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CONTROL OF THE ERRORS OF DISCRETIZATION AND IDEALIZATION IN FINITE ELEMENT ANALYSIS†

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Introduction.

Our understanding of the basic principles which control errors of discretization in finite element analysis has increased very substantially since 1980. The main milestones were:

1. Development of the theoretical basis of p-extensions (1981);
2. Understanding of the proper interplay between mesh design and assignment of polynomial degree to elements. Practical realization of exponential convergence rates, independently of the smoothness of the exact solution (1984);
3. Industrial experience with the new finite element technology known as the p- or hp-version of the finite element method: General Dynamics reported thirty- to forty-fold savings in terms of human time and large savings in computer time (1986). Lockheed reported favorably on their evaluation of error estimation and quality control capabilities of the p-version in industrial setting (1987). See [1,2].

The gains in our understanding of how to control the errors of discretization represent only half of the control necessary to ensure that a numerical model is in fact an accurate representation of the corresponding physical system. Control of the *errors of idealization* is equally important. In the following a brief overview of the main ideas of how to ensure the quality and reliability of mathematical models of structural systems is presented.

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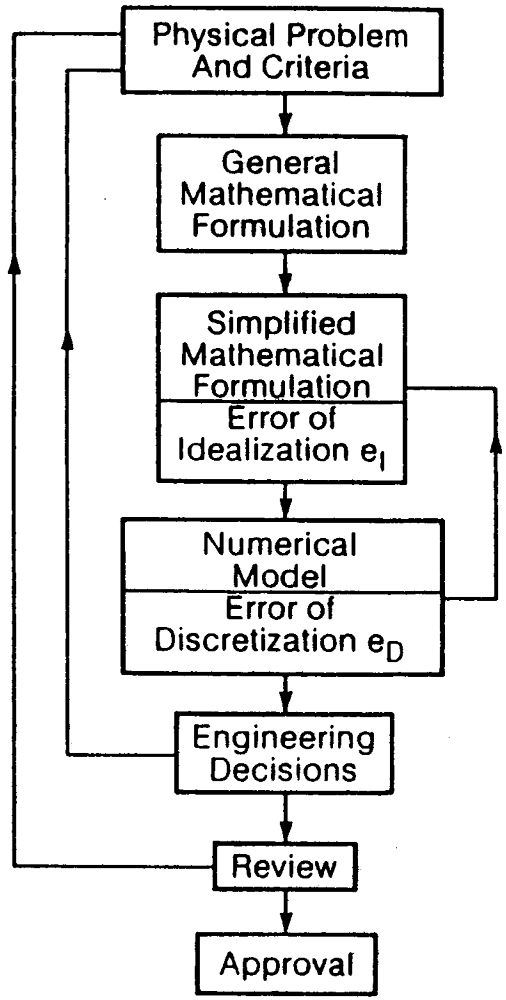
THE ERRORS OF IDEALIZATION AND DISCRETIZATION



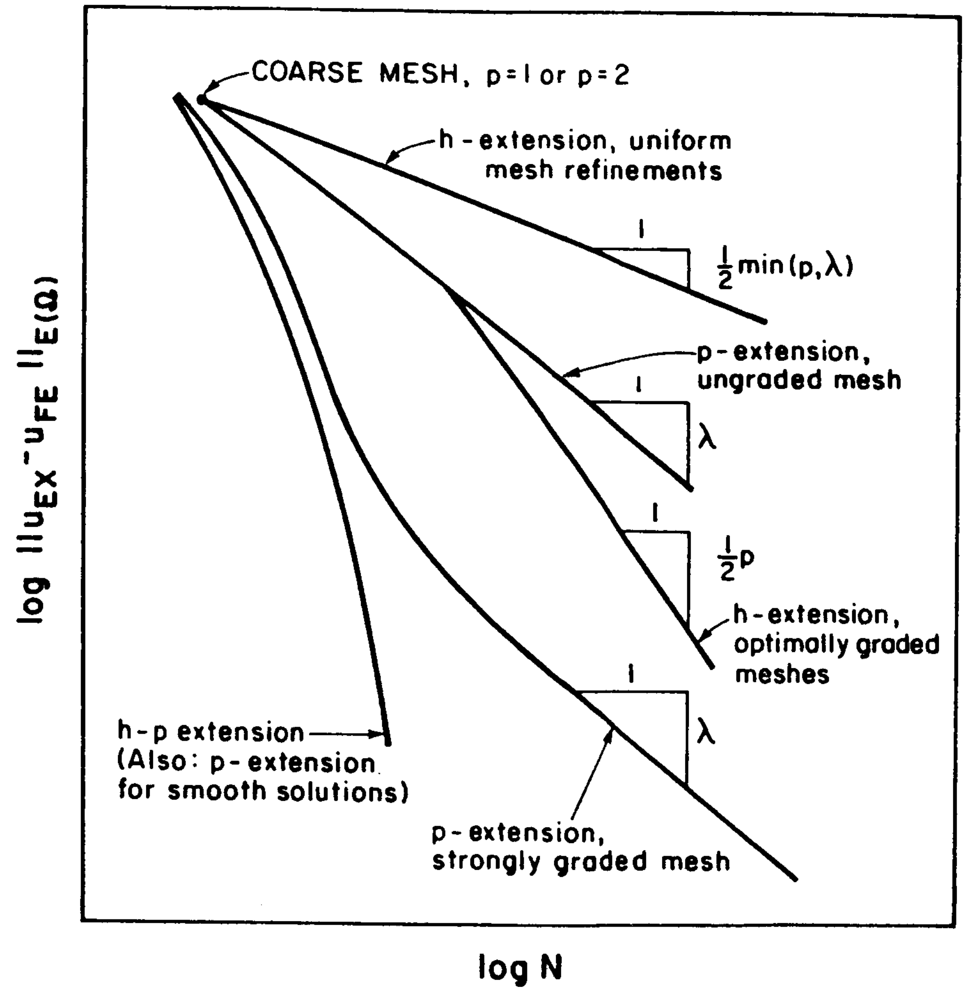
THE ERRORS OF IDEALIZATION AND DISCRETIZATION.

