpha} \right), \end{aligned} \quad (2.1d)$$

where

$$e_{\alpha\beta} = \frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}.$$

The nondimensional constants appearing in (2.1) are  $Re = U_\infty L / \nu_\infty$  the Reynolds number,  $\sigma_p$  the Prandtl number,  $\gamma$  the ratio of specific heats,  $M_\infty = U_\infty / q$  the local Mach Number with  $q$  being the speed of sound. The characteristic lengthscale  $L$  can be taken to be the local station, say  $x_0$ . We also assume that the Chapman viscosity law  $\mu = CT$ , where  $C$  is Chapman's constant, holds. The Reynolds number  $Re$  is taken to be large throughout and we set the Prandtl number to be equal to unity.

## §2.1 Pressure gradient boundary layer flows with heat transfer and insulated wall conditions.

The basic boundary layer flow is given by, with  $y = Re^{-\frac{1}{2}}Y$ ,

$$\begin{aligned} u &= U_B(x, Y), & v &= Re^{-\frac{1}{2}}V_B(x, Y), & \rho &= \rho_B(x, Y), \\ T &= T_B(x, Y) = 1/\rho_B, & p &= p_B = 1/\gamma M_\infty^2. \end{aligned} \quad (2.2)$$

For general pressure gradient boundary layers the following properties of the basic flow also hold,

$$\begin{aligned} U_B, \rho_B, T_B &\rightarrow 1 \quad \text{as } Y \rightarrow \infty, \\ U_B &\sim \lambda_1 Y + \lambda_2 Y^2 + \dots \quad \text{as } Y \rightarrow 0+, \\ \rho_B &\sim R_0 + R_1 Y + \dots \quad \text{as } Y \rightarrow 0+, \\ T_B &\sim S_0 + S_1 Y + \dots \quad \text{as } Y \rightarrow 0+, \end{aligned} \quad (2.3)$$

where the coefficients  $\lambda_1, \lambda_2$ , etc. depend on  $x_0$  and the Mach number. For insulated wall conditions  $R_1 = S_1 = 0$ , and if there is heat transfer then  $S_1 \neq 0$ . The curvature term  $\lambda_2$  depends on the pressure gradient and heat transfer. For further details of the basic flow see Stewartson (1964).

We next introduce infinitesimal disturbances of size  $\delta$  say, ( $\delta \ll 1$ ), and consider the stability of the basic flow. Since our interest is centered on the upper-branch stability of compressible boundary layer flows, the most obvious starting point is the work of Smith & Bodonyi (1982). By gradually increasing the Mach number, we find that the first significant changes due to compressibility occur when  $M_\infty^2$  becomes  $O(1)$ . This is true also for Blasius flow with heat transfer (with zero pressure gradient but  $R_1 \neq 0$ ), although for Blasius flow with insulated wall conditions ( $R_1 = \lambda_2 = 0$ ) the significant changes due to compressibility arise much earlier. This special case is treated separately later in this section.

So taking the Mach number to be  $O(1)$ , it is appropriate to consider the scalings and disturbance structure for infinitesimal perturbations of the basic flow. The basic structure is as set out in SB, see also Fig. 1, and this continues to hold for compressible flows provided we consider wavenumbers of  $O(Re^{\frac{5}{12}})$  and frequencies of  $O(Re^{\frac{1}{3}})$ , relevant to the upper-branch. There are five distinct zones Z1-Z5; the outer potential flow region Z1 of thickness  $O(Re^{-\frac{5}{12}})$ , the main part of the boundary layer Z2, a thinner inviscid region Z3 of thickness  $O(Re^{-\frac{7}{12}})$  containing the critical layer; zone Z4 the critical layer of thickness  $O(Re^{-\frac{23}{36}})$ , and finally zone Z5 the wall layer of thickness  $O(Re^{-\frac{8}{12}})$ . As in SB we consider disturbances proportional to  $E = \exp[i(\tilde{\alpha}X - \tilde{\sigma}\tau)]$ , so that

$$(u, v, p, \rho, T) = (u_B, v_B, p_B, \rho_B, T_B) + \delta[(\tilde{u}, \tilde{v}, \tilde{p}, \tilde{\rho}, \tilde{\theta})E + c.c] + \dots, \quad (2.4)$$

where c.c denotes the complex conjugate and  $x = \epsilon^5 X, t = \epsilon^4 \tau$ , and with the wavenumber  $\tilde{\alpha}$  and frequency  $\tilde{\sigma}$  expanded as

$$\tilde{\alpha} = \alpha_0 + \epsilon\alpha_1 + \dots, \quad \tilde{\sigma} = \sigma_0 + \epsilon\sigma_1 + \dots \quad (2.5)$$

Here  $\epsilon = Re^{-\frac{1}{12}}$  is a small parameter. For neutral stability  $\alpha_0, \sigma_0$  are taken to be real.

The major differences here from the work of SB arise because of the additional contributions due to the density and temperature. The disturbances now have the expansions in the various regions as:

$$(\tilde{u}, \tilde{v}, \tilde{p}, \tilde{\rho}, \tilde{\theta}) = \begin{cases} [\epsilon\bar{u}_0 + \dots, \epsilon\bar{v}_0 + \dots, \epsilon\bar{p}_0 + \dots, \epsilon\bar{\rho}_0 + \dots, \epsilon\bar{\theta}_0 + \dots] \\ \text{in Z1 where } y = \epsilon^5 \bar{y}, \\ [u_0 + \epsilon u_1 + \dots, \epsilon v_0 + \epsilon^2 v_1 + \dots, \epsilon p_0 + \epsilon^2 p_1 + \dots, \\ \rho_0 + \epsilon \rho_1 + \dots, \theta_0 + \epsilon \theta_1 + \dots] \\ \text{in Z2 where } y = \epsilon^6 Y, \\ [u^{(0)} + \epsilon u^{(1)} + \dots, \epsilon^2 v^{(0)} + \epsilon^3 v^{(1)} + \dots, \epsilon p^{(0)} + \epsilon^2 p^{(1)} + \dots, \\ \rho^{(0)} + \epsilon \rho^{(1)} + \dots, \theta^{(0)} + \epsilon \theta^{(1)} + \dots] \\ \text{in Z3 where } y = \epsilon^7 \bar{Y}, \\ [\hat{u}^{(0)} + \dots, \epsilon^3 \hat{v}^{(0)} + \dots, \epsilon \hat{p}^{(0)} + \dots, \epsilon \hat{\rho}^{(0)} + \dots, \epsilon \hat{\theta}^{(0)} + \dots] \\ \text{in Z5 where } y = \epsilon^8 \hat{y}, \end{cases} \quad (2.6)$$

together with appropriate expansions for the basic flow using (2.2),(2.3).

In the outermost potential flow region the leading order equation is now the Prandtl-Glauert equation,

$$\frac{\partial^2 \bar{p}_0}{\partial \bar{y}^2} - \alpha_0^2 (1 - M_\infty^2) \bar{p}_0 = 0.$$

Anticipating the requirement that  $M_\infty^2 < 1$  for neutral solutions and applying a boundedness condition at  $\bar{y} = \infty$  gives

$$\begin{aligned} \bar{p}_0 &= P_0 e^{-\alpha_0 \omega \bar{y}}, & \bar{v}_0 &= -i P_0 \omega e^{-\alpha_0 \omega \bar{y}}, & \bar{u}_0 &= -\bar{p}_0, \\ \gamma \bar{\rho}_0 &= \bar{p}_0, & \bar{\theta}_0 &= \gamma M_\infty^2 \bar{p}_0 - \bar{\rho}_0, \end{aligned} \quad (2.7)$$

with  $\omega = (1 - M_\infty^2)^{\frac{1}{2}}$ . Also  $P_0$  is an unknown constant.

Solutions in the other regions follow in a straightforward fashion and the important ones are given by

$$\begin{aligned} u_0 &= A_0 \frac{\partial U_B}{\partial Y}, & v_0 &= -i A_0 \alpha_0 U_B, & \rho_0 &= A_0 \frac{\partial \rho_B}{\partial Y}, \\ \theta_0 &= -\rho_0 \frac{T_B}{\rho_B}, & p_0 &= P_0, & u^{(0)} &= A_0 \lambda_1, \\ v^{(0)} &= -\frac{i \alpha_0 P_0}{R_0 \lambda_1} - i \alpha_0 A_0 (\lambda_1 \bar{Y} - c_0), & \rho^{(0)} &= -\frac{R_1 v^{(0)}}{i \alpha_0 (\lambda_1 \bar{Y} - c_0)}, \\ \theta^{(0)} &= -\frac{S_0 \rho^{(0)}}{R_0}, & p^{(0)} &= P_0, \\ v^{(1)} &= \tilde{v}^{(1)} - \frac{i \alpha_0 A_0 c_0}{\lambda_1} \left( \frac{2 \lambda_2}{\lambda_1} + \frac{R_1}{R_0} \right) \xi \ln \xi. \end{aligned} \quad (2.8)$$

Here we have put the wavespeed  $c_0 = \sigma_0 / \alpha_0$ ,  $A_0$  is an unknown constant, and  $\xi = (\lambda_1 \bar{Y} - c_0)$ . Also  $\tilde{v}^{(1)}$  contains terms which are regular as  $\xi \rightarrow 0$ . From (2.8) it is seen that as

the critical level is approached ( $\xi \rightarrow 0+$ ), the leading order solutions for the density and temperature fluctuations in zone Z3 become singular if we have heat transfer, i.e. if  $R_1 \neq 0$ . See also Lees & Reshotko (1962), and Mack (1984). For compressible flow the coefficient of the logarithmic term in the solution for  $v^{(1)}$  now has a contribution proportional to  $(R_1/R_0)$  arising from the variation of the basic density or temperature. It can be seen that the total contribution is now proportional to  $\frac{\partial}{\partial Y} (\rho_B \frac{\partial u_B}{\partial Y})|_{Y=0}$ .

