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OPTIMIZATION OF INTERPLANETARY TRAJECTORIES WITH UNPOWERED PLANETARY SWINGBYS*

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Many of the trajectories currently being examined for future planetary missions involve repeated planetary flybys with a high probability that some of these flybys are unpowered. Indeed the current baseline trajectory for Galileo involves an unpowered swingby of Venus followed by two unpowered swingbys of the Earth. This paper presents a method of both calculating and optimizing unpowered planetary swingby trajectories using a patched conic trajectory generator. Several examples of unpowered swingby trajectories are included to demonstrate the capability of solving this type of planetary mission using the formulation to be presented in this paper.

INTRODUCTION

The investigation of future planetary missions involving complex trajectory modes requires software analysis tools that are able to generate these ballistic trajectories quickly and with sufficient accuracy to enable reliable estimates of mission performance to be made. Because of constraints placed upon both spacecraft and launch vehicle capability, it is important to find the best transfer trajectory in terms of some performance parameter for a particular mission. This need translates into a requirement for some form of optimization of the free parameters of the

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trajectory such as launch date, arrival date, time of a planetary swingby, location of an impulse on a powered planetary swingby, and time and position of any deep space maneuvers. The software used to generate these trajectories should be able to automatically determine whether a planetary swingby trajectory should be powered or unpowered and should be able to determine if the addition of an intermediate deep space maneuver would result in an improvement in performance.

For preliminary mission design and feasibility studies a trajectory tool that uses a patched conic trajectory simulation for the heliocentric trajectory segments appears both adequate and desireable. A patched conic trajectory simulation is capable of producing estimates of performance which are adequate for preliminary mission design and feasibility studies. Mission performance calculated using these tools is relatively accurate for both terrestrial and small body missions but suffers somewhat in accuracy for missions to the major outer planets because of the relatively large gravity field associated with these bodies.

Use of a patched conic trajectory simulation enables a rapid generation of heliocentric trajectories using existing software code available for solving Lambert's problem and for generating state transition matrices for the trajectory segments. In addition it is relatively straightforward to generate accurate gradients or partial derivatives required in the optimization process using 'Primer Vector Theory'. This method[1], which depends upon generating a costate vector and its derivatives for each trajectory segment, has been employed in other trajectory optimization programs[2] and is well understood.

It is also possible to draw upon existing analyses to derive the conditions that allow optimization of the impulses that can occur on a powered planetary swingby trajectory. Analysis for powered planetary swingby trajectories was reported in 1973 by Walton[3] and Michael, Marchal and Culp[4] who identified a number of different classes of powered swingby trajectories where the location of the impulse or impulses that would occur on these powered swingby trajectories depended upon both the magnitude and direction of the incoming and outgoing hyperbolic excess velocity vectors. The calculation of the asymptotic values of the costate or primer vector on both the incoming and outgoing asymptotes of the hyperbola were derived by Hyde[5]. The types of unpowered swingby trajectories incorporated in the present patched conic trajectory code include both grazing and non grazing swingby trajectories with an impulse possible on either or both asymptotes, or an impulse at some finite distance from the body.

In order to calculate a particular planetary mission, it is necessary to have an initial estimate of the times of all the individual events occurring during the mission. These events would include launch date, target arrival date, the times of any planetary swingby events, and the times of any deep space maneuvers if known a priori. If a planetary event time is known, the heliocentric position and velocity of the body at that time are calculated from an ephemeris. If a maneuver event time is specified however, an estimate of the heliocentric position of the maneuver must be made available in some way.

The initial problem in a patched conic trajectory optimization is to generate conic trajectories between pairs of heliocentric position vectors. If a pair of position vectors and the transfer time between these is known, the heliocentric conic trajectory segment between these position vectors can be calculated using Lambert's theorem. Once the elements of this conic are known the heliocentric velocity vector at both endpoints of the trajectory segment are easily calculated. At a planetary body the hyperbolic excess velocity of the planet centered hyperbola is found by taking the difference between the heliocentric velocity vectors of the spacecraft and the planet. On an intermediate swingby body there is both an incoming and outgoing hyperbolic excess velocity vector defined that is used as input to the code to calculate a powered swingby maneuver. This code calculates both the optimal velocity increment that must be added by the spacecraft propulsion system and also the asymptotic value of the primer vector both before and after planetary swingby. The velocity increment that the spacecraft propulsion must add for an intermediate deep space maneuver is calculated directly by taking the difference of the heliocentric velocity vectors on either side of the maneuver.

In the above scenario in which a trajectory is defined by either powered planetary swingby trajectories or deep space maneuvers or both, the parameters that are available for optimization would include all the trajectory event times plus the position components of any deep space maneuvers. Applying primer vector theory and using the powered swingby analysis mentioned previously, it is straightforward to calculate the partial derivatives of total mission ΔV with respect to any of the trajectory event times or deep space maneuver position components. In a trajectory simulation the velocity components at the endpoints of each trajectory arc are uniquely determined by the conic trajectory connecting these endpoints and are independent of the components of velocity at the endpoints of the adjacent trajectory arcs.

UNPOWERED PLANETARY SWINGBYS

As a trajectory is being optimized, the magnitude of either a powered swingby maneuver or deep space maneuver may become very small. As a consequence a powered planetary swingby could become an unpowered swingby or a deep space maneuver could simply disappear. In the latter case, when the magnitude of a deep space maneuver becomes very small, the trajectory would be reformulated without that maneuver. However when the maneuver on a planetary swingby disappears the situation is more complex since there is now a constraint between the incoming and outgoing planetary hyperbolic excess velocity vectors in that they must be equal in magnitude. When the above occurs the analysis for powered planetary swingby trajectories does not apply and can not give the correct asymptotic values of the primer vector on the planetary hyperbolic transfer trajectory. If the primer vector is not calculated correctly, the gradients used in the optimization process are in error and the search involved in the optimization will not converge to the correct solution.