The general mathematical formulation of a physical problem is the collection of all laws of physics which apply to that problem. Unless the physical problem is very simple, it is not feasible to use the general mathematical formulation in engineering decision making processes. Such formulations are not only complicated, but also require knowledge of physical properties which are difficult and expensive to measure. For this reason, a simplified mathematical formulation is adopted, thereby incurring an error of idealization e_I.

To solve the physical problem, a numerical model is employed. The errors incurred in the numerical model are called errors of discretization e_D.

Engineering decisions based on the model can be of good quality only if both e_I and e_D are small. (We assume that roundoff errors are negligible in comparison with e_I and e_D.)

RATES OF CONVERGENCE OF h-, p-, AND hp-EXTENSIONS



RATES OF CONVERGENCE OF h-, p-, AND hp-EXTENSIONS

For a given formulation, such as the displacement formulation, finite element solutions are characterized by the finite element mesh, the polynomial degree of elements, and the mapping functions. We can create a sequence of finite element solutions by systematically refining the mesh or increasing the polynomial degree of elements, or both. The process of creating sequences of finite element solutions is called *extension*. If the extension is by mesh refinement, it is called *h-extension*; if the extension is by increase of the polynomial degree then it is called *p-extension*; if the extension is by combination of mesh refinement and increase of the polynomial degree then it is called *hp-extension*.

Errors of discretization are controlled by extensions. Properly performed extensions permit us to assess and control the quality of finite element solutions. Specifically, the following steps are recommended:

(1) Estimate the relative error in the energy norm. This is equivalent to estimating the *root-mean-square error of stresses*. Therefore estimates of error in energy norm are indicators of the overall quality of the solution. Usually, however, we are interested in quantities which are associated with points and lines in two dimensions; points, lines and surfaces in three dimensions. The error in these quantities can be large even when the error in energy norm is small. Thus additional quality control measures are necessary.

(2) Check equilibrium. It should be possible to draw free body diagrams for the entire structure or any part of the structure to check that equilibrium is satisfied to within a tolerance level which is a small fraction of the total applied load.

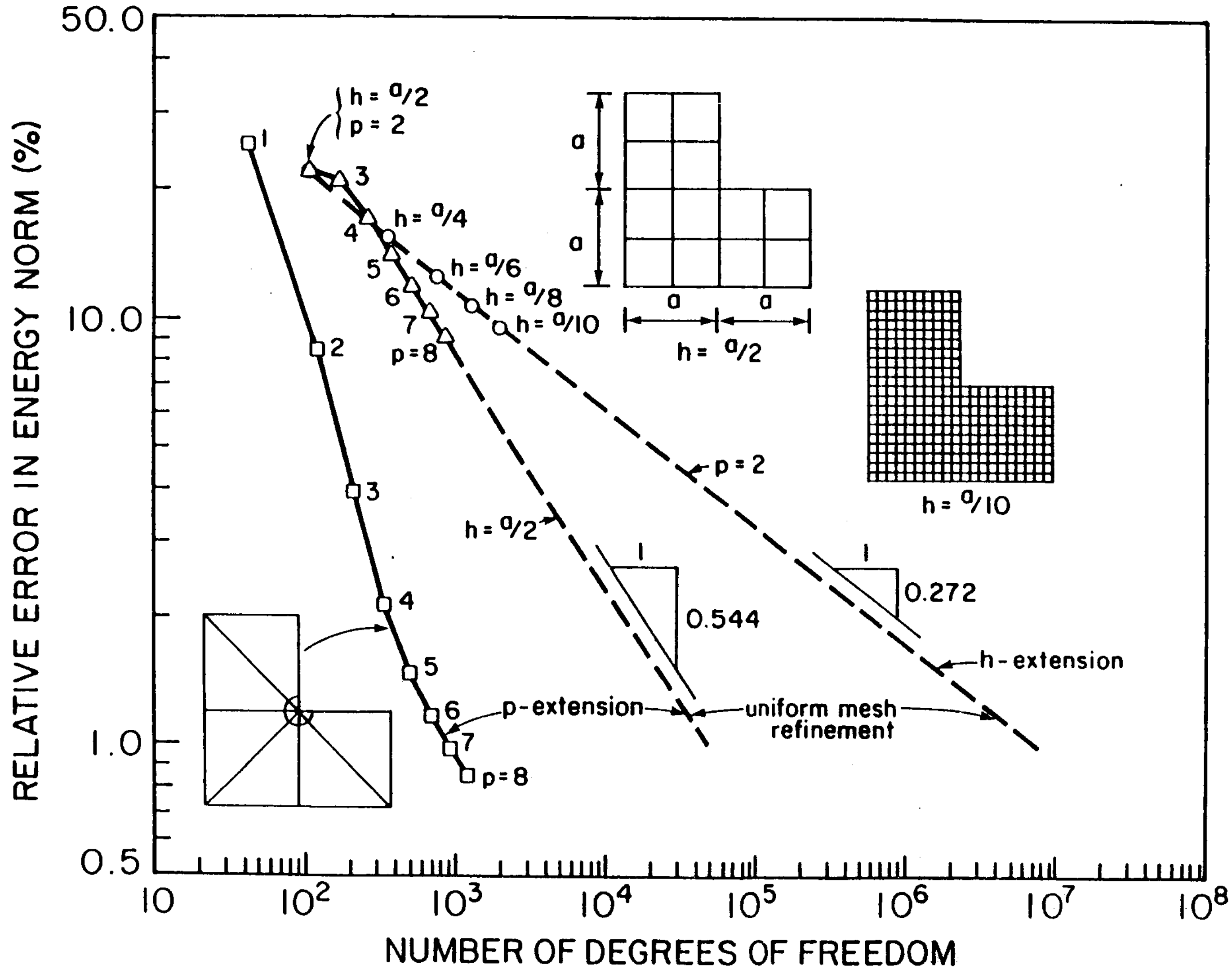
(3) Check whether the action-reaction principle is satisfied to within acceptable levels of tolerance.