Matching between zone Z1 and Z2 and requiring that the normal velocity fluctuation goes to zero at the wall then gives from (2.7),(2.8),

$$P_0 = \alpha_0 A_0 / \omega$$

and

$$P_0 = A_0 c_0 R_0 \lambda_1.$$

Hence we obtain the relation connecting the wavenumber and wavespeed

$$\alpha_0 = c_0 R_0 \lambda_1 \omega = c_0 R_0 \lambda_1 (1 - M_\infty^2)^{\frac{1}{2}}. \quad (2.9)$$

If  $M_\infty^2 > 1$  then with the solution for outgoing waves from the Prandtl-Glauert equation, the details in the other regions Z2-Z5 remain largely unaffected. However when the matching is performed, because of an extra  $i$  factor now introduced in (2.9), this shows that no neutral solutions for  $M_\infty^2 > 1$ , with these scalings, exist. With  $M_\infty^2 = 0$  in (2.9) this reduces to the relation obtained by SB.

The singularity in the density, temperature and velocity fluctuations in zone Z3 requires a thinner region the critical layer of thickness  $O(\epsilon^{\frac{23}{3}})$ . However for conciseness we summarize the main result that the solutions in Z3 continue to hold for  $\xi < 0$  provided we take  $\ln \xi \rightarrow (\ln |\xi| - i\pi)$  for  $\xi < 0$ . The details of the linear critical layer may be deduced from the nonlinear theory in section 3 below as a limiting case for small amplitudes. The wall layer is essentially the same as that in SB and gives the condition

$$v^{(1)}|_{Y=0} = \frac{i\alpha_0^2 P_0}{R_0 m \sigma_0}$$

where now

$$m = \left( \frac{R_0 \sigma_0}{\mu_B} \right)^{\frac{1}{2}} e^{-i\frac{\pi}{4}}.$$

and  $\mu_B = \mu(S_0)$  is the coefficient of viscosity at the wall.

Matching with Z3 therefore yields the second relation

$$\frac{\alpha_0 c_0^2}{\lambda_1} \left[ \frac{2\lambda_2}{\lambda_1} + \frac{R_1}{R_0} \right] (-\pi) = \frac{\alpha_0^2 c_0 \lambda_1}{\sigma_0} \left( \frac{\mu_B}{2R_0 \sigma_0} \right)^{\frac{1}{2}}. \quad (2.10)$$

Combining this with (2.9) we finally obtain the linear neutral eigenrelations for the upper-branch stability of compressible boundary layer flows,

$$c_0 = \frac{\lambda_1^{\frac{1}{2}}}{(1 - M_\infty^2)^{\frac{1}{12}} \left[ - \left( \frac{2\lambda_2}{\lambda_1} + \frac{R_1}{R_0} \right) \pi \right]^{\frac{1}{3}}} \left( \frac{\mu_B}{2R_0^2} \right)^{\frac{1}{6}}, \quad (2.11a)$$

$$\alpha_0 = \frac{R_0 \lambda_1^{\frac{3}{2}} (1 - M_\infty^2)^{\frac{5}{12}}}{\left[ - \left( \frac{2\lambda_2}{\lambda_1} + \frac{R_1}{R_0} \right) \pi \right]^{\frac{1}{3}}} \left( \frac{\mu_B}{2R_0^2} \right)^{\frac{1}{6}}, \quad (2.11b)$$

and

$$\sigma_0 = \frac{R_0 \lambda_1^2 (1 - M_\infty^2)^{\frac{1}{3}}}{\left[ - \left( \frac{2\lambda_2}{\lambda_1} + \frac{R_1}{R_0} \right) \pi \right]^{\frac{2}{3}}} \left( \frac{\mu_B}{2R_0^2} \right)^{\frac{1}{3}}. \quad (2.11c)$$

For the above relations to hold we require, in addition to the condition for subsonic flow, that the quantity  $\chi = \left( \frac{2\lambda_2}{\lambda_1} + \frac{R_1}{R_0} \right)$  be negative.

If  $M_\infty^2 = 0$ , and we set  $\mu_B = R_0 = 1, R_1 = 0$ , then the results of SB are obtained. (The latter conditions are necessary for consistency with the work of SB where a uniform basic density and temperature field is taken.) We note that the  $(1 - M_\infty^2)^{\frac{1}{2}}$  factors cannot be rescaled out of our problem because  $R_1$  depends on  $M_\infty^2$ .

Some interesting limiting cases of the relations (2.11) can be examined further. Firstly if  $\chi \rightarrow 0-$  with the Mach number fixed, then from (2.11ab)  $c_0 \sim (-\chi)^{-\frac{1}{3}}$  and  $\alpha_0 \sim (-\chi)^{-\frac{1}{3}}$ . Hence formally if  $\chi$  becomes  $O(\epsilon^3 = Re^{-\frac{1}{4}})$  then since the effective wavenumber  $\alpha$  is

$\alpha = \epsilon^{-5}\alpha_0 + \dots$ , we obtain

$$\epsilon c_0 \sim O(1), \quad \alpha \sim O(Re^{\frac{1}{2}}) \text{ and } \sigma \sim O(\epsilon^{-4}\sigma_0) \sim O(Re^{\frac{1}{2}}).$$

This implies that the critical layer moves away from the wall and into the main part of the boundary layer. The limiting neutral structure is then precisely that of classical form with inviscid scalings, i.e. the streamwise and lateral scalings are comparable to the boundary layer thickness. Also the condition  $\chi \rightarrow 0-$  implies that the above neutral modes match on to the neutral inflectional modes which come from the generalized inflexion point criterion. The above situation is similar to that occurring in the upper-branch stability of three-dimensional incompressible boundary layer flows, Bassom & Gajjar (1988). It should be noted however, (as a referee has kindly pointed out), that by varying  $\chi$  the properties of the basic flow are being changed. In Bassom & Gajjar the Rayleigh scalings were obtained by changing the orientation of the three-dimensional disturbance.

As an example if we consider Blasius flow with heat transfer, then

$$\frac{2\lambda_2}{\lambda_1} = \frac{R_1}{R_0} \text{ and } \frac{R_1}{R_0} = -D \left( \frac{1-s_w}{s_w} \right)$$

where  $D$  is some positive constant and  $s_w$  is the ratio of the prescribed wall temperature to the recovery temperature, Stewartson (1964). The condition that  $\chi < 0$  means therefore that the neutral modes exist only for subsonic flow with cooled walls ( $s_w < 1$ ). As we let  $s_w \rightarrow 1-$  then  $\chi \rightarrow 0-$ , and the neutral structure shrinks and we go over to the inviscid scalings. If we include small pressure gradients then there are some values of  $s_w$  ( $s_w < s_{w_c}$  and  $s_{w_c} > 1$ ) for which the neutral solutions exist for heated walls also. Again in the limit  $s_w \rightarrow s_{w_c}-$  we obtain the inviscid scalings.

The second limiting case concerns the limit  $M_\infty^2 \rightarrow 1-$ . From (2.11ab) we find

$$c_0 \sim (1 - M_\infty^2)^{-\frac{1}{12}}, \quad \alpha_0 \sim (1 - M_\infty^2)^{\frac{5}{12}}. \quad (2.12)$$

Formally if  $(1 - M_\infty^2)$  becomes  $O(\epsilon^{12} = Re^{-1})$  then again the critical layer moves away from the wall. But now the streamwise lengthscale increases and the frequency decreases to both become  $O(1)$ . Since also from the solutions in the outermost region Z1  $\epsilon^5 \bar{y} \sim$

$\epsilon^5 \alpha_0^{-1} (1 - M_\infty^2)^{-\frac{1}{2}}$  becomes  $O(Re^{\frac{1}{2}})$ , then this suggests a three tiered structure with a streamwise lengthscale of  $O(1)$ , and regions of lateral extents  $O(Re^{-\frac{1}{2}})$ ,  $O(1)$  and  $O(Re^{\frac{1}{2}})$ . The implications and importance of this transonic regime require further study. Professor F.T. Smith (private communication) has indicated that a similar situation arises in the transonic regime for the lower-branch modes, see also Smith (1987).

Lastly we note that if we increase the disturbance size  $\delta$  and allow for nonlinear effects then, with  $O(1)$  Mach numbers, and as in SB the critical layer becomes fully nonlinear when  $\delta$  becomes  $O(\epsilon^{\frac{7}{5}})$ . The structure and analysis for the nonlinear compressible critical layer is considered in section 3.

Before that however we return to the case of Blasius flow with insulated wall conditions. It is clear from (2.11) that with  $\lambda_2 = R_1 = 0$  the above analysis breaks down. The main reason is that the effects due to compressibility start coming in for much smaller Mach numbers, because the generalized inflexion point is in the main part of the boundary layer for  $O(1)$  Mach numbers but retreats to the wall as  $M_\infty^2 \rightarrow 0$ .

## §2.2 Blasius flow with insulated wall conditions.

When  $M_\infty^2$  becomes  $O(Re^{-\frac{1}{20}})$  then the neutral eigenrelations for the upper-branch of Blasius flow with insulated wall conditions will be significantly modified from the incompressible case. A rough argument to show this is as follows. If we consider the neutral modes satisfying the generalized point criterion, for Blasius flow with insulated wall conditions, then these satisfy the condition

$$\frac{\partial}{\partial Y} \left( \rho_B \frac{\partial U_B}{\partial Y} \right) = 0, \quad U_B = c, \text{ at } Y = Y_c, \quad (2.13)$$

where  $c$  is the phase speed of the wave, and  $\rho_B$ ,  $U_B$  are the basic density and streamwise component of velocity.

The basic velocity and density are given by

$$U_B = f'(\bar{\eta}), \quad T_B = 1/\rho_B = 1 + \frac{(\gamma - 1)}{2} M_\infty^2 (1 - f'^2), \quad (2.14a)$$

where  $f(\bar{\eta})$  is the Blasius function satisfying

$$f''' + ff'' = 0, \quad f(0) = f'(0) = 0, f'(\infty) = 1, \quad (2.14b)$$

and  $\bar{\eta}$  is related to  $Y$  by the Dorodonitsyn Howarth transformation

$$(2Cx_0)^{\frac{1}{2}}\bar{\eta} = \int_0^Y \rho_B dY. \quad (2.14c)$$

For small  $\bar{\eta}$ , (2.13) yields

$$-\bar{\eta} [2(\gamma - 1)M_\infty^2 \bar{\lambda}_1^3 + 12\bar{\lambda}_4 \bar{\eta}] \sim 0, \quad (2.15a)$$

where

$$f' \sim \bar{\lambda}_1 \bar{\eta} + \bar{\lambda}_4 \bar{\eta}^4 + \dots \text{ as } \bar{\eta} \rightarrow 0. \quad (2.15b)$$

Hence

$$\bar{\eta} \sim 0 \text{ or } \bar{\eta} \sim -\frac{(\gamma - 1)\bar{\lambda}_1^3 M_\infty^2}{6\bar{\lambda}_4}.$$

Now for small  $c$  the critical layer position is given by  $Y \sim c/\lambda_1$ , and since on the upper-branch of incompressible Blasius flow  $c \sim O(Re^{-\frac{1}{20}})$ , Smith & Bodonyi (1980), this suggests looking at the scaling  $M_\infty^2 \sim O(Re^{-\frac{1}{20}})$ .