If the powered swingby also happens to be a grazing trajectory at the constraint altitude of the swingby body when an impulse disappears, there is an additional constraint imposed on the angle between these two hyperbolic excess velocity vectors. Because of these constraints on the incoming and outgoing velocity vectors, it is apparent that the trajectory arcs on either side of an unpowered swingby body must be calculated in a manner differently from that described above. In effect a degree of freedom is lost for each of the constraints imposed by the unpowered swingby. These constraints must be satisfied by removing one or more free parameters available for optimization.

There are several ways that constraints on an unpowered swingby body can be satisfied. If there is a deep space maneuver either before or after the swingby body, the necessary degrees of freedom may be taken from the position components of the deep space maneuver and used for the swingby maneuver. Since there are three degrees of freedom associated with the position of the deep space maneuver, there results an excess of one or two degrees of freedom for the unpowered planetary maneuver that are available for optimization. These degrees of freedom may be taken as the swingby altitude and impact plane angle of either the incoming or outgoing hyperbola asymptote. If a grazing swingby transfer occurs then the altitude is constrained and could not be included in the variables available for optimization. This formulation has been employed to handle the ΔV-EGA type trajectory modes in which there is an Earth return several years following launch with an accompanying intermediate deep space maneuver. This deep space maneuver occurs around aphelion on the Earth to Earth transfer trajectory and is required to shape the return trajectory to Earth such that an increase in heliocentric energy would result following the ensuing swingby of the Earth.

The above method of handling unpowered swingby trajectories is suitable for optimizing these missions as long as there is a deep space maneuver either before or after the planetary swingby. However many of the more complex trajectory modes presently under investigation are characterized by one or more unpowered planetary swingbys with no associated deep space maneuver. The above method of handling unpowered planetary swingby trajectories fails since there is no deep space maneuver available to absorb the unpowered swingby constraints

The necessary changes in the formulation for unpowered planetary swingby trajectories have been derived by C.L.Yen and the author at JPL. These changes involve taking the degrees of freedom necessary for the unpowered swingby constraints from trajectory event times rather than from the position components of a deep space maneuver. These event times would include the time of the planetary swingby and a time associated with a trajectory event either immediately before or after the swingby. Normally, for a non-grazing unpowered swingby, the degree of freedom would come from the time of the planetary swingby itself and this time could not then be available for optimization. For a grazing unpowered swingby, an additional degree of freedom must come from one of the other event times which also could not be optimized. It is important to note that on a trajectory without deep space maneuvers the number of constraints imposed by unpowered swingbys can never be greater than the number of available event times.

GENERATION OF UNPOWERED SWINGBY TRAJECTORIES

The method used in calculating trajectories containing unpowered planetary swingbys is to first target or adjust each of the event times constrained by an unpowered swingby event so as to satisfy the swingby constraints. This is accomplished by calculating an error vector and partial derivatives of this error vector with respect to each of the constrained event times and then determining a correction to each of these times. This process is repeated until convergence is achieved or is terminated as an error if convergence is not achieved in a certain number of iterations. In the analysis to be presented only a single unpowered swingby event will be considered. Although many missions will have several consecutive unpowered swingby events, both the equations used in targeting and calculation of the primer vector can get quite complex. It is much easier to follow the formulation with only one unpowered swingby event being considered. It is then relatively easy to extend the analysis to more one unpowered swingby events.

In order to start the targeting process, an initial estimate of the constrained event times must be available. The conic trajectory on each trajectory segment that contains an unpowered swingby event is then calculated using a routine for solving Lambert's problem. A state transition matrix is then calculated for each of these trajectory segments. For the purposes of this discussion this state transition matrix is defined as four 3x3 matrices which relate variations in the spacecraft state at the terminal point of a particular trajectory segment to variations in the spacecraft state at the initial point of the trajectory arc. Denoting these matrices by $\mathbf{\Phi}_{rr}$, $\mathbf{\Phi}_{rv}$, $\mathbf{\Phi}_{vr}$, and $\mathbf{\Phi}_{vv}$ the variation of position $\delta\mathbf{X}_2$ and velocity $\delta\mathbf{V}_2$ at the final point on the arc to variations in position $\delta\mathbf{X}_1$ and velocity $\delta\mathbf{V}_1$ at the initial point are

$$\delta\mathbf{X}_2 = \mathbf{\Phi}_{rr}\,\delta\mathbf{X}_1 + \mathbf{\Phi}_{rv}\,\delta\mathbf{V}_1 \tag{1}$$

$$\delta\mathbf{V}_2 = \mathbf{\Phi}_{vr}\,\delta\mathbf{X}_1 + \mathbf{\Phi}_{vv}\,\delta\mathbf{V}_2 \tag{2}$$

where both vectors and matrices are denoted by boldface type in this paper.