(4) Observe convergence of the quantities of interest.

There are very substantial differences between the performance of h-, p-, and hp-extensions. In many cases it is not feasible to reduce the error to reasonable levels unless hp-extensions are used.

When properly refined meshes are used then the performance of p-extensions is very similar to the performance of hp-extensions up to a p-level which is problem dependent.

EXAMPLE: THE L-SHAPED DOMAIN.
CONVERGENCE OF h- AND p-EXTENSIONS.



EXAMPLE: THE L-SHAPED DOMAIN.
CONVERGENCE OF h- AND p-EXTENSIONS.

The model problem of the L-shaped domain is typical of many problems where corner singularities occur. This problem was constructed so that the exact solution is known. Specifically, the imposed loading was by tractions computed from the following stress components:

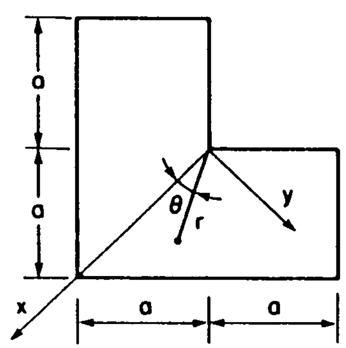
$$\sigma_x = \lambda r^{\lambda-1}\left[(2 - Q(\lambda+1))\cos(\lambda-1)\theta - (\lambda-1)\cos(\lambda-3)\theta\right] \quad (1)$$

$$\sigma_y = \lambda r^{\lambda-1}\left[(2 + Q(\lambda+1))\cos(\lambda-1)\theta + (\lambda-1)\cos(\lambda-3)\theta\right] \quad (2)$$

$$\tau_{xy} = \lambda r^{\lambda-1}\left[(\lambda-1)\sin(\lambda-3)\theta + Q(\lambda+1)\sin(\lambda-1)\theta\right]. \quad (3)$$

where r, θ are polar coordinates centered on the reentrant corner, as shown in the figure below; $\lambda = 0.544483737$ and $Q = 0.543075579$. In the finite element solutions plane strain conditions, and Poisson's ratio of 0.3 were assumed. For additional details see [3]. Note that to achieve 1% relative error in energy norm with h-extension, uniform mesh refinement and $p = 2$, 10^7 degrees of freedom are needed. The same accuracy can be achieved with p-extension on a radical mesh, with 10^3 degrees of freedom.

Assuming that the solution is by Gaussian elimination with optimal ordering of the operations, the work is proportional to $N^{3/2}$ where N is the number of degrees of freedom. Thus, the work ratio is of the order of 10^6, which is about 1 minute to 2 years.



L-SHAPED DOMAIN. p-EXTENSION ON RADICAL MESH.