If we set  $\epsilon_1 = Re^{-\frac{1}{20}}$ , and  $M_\infty^2 = \bar{M}_\infty^2 \epsilon_1$ , then the basic flow has the following properties,

$$U_B(x, Y) \sim U_{B0} + \epsilon_1 U_{B1} + \dots$$

$$V_B = O(\epsilon_1^{10}), \quad \rho_B = 1 + \epsilon_1 \rho_{B1} + \dots,$$

$$T_B = 1 + \epsilon_1 T_{B1} + \dots, \quad p_B = \epsilon_1^{-1} \bar{p}_B = \frac{\epsilon_1^{-1}}{\gamma \bar{M}_\infty^2},$$

and close to the wall with  $y = \epsilon_1^{11} \bar{Y}$ ,

$$U_B \sim \epsilon_1 a_{11} \bar{Y} + \epsilon_1^2 a_{21} \bar{Y} + \epsilon_1^3 a_{31} \bar{Y} + \epsilon_1^4 (a_{41} \bar{Y} + a_{43} \bar{Y}^3 + a_{44} \bar{Y}^4) + \dots,$$

$$\rho_B \sim 1 + \epsilon_1 b_{10} + \epsilon_1^2 b_{20} + \epsilon_1^3 (b_{30} + b_{32} \bar{Y}^2) + \dots,$$

where the constants  $a_{ij}, b_{ij}$  can be obtained explicitly from (2.14), (2.15), and in particular

$$\begin{aligned} a_{11} &= \lambda_1 = \bar{\lambda}_1 / (2Cx_0)^{\frac{1}{2}}, \quad 6a_{43} = \lambda_1^3 (\gamma - 1) \bar{M}_\infty^2, \\ a_{44} &= \lambda_4 = -\bar{\lambda}_1^2 / 24 (2Cx_0)^2, \quad 2b_{32} = \lambda_1^2 (\gamma - 1) \bar{M}_\infty^2. \end{aligned} \quad (2.16)$$

The linear disturbance structure for incompressible flows is set out in Smith & Bodonyi (1980), see also Fig.2, and this work can be extended to compressible flows in a straightforward fashion, using the properties of the basic flow as given above. For brevity however only the main points are outlined. The wavenumber and frequency now expand as

$$\begin{aligned} \alpha &= Re^{\frac{9}{20}} \tilde{\alpha} \sim Re^{\frac{9}{20}} \bar{\alpha}_0 + Re^{\frac{8}{20}} \bar{\alpha}_1 + \dots, \\ \sigma &= Re^{\frac{8}{20}} \tilde{\sigma} \sim Re^{\frac{8}{20}} \bar{\sigma}_0 + \dots, \end{aligned}$$

respectively, and  $c_0 = \sigma_0 / \alpha_0$  is the phase speed. We set  $x = \epsilon_1^9 X_1$  and  $t = \epsilon_1^8 \tau_1$  and look for disturbances proportional to  $E_1 = e^{i(\bar{\alpha} X_1 - \bar{\sigma} \tau_1)}$ . Then in the main part of the boundary layer, the disturbances have the expansions

$$\begin{aligned} \tilde{u} &= u_0 + \epsilon_1 u_1 + \dots \\ \tilde{v} &= \epsilon_1 v_0 + \epsilon_1^2 v_1 + \dots \\ \tilde{p} &= \epsilon_1 p_0 + \epsilon_1^2 p_1 + \dots, \\ \tilde{\rho} &= \epsilon_1 \rho_0 + \epsilon_1^2 \rho_1 + \dots, \\ \tilde{\theta} &= \epsilon_1 \theta_0 + \epsilon_1^2 \theta_1 + \dots, \end{aligned}$$

Substitution into the Navier-Stokes equations produces the leading order solutions

$$\begin{aligned} u_0 &= \bar{A}_0 U_{B0Y}, \quad v_0 = -i\bar{\alpha}_0 \bar{A}_0 U_{B0}, \quad p_0 = \bar{P}_0, \\ \rho_0 &= -\theta_0 = -\bar{A}_0 \frac{\partial \rho_{B1}}{\partial Y}, \end{aligned}$$

where  $\bar{A}_0, \bar{P}_0$  are constants. By writing down similar expansions for the outermost potential flow region and matching with the solutions above gives the relation,

$$\bar{P}_0 = \bar{\alpha}_0 \bar{A}_0.$$

as in the incompressible case.

Next in the thinner inviscid region which is part of the main boundary layer containing the critical layer, with  $y = \epsilon_1^{11} \bar{Y}$ , the disturbances expand as

$$\begin{aligned}\tilde{u} &= u^{(0)} + \epsilon_1 u^{(1)} + \epsilon_1^2 u^{(2)} + \epsilon_1^3 u^{(3)} + \dots, \\ \tilde{v} &= \epsilon_1^2 v^{(0)} + \epsilon_1^3 v^{(1)} + \epsilon_1^4 v^{(2)} + \epsilon_1^5 v^{(3)} + \dots, \\ \tilde{p} &= \epsilon_1 p^{(0)} + \epsilon_1^2 p^{(1)} + \dots, \\ \tilde{\rho} &= \epsilon_1^2 \rho^{(0)} + \epsilon_1^3 \rho^{(1)} + \dots, \\ \tilde{\theta} &= \epsilon_1^2 \theta^{(0)} + \epsilon_1^3 \theta^{(1)} + \dots\end{aligned}$$

The leading order solutions for  $u^{(0)}, v^{(0)}, p^{(0)}$ , are as in SB1 and produce the relation

$$\frac{i\bar{\alpha}_0 \bar{P}^{(0)}}{a_{11}} = i\bar{\sigma}_0 \bar{A}_0,$$

after requiring that the normal velocity fluctuation tends to zero at the wall. The solution for the density disturbance yields,

$$\rho^{(0)} = \frac{P^{(0)}}{\gamma \bar{P}_B} + \frac{\bar{A}_0 a_{11} 2b_{32} \bar{Y}^2}{(a_{11} \bar{Y} - c_0)}.$$

Again as for the previous case studied, the density and temperature fluctuations become quite large in the vicinity of the critical layer. The higher order solutions can be obtained by successively solving the equations at each order. As in SB1 it is necessary to proceed to  $O(\epsilon_1^5)$  in the expansions for  $\tilde{v}$ , where we find that

$$v^{(3)} = PLT - \frac{c_0^2}{a_{11}^5} (a_{11} F_0 + c_0 F_1) (a_{11} \bar{Y} - c_0) \ln(a_{11} \bar{Y} - c_0),$$

where  $PLT$  contains algebraic terms which are regular as  $\bar{Y} \rightarrow c_0/a_{11}$ , and

$$F_0 = 2i\bar{A}_0\bar{\alpha}_0a_{11}(a_{11}b_{32} + 3a_{43}), \quad F_1 = 12i\bar{\alpha}_0a_{11}a_{44}\bar{A}_0.$$

The  $F_1$  contribution to the logarithmic term is as in the incompressible case but the additional contribution due to the  $F_0$  term, arises from compressibility effects.

The critical layer needs to be introduced to smooth out the singularities in the density, temperature and velocity fluctuations, but the ensuing result is the familiar ' $-i\pi$ ' jump in the logarithmic term across the critical layer. The leading order wall layer solutions are as in SB1. After completing the matching, we finally obtain the eigenrelations,

$$\bar{\alpha}_0 = \bar{c}_0a_{11}, \quad (2.17)$$

and

$$\frac{2\bar{c}_0^2}{a_{11}^6}(-\pi)(b_{32}a_{11}^2 + 3a_{11}a_{43} + 6\bar{c}_0a_{44}) = \frac{\bar{\alpha}_0}{(2\bar{\alpha}_0^3\bar{c}_0^3)^{\frac{1}{2}}}.$$

Substituting for  $b_{32}, a_{43}$  and  $a_{44}$ , from (2.16), in terms of the scaled Mach number gives,

$$\frac{2\bar{c}_0^2\pi}{\lambda_1^6} [\lambda_1^4(\gamma - 1)\bar{M}_\infty^2 + 6\bar{c}_0\lambda_4] = -\frac{\bar{\alpha}_0}{(2\bar{\alpha}_0^3\bar{c}_0^3)^{\frac{1}{2}}}. \quad (2.18)$$

Hence for the neutral modes to exist we require also from (2.18) that

$$6\bar{c}_0 > \frac{\lambda_1^4(\gamma - 1)\bar{M}_\infty^2}{|\lambda_4|}.$$

The eigenrelations are plotted in Fig. 3a. If  $\bar{M}_\infty^2 = 0$  then (2.17), (2.18) reduce to the expressions obtained by Smith & Bodonyi (1980). Clearly the neutral solutions exist only for a certain range of wavenumbers and frequencies. If  $\bar{M}_\infty^2$  becomes large, then from (2.18) we get

$$\bar{c}_0 \sim -\frac{\lambda_1^4(\gamma - 1)\bar{M}_\infty^2}{6\lambda_4} + c_0^\dagger, \quad (2.19)$$

where  $0 < c_0^\dagger \ll 1$ . Hence for increasing Mach number the critical layer moves away from the wall. When  $\bar{M}_\infty^2$  becomes  $O(Re^{\frac{1}{20}})$  formally, i.e. the unscaled Mach number becomes

$O(1)$ , and from (2.17) and (2.19) it is seen that the wavespeed becomes  $O(1)$ , and the wavenumber and frequency become  $O(Re^{\frac{1}{2}})$ . So again the inviscid scalings are recovered. From (2.15a), (2.19) it is clear that in this limit a match with the neutral inflexional mode 1 for low  $O(1)$  Mach numbers is achieved. If we replace  $\bar{c}_0$  by  $c$  and  $\bar{M}$  by  $\bar{M}_\infty$  in (2.19) for  $O(1)$  Mach numbers, then the predicted asymptote agrees well with the computed phase shift  $c$  from the generalized inflexion point criterion, see Fig. 3b and also Fig. 9.2 in Mack (1984).