Since it is necessary to calculate the effective change in velocity at each of the endpoints as the time of either endpoint is changed, eqs. (1) and (2) must be solved with the constraint that the position of either endpoint is fixed if that endpoint is a deep space maneuver or is constrained to an ephemeris for a planetary body. In this latter case the variation of position with respect to time for a planetary body is simply the velocity of that body. Solving the above equations with the above constraints yields the following two equations relating the variation in initial or final velocity on the trajectory segment to variations in the time t_1 and t_2 at either endpoint.

$$\delta\mathbf{V}_1 = -\,\mathbf{F}\,\mathbf{V}_1\,\delta t_1 - \mathbf{E}\,\mathbf{V}_2\,\delta t_2 \tag{3}$$

$$\delta\mathbf{V}_2 = -\,\mathbf{H}\,\mathbf{V}_1\,\delta t_1 - \mathbf{G}\,\mathbf{V}_2\,\delta t_2 \tag{4}$$

where $\mathbf{V}_1$ and $\mathbf{V}_2$ are taken to be either the hyperbolic excess velocity if the endpoint is a planetary body or the heliocentric velocity is the endpoint is a deep space maneuver.

The 3x3 matrices **E, F, G,** and **H** are functions of the four state transition matrices in eqs. (1) and (2) and are given

$$\mathbf{E} = \Phi_{rv}^{-1} \tag{5}$$

$$\mathbf{F} = -\Phi_{rv}^{-1}\,\Phi_{rr} \tag{6}$$

$$\mathbf{G} = \Phi_{vv}\,\Phi_{rv}^{-1} \tag{7}$$

$$\mathbf{H} = \Phi_{vr} - \Phi_{vv}\,\Phi_{rv}^{-1}\,\Phi_{rr} \tag{8}$$

There are several properties of these matrices that are used later in calculating the primer vector but are not required in the initial targeting of the constrained event times. It can be shown that both matrices **F** and **G** are symmetric and the transpose of either matrix is equal to the original matrix,

$$\mathbf{F}^T = \mathbf{F} \tag{9}$$

$$\mathbf{G}^T = \mathbf{G} \tag{10}$$

In addition matrices **E** and **H** are equal to the negative transpose of each other, that is

$$\mathbf{E}^T = -\mathbf{H} \tag{11}$$

$$\mathbf{H}^T = -\mathbf{E} \tag{12}$$

Considering the variations in eqs. (3) and (4) as partial derivatives, the derivatives of velocity at each endpoint with respect to each of the endpoint event times are given by

$$\frac{\partial \mathbf{V}_1}{\partial t_1} = -\mathbf{F}\,\mathbf{V}_1 \tag{13}$$

$$\frac{\partial \mathbf{V}_1}{\partial t_2} = -\mathbf{E}\,\mathbf{V}_2 \tag{14}$$

$$\frac{\partial \mathbf{V}_2}{\partial t_1} = -\mathbf{H}\,\mathbf{V}_1 \tag{15}$$

$$\frac{\partial \mathbf{V}_2}{\partial t_2} = -\mathbf{G}\,\mathbf{V}_2 \tag{16}$$

On an unpowered planetary swingby trajectory the magnitudes v_{H-} and v_{H+} of the incoming and outgoing hyperbolic excess velocity vectors are equal and the first error vector e_1 can be taken as the difference in the magnitude of these two excess velocity vectors,

$$e_1 = v_{H-} - v_{H+} \tag{17}$$

If the unpowered swingby is a grazing transfer at the constrained altitude, then a second error vector e_2 is necessary to reflect this altitude constraint. Although there are several ways to define this error vector, the method used here is to define a fictitious hyperbolic velocity magnitude, v_H, that would have the same bending angle on a grazing swingby as on the actual trajectory. This excess velocity is given as a function of the bending angle, α, by

$$v_H^2 = v_0^2 \left[\frac{1 - \sin \alpha/2}{\sin \alpha/2} \right] \tag{18}$$

In eq. (18) v_0 is the circular velocity at the grazing altitude and the bending angle is defined as the angle between the incoming and outgoing excess velocity vectors. This second error vector is then taken as

$$\mathbf{e}_2 = \mathbf{v}_{H-} + \mathbf{v}_{H+} - 2\,\mathbf{v}_H \tag{19}$$

When both error vectors are driven to zero, the magnitudes of both incoming and outgoing hyperbolic excess velocity vectors are equal and the closest approach to the planet on the hyperbolic path is at the constraint altitude.

The partial derivatives of the error vectors with respect to the constrained event times are easy to calculate and can be used to drive these error vectors to zero. The derivatives of the error vectors with respect to the excess velocity vectors are first calculated from

$$\frac{\partial \mathbf{e}_1}{\partial \mathbf{V}_{H-}} = \mathbf{S}_1^T \tag{20}$$

$$\frac{\partial \mathbf{e}_1}{\partial \mathbf{V}_{H+}} = -\mathbf{S}_2^T \tag{21}$$

$$\frac{\partial \mathbf{e}_2}{\partial \mathbf{V}_{H-}} = \mathbf{S}_1^T - \frac{v_H}{v_{H-}} \left[\frac{1 + \sin \alpha/2}{\sin \alpha} \right] \mathbf{T}_1^T \tag{22}$$

$$\frac{\partial \mathbf{e}_2}{\partial \mathbf{V}_{H+}} = \mathbf{S}_2^T + \frac{v_H}{v_{H+}} \left[\frac{1 + \sin \alpha/2}{\sin \alpha} \right] \mathbf{T}_2^T \tag{23}$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ are unit vectors in the direction of the incoming and outgoing hyperbolic excess velocity vectors respectively, and $\mathbf{T}_1$ and $\mathbf{T}_2$ are unit vectors normal to and lying in the plane formed by $\mathbf{S}_1$ and $\mathbf{S}_2$ and in the direction of motion. An illustration of this unpowered swingby geometry is shown in figure 1. This figure shows the hyperbolic path of the spacecraft past the planet and indicates the orientation of the above unit vectors.