ESTIMATED AND TRUE RELATIVE ERRORS IN ENERGY NORM.

p	N	$\frac{U(\vec{u}_{FE})E}{\sigma_\infty^2 a^2 t_z}$	$\frac{U^*(\vec{u}_{EX})E}{\sigma_\infty^2 a^2 t_z}$	Est.'d 2β	True 2β	Est.'d $(e_r)_E$ (%)	True $(e_r)_E$ (%)
1	41	3.8860880	–	–	–	25.42	25.42
2	119	4.1248326	–	2.07	2.07	8.45	8.46
3	209	4.1481150	4.1599969	2.74	2.72	3.91	3.93
4	335	4.1526504	4.1543418	2.65	2.59	2.09	2.14
5	497	4.1536354	4.1540572	1.99	1.87	1.42	1.48
6	695	4.1539746	4.1542413	1.51	1.35	1.09	1.17
7	929	4.1541390	4.1543718	1.43	1.21	0.89	0.99
8	1199	4.1542378	4.1544700	1.42	1.12	0.75	0.86
∞	∞	4.1545442	4.1545442	1.09	1.09	0	0

L-SHAPED DOMAIN. p-EXTENSION ON RADICAL MESH. ESTIMATED AND TRUE RELATIVE ERRORS IN ENERGY NORM.

The relative error in energy norm is estimated from three consecutive solutions by solving for the exact strain energy $U_{EX} \stackrel{\text{def}}{=} U(\vec{u}_{EX})$ in the theoretical estimate:

$$|U_{EX} - U_p| \cong \frac{k^2}{N_p^{2\beta}}$$

where U_p is the strain energy computed from the finite element solution corresponding to degree p, N_p is the number of degrees of freedom, k and β are constants, 2β is called the asymptotic rate of convergence of the strain energy.

The estimated relative error in energy norm defined by:

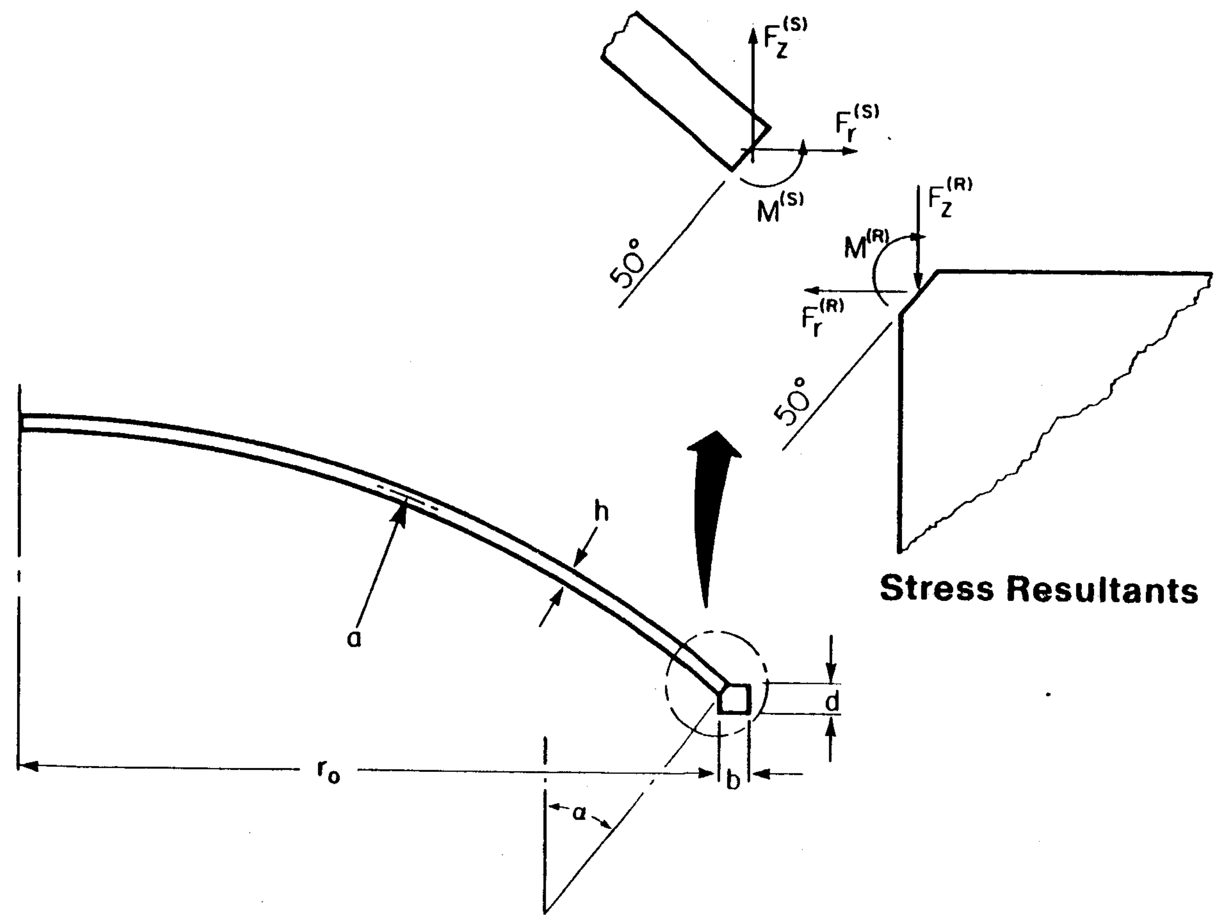
$$(e_r)_E \stackrel{\text{def}}{=} 100\sqrt{\frac{|U_{EX} - U_p|}{U_{EX}}}$$

is reasonably close to the true relative error.