### §3 Nonlinear compressible critical layer equations.

As in SB when the disturbance size  $\delta$  becomes  $O(\epsilon^{\frac{7}{3}})$  then nonlinear effects in the critical layer become important and cannot be neglected. For Blasius flow with insulated wall conditions the scalings and disturbance size to produce a nonlinear critical layer are somewhat different and are not considered here, but see Cole (1989) for a discussion of this case.

Since we are interested in wavenumbers of  $O(Re^{\frac{5}{12}})$  and frequencies of  $O(Re^{\frac{1}{3}})$  we set

$$\begin{aligned}\frac{\partial}{\partial x} &\rightarrow \epsilon^{-5}(\alpha_0 + \epsilon\alpha_1 + \dots)\frac{\partial}{\partial X} + \frac{\partial}{\partial x}, \\ \frac{\partial}{\partial t} &\rightarrow -\epsilon^{-4}(\sigma_0 + \epsilon\sigma_1 + \dots)\frac{\partial}{\partial X},\end{aligned}$$

where as in SB the slower  $\partial/\partial x$  variations do not play any significant role, and can be ignored. The constants  $\alpha_0, \alpha_1, \sigma_0, \sigma_1 \dots$ , are taken to be real, so that we are considering neutral disturbances and the wavespeed  $c_0 = \sigma_0/\alpha_0$ .

Before proceeding with the critical layer analysis, some brief details of the solutions in the other regions are necessary.

In zone Z2, the main part of the boundary layer, with  $y = \epsilon^6 Y$ , the total expansions for the flow quantities now take the form,

$$\begin{aligned}u &= U_B(x, Y) + \epsilon^{\frac{7}{3}}u_0 + \epsilon^{\frac{10}{3}}u_1 + \dots, \\ v &= \epsilon^{\frac{10}{3}}v_0 + \epsilon^{\frac{13}{3}}v_1 + \dots, \\ p &= p_B + \epsilon^{\frac{10}{3}}p_0 + \epsilon^{\frac{13}{3}}p_1 + \dots, \\ \rho &= \rho_B(x, Y) + \epsilon^{\frac{7}{3}}\rho_0 + \epsilon^{\frac{10}{3}}\rho_1 + \dots, \\ T &= T_B(x, Y) + \epsilon^{\frac{7}{3}}\theta_0 + \epsilon^{\frac{10}{3}}\theta_1 + \dots\end{aligned}\tag{3.1}$$

Substitution into the Navier-Stokes equations then produces the leading order solutions,

$$\begin{aligned}u_0 &= \tilde{A}(X)\frac{\partial U_B}{\partial Y}, \quad v_0 = -\alpha_0\frac{\partial \tilde{A}(X)}{\partial X}U_B, \quad \rho_0 = \tilde{A}\frac{\partial \rho_B}{\partial Y}, \\ \theta_0 &= -\frac{T_B}{\rho_B}\tilde{A}\frac{\partial \rho_B}{\partial Y}, \quad p_0 = P_0(X).\end{aligned}\tag{3.2}$$

The unknown disturbance function  $\tilde{A}(X)$  will be set equal to  $A_0 \cos X$  later, but for convenience we assume the more general form for now.

In the inviscid region Z3 containing the critical layer, with  $y = \epsilon^7 \bar{Y}$ , the following expansions are implied from (3.1),(3.2):

$$\begin{aligned} u &= \epsilon \lambda_1 \bar{Y} + \epsilon^2 \lambda_2 \bar{Y}^2 + \epsilon^{\frac{7}{3}} u^{(0)} + \epsilon^{\frac{8}{3}} u_{MF} + \epsilon^3 \lambda_3 \bar{Y}^3 + \epsilon^{\frac{10}{3}} u^{(1)} + \dots, \\ v &= \epsilon^{\frac{13}{3}} v^{(0)} + \epsilon^{\frac{16}{3}} v^{(1)} + \dots, \\ p &= p_B + \epsilon^{\frac{10}{3}} p^{(0)} + \epsilon^{\frac{13}{3}} p^{(1)} + \dots, \\ \rho &= R_0 + \epsilon R_1 \bar{Y} + \epsilon^{\frac{5}{3}} \rho_{MF} + \epsilon^2 R_2 \bar{Y}^2 + \epsilon^{\frac{7}{3}} \rho^{(0)} + \dots, \\ T &= S_0 + \epsilon S_1 \bar{Y} + \epsilon^{\frac{5}{3}} \theta_{MF} + \epsilon^2 S_2 \bar{Y}^2 + \epsilon^{\frac{7}{3}} \theta^{(0)} + \dots \end{aligned} \quad (3.3)$$

The suffix  $MF$  denotes mean flow terms (independent of  $X$ ) which arise because of the interaction of the nonlinear critical layer with the outer flow.

Substitution into the Navier-Stokes equation produces the solutions

$$\begin{aligned} u^{(0)} &= \tilde{A} \lambda_1, & v^{(0)} &= -\alpha_0 \tilde{A}_X \lambda_1 \bar{Y}, & p^{(0)} &= P^{(0)}, \\ \rho^{(0)} &= R_1 \tilde{A} + \frac{R_1 c_0 \tilde{A}}{(\lambda_1 \bar{Y} - c_0)}, & \theta^{(0)} &= -\frac{S_0}{R_0} \rho^{(0)}, \end{aligned} \quad (3.4)$$

where the relation

$$\frac{\alpha_0}{R_0 \lambda_1} \frac{\partial p^{(0)}}{\partial X} = \alpha_0 \tilde{A}_X c_0,$$

is required in order that the  $v^{(0)}$  solution tends to zero at the wall. The density disturbance  $\rho^{(0)}$  becomes singular at the critical layer and this gives an additional contribution to the logarithmic term at higher order. In fact we have

$$\begin{aligned} v^{(1)} &= -\frac{1}{R_0 \lambda_1} \left[ \alpha_0 \frac{\partial p^{(1)}}{\partial X} + \alpha_1 \frac{\partial p^{(0)}}{\partial X} + R_0 (c_0 \alpha_1 - \sigma_1) \frac{\partial u^{(0)}}{\partial X} \right] \\ &\quad - \bar{F}_0 \left( \bar{Y} \xi + \frac{2c_0}{\lambda_1} \xi \ln(\xi) - \frac{c_0^2}{\lambda_1} \right) \\ &\quad - \bar{F}_1 \left( \frac{\xi}{\lambda_1} \ln \xi - \frac{c_0}{\lambda_1} \right) - \alpha_0 \tilde{A}_{1X} \xi, \end{aligned} \quad (3.5)$$

where  $\xi = (\lambda_1 \bar{Y} - c_0)$ ,  $\tilde{A}_1$  is an unknown function of  $X$  and

$$\bar{F}_0 = \frac{\alpha_0 \lambda_2 \tilde{A}_X}{\lambda_1}, \quad \bar{F}_1 = \frac{\alpha_0 R_1 c_0 \tilde{A}_X}{R_0}.$$

The solution for  $u^{(1)}$  can then be obtained from the continuity equation

$$\alpha_0 \frac{\partial u^{(1)}}{\partial X} + \alpha_1 \frac{\partial u^{(0)}}{\partial X} + \frac{\partial v^{(1)}}{\partial Y} = 0.$$

Hence in the critical layer Z4, where  $y = \frac{\epsilon^7 c_0}{\lambda_1} + \epsilon^{\frac{23}{3}} \eta$ , we have from (3.3), (3.4) and (3.5),

$$\begin{aligned} u &= \epsilon c_0 + \epsilon^{\frac{5}{3}} \lambda_1 \eta + \frac{\epsilon^2 \lambda_2 c_0^2}{\lambda_1^2} + \epsilon^{\frac{7}{3}} \tilde{u}^{(1)} + \epsilon^{\frac{8}{3}} \tilde{u}^{(2)} + \epsilon^3 \frac{\lambda_3 c_0^3}{\lambda_1^3} + \epsilon^{\frac{10}{3}} \tilde{u}^{(3)} + \dots, \\ v &= \epsilon^{\frac{13}{3}} \tilde{v}^{(0)} + \epsilon^{\frac{15}{3}} \tilde{v}^{(1)} + \epsilon^{\frac{16}{3}} \tilde{v}^{(2)} + \epsilon^{\frac{17}{3}} \tilde{v}_c(X) + \epsilon^{\frac{18}{3}} \tilde{v}^{(3)} + \dots, \\ \rho &= R_0 + \frac{\epsilon R_1 c_0}{\lambda_1} + \epsilon^{\frac{5}{3}} \tilde{\rho}^{(0)} + \epsilon^{\frac{7}{3}} \tilde{\rho}^{(1)} + \dots, \\ T &= S_0 + \frac{\epsilon S_1 c_0}{\lambda_1} + \epsilon^{\frac{5}{3}} \tilde{\theta}^{(0)} + \dots, \\ p &= p_B + \epsilon^{\frac{10}{3}} \tilde{p}^{(0)} + \epsilon^{\frac{13}{3}} \tilde{p}^{(1)} + \epsilon^{\frac{14}{3}} \tilde{p}^{(2)} + \epsilon^{\frac{15}{3}} \tilde{p}^{(3)} + \dots \end{aligned} \quad (3.6)$$

Substitution into the Navier-Stokes equations then gives,

$$\begin{aligned} \tilde{v}^{(0)} &= -\frac{\alpha_0}{\lambda_1 R_0} \frac{\partial \tilde{p}^{(0)}}{\partial X}, & \tilde{v}^{(1)} &= -\alpha_0 \lambda_1 \tilde{A}_X \eta, & \tilde{u}^{(1)} &= \tilde{A} \lambda_1, \\ \frac{\partial \tilde{u}^{(2)}}{\partial \eta} &= \frac{2c_0}{\lambda_1}, & \frac{\partial \tilde{v}^{(2)}}{\partial \eta} &= 0. \end{aligned}$$