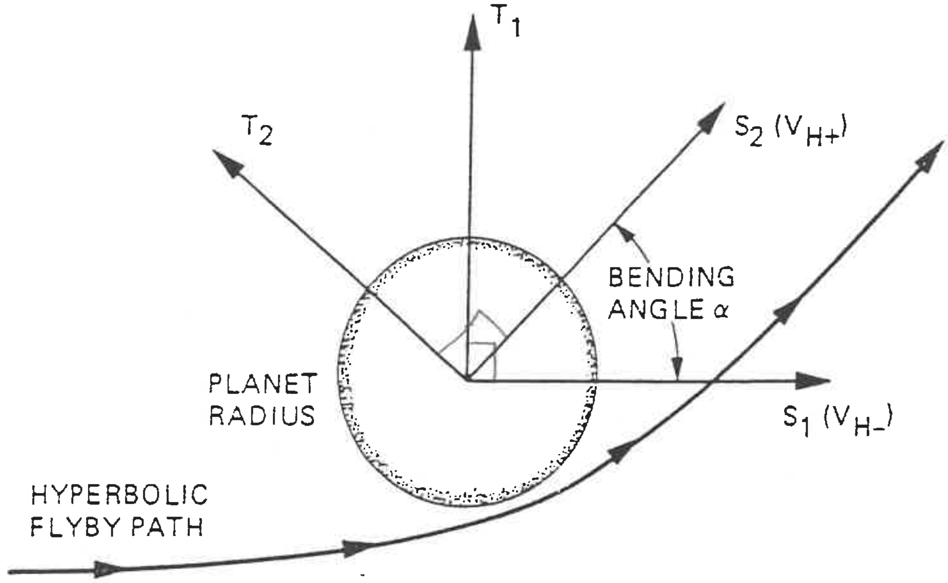

Figure 1. Planetary Flyby Geometry

The variation of either error vector with respect to the endpoint event times is next calculated using the above equations together with the partial derivatives presented in eqs. (13) to (16),

$$\frac{\partial \mathbf{e}}{\partial t_{11}} = \frac{\partial \mathbf{e}}{\partial \mathbf{V}_{H-}} \frac{\partial \mathbf{V}_{21}}{\partial t_{11}} \tag{24}$$

$$\frac{\partial \mathbf{e}}{\partial t_{22}} = \frac{\partial \mathbf{e}}{\partial \mathbf{V}_{H+}} \frac{\partial \mathbf{V}_{12}}{\partial t_{22}} \tag{25}$$

$$\frac{\partial \mathbf{e}}{\partial t_{12}} = \frac{\partial \mathbf{e}}{\partial \mathbf{V}_{H-}} \frac{\partial \mathbf{V}_{21}}{\partial t_{21}} + \frac{\partial \mathbf{e}}{\partial \mathbf{V}_{H+}} \frac{\partial \mathbf{V}_{12}}{\partial t_{12}} \tag{26}$$

In eq. (24) t_{11} represents the event time at the start of the trajectory segment immediately preceding an unpowered swingby event and in eq. (25) t_{22} represents the event time at the end of the trajectory segment following the unpowered swingby event. t_{12} and t_{21} appearing in eq. (26) are equivalent and represent the event time of the unpowered swingby event. This same notation is used to identify the velocity vectors in the above equations where the first subscript is either 1 at the start of the trajectory arc or is 2 at the end of the trajectory arc. The second subscript then identifies the particular trajectory arc being considered. For clarity in presenting this formulation, it is assumed that the unpowered swingby event is at the intersection of trajectory arcs 2 and 3. In general only those partial derivatives corresponding to constrained event times that are actually necessary for targeting need to be calculated. In the advent of more than one unpowered swingby it is straightforward to chain together the trajectory arcs and error vector equations so that all the unpowered flyby events are targeted simultaneously.

It can be quite difficult to actually target and drive the above error vector components to zero, however, if these error vector components are large. In the present software simulation code the initial estimates of the constrained event times are normally available from trajectories in which the impulse on a powered trajectory has gone to zero. As a consequence the initial errors are generally small and targeting is usually successful in a relatively few steps using a simple Newton-Raphson iteration scheme. Targeting is accomplished by solving the set of linear equations for the corrections to the constrained times required to drive the error vector components to zero. Since this is inherently a very non-linear problem, this correction process must be applied iteratively until the correction times are less than some prescribed tolerance.

PRIMER VECTOR CALCULATION

In the preceding section the steps necessary to generate a trajectory containing an unpowered planetary swingby event have been presented. In order to optimize the remaining free or unconstrained variables it is necessary to calculate the primer vector along the trajectory arcs containing unpowered swingbys so that the gradients or partial derivatives with respect to the free variables can be calculated. Calculation of these derivatives are not as straightforward as it might appear since the primer vector must be calculated indirectly for an unpowered swingby event. This indirect calculation requires that the primer vectors and primer vector derivatives must be calculated at the endpoints on the unconstrained trajectory arcs before they can be calculated on the constrained trajectory arcs.