GIRKMANN'S PROBLEM: SPHERICAL SHELL WITH EDGE RING.

$a = 919.2$ in; $h = 2.36$ in; $\alpha = 40°$; $b = 23.64$ in; $d = 19.68$ in

(Not to scale.)



GIRKMANN'S PROBLEM: SPHERICAL SHELL WITH EDGE RING

To demonstrate control of the error of idealization, let us consider the problem of a spherical shell with an edge ring posed by Girkmann in 1956 [4] and quoted by Timoshenko and Woinowsky-Krieger in 1959 [5]. The shell is loaded by gravity which was modeled by Girkmann as uniformly distributed vertical load acting on the middle surface. The goal of computation is to determine the reactions F_r, and M. The following results were obtained by Girkmann:

$$F_r = -8.95 \text{ lbf/in.}, \qquad M = -24.84 \text{ lbf in./in.}$$

He idealized the edge ring as a circular beam, estimated the radial and torsional stiffnesses of this ring and computed the reactions by equating the displacements of the ring and the shell.

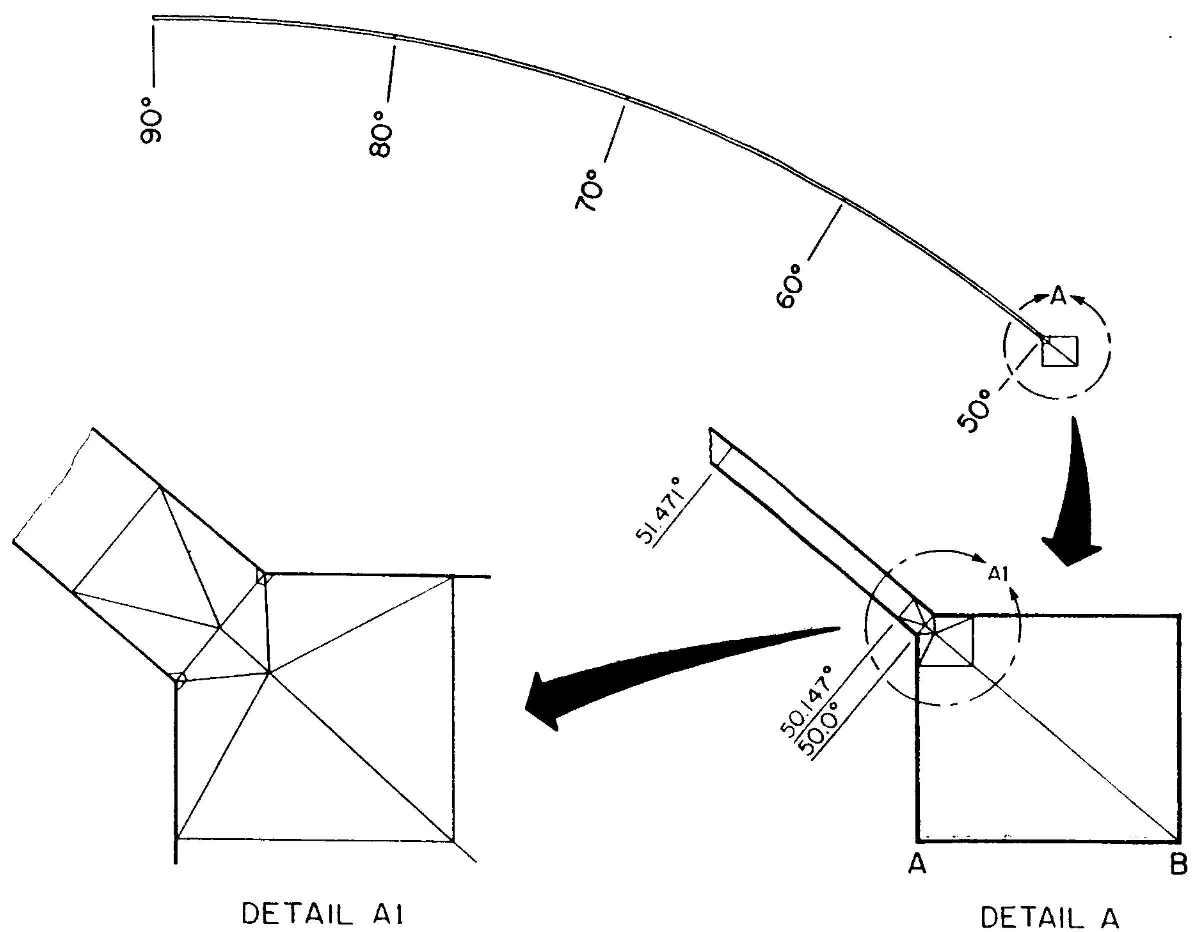
This model problem is representative of an important class of problems in stuctural mechanics: problems which involve stiffened and reinforced plates and shells, and shell intersections.