At the next order, the governing equations are

$$\begin{aligned} R_0 \left[ \frac{\partial}{\partial X} (\alpha_0 \tilde{u}^{(3)} + \alpha_1 \tilde{u}^{(1)}) + \frac{\partial \tilde{v}^{(3)}}{\partial \eta} \right] + \frac{R_1 c_0}{\lambda_1} \left[ \alpha_0 \frac{\partial \tilde{u}^{(1)}}{\partial X} + \frac{\partial \tilde{v}^{(1)}}{\partial \eta} \right] \\ + \lambda_1 \eta \alpha_0 \frac{\partial \tilde{\rho}^{(0)}}{\partial X} + \tilde{v}^{(0)} \frac{\partial \tilde{\rho}^{(0)}}{\partial \eta} = 0, \end{aligned}$$

together with

$$\begin{aligned} R_0 \left[ \lambda_1 \eta (\alpha_0 \frac{\partial \tilde{u}^{(3)}}{\partial X} + \alpha_1 \frac{\partial \tilde{u}^{(1)}}{\partial X}) + \alpha_0 (\tilde{u}^{(1)} \frac{\partial \tilde{u}^{(2)}}{\partial X} + \tilde{u}^{(2)} \frac{\partial \tilde{u}^{(1)}}{\partial X}) + \right. \\ \left. \tilde{v}^{(0)} \frac{\partial \tilde{u}^{(3)}}{\partial \eta} + \tilde{v}^{(1)} \frac{\partial \tilde{u}^{(2)}}{\partial \eta} + \tilde{v}^{(2)} \frac{\partial \tilde{u}^{(1)}}{\partial \eta} + \tilde{v}^{(3)} \lambda_1 \right] \\ + \frac{R_1 c_0}{\lambda_1} [0] + \tilde{\rho}^{(0)} \tilde{v}^{(0)} \lambda_1 = -\alpha_0 \frac{\partial \tilde{p}^{(3)}}{\partial X} + \frac{\partial}{\partial \eta} (\mu_B \frac{\partial \tilde{u}^{(3)}}{\partial \eta}), \end{aligned}$$

and

$$\gamma p_B \left[ \alpha_0 \lambda_1 \eta \frac{\partial \tilde{\rho}^{(0)}}{\partial X} + \tilde{v}^{(0)} \frac{\partial \tilde{\rho}^{(0)}}{\partial \eta} \right] + \frac{\mu_B R_0}{M_\infty^2} \frac{\partial^2 \tilde{\theta}^{(0)}}{\partial \eta} = 0,$$

and finally,

$$\tilde{\rho}^{(0)} S_0 + R_0 \tilde{\theta}^{(0)} = 0.$$

The pressure terms in (3.6) are independent of  $\eta$  to the order indicated. It is at this order that the first non-simple solutions are generated which smooth out the irregularities in  $\rho^{(0)}$  and  $v^{(1)}$ . Using the previous solutions we obtain after some manipulation the nonlinear compressible critical layer equations,

$$\alpha_0 \lambda_1 \eta R_0 \frac{\partial}{\partial X} \left( \frac{\partial \tilde{u}^{(3)}}{\partial \eta} \right) + R_0 \tilde{v}^{(0)} \frac{\partial^2 \tilde{u}^{(3)}}{\partial \eta^2} - \mu_B \frac{\partial^3 \tilde{u}^{(3)}}{\partial \eta^3} = \lambda_1^2 \eta \alpha_0 \frac{\partial \tilde{\rho}^{(0)}}{\partial X} + \lambda_1 C \frac{\partial^2 \tilde{\theta}^{(0)}}{\partial \eta^2}, \quad (3.7a)$$

where  $\tilde{\rho}^{(0)}$  satisfies

$$\alpha_0 \lambda_1 \eta \frac{\partial \tilde{\rho}^{(0)}}{\partial X} + \tilde{v}^{(0)} \frac{\partial \tilde{\rho}^{(0)}}{\partial \eta} - \mu_B S_0 \frac{\partial^2 \tilde{\rho}^{(0)}}{\partial \eta^2} = 0. \quad (3.7b)$$

We note that for a more general viscosity law, the Chapman constant  $C$  in (3.7a) is replaced by  $\frac{d\mu(S_0)}{dT}$ .

From (3.4)-(3.6) the boundary conditions are

$$\tilde{\rho}^{(0)} \sim R_1 \eta + \bar{\Delta}^\pm + \frac{R_1 \tilde{A} c_0}{\lambda_1 \eta} \quad \text{as } \eta \rightarrow \pm\infty, \quad (3.8a)$$

$$\frac{\partial \tilde{u}^{(3)}}{\partial \eta} \sim 2\lambda_2 \eta + \bar{H}^\pm + c_0 \tilde{A} \left( \frac{2\lambda_2}{\lambda_1} + \frac{R_1}{R_0} \right) \frac{1}{\eta} \quad \text{as } \eta \rightarrow \pm\infty. \quad (3.8b)$$

Also matching with Z3 gives

$$\tilde{v}^{(0)} = -\alpha_0 \tilde{A}_X c_0,$$

so that if we set  $\tilde{A} = A_0 \cos X$  and introduce the normalizations

$$\begin{aligned} \eta &= gZ, & g^2 &= \frac{c_0 A_0}{\lambda_1}, & \lambda_c &= \mu_B / (g^3 \lambda_1 R_0 \alpha_0), \\ \frac{\partial \tilde{u}^{(3)}}{\partial \eta} &= \frac{c_0 A_0}{g} \left( \frac{2\lambda_2}{\lambda_1} + \frac{R_1}{R_0} \right) \Omega, & \tilde{\rho}^{(0)} &= R_1 g \Pi, \end{aligned} \quad (3.9)$$

then (3.7),(3.8) become

$$Z \frac{\partial \Omega}{\partial X} + \sin X \frac{\partial \Omega}{\partial Z} - \lambda_c \frac{\partial^2 \Omega}{\partial Z^2} = J \left( Z \frac{\partial \Pi}{\partial X} - \lambda_c \frac{\partial^2 \Pi}{\partial Z^2} \right), \quad (3.10a)$$

and

$$Z \frac{\partial \Pi}{\partial X} + \sin X \frac{\partial \Pi}{\partial Z} - \lambda_c \frac{\partial^2 \Pi}{\partial Z^2} = 0, \quad (3.10b)$$

with the boundary conditions

$$\Omega \sim (1 - J)Z + \frac{\cos X}{Z} + H^\pm \quad \text{as } Z \rightarrow \pm\infty, \quad (3.10c)$$

$$\Pi \sim Z + \frac{\cos X}{Z} + \Delta^\pm \quad \text{as } Z \rightarrow \pm\infty. \quad (3.10d)$$

Here the parameter  $J = R_1 \lambda_1 / (2\lambda_2 R_0 + \lambda_1 R_1)$ . The set (3.10a-d) constitutes an extension to compressible flow of the Haberman (1972) equation. The nonlinear compressible critical layer equation differs from the usual Haberman equation because of the extra forcing term on the right hand side of (3.10a). The parameter  $J$  depends on  $R_1$ , the basic density gradient at the wall and  $\lambda_2$  the curvature of the basic flow. For problems involving heat transfer  $J$  is nonzero. Also as in the linear theory the above analysis applies equally to Blasius flow with heat transfer when  $J = \frac{1}{2}$ .

The important quantity as far as the eigenvalue problem is concerned is the velocity jump and phase shift  $\phi$ , resulting from the set (3.10).

If we put  $\Omega = \partial U / \partial Z$ , then from (3.10c),

$$U \sim (1 - J) \frac{Z^2}{2} + H^\pm Z + \cos X \ln |Z| + U^\pm(X) \quad \text{as } Z \rightarrow \pm\infty.$$

The phase shift  $\phi$  is then given by

$$\phi = \frac{1}{\pi} \int_0^{2\pi} \sin x (U^+ - U^-) dx.$$

and as in Haberman (1972), SB, the following identity can be derived,

$$\phi + J\phi_H = 2\lambda_c(H^+ - H^-). \quad (3.11)$$

Here  $\phi_H(\lambda_c)$  is the phase shift from the Haberman problem, with  $J = 0$  above.

The problem (3.10) has to be solved numerically to determine  $\phi$ , but as far as the eigenvalue problem is concerned we note firstly that  $\ln \xi \rightarrow (\ln |\xi| + i\phi)$  for  $\xi < 0$ , in (3.5). Then since the leading order solutions outside the critical layer are determined from linear equations, the eigenrelations can be obtained as in section 2 with now  $-\pi$  being replaced by  $\phi$ .

This then yields the relations for neutral stability,

$$c_0 = \frac{\lambda_1^{\frac{1}{2}}}{(1 - M_\infty^2)^{\frac{1}{12}} \left[ \left( \frac{2\lambda_2}{\lambda_1} + \frac{R_1}{R_0} \right) \phi \right]^{\frac{1}{3}}} \left( \frac{\mu_B}{2R_0^2} \right)^{\frac{1}{6}}, \quad (3.12a)$$

$$\alpha_0 = \frac{R_0 \lambda_1^{\frac{3}{2}} (1 - M_\infty^2)^{\frac{5}{12}}}{\left[ \left( \frac{2\lambda_2}{\lambda_1} + \frac{R_1}{R_0} \right) \phi \right]^{\frac{1}{3}}} \left( \frac{\mu_B}{2R_0^2} \right)^{\frac{1}{6}}, \quad (3.12b)$$

and

$$\sigma_0 = \frac{R_0 \lambda_1^2 (1 - M_\infty^2)^{\frac{1}{3}}}{\left[ \left( \frac{2\lambda_2}{\lambda_1} + \frac{R_1}{R_0} \right) \phi \right]^{\frac{2}{3}}} \left( \frac{\mu_B}{2R_0^2} \right)^{\frac{1}{3}}. \quad (3.12c)$$

Clearly for neutral stability we require the condition  $M_\infty^2 < 1$ , i.e. subsonic flow and

$$\phi\chi = \phi \left( \frac{2\lambda_2}{\lambda_1} + \frac{R_1}{R_0} \right) > 0. \quad (3.13)$$

Finally since the equation for the density disturbance  $\Pi$  is the Haberman equation,  $(\Delta^+ - \Delta^-)$  is non zero, and hence there is a jump in the mean density across the critical layer. As the parameter  $\lambda_c$  goes to infinity we recover the linear results since  $\phi \rightarrow -\pi$ .