The only information available about the primer vector on an unpowered planetary swingby is the way that the primer vector is propagated through the swingby. Using the methods discussed by Hyde[5] it is possible to derive a relationship between the asymptotic values of the primer vector on the incoming and outgoing asymptotes of the planet centered swingby hyperbola. These relationships involve one unknown constant if the swingby is above the constrained swingby altitude and two unknown parameters if the swingby is on a grazing transfer at the constrained swingby altitude. Without going through the derivations, the incoming primer vector $\mathbf{p}_-$ and outgoing primer vector $\mathbf{p}_+$ are given by

$$\mathbf{p}_- = (A \sin \delta/2 + B \cot \delta/2)\, \mathbf{S}_1 - A \sec \delta/2\, \mathbf{T}_1 \qquad (27)$$

$$\mathbf{p}_+ = \left(-A \sin\tfrac{\alpha}{2}\left[\frac{3 - \sin\frac{\alpha}{2}}{1 + \sin\frac{\alpha}{2}}\right] + B \cot\tfrac{\alpha}{2}\right) \mathbf{S}_2 - A \sec\tfrac{\alpha}{2}\, \mathbf{T}_2 \tag{28}$$

where A and B are the two unknown constants mentioned above and $\mathbf{S}_1, \mathbf{T}_1$ and $\mathbf{S}_2, \mathbf{T}_2$ are unit vectors defined previously in eqs. (20) to (23). The above equations can be simplified and rewritten as,

$$\mathbf{p}_- = a_1 \mathbf{S}_1 + a_2 \mathbf{T}_1 \tag{29}$$

$$\mathbf{p}_+ = a_1 \mathbf{S}_2 + a_2 \left[\mathbf{T}_2 + \frac{2 \sin\alpha}{1 + \sin\frac{\alpha}{2}} \mathbf{S}_2 \right] \tag{30}$$

where a_1 and a_2 are related to the unknown constants A and B by,

$$a_2 = - A \sec\tfrac{\alpha}{2} \tag{31}$$

$$a_1 = A \sin\tfrac{\alpha}{2} + B \cot\tfrac{\alpha}{2} \tag{32}$$

On a non-grazing unpowered swingby both A and a_2 are zero and result in the asymptotic primer vectors being aligned with the incoming and outgoing excess velocity vectors and having a magnitude less than unity. These same conditions in a slightly different form were derived previously by the author[6] and used in a simulation of low-thrust unpowered swingby trajectories.

In order to determine the primer vectors for these trajectories it is necessary to determine values for the unknown parameters a_1 and a_2 appearing in eqs. (31) and (32) above. This requirement dictates a corresponding set of conditions that must be satisfied at the constrained event times. These conditions are found to be equivalent to the transversality conditions required to optimize these points as if they were unconstrained event times. The conditions to be satisfied at an event time dictate that the partial derivatives of the performance parameter be zero with respect to that particular event time. In this analysis this performance parameter is taken to be the sum of the magnitudes of the velocity impulses that must be added by both launch vehicle and spacecraft propulsion systems. This minimization of total trajectory ΔV is a classic definition of a performance index and can be easily extended to mass optimizations of various types. The derivatives of total mission ΔV discussed before are easily derived from the transversality conditions associated with the optimization formulation. The partial derivative with respect to launch date, t_L, for example, is

$$\frac{\partial \Delta V}{\partial t_L} = \mathbf{V}_{HL}^T \dot{\mathbf{p}}_L \tag{33}$$

where $\mathbf{V}_{HL}$ and $\dot{\mathbf{p}}_L$ are values of the hyperbolic excess velocity and primer vectors at launch. The corresponding partial derivative with respect to target arrival date, t_A, is

$$\frac{\partial \Delta V}{\partial t_A} = - \mathbf{V}_{HA}^T \dot{\mathbf{p}}_A \tag{34}$$

with $\mathbf{V}_{HA}$ and $\dot{\mathbf{p}}_A$ the same quantities at arrival. The partial derivatives with respect to either planet swingby time or deep space maneuver time are interior events and contain two terms with the velocity and primer vectors calculated both before and after the event. The partial derivative with respect to a planet swingby time, t_S, is

$$\frac{\partial \Delta V}{\partial t_S} = \mathbf{V}_{H+}^T \dot{\mathbf{p}}_+ - \mathbf{V}_{H-}^T \dot{\mathbf{p}}_- \tag{35}$$

where $\mathbf{V}_{H-}$ and $\mathbf{V}_{H+}$ are taken to be the excess velocity vectors before and after swingby. The corresponding partial derivative with respect to a maneuver time, t_m, is similar and is given by

$$\frac{\partial \Delta V}{\partial t_m} = \mathbf{V}_{m+}^T \dot{\mathbf{p}}_{m+} - \mathbf{V}_{m-}^T \dot{\mathbf{p}}_{m-} \tag{36}$$

where $\mathbf{V}_{m-}$ and $\mathbf{V}_{m+}$ are the heliocentric velocity vectors before and after the deep space maneuver.

On a trajectory arc where the primer vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ are given at both endpoints, the derivative of the primer vector at these endpoints is calculated from

$$\dot{\mathbf{p}}_1 = \mathbf{F}\, \mathbf{p}_1 + \mathbf{E}\, \mathbf{p}_2 \tag{37}$$

and

$$\dot{\mathbf{p}}_2 = \mathbf{H}\, \mathbf{p}_1 + \mathbf{G}\, \mathbf{p}_2 \tag{38}$$

where the 3x3 matrices **E, F, G** and **H** were previously defined in eqs. (5) through (8). Using eqs. (33) through (36) in conjunction with the above two equations, a set of linear equations can be derived in the unknown parameters a_1 and a_2 equal to the number of unknown parameters. These equations are solved for the unknown parameters which will then allow the primer vector to be calculated on both sides of the unpowered swingby trajectory. After these primer vectors are calculated, eqs. (37) and (38) above are used to calculate the primer vector derivatives on the trajectory segments containing an unpowered swingby at one of the endpoints.

These primer vector derivatives are then employed to calculate the remaining gradients of the unconstrained event times.