We have solved this problem as a problem of three dimensional elasticity, taking advantage of the rotational symmetry. We applied half of the uniformly distributed vertical load to the upper surface of the shell, half to the lower surface. We assumed two kinds of support conditions: For the first condition the base of the ring is simply supported. For the second condition uniformly distributed normal tractions are applied to the base of the ring so that the shell is in equilibrium.

The superscript (S) refers to the shell and the superscript (R) refers to the ring.

GIRKMANN'S PROBLEM: SPHERICAL SHELL WITH EDGE RING.

22-element mesh.



Mesh M22

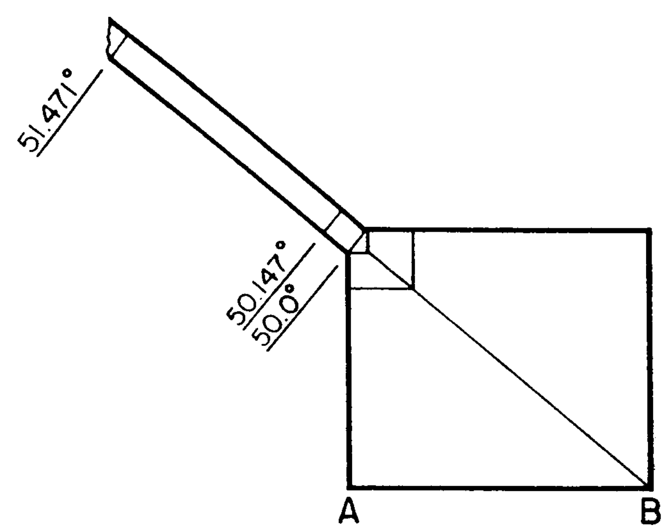
GIRKMANN'S PROBLEM: SPHERICAL SHELL WITH EDGE RING.
22-element mesh.

This figure, drawn to scale, shows the true aspect ratios in Girkmann's problem. The radius to thickness ratio is 389.5. Essentially, the spherical shell is connected to a solid body.

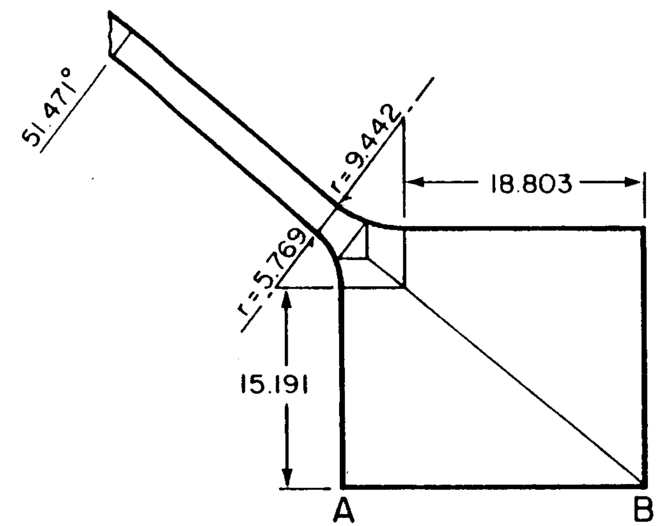
This figure also shows one of the mesh designs, called Mesh M22. Note the similarity in grading with the L-shaped domain.

GIRKMANN'S PROBLEM: SPHERICAL SHELL WITH EDGE RING.

Two 11-element meshes. Detail of the shell-ring junction.



Mesh M11



Mesh M11F

GIRKMANN'S PROBLEM: SPHERICAL SHELL WITH EDGE RING
Two 11-element meshes. Detail of the shell-ring junction.

This figure shows finite element meshes for the shell-ring junction. Both meshes consist of 11 finite elements. In the case of mesh M11, the junction has two sharp reentrant corners. In the case of mesh M11F, the corners have been rounded by fillets as shown. The exact solution is much smoother in the case of mesh M11F than in the case of mesh M11.

GIRKMANN'S PROBLEM: SPHERICAL SHELL WITH EDGE RING.

Estimated relative errors in energy norm $(e_r)_E$ (percent).

p	N	Mesh M11	Mesh M11F	N	Mesh M22
1	40	54.56	54.95	56	55.36
2	101	18.12	18.11	156	18.09
3	164	6.71	6.51	274	6.52
4	249	4.54	4.36	436	4.08
5	356	2.42	2.07	642	1.98
6	485	1.17	0.57	892	0.54
7	636	0.80	0.16	1186	0.11
8	809	0.57	0.05	1524	0.03

GIRKMANN'S PROBLEM: SPHERICAL SHELL WITH EDGE RING

Estimated relative errors in energy norm $(e_r)_E$.

The estimated relative errors in the energy norm are less than one percent for $p \geq 7$ for all three meshes, as shown in the table. If we were to judge the solutions only on the basis of the error in the energy norm, then we would have no reason to reject either sequence of solutions.