#### §4 Results and Discussion.

The problem (3.10) was solved numerically using a Fourier series for  $\Pi$  and  $\Omega$  and solving a truncated system using second order central differencing. The discrete equations were solved iteratively using relaxation especially for the small  $\lambda_c$  values. The usual checks were made on grid size independence, the number of Fourier terms used etc. The phase shift  $\phi$  and the vorticity jump ( $H^+ - H^-$ ) are shown in Fig. (4ab) for several values of  $J$ . For  $J = 0$  our results agree with those of Haberman (1972) and SB. Solutions were obtained for both positive and negative values of  $J$ . For  $J$  becoming large and negative,  $\phi$  becomes much more negative than the  $J = 0$  case. For  $J$  positive however, for a fixed value of  $\lambda_c$  the phase shift is larger than the equivalent Haberman value. Our calculations also indicate that there exists critical values of  $J = J_c > 0$  such that for  $J > J_c$ , the phase shift becomes positive for some values of  $\lambda_c$ .

Clearly  $J < 0$  implies either  $R_1/R_0 > 0$  and  $\chi < 0$ , or  $R_1/R_0 < 0$  and  $\chi > 0$ . Our results indicate that no solutions are possible with the latter case since from (3.13) we also require  $\phi > 0$ . The former case implies a positive basic density gradient (and hence a negative temperature gradient) at the wall, and applies only for accelerated flows.

If on the other hand  $J > 0$ , then we require that  $R_1/R_0 > 0$  and  $\chi > 0$  and hence  $\phi > 0$ , or  $R_1/R_0 < 0$  and  $\chi < 0$  (and  $\phi < 0$ ). The numerical results show that there are solutions with positive  $\phi$  if  $J > J_c > 0$  and  $0 < \lambda_c < \lambda_{crit}(J)$ , where  $\lambda_{crit}$  is the value of  $\lambda_c$  such that  $\phi(J, \lambda_c) = 0$ . Since  $R_1/R_0 > 0$  this implies heated walls only. The other case with  $\phi < 0$  has solutions only for  $J < J_c$ ,  $\lambda_{crit} < \lambda_c$ , and requires cooled walls.

The nonlinear neutral curves will differ considerably depending on the above cases. Some sample curves of the nonlinear neutral amplitude against frequency are sketched in Fig. 5. It can be seen in Fig. 5a that if  $J < J_c$  then the behaviour of  $A_0$  is qualitatively similar to that for the incompressible (SB) case. If however  $J_c < J$  and  $\phi < 0$  then (Fig. 5b) the amplitude reaches a peak and decreases to zero as the neutral frequency increases. Finally if  $J_c < J$  and  $\phi > 0$ , then Fig. 5c indicates the possible behaviour. We note that these latter solutions are not present in the linear case as the amplitude approaches zero. There is a critical value of the neutral frequency below which the nonlinear solutions disappear.

In the limit  $J \rightarrow \infty$ ,  $\lambda_c = O(1)$ , a limiting solution of (3.10) can be found in the form

$$\Omega = J\Omega_1 + \dots, \quad \phi = J\tilde{\phi}_1 + \dots,$$

where  $\Omega_1$  satisfies

$$Z\Omega_{1X} + \sin x\Omega_{1Z} - \lambda_c\Omega_{1ZZ} = Z\Pi_X - \lambda_c\Pi_{ZZ} \quad (4.1a)$$

with the boundary conditions,

$$\Omega_1 \sim -Z + \tilde{H}^\pm \quad \text{as } Z \rightarrow \pm\infty. \quad (4.1b)$$

The phase shift from the solution of (4.1) is shown in Fig. 6 and this is in good agreement with the numerical solutions of the full equations, see Fig. 7. For  $|J|$  large the above predicts large mean vorticity gradients and phase shifts across the critical layer.

The limit  $\lambda_c \rightarrow \infty$ ,  $|J| \rightarrow \infty$ ,  $J = O(\lambda_c)$ , can be analyzed in the same manner as the linear limit with  $J = O(1)$ . This then reproduces the linear result that  $\phi \rightarrow -\pi$  as  $\lambda_c \rightarrow \infty$ , with  $J = O(\lambda_c)$ .

The last case which merits further consideration is the limit  $\lambda_c \rightarrow 0$  when the critical layer becomes strongly nonlinear. Our numerical results indicate and also the corresponding incompressible limit suggests expansions of the form,

$$\Omega = \tilde{\Omega}_0 + \lambda_c\tilde{\Omega}_1 + \dots, \quad \Pi = \tilde{\Pi}_0 + \lambda_c\tilde{\Pi}_1 + \dots \quad (4.2)$$

Since  $\Pi$  satisfies the Haberman (1972) equation, with the same boundary conditions, the results of Haberman (1972), Brown & Stewartson (1978), SB, still hold and we get

$$\tilde{\Pi}_0 = K(\eta_*) \text{ where } \eta_* = \frac{1}{2}Z^2 + \cos X. \quad (4.3)$$

The function  $K(\eta_*)$  is determined from a secularity condition at higher order. In fact

$$\begin{aligned} K(\eta_*) &= N_0 + \int_1^{\eta_*} K'(t) dt & \eta_* > 1 \\ K(\eta_*) &= N_0 & \eta_* < 1, \\ \text{where } K'(\eta_*) &= \frac{\pm 2\pi}{2^{\frac{1}{2}} \int_0^{2\pi} (\eta_* - \cos x)^{\frac{1}{2}} dx} & \text{for } \eta_* > 1. \end{aligned} \quad (4.4)$$

Here  $N_0$  is a constant, and (4.4) expresses the fact that the density is constant inside the closed streamlines  $\eta_* < 1$ , and is continuous across the boundaries, see SB for further details.

After substituting (4.2) into (3.10) the equation for  $\tilde{\Omega}_0$  produces

$$\tilde{\Omega}_0 = JK'(\eta_*) \cos X + G_0(\eta_*), \quad (4.5)$$

where  $G_0$  is an unknown function of  $\eta_*$  but again determined from a viscous secularity condition at higher order. The equation for  $\tilde{\Omega}_1$  is

$$\frac{\partial \tilde{\Omega}_1}{\partial X} = -J \sin X \frac{\partial \tilde{\Pi}_1}{\partial \eta_*} - \frac{\partial}{\partial \eta_*} \left( Z \frac{\partial \tilde{\Omega}_0}{\partial \eta_*} \right),$$

with

$$\begin{aligned} \tilde{\Pi}_1 &= \int_0^X \frac{\partial}{\partial \eta_*} (2^{\frac{1}{2}} (\eta_* - \cos t)^{\frac{1}{2}} K') dt + K_1(\eta_*) & \eta_* > 1, \\ &= N_1 & \eta_* < 1, \end{aligned}$$

where  $N_1$  is a constant and  $K_1$  is an unknown function of  $\eta_*$ . Hence using the above expression for  $\tilde{\Pi}_1$ , and applying a periodicity condition on  $\tilde{\Omega}_1$  gives for  $\eta_* > 1$ ,

$$\begin{aligned} -J \int_0^{2\pi} \sin x \int_0^{x_1} \frac{\partial^2}{\partial \eta_*^2} (2^{\frac{1}{2}} (\eta_* - \cos x_1)^{\frac{1}{2}} K') dx_1 dx \\ + \frac{\partial}{\partial \eta_*} \int_0^{2\pi} 2^{\frac{1}{2}} (\eta_* - \cos x)^{\frac{1}{2}} \frac{\partial \tilde{\Omega}_0}{\partial \eta_*} dx = 0. \end{aligned}$$

After using the boundary conditions as  $\eta_* \rightarrow \infty$  this now gives

$$G'_0 = K' \left[ (1 - J) + J \frac{\int_0^{2\pi} \frac{\cos x}{2(\eta_* - \cos x)^{\frac{1}{2}}} dx}{\int_0^{2\pi} (\eta_* - \cos x)^{\frac{1}{2}} dx} \right]. \quad (4.6)$$

For  $\eta_* < 1$  the approach adopted by SB, Brown-Stewartson (1978), and Goldstein & Hultgren (1988) gives  $G'_0 = 0$  inside the closed streamlines. From (4.6)  $G'_0$  becomes logarithmically singular at the boundaries of the closed streamlines as  $\eta_* \rightarrow 1+$ , and we find that

$$G'_0 \sim -\frac{K'(1)J}{8} \ln\left(\frac{\eta_* - 1}{16}\right) + K'(1)\left(1 - \frac{3J}{2}\right) \quad \text{as } \eta_* \rightarrow 1+.. \quad (4.7)$$

Viscous layers are thus necessary near the boundaries of the cat's eye to resolve the discontinuity in the vorticity gradient. Although the vorticity gradient is discontinuous also near the boundary  $\eta_* = 1$  in the incompressible case, here there is a much stronger discontinuity because of heat transfer. Details of the viscous layer calculation are omitted here, but in summary we find that the phase shift from the viscous layer alone is

$$-\lambda_c \pi [J(\frac{\ln 16}{16} + \frac{1}{3}) - (1 - \frac{3J}{2})]$$

across each layer. The phase shift across the inviscid region, from  $\eta_* = 1$  to  $\eta_* = \infty$ , above the cat's eye is

$$\lambda_c [\frac{C^{(1)}}{2}(1 - 2J) - \pi(1 - \frac{3J}{2}) + C^{(2)}J - \frac{2\pi}{3}J + \frac{\pi}{16}J \ln(16)].$$

The same phase shift occurs below the cat's eye. Here  $C^{(1)}$  is the constant appearing in the incompressible theory, see SB, and

$$C^{(2)} = \frac{2^{\frac{1}{2}}}{2\pi} \int_1^\infty K'^2 \int_0^{2\pi} \frac{\cos x}{(\eta_* - \cos x)^{\frac{1}{2}}} dx d\eta_*.$$

Our calculations give

$$C^{(1)} = -5.516, \quad C^{(2)} = .194 \quad .$$

Thus the total phase shift is found to be given by

$$\phi \sim \lambda_c [C^{(1)}(1 - 2J) + 2C^{(2)}J - 2\pi J] \quad \text{as } \lambda_c \rightarrow 0. \quad (4.8)$$

The above calculation neglects the smaller contributions from the edges of the cat's eyes at  $X = 0, 2\pi$ .

For  $J = 0$  the above asymptote agrees with that in SB. For  $J \neq 0$ , the last term in (4.8) represents a contribution from the viscous layers which does not cancel out, in contrast to the incompressible case.

Comparisons of (4.8) with the full numerical calculations are encouraging and the general trend for different values for  $J$  is correctly predicted, see Fig. 8.

For fixed Mach number and as  $\lambda_c \rightarrow 0$  (3.12),(4.8) shows that the wavespeed and wavenumber increase and we recover exactly the scalings of SB and Bodonyi, Smith & Gajjar (1983) for the strongly nonlinear critical layer problem.