Upon substituting expressions for the primer derivatives in eqs. (37) and (38) into eqs. (33) through (36) the following equations are derived. Recalling that only a single unpowered swingby is being considered at the common endpoint of trajectory segments 2 and 3, the condition to be satisfied at the initial point on trajectory segment 2 is,

$$\mathbf{V}_{21}^{T}\,\dot{\mathbf{p}}_{21} = \left[\,\mathbf{F}_{2}^{T}\,\mathbf{V}_{12}\,\right]^{T}\mathbf{p}_{12} + \left[\,\mathbf{E}_{2}^{T}\,\mathbf{V}_{12}\,\right]^{T}\mathbf{p}_{-} \qquad (39)$$

where the same notation is used as in eqs. (24) through (26). In the above equation both $\dot{\mathbf{p}}_{21}$ and $\mathbf{p}_{12}$ must have been previously calculated. If the first point is actually the initial launch point, then the term to the left of the equal sign in the above equation does not exist and the terms on the right hand side of the equation are equal to zero.

The corresponding condition to be satisfied at the final point of trajectory segment 3 is given by

$$\left[\,\mathbf{H}_{3}^{T}\,\mathbf{V}_{23}\,\right]^{T}\mathbf{p}_{+} + \left[\,\mathbf{G}_{3}^{T}\,\mathbf{V}_{23}\,\right]^{T}\mathbf{p}_{23} = \mathbf{V}_{14}^{T}\,\dot{\mathbf{p}}_{14} \qquad (40)$$

where both $\mathbf{p}_{-}$ and $\mathbf{p}_{+}$ in the above two equations and in the first equation below were previously defined in eqs. (29) and (30). If this last event time corresponds to arrival, then the term on the right hand side of the above equation does not exist and the terms on the left hand side of the equation are equal to zero. The values of $\mathbf{p}_{23}$ and $\dot{\mathbf{p}}_{14}$ must also be known a priori in order to use the condition in eq. (40). In both eqs. (39) and (40) the events may correspond to any event other than another unpowered planetary swingby.

The condition to be satisfied for the unpowered swingby point is more complicated since it involves the two adjacent trajectory segments. Substituting the expressions for the primer vector derivatives in eqs. (37) and (38) into eq. (35) yields

$$\left[\,\mathbf{H}_{2}^{T}\,\mathbf{V}_{22}\,\right]^{T}\mathbf{p}_{12} + \left[\,\mathbf{G}_{2}^{T}\,\mathbf{V}_{22}\,\right]^{T}\mathbf{p}_{-} = \left[\,\mathbf{F}_{3}^{T}\,\mathbf{V}_{13}\,\right]^{T}\mathbf{p}_{+} + \left[\,\mathbf{E}_{3}^{T}\,\mathbf{V}_{13}\,\right]^{T}\mathbf{p}_{23} \qquad (41)$$

Although the above three expressions appear formidable, they can be simplified considerably by using the matrix relationships presented in eqs. (9) through (12) together with the partial derivatives developed in eqs. (13) through (16). After substitution, the above three equations can be rewritten as

$$\left[\frac{\partial \mathbf{V}_2}{\partial t_1}\right]_2^T \mathbf{p}_- = \left[\frac{\partial \mathbf{V}_1}{\partial t_1}\right]_2^T \mathbf{p}_{12} + \mathbf{V}_{21}^T \dot{\mathbf{p}}_{21} \tag{42}$$

corresponding to eq. (39) for the initial point on the second segment,

$$\left[\frac{\partial \mathbf{V}_1}{\partial t_2}\right]_8^T \mathbf{p}_+ = \left[\frac{\partial \mathbf{V}_2}{\partial t_2}\right]_8^T \mathbf{p}_{28} + \mathbf{V}_{14}^T \dot{\mathbf{p}}_{14} \tag{43}$$

for the final endpoint on segment 3 corresponding to eq. (40), and

$$\left[\frac{\partial \mathbf{V}_2}{\partial t_2}\right]_2^T \mathbf{p}_- - \left[\frac{\partial \mathbf{V}_1}{\partial t_1}\right]_8^T \mathbf{p}_+ = \left[\frac{\partial \mathbf{V}_1}{\partial t_2}\right]_2^T \mathbf{p}_{12} - \left[\frac{\partial \mathbf{V}_2}{\partial t_1}\right]_8^T \mathbf{p}_{23} \tag{44}$$

for the intermediate unpowered swingby point corresponding to eq. (41).

When two or more consecutive unpowered swingby events occur, the above equations become more complicated since either the initial endpoint on segment 2 or the final endpoint on segment 3 may be an unpowered swingby point. If this occurs the expression for the primer vector appearing on the right hand side of equation (44) must be changed to reflect the additional unknown parameters associated with the other unpowered swingby. The derivation of these additional terms are straightforward and are not presented in this paper.

The only equations that actually need to be calculated are those that are required for either the targeting or for calculation of the primer vector. The solution of the above set of linear equations yields values for the unknown parameters a_1 and a_2 which are sufficient for calculating the asymptotic values of the primer vector on the swingby trajectory. Once the primer vector has been calculated, the primer vector derivatives are calculated from eqs. (37) and (38).

The actual implementation of the logic in a software program required to calculate multiple unpowered planetary swingbys can be quite complicated however; particularly so if one or more swingbys have a closest approach at the constrained altitude. The major difficulty arises when the above occurs and involves the selection of additional event times required for the targeting. Since there is no a priori method of identifying which of the available event times is best to be constrained, it may be necessary to make several trial trajectory calculations to determine the most appropriate event times to use.

EXAMPLES OF UNPOWERED FLYBY MISSIONS

Several examples of a CASSINI mission to Saturn having unpowered planetary swingbys have been calculated using the above formulation. The first mission is an 8.5 year ΔV-EGA trajectory launched in 1995 which includes an additional Jupiter gravity assist flyby. An ecliptic projection of this trajectory is shown in figure 2 and some of the pertinent trajectory events and features are given in table I. There is a deep space maneuver on this trajectory around 388 days after launch and this is followed by an unpowered grazing flyby of the Earth at 780 days after launch. 4.5 years after launch there is an unpowered flyby of Jupiter with an unconstrained closest approach to Jupiter of about 23 Jupiter radii.