STRESS RESULTANTS AT THE SHELL-RING INTERFACE

Mesh M11. Simple support.

p	$F_r^{(S)}$ (lbf/in)	$F_r^{(R)}$ (lbf/in)	$F_z^{(S)}$ (lbf/in)	$F_z^{(R)}$ (lbf/in)	$M^{(S)}$ (in lbf/in)	$M^{(R)}$ (in lbf/in)
2	−84.6	−86.6	100.1	43.0	−234.5	−75.2
3	−92.6	−84.8	95.3	49.5	−235.7	−112.8
4	−92.9	−91.4	95.3	55.8	−234.2	−123.8
5	−93.1	−90.0	95.3	63.5	−230.0	−157.9
6	−93.1	−90.7	95.3	69.6	−229.2	−172.2
7	−93.1	−91.2	95.3	74.3	−229.0	−182.4
8	−93.1	−91.4	95.3	77.5	−229.0	−189.9
∞	(?)	(?)	95.3	95.3	(?)	(?)

STRESS RESULTANTS AT THE SHELL-RING INTERFACE

Mesh M11. Simple support.

The superscripts (S) and (R) refer to whether the resultants were computed for the shell or the ring, respectively.

The value of F_z is known from equilibrium. It is 95.3 lbf/in. Obviously F_z, computed from the finite element solutions, must converge to this value. The values of F_r and M are not known. Therefore the corresponding limit values, indicated by question marks, must be inferred from the sequence of values computed from the finite element solutions.

It is always possible to compute at least some functionals from the finite element solutions, the exact values of which are known, or can be computed from the input data. In this example F_z is such a functional. Such functionals are useful indicators of the quality of similar functionals computed from the finite element solutions.

This table shows that the action-reaction principle is not satisfied at p=8 for the vertical forces ($F_z^{(S)} \neq F_z^{(R)}$) and the moments ($M^{(S)} \neq M^{(R)}$). Nevertheless, very slow convergence is evident. The radial force F_R very nearly satisfies the action-reaction principle. Interestingly, $F_z^{(S)}$ is very accurate.

STRESS RESULTANTS AT THE SHELL-RING INTERFACE

Mesh M11F. Simple support.

p	$F_r^{(S)}$ (lbf/in)	$F_r^{(R)}$ (lbf/in)	$F_z^{(S)}$ (lbf/in)	$F_z^{(R)}$ (lbf/in)	$M^{(S)}$ (in lbf/in)	$M^{(R)}$ (in lbf/in)
2	−114.0	−95.2	47.0	90.2	−253.9	−143.9
3	−91.8	−90.6	97.6	97.3	−249.6	−254.8
4	−91.3	−93.2	96.4	96.0	−249.3	−258.8
5	−92.4	−93.1	95.1	94.7	−244.7	−244.1
6	−92.3	−92.1	95.3	95.6	−243.9	−245.4
7	−92.3	−92.4	95.3	95.1	−243.9	−243.2
8	−92.3	−92.3	95.3	95.3	−243.9	−244.2
∞	(?)	(?)	95.3	95.3	(?)	(?)

STRESS RESULTANTS AT THE SHELL-RING INTERFACE

Mesh M11F. Simple support.

This table shows that introducing fillets in the numerical model has a very large effect on the quality of the solution not only from the point of view of convergence in the energy norm but also from the point of view of satisfying the action-reaction principle. The question arises: Which is the better idealization, the stiffened shell with the rounded fillets or the sharp corners? Clearly, use of the rounded fillets simplifies the analysis considerably.

Note that F_r, F_z and M satisfy the action-reaction principle to three significant digits for $p = 8$. The moments deviate from their average by only 0.06%.

STRESS RESULTANTS AT THE SHELL-RING INTERFACE

Mesh M22. Simple support.

p	$F_r^{(S)}$ (lbf/in)	$F_r^{(R)}$ (lbf/in)	$F_z^{(S)}$ (lbf/in)	$F_z^{(R)}$ (lbf/in)	$M^{(S)}$ (in lbf/in)	$M^{(R)}$ (in lbf/in)
2	−73.3	−94.8	129.3	75.1	−253.2	−214.1
3	−92.9	−90.4	86.8	88.3	−240.6	−218.5
4	−95.5	−91.4	86.6	90.4	−227.9	−221.8
5	−95.5	−92.2	92.6	90.2	−230.9	−221.1
6	−93.7	−92.5	95.9	91.9	−230.2	−222.1
7	−92.8	−92.6	96.0	92.5	−228.8	−223.2
8	−92.9	−92.7	95.4	92.9	−228.5	−224.1
∞	(?)	(?)	95.3	95.3	(?)	(?)

STRESS RESULTANTS AT THE SHELL-RING INTERFACE
Mesh M22. Simple support.

This table shows that isolation of the reentrant corners by rings of small elements significantly improves the quality of the solution judged from the points of view of satisfaction of equilibrium and the action-reaction principle. Nevertheless, addition of the rings is not as effective as introduction of the fillets, even though the number of degress of freedom increased from 809 to 1524 at p=8.

In particular, if we assume once again that the solution is performed by Gaussian elimination, with optimal ordering of the operations, the work ratio in the solution phase is $(1524/809)^{1.5} = 2.6$. The work ratio in the stiffness generation phase is $22/11 = 2$. The work in the input phase, the human time, is generally the costliest operation but also the most difficult one to quantify. Nevertheless, this also favors mesh M11F to mesh M22.