Next for fixed properties of the basic flow, the numerical results show that there are cases when  $\phi \rightarrow \pm 0$  as  $\lambda_c \rightarrow \pm \lambda_{crit}(J)$ , for  $O(1)$  values of  $\lambda_{crit}$ . Then (3.9) together with (3.12) shows that  $A_0 \sim \alpha_0^{-\frac{2}{3}} c_0^{-1}$  and  $c_0 \sim |\phi|^{-\frac{1}{3}}, \alpha_0 \sim |\phi|^{-\frac{1}{3}}$ . Thus formally in the limit when  $c_0$  becomes  $O(\epsilon^{-1})$  the disturbance amplitude ( $\epsilon^{\frac{7}{3}} A_0$ ) falls to become  $O(Re^{-\frac{1}{3}})$ . In this regime the Rayleigh scalings are obtained but with a viscous nonlinear critical layer and with zero phase shift across the critical for the leading order stability problem.

As  $\chi \rightarrow 0$ , for  $\lambda_c = O(1)$ , the situation is different from the linear case. Since  $\chi \rightarrow 0$  implies  $|J| \rightarrow \infty$  for  $R_1/R_0 \neq 0$ , we have that  $\chi\phi \rightarrow R_1\tilde{\phi}/R_0$  to leading order. From (3.13) we need  $R_1/R_0$  positive and so this limit applies for heated walls only. In order to retrieve the Rayleigh scalings of the linear theory the limiting process  $\lambda_c \rightarrow \infty, J \rightarrow \infty$ , but  $J = O(\lambda_c)$  is necessary.

Finally the last case concerns the limit  $M_\infty^2 \rightarrow 1 -$ . Again for fixed  $\lambda_c = O(1)$ , the wavespeed and the wavenumber become  $O(1)$  when  $(1 - M_\infty^2) \sim O(1/Re)$ . But from (3.9) we have  $A \sim \alpha_0^{-\frac{2}{3}} c_0^{-1}$ , so that in the above limit  $\epsilon^{\frac{7}{3}} A$  becomes  $O(1)$ . Thus in the transonic regime, for the nonlinear problem, the disturbance amplitude becomes  $O(1)$  comparable with the basic flow. The earlier comments regarding the three-tiered structure still hold and again this special case deserves further study.

## §5 Summary.

We have shown how the work of SB,SB1 can be extended to compressible flows. In particular the results of SB continue to hold for compressible flows with insulated wall conditions until the Mach number becomes  $O(1)$ , but for Blasius flow with insulated wall conditions the effects of compressibility start coming in when the Mach number reaches an  $O(Re^{-\frac{1}{40}})$  value. In the compressible theory the density and temperature fluctuations play an important role in determining the properties of the nonlinear compressible critical layer and hence the neutral modes.

Several results with possible important practical implications are that the density and temperature disturbances are quite large near the critical layer, and in the nonlinear theory the interaction of the critical layer with the outer flow produces mean flow distortions in both the temperature and density fields. Also with increased disturbance amplitudes the nonlinear critical layer takes on a form similar to that of incompressible flows with open and closed streamlines.

We have seen also how the classical inviscid scalings for the neutral inflexional waves are retrieved as the quantity  $D(\rho_B D U_B)$  goes to zero. The nonlinear disturbance structure for this limit, and also that for the strongly nonlinear critical layer, is currently being investigated. This should provide further new results on the nonlinear aspects of compressible boundary layer stability. In the strongly nonlinear case, as in SB and Bodonyi, Smith & Gajjar (1983), the influence of the transverse pressure gradients across the critical layer is not negligible, and in addition the effects of the varying density and temperature also has to be taken into consideration. How this affects the nonlinear neutral states remains to be seen.

Finally we emphasize that although equilibrium critical layers are a special case, the current work provides the scalings and structures for investigating the possibly more important problem of spatially and temporally growing modes.

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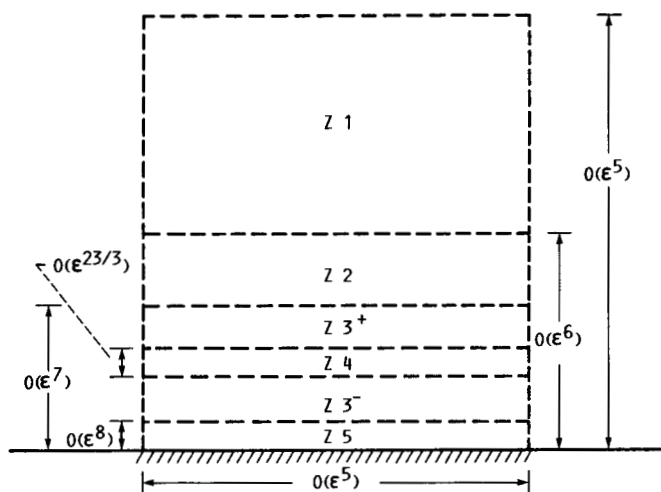


FIGURE 1. - SKETCH OF THE VARIOUS REGIONS FOR GENERAL BOUNDARY LAYER FLOWS. Z1 IS THE OUTER POTENTIAL FLOW REGION, Z2 THE MAIN PART OF OF THE BOUNDARY LAYER, Z3 THE THINNER ADJUSTMENT REGION CONTAINING THE CRITICAL LAYER Z4, AND Z5 IS THE WALL LAYER.

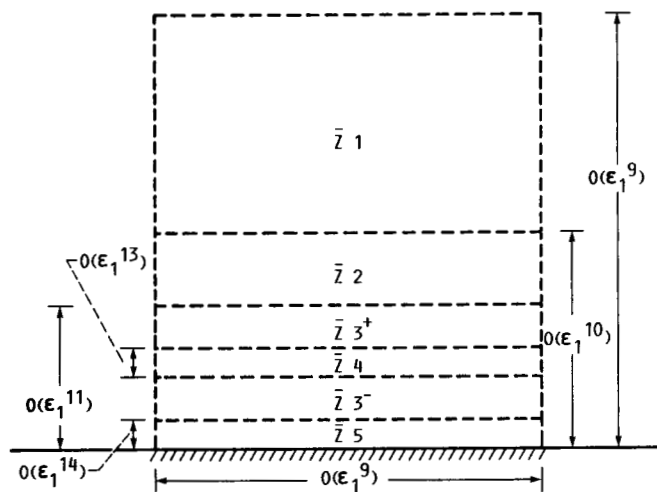
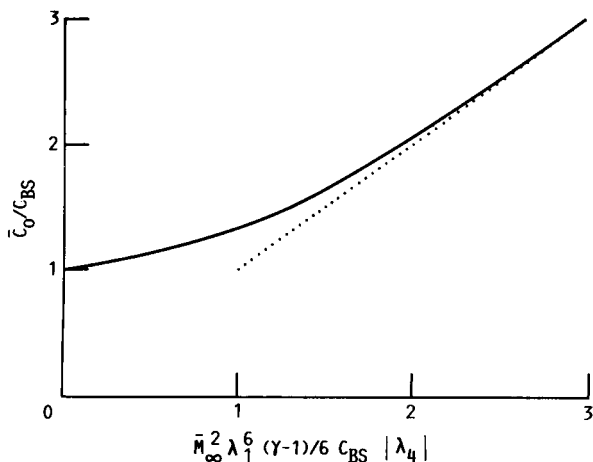
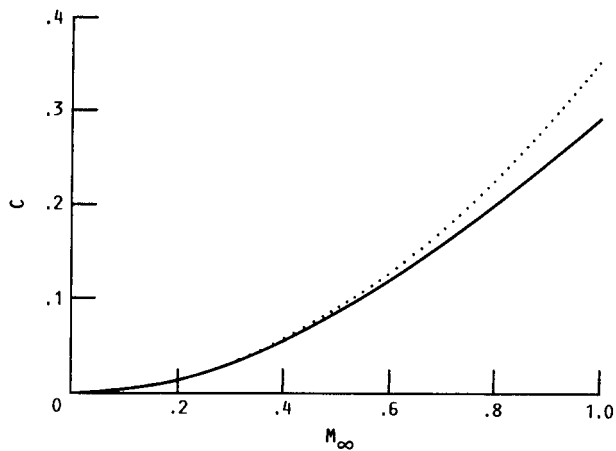


FIGURE 2. - SKETCH OF THE DIFFERENT REGIONS FOR BLASIUS FLOW WITH INSULATED WALL CONDITIONS. Z-bar 1 IS THE OUTER POTENTIAL FLOW REGION, Z-bar 2 THE MAIN PART OF THE BOUNDARY LAYER, Z-bar 3 THE THINNER ADJUSTMENT REGION CONTAINING THE CRITICAL LAYER Z-bar 4, AND Z-bar 5 IS THE WALL LAYER.

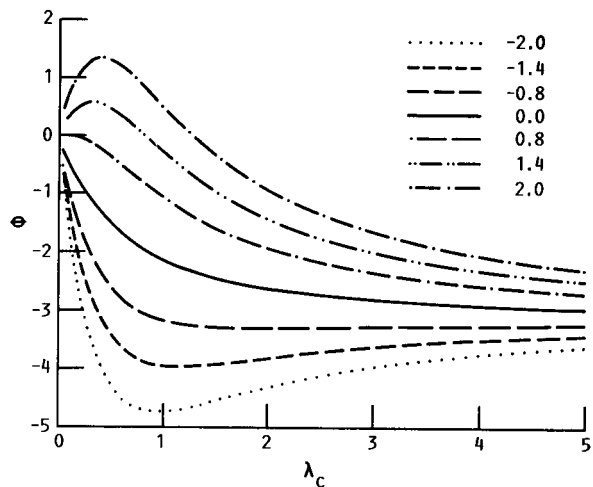


(a) PLOT OF THE NEUTRAL WAVESPEED  $\bar{c}_0$  AS A FUNCTION OF THE SCALED MACH NUMBER  $\bar{M}_\infty$ .  $c_{BS}$  IS THE VALUE OF  $\bar{c}_0$  WHEN  $\bar{M}_\infty = 0$ , i.e. THE SMITH & BODONYI (1980) CASE. THE DOTTED LINE IS THE INVISCID LIMIT FOR INCREASING  $\bar{M}_\infty$ , SEE SECTION (2.19).