The bending angle at Earth swingby is 47 degrees and the values of the unknown constants a_1 and a_2 were determined to be 0.211 and 0.027 respectively. The bend angle at Jupiter is nearly 85 degrees and the value of the unknown constant a_1 for this swingby is -0.072, the value of a_2 being zero since the swingby altitude was unconstrained.

Both swingby event times and the time of the deep space maneuver were used to target the unpowered swingbys. The remaining free parameters which were then optimized were the vector components of the deep space maneuver and the launch date. The flight time was constrained to 8.5 years and consequently the arrival date was constrained and not available to be optimized.

The second example is of a CASSINI mission to Saturn launched in 1994 that has an unpowered swingby of Venus, two unpowered swingbys of the Earth, and a final unpowered swingby of Jupiter. All of these planetary flyby trajectories are above the constraint altitude so that only the time of the unpowered swingbys are required for targeting. This second example also has a flight time constraint of 8.5 years and the only free parameter available for optimization is the launch date. An ecliptic projection of this trajectory is shown in figure 3 and some pertinent trajectory parameters are presented in table II.

In this trajectory the Venus flyby occurs about 165 days after launch, this is followed by Earth swingbys at 488 days and 1336 days after launch. The Earth return trajectory is not a resonant return and Earth return actually occurs after a heliocentric transfer angle of 475 degrees. Since this return is non resonant, the Earth to Earth transfer arc must be in the orbital plane of the Earth. The Jupiter flyby occurs about 5.3 years after launch and closest approach to Jupiter is around 54 Jupiter radii. The respective hyperbolic bend angles at each body are 33 degrees at Venus, 25 degrees and 29 degrees at Earth, and 38 degrees at Jupiter. The values of the four unknown parameters used in calculating the primer vectors is chronologically, -0.064, 0.026, - 0.052, and -.041 for swingbys of Venus, Earth and Earth and Jupiter respectively.

An unconstrained quasi-Newton minimization routine[7] was used to optimize these trajectories. This routine has quadratic convergence when near the solution and the final value of the gradients as indicated in tables I and II is less than around 10^{-7} km/s/AU for maneuver position and less than 10^{-9} km/s/day for launch date. The formulation for the gradients was also checked by calculating the derivatives of the constrained swingby event times and the time of the deep space maneuver. In all cases these derivatives were less than 10^{-15} km/s/day. This value, for all intents and purposes, is zero for the double precision formulation used in solving these trajectories.

SUMMARY AND CONCLUSIONS

This paper has presented the formulation for optimizing patched conic trajectories containing unpowered planetary flyby trajectories using primer vector theory. Although this patched conic approximation for heliocentric trajectories is not highly accurate, especially for missions involving any of the major outer planets, it is sufficient to provide relatively reliable projections of spacecraft performance for preliminary mission definition studies. The use of primer vector theory in conjunction with both powered and unpowered swingby approximations results in a powerful mission design and analysis tool.

Although there are more sophisticated trajectory formulations such as the Pseudostate Theory presented by S.Wilson8 and Multi-Conic Theory by Byrnes[9] that model the three body effects more accurately than a patched conic trajectory formulation, these trajectory propagators make it much more difficult to apply such techniques as the maximum principle and primer vector theory to the optimization of the trajectory. Thus, while it is relatively straightforward to include a mixture of both powered and unpowered swingby trajectories and to apply tests to see if the addition of a deep space maneuver would improve mission performance in a patched conic program, it would be much more difficult, but certainly not impossible, to include the same capability in these more accurate trajectory formulations.

The increased emphasis on swingby trajectories of both planetary bodies and satellites in future planetary missions has resulted in the need for additional trajectory analysis tools that are capable of analyzing these complex trajectories. Such tools as those developed by Byrnes, D'Amario, Stanford and Sergeyevsky[10,11,12] are being actively used for the Galileo project and other outer planet mission studies. However, for extensive preliminary mission studies where many different and possibly complex missions must be examined, the use of a patched conic trajectory simulation has advantages in both calculation speed and in adaptability to changes or additions to the formulation.

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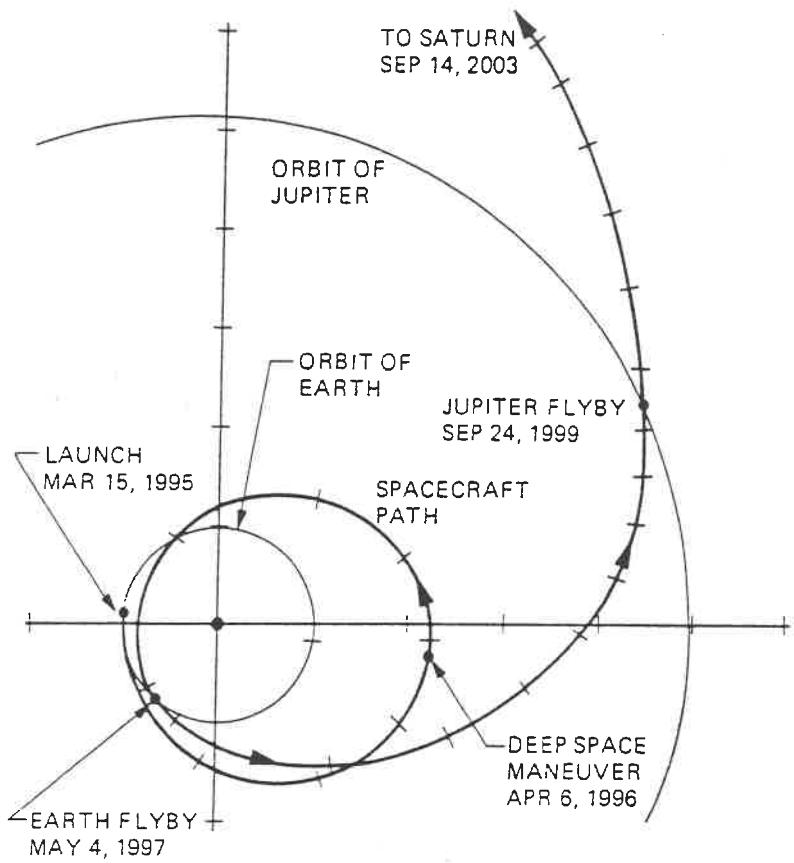