STRESS RESULTANTS AT THE SHELL-RING INTERFACE

Mesh M22. Equilibrium loading.

p	$F_r^{(S)}$ (lbf/in)	$F_r^{(R)}$ (lbf/in)	$F_z^{(S)}$ (lbf/in)	$F_z^{(R)}$ (lbf/in)	$M^{(S)}$ (in lbf/in)	$M^{(R)}$ (in lbf/in)
5	−105.4	−103.7	95.1	93.6	−7.89	−7.02
6	−105.7	−104.0	95.7	93.9	−7.88	−7.14
7	−105.3	−104.2	95.4	94.2	−7.79	−7.24
8	−105.1	−104.4	95.2	94.3	−7.76	−7.32
∞	(?)	(?)	95.3	95.3	(?)	(?)

STRESS RESULTANTS AT THE SHELL-RING INTERFACE

Mesh M22. Equilibrium loading.

When the shell is held in equilibrium by uniformly-distributed normal tractions applied at the base of the ring, the radial force and the moment change very significantly as compared with the simply supported condition. The question arises: Which idealization of the support is more nearly representative of the support of the real shell? Usually, the support has elastic properties. The sensitivity of the stress resultants to support conditions indicates that the elastic properties of the support must be taken into consideration.

GIRKMANN'S PROBLEM

Summary of results.

	Force F_r (lbf)	Moment M (lbf in/in)
Girkmann/Timoshenko :	-8.95	-24.84
Finite element analysis :		
(a) Simple support	-92.8 ± 0.1	-226.3 ± 2.2
(b) Equilibrium loading	-104.8 ± 0.4	-7.54 ± 0.22

GIRKMANN'S PROBLEM
Summary of Results.

This example clearly shows that certain idealizations can lead to very bad results even in the hands of internationally recognized experts. The difficulty is caused by the fact that the stiffness of the ring cannot be computed with sufficient accuracy by the simple methods of idealization based on standard techniques of structural engineering.

We claim only that the results of the finite element analyses are accurate with respect to solutions based on the three dimensional theory of elasticity. The tolerance levels indicated are estimates based on the observed discontinuity of stress resultants.

We have not considered the effects of deformation on equilibrium or the possibility of yielding. Proper control of the errors of idealization would require us to investigate whether all of the assumptions incorporated in the linear theory of elasticity hold in this particular case. Since we modeled the shell-ring junction with sharp corners, we know that at least some local yielding may occur. In view of the fact that the shell is very thin, we expect geometric nonlinearities to have a significant effect on the computed data.

SUMMARY AND CONCLUSIONS.

(1) Errors of discretization are estimated and controlled by extensions.

(2) Errors of idealization are also estimated and controlled by extensions. Theoretical representations of physical systems constitute a natural hierarchic order. We seek the lowest level in this order which accounts for all essential details.

(3) The use of sharp reentrant corners should be avoided. It is generally less expensive to account for fillets than to omit them.

(4) Smallness of error in energy norm does not quarantee that the solution is good. We must also investigate equilibrium, satisfaction of the action-reaction principle and convergence of the quantities of interest.

SUMMARY AND CONCLUSIONS.

(1) Errors of discretization are estimated and controlled by extensions: A *hierarchic sequence of discretizations* is created and the corresponding finite element solutions are computed. The error in energy norm can be estimated accurately. The error in other quantities is estimated by observing convergence to their limiting values. P-extensions and properly designed meshes provide the most efficient and most reliable means for error estimation and quality control procedures. For surveys of the key theoretical results see [6,7]. For additional engineering data see [3,8,9, 10].

(2) Errors of idealization are also estimated and controlled by extensions: The various theories constitute a natural hierarchic order. In structural mechanics, the lowest levels of the hierarchy are the engineering theories of strength of materials. The highest levels are fully three-dimensional representations which account for all geometric details, material and geometric nonlinearities; the mechanical properties of support conditions, etc. In general, we first solve a problem by the simplest available means, much the same way as Girkmann solved the stiffened shell problem. But then we must continue to higher levels of idealization until we can ascertain that no essential details have been overlooked. We have shown that the Girkmann/Timoshenko model is in substantial error with respect to a fully three-dimensional model based on the theory of elasticity. Our solution cannot be accepted as an accurate representation of the structural response of this shell until we estimate the effect of geometric nonlinearities and check whether the stresses exceed the elastic limit. We have seen that the mechanical properties of the support have a large effect on the stress resultants at the shell-ring intersection.

(3) Sharp corners rather than fillets are often used in finite element models because analysts believe that minor topological details do not affect the structural response significantly. While this is usually true, the performance of h- and p-extensions is significantly affected by singularities. We have seen that it is better to avoid the use of sharp reentrant corners than to compensate for the consequent discretization error by mesh refinement. We note that real structures usually do not have sharp corners.

(4) Smallness of error in energy norm does not guarantee that the quality of the finite element solution is good when the purpose of computation is other than determination of the strain energy. In checking the quality of finite element solutions it is essential to investigate equilibrium, satisfaction of the action-reaction principle, and convergence of the quantities of interest.

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