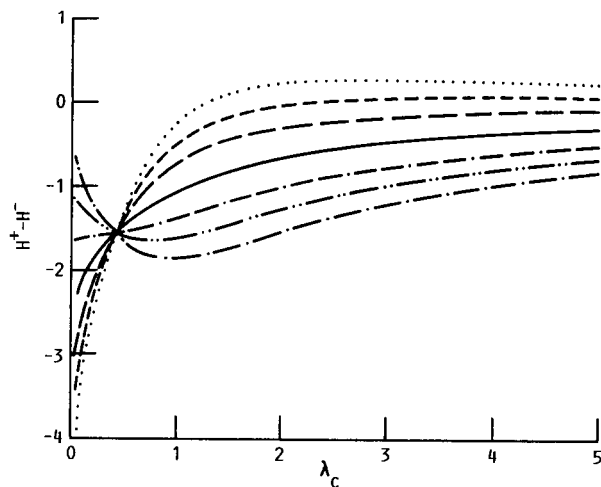


(b) THE SOLID LINE IS THE NEUTRAL PHASE SPEED  $c$  FROM THE GENERALIZED INFLEXION POINT CRITERION AND THE DASHED LINE THE ASYMPTOTE (2.19) REWRITTEN FOR (1) MACH NUMBERS, SEE SECTION 2.

FIGURE 3.

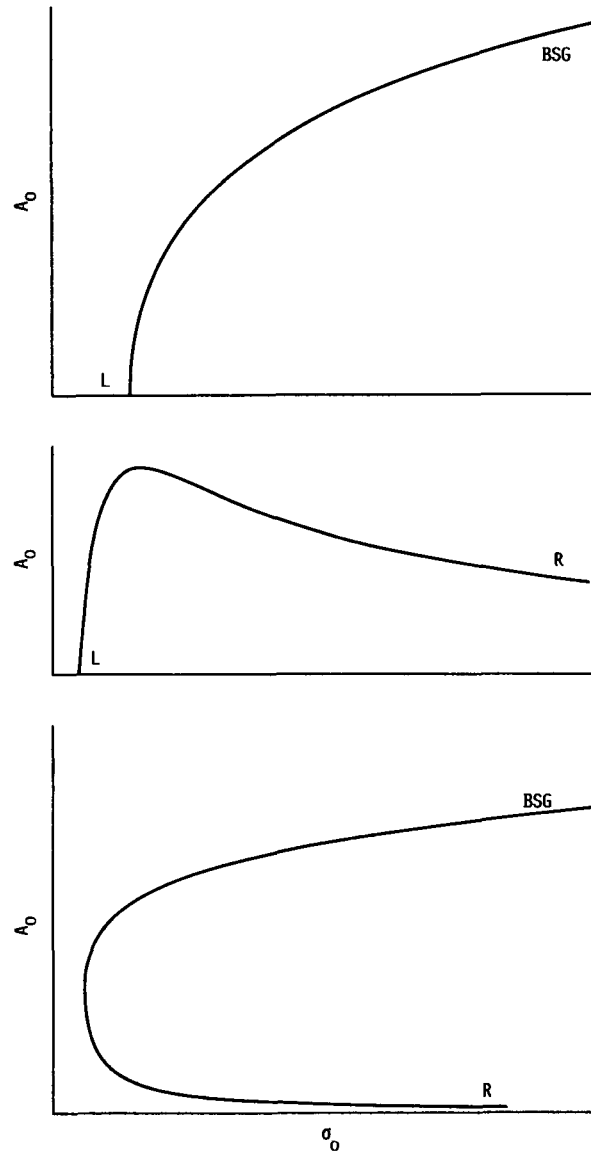


(a) THE COMPUTED PHASE SHIFT  $\Phi$  AS A FUNCTION OF  $\lambda_c$  FOR VARIOUS VALUES OF THE PARAMETER  $J$  IN EQUATION (3.10). THE SOLID CURVE IS THAT FOR  $J = 0$ , i.e. THE HABERMAN (1972) CASE.



(b) THE VORTICITY JUMP  $H^+ - H^-$  AS A FUNCTION OF  $\lambda_c$  FOR THE SAME VALUES OF  $J$  AS IN FIGURE 4.

FIGURE 4.



(a-c) A SKETCH OF THE POSSIBLE NEUTRAL CURVES FOR THE NONLINEAR THEORY.  $A_0$  IS THE NEUTRAL AMPLITUDE,  $\sigma_0$  IS THE NEUTRAL FREQUENCY. L,R,BSG DENOTE THE LINEAR, RAYLEIGH, AND BODONYI, SMITH & GAJJAR (1983) LIMITS AND SCALINGS RESPECTIVELY. (a)  $J < J_c$ , (b)  $J_c < J$  AND  $\lambda_{CRIT} < \lambda_c$ , (c)  $J_c < J$  AND  $0 < \lambda_c < \lambda_{CRIT}$ .

FIGURE 5.

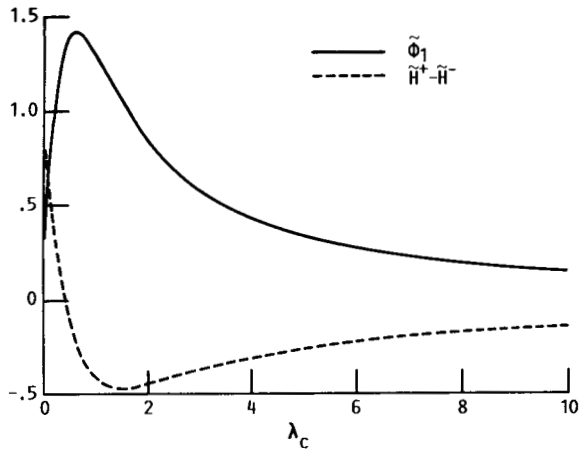


FIGURE 6. - THE PHASE SHIFT  $\tilde{\phi}_1(\lambda_c)$  AND THE VORTICITY JUMP  $(\tilde{H}^+ - \tilde{H}^-)$  FOR THE CASE OF  $|J| \gg 1$ , SEE SECTION 4.

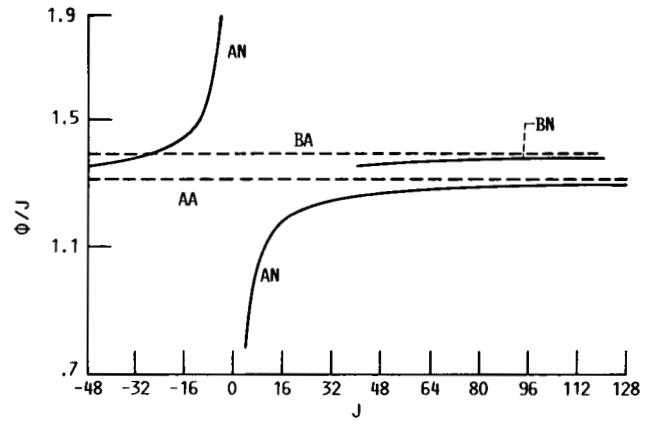


FIGURE 7. - A COMPARISON OF THE PHASE SHIFT FROM THE FULL NUMERICAL SOLUTIONS OF (3.10) WITH THE ASYMPTOTE FOR LARGE  $J$ . AN, BN DENOTE THE NUMERICAL SOLUTIONS FOR  $\lambda_c = 1.0, 0.5$  RESPECTIVELY, AND AA, BA THE CORRESPONDING ASYMPTOTES FOR LARGE  $J$ .

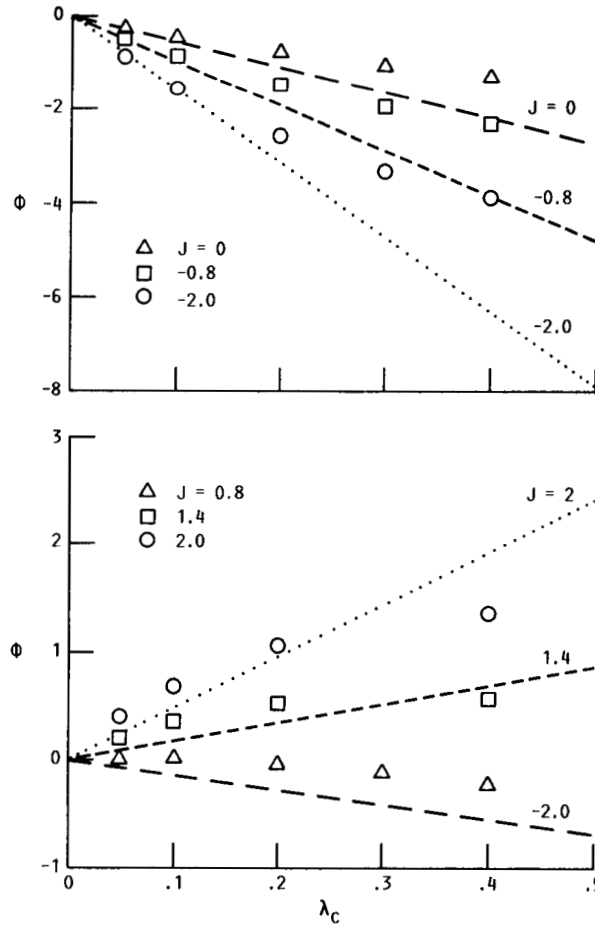


FIGURE 8. - A COMPARISON OF THE CALCULATED PHASE SHIFT WITH THE ASYMPTOTE (4.8) FOR SMALL VALUES OF  $\lambda_c$ , FOR DIFFERENT VALUES OF  $J$ . THE DASHED CURVES ARE THE ASYMPTOTES AND THE SYMBOLS ARE THE PHASE SHIFTS FROM THE NUMERICAL COMPUTATIONS.

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16. Abstract <p>The upper-branch linear and nonlinear stability of compressible boundary layer flows is studied using the approach of Smith &amp; Bodonyi (1982) for a similar incompressible problem. Both pressure gradient boundary layers and Blasius flow are considered with and without heat transfer, and the neutral eigenrelations incorporating compressibility effects are obtained explicitly. The compressible nonlinear viscous critical layer equations are derived and solved numerically and the results indicate some solutions with positive phase shift across the critical layer. Various limiting cases are investigated including the case of much larger disturbance amplitudes and this indicates the structure for the strongly nonlinear critical layer of the Benney-Bergeron (1969) type. Finally we show also how a match with the inviscid neutral inflexional modes arising from the generalized inflexion point criterion, is achieved.</p>					
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