Figure 2. Ecliptic View 1995 ΔV-EJGA Cassini Trajectory

Table I. 1995 Cassini ΔV-EJGA (2+)

Launch C_3	27.68 km^2/s^2
Launch declination	-23.3 deg
Launch right ascension	262.5 deg
Arrival excess velocity, VHP	3.938 km/s
Arrival declination	20.4 deg
Arrival right ascension	57.7 deg
Total flight time	3105 days 8.50 years
Total ΔV	5.468 km/s
Post launch ΔV	1.064 km/s

EVENT	DATE	ΔV km/s	Altitude	$\partial\Delta V/\partial t$ km/s/day
Earth launch	Mar 15, 1995	4.404		$2.2\ 10^{-11}$
Deep space maneuver	Apr 6, 1996	0.595		$2.0\ 10^{-16}$
Earth swingby	May 4, 1997	.000	300 km	$2.0\ 10^{-17}$
Jupiter swingby	Sep 24, 1999	.000	22.794 r_j	$-4.8\ 10^{-13}$
Saturn arrival	Sep 14, 2003	0.468		$-4.3\ 10^{-4}$

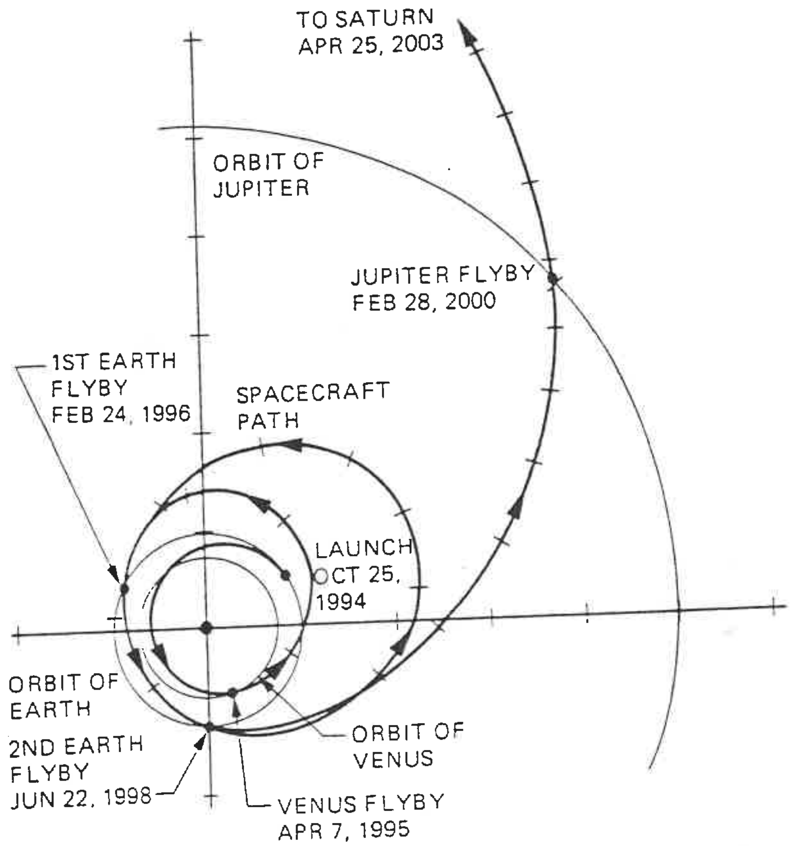

Figure 3. Ecliptic View 1994 VEEJGA Cassini Trajectory

Table II. 1994 Cassini VEEJGA (2,3)

Launch C_3	17.66 km^2/s^2 → 4.202 km/s
Launch declination	-8.8 deg
Launch right ascension	296.8 deg
Arrival excess velocity, VHP	5.194 km/s
Arrival declination	19.9 deg
Arrival right ascension	53.9 deg
Total flight time	3105 days 8.50 years
Total ΔV	4.667 km/s
Post launch ΔV	0.682 km/s

EVENT	DATE	ΔV km/s	Altitude	$\partial\Delta V/\partial t$ km/s/day
Earth launch	Oct 25, 1994	3.985		$-4.2\ 10^{-10}$
Venus swingby	Apr 7, 1995	.000	4785 km	$8.5\ 10^{-17}$
Earth swingby	Feb 24, 1996	.000	4727 km	$2.0\ 10^{-16}$
Earth swingby	Jun 22, 1998	.000	2630 km	$9.3\ 10^{-17}$
Jupiter swingby	Feb 28, 2000	.000	54.589 r_j	$-6.0\ 10^{-18}$
Saturn arrival	Apr 25, 2003	0.682		$-9.4\ 10^{-4}$